Characterization of the painting rule for multi-source minimal cost spanning tree problems

Bergantiños, Gustavo and Navarro, Adriana

Universidade de Vigo

12 April 2019

Characterization of the painting rule for multi-source minimal cost spanning tree problems

G. Bergantiños, A. Navarro-Ramos*

Economics, Society and Territory. Facultad de Economía, Campus Lagoas-Marcosende, s/n, Universidade de Vigo, Vigo, Pontevedra, Spain

Abstract

In this paper we provide an axiomatic characterization of the painting rule for minimum cost spanning tree problems with multiple sources. The properties we need are: cone-wise additivity, cost monotonicity, symmetry, isolated agents, and equal treatment of source costs.

Keywords: minimum cost spanning tree problems with multiple sources, painting rule, axiomatic characterization.

1. Introduction

The multi-source minimal cost spanning tree problems consider a group of agents that needs services provided by multiple sources. Agents do not care if they are connected directly or indirectly to the sources but they need to be connected to all of them. Every connection entails a cost. These situations are an extension of the classical minimum cost spanning tree problem with one source.

There are two objectives in these problems. The first one is to find a cost minimizing network which connects all the agents with all sources. Such a network is a tree, and it can be computed using the same algorithms as in the classical problem.

Once the tree is obtained, the second issue is how to allocate the cost of such tree among the agents. Some recent papers have studied rules for the multi-source problem. Bergantiños et al. [4] extend the definition of the folk rule following four definitions and present some axiomatic characterizations. Bergantiños and Navarro-Ramos [2] extend the definition of the painting rule to the case of multiple sources and prove that it also coincides with the extension of folk rule. Bergantiños and Lorenzo [1] consider several family of rules obtained through Kruskal's algorithm.

*Corresponding author

Email addresses: gbergant@uvigo.es (G. Bergantiños), adnavarro@uvigo.es (A. Navarro-Ramos)

The objective of the paper is to provide an axiomatic characterization of the painting rule for multi-source cost spanning tree problems. We do it using the properties of cone-wise additivity, cost monotonicity, symmetry, isolated agents, and equal treatment of source costs. The first three properties are quite standard in the literature and are defined as in the classical minimum cost spanning tree problem. Cost additivity says that the rule should be additive on the cost function when restricted to cones. Cost monotonicity says that if some connection costs increase and the rest (if any) remain the same, no agent should end up better off. If two agents are symmetric with respect to their connection costs, symmetry says that both agents should pay the same.

The isolated agents property is inspired by the property introduced in Bergantiños et al. [3]. Nevertheless, the extension is not as straightforward as with the previous ones. An agent is called isolated when her connection cost to any other agent is the same. Besides such connection cost is larger than any other connection cost in which such agent is not involved. If there is a way of connecting all sources to one another for free (not necessarily directly), an isolated agent should only pay her connection cost to any node.

Equal treatment of source costs was introduced in Bergantiños et al. [4], and it is a property defined only for the case of multiple sources. It says that if the cost between two sources increases, then all agents should be affected in the same way.

The paper is organized as follows. Section 2 introduces minimum cost spanning tree problems with multiple sources. Section 3 introduces two definitions of the folk rule for the multi-source problem. Section 4 gives the axiomatic characterization.

2. The model

We introduce the model following the same notation as in Bergantiños and Navarro-Ramos [2].

Let $N = \{1, ..., n\}$ be the set of agents. Let $M = \{a_1, ..., a_m\}$ be the set of sources. We assume that each agent want to be connected to all the sources. The *cost matrix* $C = (c_{ij})_{i,j \in N \cup M}$ over $N \cup M$ represents the cost of the direct link between any pair of nodes, with $c_{ji} = c_{ij} \geq 0$ and $c_{ii} = 0$, for all $i, j \in N \cup M$. $\mathcal{C}^{N \cup M}$ is the set of all cost matrices over $N \cup M$.

