On nesting nonhomothetic preferences

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Abstract

We consider nested utility function with nonhomothetic subutility functions. We express demand functions for aggregate goods in terms of marshalian and hicksian demands associated with the standard consumer problem of maximization of aggregate utility function. We also presents a simple method of calibrating such a demand system.

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1 Introduction

This paper considers consumers facing the standard problem of maximizing utility function in the form $u(x) = u(v_1(x_1), \dots, v_k(x_k))$, where $x = \mathrm{col}(x_1, \dots, x_k)$ and x_i is a vector of goods belonging to i-th group of goods and $v_i(x_i)$ is any sub-utility function, possibly nonhomothetic, satisfying standard regularity conditions. Such a problem arise naturally in general equilibrium modeling, e.g. in models with endogenous leisure choice. In such a case x_1 would represent a vector of consumption goods and x_2 a vector of differentiated labor supply. One requires a nonhomothetic subutility v_1 in order to obtain empirically valid consumption demands, since in case of homothetic subutility $v_1(x_1)$ all income elasticities would equal 1.

Weak separation of utility function between goods in different groups allows for considering two stage budgeting problem, where in the first stage consumers split total expenditures between different groups of goods and in the second stage choose optimal consumption of goods in given group subject to total expenditure on given group of goods. However in the nonhomothetic case the first stage problem is not standard, since the budget constraint is not linear. This greatly complicates optimal solution to the first stage budgeting problem. For example if the aggregate utility is the Cobb-Douglas function, then optimal expenditure on given group of goods is no longer constant share of total expenditure.

In order to obtain empirically reasonable demands one requires appropriately flexible utility functions. However a set of available tractable utility functions is very small, actually restricted to the CES utility function. Obtaining a less restrictive demands requires considering demand systems obtained applying the Roy's identity to an indirect utility function or the Sheppard's lemma to an expenditure function. A set of such demand system is very wide and includes among others locally regular Translog or AIDS systems and globally regular[1] CDES and AIDADS systems. Demand systems generated by indirect utility function are especially convenient, since this technique allows for obtaining simple, flexible, globally regular demands[2]. However in such cases we are not able to maximize aggregate utility directly, since the utility function is not expressed in closed form.

To overcome this difficulty we are going to express solution to the first stage problem in terms of solution to the standard utility maximization problem, i.e. maximization of given utility function subject to linear budget constraint. In this way exact function form of a utility function is not required. We also show that elasticity of substitution between aggregate goods and income elasticity can be expressed in terms of income and substitution elasticity associated with the standard optimization problem. This allows for a simple calibration of parameters of the utility function u based on market shares, income and substitution elasticity between aggregate goods.

There is little literature on aggregating nonhomothetic utility functions. To our knowledge only McDougall (2003) considered such a problem an derived solution to the first stage problem in case of Cobb-Douglas aggregate utility and CDES subutility functions. However his method cannot be applied to any more general aggregate utility function.

[1] A demand system is globally regular, if is homogenous and the Slutsky's matrix is symmetric and the negatively semidefinite Slutsky's matrix for any positive prices and incomes. If these conditions are satisfied only at given prices and expenditure, then a demand system is call locally regular.

[2] See Cooper and McLaren (2006).

In section 2 we repeat results in standard consumer theory and introduce notation. In section 3 we solve the first stage problem, elasticity of substitution between aggregate goods and income elasticity are considered in sections 4 and 5. In section 6 we consider calibration of utility parameters. Finally section 7 concludes.

