# Effective Demand and Quantity Constrained Growth: A Simple Two-Sector Disequilibrium Approach

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**Abstract**

In this study, we construct a simple disequilibrium growth model to explore the dynamic property of effective demand. This study's main concern is the effect of the quantity constraint: How do the quantities of consumption and investment goods demand and the productive capacity affect capital accumulation? To answer this, we build a two-sector growth model with quantity constraints. One interesting result is that consumption goods demand enhances capital accumulation when the capital is sufficiently accumulated but impedes it when the capital is insufficient. The latter case is shown as a shrinking path by graphical analysis and a numerical experiment.

**Keywords**: Disequilibrium macroeconomics, Non-Walrasian analysis, Economic growth, Two-sectors, Quantity constraints

**JEL Classification**: E12, O41

## 1 Introduction

Business cycles and economic growth (or capital accumulation) are the main themes of macroeconomics. Several works have analyzed these dynamic phenomena, and the standard viewpoint today is the so-called equilibrium economics, in which the price must be adjusted until the market is in equilibrium. Although the equilibrium approach represented by "DSGE (Dynamic Stochastic General Equilibrium)" models flourishes in economic dynamics,[1] this approach overlooks an important issue: quantity constraint.

Once incomplete price adjustment is accepted, the realized transaction quantity in the market could be different from the notional demand or supply, which are derived from optimization problems. This quantity constraint in one market has a spillover effect into another market. In other words, a firm rationed in the goods market would express small labor demand, and households rationed in the labor market would reduce their consumption.

The concept of quantity constraint is specifically illustrated by the so-called disequilibrium (or non-Walrasian) approach. In disequilibrium economics, goods transactions are executed before the price adjustment is completed, and the transaction quantity is

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[1]See Christiano et al. (2018) and Galí (2018).

adjusted to match *effective* demand and supply, which are in accordance with the quantity constraint in each market.[2] A simple expression for the realized transaction of goods $i$ $\tilde{x}_i$ is as follows:

$$\tilde{x}_i = \min\{x_i^s(P, \tilde{x}_{-i}), x_i^d(P, \tilde{x}_{-i})\}, \quad \forall i,$$

where $-i$ is a set of goods index except $i$, and $P$ is the prevailing price vector. Subscription $s$ means supply, and $d$ is demand. As the demand quantity plays an important role in determining the realized transaction compared with the market equilibrium theory, the disequilibrium approach should be one interpretation of Keynes (1936).[3]

As Backhouse and Boianovsky (2013) comments, the disequilibrium approach has been forgotten these days. After the crisis of 2008, however, some researchers reconsidered the worth of disequilibrium to explain secular stagnation; see Mankiw and Weinzierl (2011) and Michaillat and Saez (2015).[4] One characteristic of disequilibrium dynamics is the persistence of the disequilibrium regime, such as Keynesian unemployment.[5] Furthermore, the adjustment of expectation, prices, and quantities in a disequilibrium sometimes induces an endogenous cycle (Bénassy, 1984), which means that this approach is able to explicitly illustrate business fluctuations as regime transitions. These merits of disequilibrium dynamic models in explaining dynamic phenomena imply the need for more analyses of such models.

This study explores a new dynamic characteristic of disequilibrium macroeconomic models: an ample goods demand is NOT a sufficient condition for a desirable capital accumulation. In other words, the large consumption goods demand impedes the reproduction of investment goods when the capital is in shortage. Although this reasoning is intuitive, the quantity constraint plays an important role in explaining it.

In this study, we build a simple two-sector growth model, à la Uzawa (1963) and Inada (1963), which has two types of goods sectors (consumption and investment) and expresses capital accumulation as the reproduction of investment goods.[6] Our model is partly neoclassical in the sense that we use a neoclassical production function and ignore the effect of money on goods trading. However, the goods/labor transaction is quantity constrained, and the demand-supply gap in each market determines the regime, unlike the neoclassical (or equilibrium) theory. The capital accumulation in each sector and wage dynamics are affected by these realized transactions, not the notional or intended transactions. The basis of the neoclassical growth model is used according to Solow (1988): growth or accumulation in the long-term is approximated by simple neoclassical

[2]Early studies in macroeconomics include Barro and Grossman (1971), Malinvaud (1977, 1980), Korliras (1975), Hildenbrand and Hildenbrand (1978), and Muellbauer and Portes (1978). Bénassy (1986) and Böhm (1989, 2017) introduce analytical methods, and Backhouse and Boianovsky (2013) introduce the history of disequilibrium approach.