A *multi-source minimal cost spanning tree problem* (briefly, multi-source *mcstp* or a *problem*) is a triple (N, M, C) where N is the set of agents, M is the set of sources, and $C \in \mathcal{C}^{N \cup M}$ is the cost matrix. If $c_{ij} \in \{0, 1\}$, for all $i, j \in N \cup M$, then (N, M, C) is called a *simple problem*.

An *edge* is a non-ordered pair (i, j) such that $i, j \in N \cup M$. A *network* g is a subset of edges. The cost associated with a network g is defined as $c(N, M, C, g) = \sum_{(i,j) \in g} c_{ij}$. When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N, M, C, g)$.

Given a network g and any pair of nodes i and j, a *path* from i to j in g is a sequence of distinct edges $g_{ij} = \{(i_{h-1}, i_h)\}_{h=1}^{q}$ satisfying that $(i_{h-1}, i_h) \in g$

for all $h = 1, ..., q$, $i = i_0$ and $j = i_q$. A *cycle* is a path from i to i with at least two edges. A *tree* is a graph without cycles that connects all the elements of $N \cup M$. Given $S \subset N \cup M$, g_S denotes the restriction of g to nodes in S.

Two nodes i, j are *connected* in g if there exists a path from i to j in g. A subset of nodes $S \subseteq N \cup M$ is a *connected component* on g if every $i, j \in S$ are connected in g and S is maximal, *i.e.*, for each $T \in N \cup M$ with $S \subsetneq T$ there exist $k, l \in T$, $k \neq l$, such that k and l are not connected in g.

Let (N, M, C) be a simple problem. The network induced by the edges with zero cost is denoted by $g^{0,C} = \{(i, j) : i, j \in N \cup M \text{ and } c_{ij} = 0\}$. $S \subseteq N \cup M$ is a *C-component* if S is a connected component on $g^{0,C}$.

A *minimal tree* (briefly, *mt*) connects all agents to the sources at the lowest cost. Several algorithms (for instance, Kruskal [7] and Prim [9]) enable us to compute a *mt*. We denote by $m(N, M, C)$ the cost of any *mt* in (N, M, C).

Let (N, M, C) be a problem and t a minimal tree in (N, M, C). For each $i, j \in N \cup M$, t_{ij} is the unique path in t joining i and j. Bird [5] defines the *minimal network* associated with the minimal tree t as the problem (N, M, C^t), where $c^t_{ij} = \max_{(k,l)\in t_{ij}} c_{kl}$. It is well known that C^t is independent of t. Then, the *irreducible problem* (N, M, C^*) of (N, M, C) is defined as the minimal network associated with any minimal tree in (N, M, C).

After obtaining a minimal tree, sometimes it is necessity to divide its cost among the agents. A cost allocation rule (briefly, a rule) is a map f that associates a vector $f(N, M, C) \in \mathbb{R}^N$ with each problem (N, M, C) such that $\sum_{i\in N} f_i(N, M, C) = m(N, M, C)$. The element $f_i(N, M, C)$ denotes the payment of agent $i \in N$.

3. Extensions of the folk rule

In the classical minimum cost spanning tree problem, the most popular rule is folk rule. Bergantiños et al. [4] extend the definition of the folk rule to the multi-source problem and provide several ways to obtain it. One of them is through cone-wise decomposition.

For each problem (N, M, C), there exists a positive number $m(C) \in \mathbb{N}$, a sequence $\{C^q\}_{q=1}^{m(C)}$ of simple cost matrices, and a sequence $\{x^q\}_{q=1}^{m(C)}$ of non-negative real numbers satisfying two conditions:

1. $C = \sum\limits_{q=1}^{m(C)} x^q C^q$.
2. Take $q \in \{1, \ldots, m(C)\}$ and $\{i, j, k, l\} \subset N \cup M$. If $c_{ij} \leq c_{kl}$, then $c^q_{ij} \leq c^q_{kl}$.

This means that any cost matrix can be written as a non-negative combination of simple problems. This is an adaptation of a result of Norde et al. [8] for the problem with one source.