2 Consumer choice

We consider a consumer faced with possible consumption bundles in consumption set $X = \mathbb{R}^K_+$, where $\mathbb{R}^K_+$ is the nonnegative orthant in $\mathbb{R}^K$. The consumer is assumed to have preferences on the consumption bundles in X given by the utility function $u : X \to R$, which is assumed to be a strictly quasi-concave, increasing and differentiable function. Let M be the fixed amount of money available to a consumer, and let $p = \mathrm{col}(p_1, \ldots, p_K)$ be the vector of prices of goods, additionally let $p_i > 0$ for all $i = 1, 2, \ldots, K$. The consumer chooses an affordable consumption bundle $x = \mathrm{col}(x_1, \ldots, x_K) \in X$ that maximizes utility function u:

$$\begin{aligned} &\max_{x\in X} u(x) \\ &s.t. \sum_i p_i x_i \le M \end{aligned} \tag{1}$$

Let the utility u is weakly separable in the subvectors $\{x_1, \ldots, x_k\}$, where $x_i = \mathrm{col}(x_{i1}, \ldots, x_{in_i}) \in \mathbb{R}^{n_i}_+ \doteq X_i$, $i = 1, 2, \ldots, k$, and $n_1 + \cdots + n_k = K$, that is

$$u(x) = v(v_1(x_1), \ldots, v_k(x_k)) \tag{2}$$

for some functions $v_i : \mathbb{R}^{n_i}_+ \to R$, $i = 1, \ldots, k$. Subutility functions v_i, $i = 1, \ldots, k$, and aggregate utility function, v are assumed to be strictly quasi-concave, increasing and differentiable functions. Then also the utility function u posses these properties. Similarly, let $p_i = \mathrm{col}(p_{i1}, \ldots, p_{in_i})$ is a vector of prices of goods in k-th group and let $p = \mathrm{col}(p_1, \ldots, p_k)$ is a vector of prices of all goods.

Let x^* is a solution to the problem (1) with weakly separable utility function (2). Let M_i^* is an optimal expenditure on goods in group i,

$$M_i^* = \sum_j p_{ij} x_{ij}^*$$

Then x_i^* solves also

$$\max_{x_i \in X_i} v_i(x_{i1}, \ldots, x_{ik_i})$$
$$s.t. \sum_j x_{ij} p_{ij} \le M_i^* \tag{3}$$

Since utility function u is strictly quasi-concave, consumer preferences are strictly convex, and solution to (1) is unique. Similarly, solution to (3) given subgroup expenditure level, M_i^*, is also unique. Hence, given optimal expenditure on goods in group i we can find optimal consumption of goods in i-th group maximizing subutility function subject to the standard budget constraint.

Let $x_i^m(p_i, M_i^*)$ is an optimal consumption bundle in i-th group given prices of goods, p_i, and total expenditure on goods in i-th group, M_i^*. Let $w_i(p_i, M_i^*)$ is an indirect utility function associated with the consumer choice problem (3), i.e.

$$w_i(p_i, M_i^*) = v_i(x_{i1}^*(p_i, M_i^*), \ldots, x_{in_i}^*(p_i, M_i^*))$$

and let $e_i(p_i, u)$ is an expenditure function associated with (3), that is minimum cost of achieving a fixed level of subutility

$$e_i(p_i, u) = \min_{x_i \in X_i} \sum_j x_{ij} p_{ij} \qquad\qquad s.t.\ v_i(x_i) \ge u$$

Now we can write the overall maximization problem of the consumer (1) as

$$\max_{v_1, \ldots, v_k} v(v_1, \ldots, v_k)$$
$$s.t. \sum_i e_i(p_i, v_i) \le M \tag{4}$$

where v_i is level of subutility from consumption goods in i-th group. Let v_i^*, $i = 1, \ldots, k$ solves (4). Then $M_i^* = e(p_i, v_i^*)$. Since there exists exactly one solution to (1), solution to (4) is unique.

3 The first stage problem

Let $Y_i = \{v_i(x_i), x_i \in X_i\}$. Let $f_i : Y_i \to \mathbb{R}_+$ is any strictly increasing, differentiable function, such that $f_i(Y_i) = \mathbb{R}_+$. Then there exist strictly increasing, differentiable functions $g_i(x)$ such that $g_i(f_i(x)) = x$ and we can write the problem (4) as

$$\begin{aligned} \max_{\tilde{x}_i \in \mathbb{R}_+} \; & \tilde{v}(\tilde{x}_1, \ldots, \tilde{x}_k) \\ s.t. \sum_i & e_i(p_i, g_i(\tilde{x}_i)) \leq M \end{aligned} \tag{5}$$

where $\tilde{x}_i = f_i(v_i)$ and

$$\tilde{v}(\tilde{x}_1, \ldots, \tilde{x}_k) \doteq v(g_1(\tilde{x}_1), \ldots, g_k(\tilde{x}_k)) : \mathbb{R}_+^k \to R$$

Observe that the function $\tilde{v}$ is strictly quasi-concave, increasing, and differentiable, since $\tilde{v}$ is a composition of increasing, strictly quasi-concave, differentiable function v and strictly increasing, differentiable functions g_i.