[3]See Clower (1965) and Leijonhufvud (1968). Tobin (1993) criticizes the so-called new Keynesian school, as the author overlooks the quantity constraint. Note that the disequilibrium theory has another context: it supplies a rigorous model of temporary equilibrium in Hicks (1939); see Grandmont (1972, 1983) and Grandmont and Laroque (1976).

[4]Although they refer to disequilibrium models such as in Barro and Grossman (1971), their analytical tools are not of the disequilibrium type but are new Keynesian with sticky prices.

[5]Böhm (1978), Honkapohja (1980), Malinvaud (1980), Blad and Zeeman (1982), and Picard (1983) adopt dynamic analysis into disequilibrium economics, and they show that the steady state would be in a disequilibrium regime.

[6]As Murakami (2018) pointed out, not many studies utilize two-sector (post) Keynesian models. The representative studies include Okishio (1967), Sato (1985), Dutt (1988), Hori (1998), and Murakami (2018). For disequilibrium macroeconomics, Fourgeaud, Lenclud and Michel (1981) construct a static two-sector model.

growth models such as in Solow (1956), but we need to conduct a medium-term analysis to illustrate it around the steady state. Therefore, we combine a neoclassical growth model and a disequilibrium cycle model in this study. The steady state in the long term is of neoclassical type, but the dynamics are affected by disequilibrium adjustment.[7]

The rest of this paper is organized as follows. Section 2 presents the framework of the two-sector quantity constrained model. The economy is characterized by the regime dividing, and the regime is determined by the accumulated capital and prevailing real wage. In section 3, we formulate a dynamical system of capital accumulation. Our system is autonomous of the capital in each sector and the real wage, which are controlled by the realized investment and wage adjustment. In section 4, we specify the function and analyze the dynamic property of our system. Stability of the steady states is confirmed, and we find the existence of a "shrinking path," on which capital accumulation is impeded by excess consumption goods demand. In section 5, we summarize the analysis.

# 2 The Model

In this section, we construct a static model. Before the analysis, we set the following rules regarding the mathematical conditions and notations; all the functions in this paper are twice continuously differentiable; let $\dot{x}$ denote the time derivative of $x$, or $\dot{x} = dx/dt$; let $f_i$ denote the partial derivative of function $f$ with respect to the $i$th variable, that is, $f_1 = \partial f(x_1, x_2, x_3)/\partial x_1$.

The model consists of households, firms, and the government. For simplicity, we assume that the households are identical and each sector has one representative firm. In this study, the economic agents deal with labor, consumption goods, investment goods, and money. The quantity variables such as consumption and capital are per capita variables denominated by the population (or inelastic labor supply) $L^s$.

Our crucial assumptions are as follows. First, the accumulated capital of each sector is irreversible. Second, the households have no forward-looking expectation. Third, money is only a buffer for the transactions and plays no role in determining the realized transaction.[8]

## 2.1 Households

The households supply one unit labor inelastically and demand each good and money. We assume the population growth rate is constant:

$$\frac{\dot{L}^s}{L^s} = n. \tag{2.1}$$

[7]Many disequilibrium growth models use a neoclassical basis; Ito (1980) extends Solow (1956); Ginsburgh et al. (1985) extends an optimal growth model; and Weddepohl and Yildirim (1993) use an overlapping-generations framework. As extensions of Ito (1980), Sgro (1984) builds a monetary growth model, and van Marrewijk and Verbeek (1993, 1994) utilize a two-sector framework.

[8]Expectation is an important matter for the disequilibrium theory, but it has not been explored well. Neary and Stiglitz (1983) argue that a pessimistic expectation enlarges a disequilibrium region. Lorentz and Lohmann (1996) show that different expectation formulations cause a drastic change in the dynamic property such as chaos. We use a static expectation model following Uzawa (1963) and ignore these problems to construct a starting point.

Let $j = I, C$ denote the sector index ($I$: investment sector, $C$: consumption sector). The budget constraint is written as follows:

$$P_C c^d + P_I(i_I^d + i_C^d) + \dot{m}^d + mn = We + R_I k_I + R_C k_C, \tag{2.2}$$

where $P_j$ is the price of goods $j$, $c^d$ is consumption demand, $i_j^d$ is investment demand for sector $j$, $m$ is the holding money per capita, $W$ is the nominal wage, $e$ is employment rate, $R_j$ is the rate of return of the holding capital on sector $j$, and $k_j$ is the holding capital of sector $j$ per capita. The money demand is expressed as the net increase in the money holding per capita, $\dot{m}^d$.