Let (N, M, C) be a simple problem and $P = \{S_1, ..., S_p\}$ the partition of $N \cup M$ in C-components. Bergantiños et al. [4] define the folk rule F for simple problems as follows.

$$F_i(N,M,C)=\begin{cases}\dfrac{|S_k\in P:S_k\cap M\neq\emptyset|-1}{|N|}, & \text{if } S(i,P)\cap M\neq\emptyset\\ \dfrac{1}{|S(i,P)|}+\dfrac{|S_k\in P:S_k\cap M\neq\emptyset|-1}{|N|}, & \text{otherwise,}\end{cases}$$

where $S(i,P)$ is the element of P to which i belongs to. Then, the folk rule for a general problem (N,M,C) is defined as

$$F(N,M,C)=\sum_{q=1}^{m(C)}x^qF(N,M,C^q).$$

Bergantiños et al. [3] study a general framework of connection problems involving a single source, which contains classical minimum cost spanning tree problems. They propose a cost allocation rule, called the painting rule because it can be interpreted through a painting story. They also give some axiomatic characterizations of the rule. They prove that the painting rule coincides with the folk rule in classical *mcstp*.

Bergantiños and Navarro-Ramos [2] extend the definition of the painting rule to problems with multiple sources. They also prove that it coincides with the folk rule in the multi-source problem.

Next, we present the two-phase algorithm introduced in Bergantiños and Navarro-Ramos [2] that induce the painting rule. Given a problem (N,M,C) and a minimal tree t in (N,M,C), let $P(t_M)=\{S_1,...,S_{m(t)}\}$ denote the partition of M in connected components induced by t_M.

Phase 1: Constructing the tree. Start with $t^0=t$. Assume that stage β is defined for all $\beta\leq\delta-1$.

Stage δ:

- If $P(t_M^{\delta-1})=\{M\}$. The algorithm ends and $t^*=t^{\delta-1}$.
- If $P(t_M^{\delta-1})\neq\{M\}$. Let

  $$E(t^{\delta-1})=\{(i_{h-1},i_h)\}_{h=1}^q$$

  be the unique path from $\bigcup_{r=1}^{\delta}S_r$ to $S_{\delta+1}$ in $t^{\delta-1}$, with $i_0\in\bigcup_{r=1}^{\delta}S_r$, $i_q\in S_{\delta+1}$, $i_1\notin\bigcup_{r=1}^{\delta}S_r$ and $i_{q-1}\notin S_{\delta+1}$.

  Let (i,j) be the most expensive edge in $E(t^{\delta-1})$ (if there are several edges, select just one). Namely, $c_{ij}=\max_{(k,l)\in E(t^{\delta-1})}\{c_{kl}\}$. Now,

  $$t^\delta=t^{\delta-1}\backslash(i,j)\cup(i_0,i_q).$$

Phase 2: Painting the tree. Let t^* be the tree obtained in Phase 1. Start with

- $e_i^0\ (C,t^*)=\emptyset$ for all $i\in N$. In general, $e_i^\delta(C,t^*)$ denotes the edge of t^* assigned to agent i at stage δ. Agent i will pay part of the cost of this edge.

- $c^0(C,t^*) = 0$ and $c^\delta(C,t^*)$ represents the part of the cost of each edge that it is paid at stage δ.
- $p_i^0(C,t^*) = 0$ for all $i \in N$. In general, $p_i^\delta(C,t^*)$ is the cost that agent i pays at stage δ.
- $E^0(C,t^*) = t^* \backslash t_M^*$ and $E^\delta(C,t^*)$ is the set of unpaid edges of $t^* \backslash t_M^*$ at stage δ.

When no confusion arises, we will write e_i^δ, $e_i^\delta(C)$ or $e_i^\delta(t^*)$ instead of $e_i^\delta(C,t^*)$. We will do the same with $c^\delta(C,t^*)$, $p_i^\delta(C,t^*)$ and $E^\delta(C,t^*)$. Assume that stage β is defined for all $\beta \leq \delta - 1$.