Proposition 3.1. *For any strictly quasi-concave, continuous, increasing function $\tilde{v} : \mathbb{R}_+^k \to R$, and for any strictly increasing, continuous functions g_i, $i = 1, \ldots, k$, there exists exactly one solution to (5) for any $M > 0$, $p_i > 0$, $i = 1, \ldots, k$. Additionally the budget constraint is satisfied with equality.*

We can interpret $\tilde{x}_i$ as an aggregate good index and define marshalian demand functions $x_i^m(p, M)$ which gives optimal consumption of aggregate good $\tilde{x}_i$ at given prices p and total expenditure M. Observe that $x_i^m(p, M)$ is a homogenous of degree zero function in p and M, since all expenditure functions $e_i(p_i, g_i(x_i))$ are homogenous of degree 1 in prices. Let us define aggregate price index, $\tilde{p}_i$, of i-th type aggregate good as

$$\tilde{p}_i = \frac{e_i(p_i, g_i(\tilde{x}_i))}{\tilde{x}_i}$$

Generally, the price index $\tilde{p}_i$ depends not only on i-th group prices, p_i, but also depends on level of consumption of aggregate good $\tilde{x}_i$. In case of homothetic subutility function v_i we can take $f_i(v_i) = e_i(p_i^0, v_i)$, which is a strictly increasing function, and $g_i(x_i) = w_i(p_i^0, x_i)$, where $p_i^0 > 0$ is arbitrary base price vector. Then $g_i(f_i(v_i)) = v_i$ and

$$\tilde{p}_i = \frac{e_i(p_i, v_i)}{e_i(p_i^0, v_i)} = \frac{e_i^1(p_i) \times e_i^2(v_i)}{e_i^1(p_i^0) \times e_i^2(v_i)} = \frac{e_i^1(p_i)}{e_i^1(p_i^0)}$$

since in case of homothetic utility function v_i, there exists functions e_i^1, e_i^2, such that $e_i(p_i, v_i) = e_i^1(p_i) \times e_i^2(v_i)$. However, in case of non-homothetic subutility function v_i, by the Gorman's theorem, there does not exists any variable transformation given by the function f_i, such that price index $\tilde{p}_i$ is a function of i-th group of goods prices, p_i.

Let us consider a standard consumer's problem

$$\begin{aligned} &\max_{\tilde{x}\in\mathbb{R}^k_+} \tilde{v}(\tilde{x}_1,\ldots,\tilde{x}_k) \\ &s.t. \sum_i \tilde{p}_i\tilde{x}_i \leq M \end{aligned} \tag{6}$$

where utility function $\tilde{v}$ satisfies conditions from proposition 3.1. Then, for any $\tilde{p} > 0$, $M > 0$, there exists exactly one solution to (6). Let $x_i^{ms}(\tilde{p}, M)$, $x_i^{hs}(\tilde{p}, u)$, $w^s(\tilde{p}, M)$, $e^s(\tilde{p}, u)$ are respectively a marshalian demand function, a hicksian demand function, an indirect utility, and an expenditure function associated with the problem (6).

We are going to express solution to (5) in terms of solution to (6). We have

Theorem 3.2. *Let the utility function $\tilde{v}(\tilde{x})$ is a $\mathcal{C}^2$, differentiably strictly quasi-concave, differentiably strictly increasing function. Let the indirect utility function w^s is differentiably strictly quasi-convex. Let for $i = 1,\ldots,k$, $e_i(p_i, g_i(\tilde{x}_i))$ is differentiably increasing function. Let $x_i^{ms}(p, M) > 0$ is a marshalian demand at prices p and total expenditure M and let $x_i^{hs}(p, u)$ denotes hicksian demand at prices p and aggregate utility level u, associated with the standard consumer's problem (6). Then solution to the problem (5) is given implicitly by*

$$\tilde{x}_i^* = x_i^{ms}(\hat{p}, \hat{M}) \qquad\qquad \tilde{x}_i^* = x_i^{hs}(\hat{p}, u) \tag{7}$$

where $\hat{p} = \mathrm{col}(\hat{p}_1,\ldots,\hat{p}_k)$, and

$$\hat{p}_i = \frac{\partial e_i(p_i, g_i(\tilde{x}_i^*))}{\partial \tilde{x}_i} \qquad\qquad \hat{M} = M + \sum_i \tilde{x}_i^* \times (\hat{p}_i - \tilde{p}_i)$$

Additionally there exists exactly one solution to (7) for any prices $p > 0$, total income M and utility level u.

If all subutility functions are homothetic, and $f_i(v_i) = e_i(p_i^0, v_i)$, then $\hat{p}_i = \tilde{p}_i$ and demand system (7) is reduced to the standard demand.

We assume that the we know demands associated with the standard maximization problem for a utility function $v(g_1(\tilde{x}_1,\ldots,g_k(\tilde{x}_k)$, not with the utility function $v(\tilde{v}_1,\ldots,\tilde{v}_k)$. One can easily derive solution to the problem (5) if standard demands associated with the utility function $v(\tilde{v}_1,\ldots,\tilde{v}_k)$ are known. However in this case elasticity of substitution between aggregate goods cannot be expressed as a simple function of elasticity of substitution associated with the standard demands.

4 Elasticity of substitution

The partial (Allen-Uzawa) elasticity of substitution of the i'th and the j'th aggregate good, $i, j = 1, \ldots, k$, at given consumption bundle $\tilde{x}$ is

$$\sigma_{ij}^{AES}(\tilde{x}) = \frac{\sum_k \tilde{v}_i(\tilde{x})\tilde{x}_i}{\tilde{x}_i\tilde{x}_j} \times \frac{|B_{ij}(\tilde{x})|}{|B(\tilde{x})|}$$

where $|B(\tilde{x})|$ is the determinant of the bordered hessian matrix

$$B(\tilde{x}) = \begin{bmatrix} 0 & \tilde{v}_1(\tilde{x}) & \cdots & \tilde{v}_k(\tilde{x}) \\ \tilde{v}_1(\tilde{x}) & \tilde{v}_{11}(\tilde{x}) & \cdots & \tilde{v}_{1k}(\tilde{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{v}_k(\tilde{x}) & \tilde{v}_{k1}(\tilde{x}) & \cdots & \tilde{v}_{kk}(\tilde{x}) \end{bmatrix}$$

and $|B_{ij}(\tilde{x})|$ is the determinant of the cofactor of $\tilde{v}_{ij}(\tilde{x})$. Let $\tilde{x}^*$ is an optimal consumption bundle at prices p and total expenditure M. Let u is an aggregate utility level u obtained from consumption of $\tilde{x}$. From the proposition (3.2) we have $\tilde{x}^* = x^{hs}(\hat{p}, u)$, hence $\tilde{x}$ solves the standard optimization problem (6) at prices $\hat{p}$ and utility level u. Using the fact that for the standard optimization problem the Allen-Uzawa elasticity can be expressed in terms of expenditure function (Uzawa (1962))

$$\frac{\sum_k \tilde{v}_i(\tilde{x}^*)\tilde{x}_i^*}{\tilde{x}_i^*\tilde{x}_j^*} \times \frac{|B_{ij}(\tilde{x}^*)|}{|B(\tilde{x}^*)|} = \frac{e_{ij}^s(\hat{p}, u)e^s(\hat{p}, u)}{e_i^s(\hat{p}, u)e_j^s(\hat{p}, u)}$$

and applying the Sheppard's identity for the standard optimization problem we have

$$\sigma_{ij}^{AES}(p, M) = \frac{\partial x^{sh}(\hat{p}, u)}{\partial \hat{p}} \times \frac{\hat{M}}{\tilde{x}_i^*\tilde{x}_j^*} = \sigma_{ij}^{AES,s}(\hat{p}, \hat{M}) \tag{8}$$

where $\sigma_{ij}^{AES,s}(\hat{p}, \hat{M})$ is the Allen-Uzawa elasticity of substitution for the standard optimization problem evaluated at prices $\hat{p}$ and total expenditure $\hat{M}$.