Stage δ:

- For each $i \in N$, let e_i^δ be the first edge in the unique path in t^* from i to M belonging to $E^{\delta-1}$. If all edges in such path are not in $E^{\delta-1}$, take $e_i^\delta = \emptyset$.
- For each $(i,j) \in E^{\delta-1}$, let

$$N_{ij}^\delta = \{k \in N : e_k^\delta = (i,j)\}$$

and

$$c^\delta = \min \left\{ c_{ij} - \sum_{r=0}^{\delta-1} c^r : (i,j) \in E^{\delta-1} \right\}.$$

- For each $i \in N$,

$$p_i^\delta = \begin{cases} \dfrac{c^\delta}{\left| N_{e_i^\delta}^\delta \right|}, & \text{if } e_i^\delta \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

- Now,

$$E^\delta = \left\{ (i,j) \in E^{\delta-1} : \sum_{r=0}^{\delta} c^r < c_{ij} \right\}.$$

This procedure ends when we find a stage $\gamma(C,t^*)$ ($\gamma(C)$, $\gamma(t^*)$ or γ when no confusion arises) such that $E^\gamma = \emptyset$.

Stage $\gamma + 1$:

$$p_i^{\gamma+1} = \frac{c(t_M^*)}{|N|}.$$

For each problem (N,M,C), each *mt* t, and each $i \in N$, the panting rule $f_i^{P,t}$ is defined as

$$f_i^{P,t}(N,M,C) = \sum_{\delta=1}^{\gamma+1} p_i^\delta(C,t^*).$$

Even this definition could depend on t and t^*, Bergantiños and Navarro-Ramos [2] show that the $f^{P,t}$ coincides with F for every t, t^*, and (N,M,C). Henceforth, we denote the painting rule also by F.

4. An axiomatic characterization

This section presents an axiomatic characterization of the painting rule. In their Corollary 1, Bergantiños et al. [3] characterize the folk rule in classical *mcstp* with the properties of cost monotonicity, symmetry, cone-wise additivity, and isolated agents. We extend this characterization to the case of multiple sources by considering these four axioms and adding a new one called equal treatment of source costs. The definition of the properties of cost monotonicity, symmetry, and cone-wise additivity in the case of multiple sources is the same as in the classical case. The definition of isolated agents multi-source *mcstp* is not so straightforward. Equal treatment of source costs is a property defined only in the case of multiple sources.

A rule f for a problem (N, M, C) satisfies:

• *Cone-wise additivity (CA).* Let (N, M, C) and (N, M, C') be two problems satisfying that there is an order σ over the set of edges of $N \cup M$ such that for all $i, j, k, l \in N \cup M$ satisfying that $\sigma(i, j) < \sigma(k, l)$, then $c_{ij} \leq c_{kl}$ and $c'_{ij} \leq c'_{kl}$. Thus, $f(N, M, C + C') = f(N, M, C) + f(N, M, C')$.

CA says that the rule should be additive on the cost function C when restricted to cones.

• *Cost monotonicity (CM).* For all (N, M, C) and (N, M, C') such that $C \leq C'$, then $f(N, M, C) \leq f(N, M, C')$.

CM says that if a certain number of connection costs increase and the rest (if any) remain the same, no agent should end up better off.

• *Symmetry (SYM).* For all (N, M, C) and all $i, j \in N$ such that $c_{ik} = c_{jk}$, $\forall k \in (N \cup M) \backslash \{i, j\}$, then $f_i(N, M, C) = f_j(N, M, C)$.

If two agents are symmetrical with respect to their connection costs, *SYM* says that they should pay the same.

The next property is inspired by the isolated agents property introduced in Bergantiños et al. [3] for source connection problems.

An agent $i \in N$ is called *isolated* in a problem (N, M, C) if $c_{ij} = x$, for all $j \in (N \cup M) \backslash \{i\}$ and $c_{jk} \leq x$, for all $j, k \in (N \cup M) \backslash \{i\}$. Notice that if agent i is isolated, then agent i does not benefit from connecting to the sources through agents in $N \backslash \{i\}$.