Similarly, the Morishima elasticity of substitution, σ_{ij}^{MES} at given consumption prices p and total expenditure, M is given by

$$\begin{aligned} \sigma_{ij}^{MES}(p, M) &= \frac{\tilde{v}_j(\tilde{x}^*)}{\tilde{x}_i^*} \times \frac{|B_{ij}(\tilde{x}^*)|}{|B(\tilde{x}^*)|} - \frac{\tilde{v}_j(\tilde{x^*})}{\tilde{x}_j^*} \times \frac{|B_{ij}(\tilde{x^*})|}{|B(\tilde{x}^*)|} \\ &= \frac{\tilde{v}_j(\tilde{x}^*)\tilde{x}_j^*}{\tilde{v}_i(\tilde{x}^*)\tilde{x}_i^*} \times (\sigma_{ij}^{AES}(\tilde{x}^*) - \sigma_{jj}^{AES}(\tilde{x}^*)) = \sigma_{ij}^{MES,s}(\hat{p}, \hat{M}) \end{aligned}$$

where $\sigma_{ij}^{MES,s}(\hat{p}, \hat{M})$ is the Morishima elasticity of substitution for the standard optimization problem evaluated at prices $\hat{p}$ and total expenditure $\hat{M}$.

5 Income elasticity

Let $\tilde{x} = x^m(p, M)$ is a solution to (5) at prices p and total income M. Differentiating with respect to M yields

$$\eta_i(p, M) \doteq \frac{\partial x_i^m(p, M)}{\partial M} \times \frac{M}{\tilde{x}_i} = \sum_j \varepsilon_{ij}^s(\hat{p}, \hat{M}) \times \hat{\zeta}_j(p, M) + \eta_i^s(\hat{p}, \hat{M}) \times \hat{\theta}(p, M)$$

where $\varepsilon_{ij}^s(\hat{p}, \hat{M})$ and $\eta_i^s(\hat{p}, \hat{M})$ are respectively marshalian price elasticity and income elasticity for the demand associated with the standard consumer problem (6) at prices $\hat{p}$ and total income $\hat{M}$ and

$$\hat{\zeta}_j = \frac{\partial \hat{p}_j}{\partial M} \times \frac{M}{\hat{p}_j} \qquad \tilde{\zeta}_j = \frac{\partial \tilde{p}_j}{\partial M} \times \frac{M}{\tilde{p}_j} \qquad \hat{\theta} = \frac{\partial \hat{M}}{\partial M} \times \frac{M}{\hat{M}}$$

Differentiating $\tilde{p}$ and $\hat{M}$ with respect to M yields

$$\hat{\theta} = \Big(1 - \sum_i (\eta_i + \tilde{\zeta}_i) \times s_i\Big) \times \frac{M}{\hat{M}} + \sum_i (\eta_i + \hat{\zeta}_i) \times \hat{s}_i$$

where

$$\tilde{s}_i = \frac{\tilde{x}_i \tilde{p}_i}{M} \qquad \hat{s}_i = \frac{\tilde{x}_i \hat{p}_i}{\hat{M}}$$

We have $\sum_i \eta_i \hat{s}_i = -\sum_i \hat{s}_j \hat{\zeta}_j + \hat{\theta}$ since the Engel and Cournot aggregation hold for demand $x^{ms}(\hat{p}, \hat{M})$. Then $\sum_i \eta_i \hat{s}_i = (1 - \sum_i (\eta_i + \tilde{\zeta}_i) \times s_i) \times \frac{M}{\hat{M}} + \sum_i \eta_i \hat{s}_i$ and

$$\sum_i (\eta_i + \tilde{\zeta}_i) s_i = 1 \tag{9}$$

Condition (9) implies, that the Engel aggregation condition holds for aggregated goods only if $\sum_i \tilde{\zeta}_i s_i = 0$. Condition (9) also implies implies, that $\hat{\theta} = 1 + \sum_i (\hat{\zeta}_i - \tilde{\zeta}_i) \times \hat{s}_i$. Further

$$\tilde{\zeta}_i = \eta_i \times (\frac{\hat{p}_i}{\tilde{p}_i} - 1) \qquad \hat{\zeta}_i = \eta_i \times \frac{\partial^2 e_i(p_i, g_i(\tilde{x}_i))}{\partial \tilde{x}_i \partial \tilde{x}_i} \times \frac{\tilde{x}_i}{\hat{p}_i}$$

We have

$$\frac{\partial^2 e_i(p_i, g_i(\tilde{x}_i))}{\partial \tilde{x}_i \partial \tilde{x}_i} = \frac{\partial^2 e_i(p_i, g_i(\tilde{x}_i))}{\partial \tilde{v}_i^2} \times \Big(\frac{\partial g_i(\tilde{x}_i)}{\partial \tilde{x}_i}\Big)^2 + \hat{p}_i \times \frac{\partial^2 g_i(\tilde{x}_i)}{\partial \tilde{x}_i^2}$$

Let $\tilde{x}_i = e_i(p_i^0, \tilde{v}_i)$. Then $g_i(\tilde{x}_i) = w_i(p_i^0, \tilde{x}_i)$ and $\partial g_i(\tilde{x}_i)/\partial \tilde{x}_i = \partial w_i(p_i^0, \tilde{x}_i)/\partial M$, $\partial^2 g_i(\tilde{x}_i)/\partial \tilde{x}_i^2 = \partial^2 w_i(p_i^0, \tilde{x}_i)/\partial M^2$. Since $w_i(p_i, e_i(p_i, \tilde{v}_i)) = \tilde{v}_i$, thus, $\partial e_i(p_i, \tilde{v}_i)/\partial \tilde{v}_i = (\partial w_i(p_i, e_i(p_i, \tilde{v}_i))/\partial M)^{-1}$, and

$$\frac{\partial^2 e_i(p_i, g_i(\tilde{x}_i))}{\partial \tilde{v}_i \partial \tilde{v}_i} = -\frac{\partial^2 w_i(p_i, e_i(p_i, g_i(\tilde{x}_i)))/\partial M^2}{(\partial w_i(p_i, e_i(p_i, g_i(\tilde{x}_i)))/\partial M)^2} \times \hat{p}_i$$

Observe that

$$\frac{\partial^2 e_i(p_i^0, g_i(\tilde{x}_i))}{\partial \tilde{x}_i \partial \tilde{x}_i} = -\frac{\partial^2 w_i(p_i^0, \tilde{x}_i)/\partial M^2}{(\partial w_i(p_i^0, \tilde{x}_i)/\partial M)^2} \times \hat{p}_i \times \left(\frac{\partial g_i(\tilde{x}_i)}{\partial \tilde{x}_i}\right)^2 + \hat{p}_i \times \frac{\partial^2 g_i(\tilde{x}_i)}{\partial \tilde{x}_i^2} = 0$$

hence $\hat{\zeta}(p_0, M) = 0$. Additionally, $\tilde{p}_i(p_i^0, M) = \hat{p}_i(p_i^0, M) = 1$, hence $\tilde{\zeta}_i(p_i^0, M) = 0$. Finally

$$\eta_i(p^0, M) = \eta_i^s(\mathbf{1}_k, M) \tag{10}$$

where $\mathbf{1}_k$ is a $k \times 1$ vector with all elements equal 1.