• *Isolated agents (IA).* For all (N, M, C) such that for all $k, l \in M$, there is a path from k to l, g_{kl}, such that $c(g_{kl}) = 0$, $f_i(N, M, C) = x$, for every isolated agent $i \in N$.

If there is a way of connecting all sources to one another for free (not necessarily directly), an isolated agent should only pay her connection cost to any node.

• *Equal treatment of source costs (ETSC).* For each pair of problems (N, M, C) and (N, M, C') such that there exist $k, l \in M$, $k \neq l$, such that $c_{kl} < c'_{kl}$ and $c_{ij} = c'_{ij}$ otherwise, then $f_i(N, M, C') - f_i(N, M, C) = f_j(N, M, C') - f_j(N, M, C)$, for each $i, j \in N$.

This property was introduced in Bergantiños et al. [4]. It says that if the cost between two sources increases, then all agents should be affected in the same way.

In the next theorem, we present the characterization of the painting rule.

Theorem 1. *The painting rule is the unique rule satisfying CA, CM, SYM, IA and ETSC.*

Proof. First we prove that the painting rule satisfies the five properties. Bergantiños et al. [4] proved that the folk rule satisfies *CA*, *CM*, *SYM* and *ETSC*. Bergantiños and Navarro-Ramos [2] proved that F coincides with the folk rule. Then, F satisfies *CA*, *CM*, *SYM* and *ETSC*.

We now prove that F satisfies *IA*. Let $i \in N$ be an isolated agent for a problem (N, M, C). Let t be a minimal tree for (N, M, C). We can take t in such a way that no agent in $N \backslash \{i\}$ is connected to any source through agent i. Namely, for each $j \in N \backslash \{i\}$ and each $k \in M$, $i \notin t_{jk}$.

Since there is a path at cost zero to join together every two sources, the tree obtained in Phase 1, t^*, is such that $c(t^*_M) = 0$.

We now apply Phase 2. Since no agent is connected to the source through agent i and $c_{ik} = x \geq c_{jk}$, $\forall j, k \in (N \cup M) \backslash \{i\}$, we have that, for each $\delta = 1, ..., \gamma$, $e_i^\delta = (i, i^M)$ and $e_j^\delta \neq (i, i^M)$, for all $j \in N \backslash \{i\}$.

Then,

$$F_i(N, M, C) = c_{ii^M} + \frac{c(t^*_M)}{|N|} = x + 0 = x.$$

Thus, F satisfies *IA*.

We now prove the uniqueness. Let f be a rule satisfying the properties of Theorem 1. By *CA*, it is enough to prove that $f = F$ in simple problems.

Let (N, M, C) be a simple problem and $P = \{S_1, ..., S_p\}$ the set of C-components. Consider the next cost function:

$$c'_{ij} = \begin{cases} c_{ij}, & \text{if } \{i, j\} \cap N \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

We have a simple problem (N, M, C') such that all sources are connected to one another at cost zero and $C \geq C'$.

For each $S_k \in P$ such that $S_k \cap M = \emptyset$, we define a pair of cost function as follows:

$$c^k_{ij} = \begin{cases} 1, & \text{if } \{i, j\} \cap S_k \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and

$$c'^k_{ij} = \begin{cases} c'_{ij}, & \text{if } \{i, j\} \cap S_k \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

We first analyze how f works on (N, M, C^k) and (N, M, C'^k). Let $S_k \in P$ with $S_k \cap M = \emptyset$ and $i \in N$.