6 A note on calibration

Equations (8) and (10) allows for calibrating aggregate demand based on shares of aggregate goods in total expenditure, elasticity of substitution and income elasticity. Let M^0 and p^0 is a base level of total expenditures and prices. Then consumption of an i-th aggregate good is $x_i^M(p^0, M) = x_i^{ms}(\mathbf{1}_k, M)$. Since aggregate prices are equal 1, share of an i-th aggregate goods in total expenditure is equal

$$s_i^0 = \frac{1}{M} \times x_i^{ms}(\mathbf{1}_k, M)$$

From (8) and (10) we have also

$$\sigma_{ij}^{AES}(p^0, M) = \sigma_{ij}^{AES,s}(\mathbf{1}_k, M) \qquad\qquad \eta_i(p^0, M) = \eta_i^s(\mathbf{1}_k, M)$$

Finally one can calibrate sub-utility functions based on conditional demands.

7 Conclusions

We have shown that a solution the first stage utility maximization problem can be expressed in terms of demand associated with the standard optimization problem. This technique allows to find optimal consumption of aggregate goods when only marshalian or hicksian demand associated with the standard maximization of aggregate utility is known, hence is applicable to wide set of aggregate utility functions.

We have also derived elasticity of substitution between aggregate goods and income elasticity. Elasticity of substitution and income elasticity are equal to income and substitution elasticity of the standard demands allowing for simple calibration of parameters of utility functions. However in case of income elasticity this property holds only at base prices.

A Proof of proposition 3.1

This proof is a simple extension of similar proposition in case of the standard utility maximization problem.

Since functions $e_i(p_i, g_i(\tilde{x}_i))$ are increasing in $\tilde{x}_i$ for any $i = 1, \ldots, k$, the function $\sum_i e_i(p_i, g_i(\tilde{x}_i))$ is increasing in all $\tilde{x}_i$, hence quasi-convex. This implies that the set $C \doteq \{\tilde{x} \in \mathbb{R}^k_+ : \sum_i e_i(p_i, g_i(\tilde{x}_i)) \leq M\}$ is convex and nonempty for any $M > 0$. Since $\tilde{v}$ is strictly quasi-convex, there exists at most one solution to (5).

For $i = 1, \ldots, k$, $e_i(p_i, g_i(\tilde{x}_i)) \leq M$, since $e_i(p_i, g_i(\tilde{x}_i)) \geq 0$. Hence, if $\tilde{x} \in C$, then $\tilde{x}_i \leq f_i(w_i(p_i, M))$, and the set C is bounded. The set C is also closed since expenditure functions $e_i(p_i, g_i(x_i))$ are continuous, hence compact. By the Weierstrass' Extreme Value Theorem, there exists a solution, x^* to (5).

Let $\tilde{x}^*$ solves (5) and $\sum_i e_i(p_i, g_i(\tilde{x}^*_i)) < M$. Then, there exists $\tilde{x}' > \tilde{x}^*$ and $\tilde{x}' \in C$. We have $\tilde{v}(\tilde{x}^*) = \tilde{v}(\tilde{x}')$, since the function $\tilde{v}$ is increasing. Let $\alpha \in (0, 1)$ is any scalar. Then, by the strong quasi-concavity of $\tilde{v}$, $\tilde{v}(\alpha\tilde{x}^* + (1-\alpha)\tilde{x}') > \tilde{v}(\tilde{x}^*)$. This contradicts optimality of $\tilde{x}^*$.

B Proof of proposition 3.2

Assumptions imply that $x_i^{ms}(p, M)$ is a $\mathcal{C}^1$ function.

The utility function $\tilde{v}(\tilde{x})$ satisfies for all $\tilde{x} \in \mathbb{R}^k_{++}$

$$\tilde{v}(\tilde{x}) = \min_{q \in \mathbb{R}^k_+} w^s(q, 1) \qquad s.t. \sum_i q_i \tilde{x}_i = 1 \tag{11}$$

There exists exactly one solution to this problem, since w^s is strictly quasi-convex and set of feasible price vectors is compact. Let $\mathcal{L}^s$ is a lagrangian associated with (11)

$$\mathcal{L}^s = w^s(q, 1) - \mu(\sum_i q_i \tilde{x}_i - 1)$$

Let q^* solves (11). Then there exists a lagrange multiplier μ^*, such that

$$\frac{\partial \tilde{w}(q^*, 1)}{\partial q_i} = \mu^* x_i > 0 \qquad \sum_i q^*_i x_i = 1$$