- On (N, M, C^k). If $i \in S_k$, i is an isolated agent. By *IA*, $f_i(N, M, C^k) = 1$, for all $i \in S_k$. Besides, $m(N, M, C^k) = |S_k|$. Since all agents in $N \backslash S_k$ are symmetric, $f_i(N, M, C^k) = 0$, for all $i \notin S_k$. This is,

$$f_i(N, M, C^k) = \begin{cases} 1, & \text{if } i \in S_k \\ 0, & \text{otherwise.} \end{cases}$$

- On (N, M, C'^k). We have that $C'^k \leq C^k$. If $i \notin S_k$, by *CM*, $f_i(N, M, C'^k) \leq f_i(N, M, C^k) = 0$. It is straightforward to see that if a rule satisfies *CM* and *SYM*, then it should be non-negative. Then, $f_i(N, M, C^k) = 0$ if $i \notin S_k$. All agents on S_k are symmetric and $m(N, M, C'^k) = 1$. Thus,

$$f_i(N, M, C'^k) = \begin{cases} \dfrac{1}{|S_k|}, & \text{if } i \in S_k \\ 0, & \text{otherwise.} \end{cases}$$

Take $i \in N$ and let $S(i, P)$ denote the C-component to which i belongs. We consider two cases:

- $S(i, P) \cap M = \emptyset$. Since $C' \geq C'^k$ and *CM*,

$$f_i(N, M, C') \geq f_i(N, M, C'^k) = \frac{1}{|S(i, P)|}.$$

- $S(i, P) \cap M \neq \emptyset$. Since a rule satisfying *CM* and *SYM* should be non-negative, namely $f_i(N, M, C') \geq 0$.

Taking into account that $m(N, M, C') = |S_k \in P : S_k \cap M = \emptyset|$ and

$$\sum_{i \in N} f_i(N, M, C') \geq \sum_{i \in N | S(i,P) \cap M = \emptyset} \frac{1}{|S(i, P)|} = |S_k \in P : S_k \cap M = \emptyset|,$$

we conclude that

$$f_i(N, M, C') = \begin{cases} \dfrac{1}{|S(i, P)|}, & \text{if } S(i, P) \cap M = \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Finally, notice that C could be obtained from C' by increasing the connection costs among the sources. Thus, by applying *ETSC* several times (once by each pair $k, l \in M$ such that $c_{kl} > 0$) we deduce that for all $i, j \in N$,

$$f_i(N, M, C) - f_i(N, M, C') = f_j(N, M, C) - f_j(N, M, C').$$

Fix $i \in N$,

$$\begin{aligned} |N|[f_i(N, M, C) - f_i(N, M, C')] &= \sum_{j \in N} [f_j(N, M, C) - f_j(N, M, C')] \\ &= \sum_{j \in N} f_j(N, M, C) - \sum_{j \in N} f_j(N, M, C') \\ &= |P| - 1 - (|P| - |S_k \in P : S_k \cap M \neq \emptyset|) \\ &= |S_k \in P : S_k \cap M \neq \emptyset| - 1. \end{aligned}$$

Thus,

$$\begin{aligned} f_i(N,M,C) =& \frac{|S_k \in P : S_k \cap M \neq \emptyset| - 1}{|N|} + f_i(N,M,C') \\ =& \begin{cases} \dfrac{|S_k \in P : S_k \cap M \neq \emptyset| - 1}{|N|}, & \text{if } S(i,P) \cap M \neq \emptyset \\ \dfrac{1}{|S(i,P)|} + \dfrac{|S_k \in P : S_k \cap M \neq \emptyset| - 1}{|N|}, & \text{otherwise,} \end{cases} \end{aligned}$$

Therefore, $f(N,M,C) = F(N,M,C)$. □

In the next proposition we prove that all properties are needed in the previous characterization.

Proposition 1. *The properties used in Theorem 1 are independent.*

Proof. CA is independent of the other properties. Consider the rule f^e defined in Bergantiños et al. [4] when they prove that CA is independent of the properties they use in Theorem 2. f^e satisfies all properties but CA.

CM is independent of the other properties. Given a problem (N,M,C), let t be a mt of (N,M,C) and t^* a mt of (N,M,C^*) obtained through Phase 1. We now consider the following classical problem $(N_0, \overline{C})$, where $\bar{c}_{0i} = \max\{c^*_{kl} : (k,l) \in t^*_{ij}$ for some $j \in M$ and $k,l \in N\}$ and $\bar{c}_{ij} = c^*_{ij}$, for all $i,j \in N$.

For a classical problem with a single mt, Bird [5] proposed a rule called the *Bird rule*. This rule is obtained by requiring each agent to pay the total cost of the first edge in her unique path to the source. Dutta and Kar [6] extended the Bird rule when there is more than one mt (an extension we denote as B). This rule is the average of the allocations given by the Bird rule on all the minimal trees associated with Prim's algorithm. We now extend it to our setting in the following way:

$$f^B(N,M,C) = B(N_0, \overline{C}) + \frac{c(t^*_M)}{|N|}.$$

f^B satisfies all properties but CM.

SYM is independent of the other properties. For each problem (N,M,C) and each $\delta = 1, ..., n+m-1$, let (i^δ, j^δ) denote the edge selected by Kruskal's algorithm at stage δ and g^δ be the set of all edges selected according to Kruskal's algorithm until stage δ (included). Besides $P(g^\delta)$ denotes the partition of $N \cup M$ in connected components induced by g^δ.

Given a partition P we define the function α as

$$\alpha_i(P) = \begin{cases} \dfrac{|S_k \in P : S_k \cap M \neq \emptyset| - 1}{|N|} + 1, & \text{if } S(i,P) \cap M = \emptyset \\ & \text{and } i \leq j, \forall j \in S(i,P) \\ \dfrac{|S_k \in P : S_k \cap M \neq \emptyset| - 1}{|N|}, & \text{otherwise,} \end{cases}$$

Thus we define the rule f^α such that for each problem (N, M, C) and each $i \in N$,

$$f_i^\alpha(N, M, C) = \sum_{\delta=1}^{n+m-1} c_{i^\delta j^\delta}[\alpha_i(P(g^{\delta-1})) - \alpha_i(P(g^\delta))].$$

f^α satisfies all properties but *SYM*.

IA is independent of the other properties. Let E be the rule in which the cost of the minimal tree is divided equally among all agents. Namely, for each problem (N, M, C) and each $i \in N$,

$$E_i(N, M, C) = \frac{m(N, M, C)}{|N|}.$$

This rule satisfies all properties but *IA*.

ETSC is independent of the other properties. Let (N, M, C) be a problem. If $N = \{1, 2\}$ and $M = \{a_1, a_2\}$, let us define the sets $N' = \{1, 2, a_2\}$ and $M' = \{a_1\}$. Then, for every $i \in N$, we define the rule

$$f_i(N, M, C) = \begin{cases} F_i(N', M', C) + \dfrac{F_{a_2}(N', M', C)}{2}, & \text{if } N = \{1, 2\} \text{ and } M = \{a_1, a_2\} \\ F_i(N, M, C), & \text{otherwise.} \end{cases}$$

This rule satisfies all properties but *ETSC*. □

We end this paper by comparing our characterization with other results of the literature.

Bergantiños et al. [3] characterizes the folk rule in classical minimum cost spanning tree problems with *CA*, *CM*, *SYM*, and *IA*. If we restrict to classical minimum cost spanning tree problems we realize that our CA coincides with the property of *CA* as it was defined in Bergantiños et al. [3]. The same happens with the properties of *CA*, *SYM*, and *IA*. Besides *ETSC* says nothing in classical minimum cost spanning tree problems because only applies when we have several sources. Thus, our result is an extension of the characterization of Bergantiños et al. [3] to the case of multiple sources.

Bergantiños et al. [4] provide two characterizations of the folk rule in multi-sources *mcstp*. The next properties are used in such characterizations.

• *Independence of irrelevant trees (IIT).* For each (N, M, C) and (N, M, C'), if they have a common minimal tree t such that $c_{ij} = c'_{ij}$ for each $(i, j) \in t$, then $f(N, M, C) = f(N, M, C')$.

This property requires the cost allocation chosen by a rule to depend only on the edges that belong to a minimal tree.

• *Core selection (CS).* For each (N, M, C) and each $S \subseteq N$, $\sum_{i \in S} f_i(N, M, C) \leq m(S, M, C)$.