On the other hand if q^* and μ^* solves these conditions, then q^* is a global optimum, since the differentiable strictly quasi-convex function w^s is also pseudo-convex function. Hence

$$\mu^* = \sum_i q^*_i \frac{\partial \tilde{w}(q^*, 1)}{\partial q_i}$$

Let as consider q^* and μ^* as functions of $\tilde{x}$, $q^* = q(x)$, $\mu^* = \mu(x)$. The envelope theorem implies

$$\frac{\partial \tilde{v}(\tilde{x})}{\partial \tilde{x}_i} = -\mu(\tilde{x}) q_i(\tilde{x}) \qquad \mu(x) = -\sum_i x_i \frac{\partial \tilde{v}(\tilde{x})}{\partial \tilde{x}_i} \tag{12}$$

Then, taking $y = x^{ms}(-z/\mu^*, 1)$ for given μ^* and any $z \in \mathrm{dom}(\partial \tilde{v}(y)/\partial \tilde{x})$, we obtain $\partial \tilde{v}(y)/\partial \tilde{x}_i = z_i$.

Let $\mathcal{L}$ is lagrangian associated with (5) is

$$\mathcal{L} = \tilde{v}(\tilde{x}_1, \ldots, \tilde{x}_k) - \lambda(\sum_i \tilde{p}_i \tilde{x}_i - M)$$

Let $\tilde{x}^{**}$ solves (5). Since $e_i(p_i, g_i(\tilde{x}_i^{**})$ is differentiably strictly increasing, $\partial e_i(p_i, g_i(\tilde{x}_i^{**})/\partial \tilde{x}_i > 0$ and gradient of $\sum_i \tilde{p}_i \tilde{x}_i$ evaluated at $\tilde{x}^{**}$ has rank 1. Then there exists a lagrange multiplier λ^{**}, such that

$$\frac{\partial \tilde{v}(\tilde{x}^{**})}{\partial \tilde{x}_i} = \lambda^{**} \times \hat{p}_i^{**} \qquad \sum_i \tilde{p}_i^{**} \tilde{x}_i^{**} = M \tag{13}$$

where $\tilde{p}_i^{**}$ and $\hat{p}_i^{**}$ are values of $\tilde{p}_i$ and $\hat{p}_i$ evaluated at $\tilde{x}^{**}$. On the other hand if $\tilde{x}^{**}$, λ^{**}, $\tilde{p}^{**}$, $\hat{p}^{**}$ solves these conditions, then $\tilde{x}^{**}$ is a global optimum, since the differentiable strictly quasi-concave function $\tilde{v}$ is also pseudo-concave function and $\sum_i \tilde{p}_i \tilde{x}_i$ is a quasi-concave function since is increasing in $\tilde{x}$. From (12) we have

$$\mu(x^{**}) = -\lambda^{**} \times M - \lambda^{**} \sum_i x_i^{**} \times (\hat{p}_i^{**} - \tilde{p}_i^{**})$$

hence, x^{**} also solves

$$x^{**} = x^{ms}(\hat{p}_i^{**}, M + \sum_i x_i^{**} \times (\hat{p}_i^{**} - \tilde{p}_i^{**})) \tag{14}$$

For given λ^{**}, $\tilde{p}^{**}$, $\hat{p}^{**}$, and M, x^{**} defined by (14) is a solution to (13). But since there exists exactly one solution to (5), equation (14) determines uniquely x^{**}, which is also solution to (5).

Let $\tilde{x}^*$ solves (5). Then $\tilde{x}^*$ also solves (6) at prices $\hat{p}$ and income $\hat{M}$. Let $u = \tilde{v}(\tilde{x}^*)$ is a utility level from consumption of consumption bundle $\tilde{x}^*$. Then $x^* = x^{hs}(\hat{p}, u)$, since $\tilde{x}^*$ delivers the same utility level in case of problem (5) and (6).

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