CS implies that no coalition of agents would be better off by constructing their own minimal tree.

• *Separability (SEP)*. For each (N, M, C) and each $S \subseteq N$, if $m(N, M, C) = m(S, M, C) + m(N \backslash S, M, C)$, then

$$f_i(N, M, C) = \begin{cases} f_i(S, M, C), & \text{if } i \in S, \\ f_i(N \backslash S, M, C), & \text{if } i \in N \backslash S. \end{cases}$$

Two subsets of agents, S and $N \backslash S$ can be connected to all the sources either separately or jointly. This property implies that if the minimal costs in two situations are the same, agents will pay the same in both circumstances.

These properties are related with some properties used in our characterization. The following proposition summarizes such relations.

Proposition 2. *(i) CM implies IIT.*

(ii) CS implies IA.

(iii) SEP implies IA.

Proof. (i) It has been proved in Bergantiños et al. [4].

(ii) Suppose that $i \in N$ is an isolated agent in (N, M, C). Then $m(N, M, C) = m(N \backslash \{i\}, M, C) + x$. By CS $\sum_{j \in N \backslash \{i\}} f_j(N, M, C) \leq m(N \backslash \{i\}, M, C)$ and $f_i(N, M, C) \leq x$. Thus, $f_i(N, M, C) = x$.

(iii) It is similar to Case (ii). Bergantiños et al. [4] □

As in Theorem 1, Bergantiños et al. [4] use *CA*, *SYM*, and *ETSC* in both characterizations of the folk rule in multi-sources minimum cost spanning tree problems.

They also use *IIT* and complete one characterization with *CS* and the other with *SEP*. By Proposition 2 the three characterizations are unrelated. Namely, no characterization is a consequence of another.

Apart from this, the proof of uniqueness in the characterization of this paper and the proof of uniqueness in the characterizations of Bergantiños et al. [4] are also unrelated. In all three cases the first step is the same. By *CA* we can consider only simple games. But now the arguments are completely different. In this paper we consider the problems C', C^k, and C'^k and depending on how a rule works in such problems uniqueness is obtained. Bergantiños et al. [4] obtain uniqueness by considering the expression of the folk rule as an obligation rule.

Acknowledgments

This work is partially supported by the Spanish Ministerio de Economía y Competitividad [grant number ECO2017-82241-R] and Consejo Nacional de Ciencia y Tecnología - CONACyT [grant number 438366].

References

[1] G. Bergantiños and L. Lorenzo. Cost additive rules in minimum cost spanning tree problems with multiple sources. Mimeo, Universidade de Vigo, 2019.

[2] G. Bergantiños and A. Navarro-Ramos. The folk rule through a painting procedure for minimum cost spanning tree problems with multiple sources. Mathematical Social Sciencies, forthcoming, 2019.

[3] G. Bergantiños, M. Gómez-Rúa, N. Llorca, M. Pulido, and J. Sánchez-Soriano. A new rule for source connection problems. *European Journal of Operational Research*, 234(3):780–788, 2014.

[4] G. Bergantiños, Y. Chun, E. Lee, and L. Lorenzo. The folk rule for minimum cost spanning tree problems with multiple sources. Mimeo, Universidade de Vigo, 2019.

[5] C. G. Bird. On cost allocation for a spanning tree: a game theoretic approach. *Networks*, 6(4):335–350, 1976.

[6] B. Dutta and A. Kar. Cost monotonicity, consistency and minimum cost spanning tree games. *Games and Economic Behavior*, 48(2):223–248, 2004.

[7] J. B. Kruskal. On the shortest spanning subtree of a graph and the traveling salesman problem. *Proceedings of the American Mathematical society*, 7(1): 48–50, 1956.

[8] H. Norde, S. Moretti, and S. Tijs. Minimum cost spanning tree games and population monotonic allocation schemes. *European Journal of Operational Research*, 154(1):84–97, 2004.

[9] R. C. Prim. Shortest connection networks and some generalizations. *Bell Labs Technical Journal*, 36(6):1389–1401, 1957.