Households determine the quantity of consumption, purchase of investment good (gross increase of the holding capital), and net increase of holding money under the budget constraint. The saving consists of the capital and money.

Note that the right-hand side of eq.(2.2) includes $e$, or realized employment, not the (notional) labor supply. This implies that households express the goods demand considering the realized employment and realized capital return. Therefore, the expressed demand must be effective demand.

The households' activities are completely defined when the goods demand functions $c^d$, $i_I^d$, and $i_C^d$ are defined. In this study, we impose the following two assumptions on the goods demand to obtain the uniqueness of the temporary equilibrium.

**Assumption 1.** The expressed consumption demand depends only on the income from the consumption sector, and the demand has an upper limit $c^M$:

$$c^d = c^d(we, r_C k_C),\ c^d(0,0) \geq 0,\ 0 < c_2^d \leq c_1^d < 1, \tag{2.3}$$

$$\exists c^M > 0, c^d < c^M, \forall we, r_C k_C, \tag{2.4}$$

where $r_j = R_j/P_j$ and $w = W/P_C$.

The above equations imply that the marginal propensity to consume is below one, and households prefer expending their labor income on consumption. For simplicity, we exclude the capital income from the $I$-sector. This simplification is not a problem when dividing the regimes on the state space: it only affects the area of each regime.

**Assumption 2.** The expressed investment demand for each sector is written as the following investment function:

$$i_j^d = i_j^d(r_j, k_j),\quad i_j^d(0, k_j) > 0,\quad \frac{\partial i_j^d}{\partial r_j} > 0,\quad \frac{\partial^2 i_j^d}{\partial r_j^2} < 0,\quad \forall k_j > 0,\quad j = I, C. \tag{2.5}$$

Households determine the portfolio of the increase in saving by considering the rate of return and holding capitals today. A high rate of return induces the purchase of investment goods.

## 2.2 The government

The government creates money and purchases goods.[9] In this study, we do not analyze the impact of government policy in detail. For simplicity, we impose the following assumption.

[9] Of course, the government's budget can be offset by tax and bonds, and these resources affect the effective goods demand; see Böhm (2017). However, the method of securing the budget does not play an important role in the dynamic property of our model.

**Assumption 3.** The government only purchases a fixed quantity of consumption goods $g_C$, and the purchase is financed by creating money $\dot{m}^s + nm$:

$$P_C g_C = \dot{m}^s + nm, \quad g_C = \text{const.} \tag{2.6}$$

Besides, a government purchase is not rationed, which means the consumption demand constraint is imposed only on households.

Therefore, goods demand $y_j^d$ is written as follows:

$$y_I^d = i_I^d + i_C^d, \tag{2.7}$$
$$y_C^d = c^d + g_C^d. \tag{2.8}$$

## 2.3 Firms

Each sector has one representative firm, which has a homothetic production technology. We assume that the investment goods sector needs only capital for production, which means that households are employed only in the consumption goods sector.

Each firm produces each good using production factors, and all the profit is distributed to the households. The distribution is written as follows:

$$r_I k_I = F_I(k_I), \tag{2.9}$$
$$r_C k_C = F_C(e, k_C) - we. \tag{2.10}$$

Production function $F_j$ is assumed to have homotheticity, so $F_I$ should have a proportional form:

$$F_I = u\beta k_I, \; \beta = \text{const,} \tag{2.11}$$

where $u \in [0, 1]$ is the utilization rate, and $\beta > 0$ is the production capital ratio with full utilization. The firm of the $I$-sector controls $u$ to adjust production to the effective investment demand, $y_I^d$. The production function of the $C$-sector or $F_C(e, k_C)$ is assumed to be of the so-called neoclassical type:

$$F_C(0, 0) = 0, \quad F_{Cj} > 0, \quad F_{Cjj} < 0, \quad \lim_{e\to\infty} F_{C1} = 0, \quad \lim_{e\to+0} F_{C1} = \infty, \quad j = 1, 2. \tag{2.12}$$

The firm of the $C$-sector controls employment $e$ to maximize the present profit by considering the effective goods demand $y_C^d$, as the capital stock is given in this static model. The limitation constraints, that is, the so-called Inada condition, refer only to $e$ since the conditions for marginal productivity of $k_C$ are not necessary for our model. The firm solves the following profit maximizing problem:

$$\max_{l^d} F_C(l^d, k_C) - wl^d \; \text{ subject to } \; F_C \le y_C^d. \tag{2.13}$$

The solution $l^d$ for this problem is the expressed labor demand, and it has two forms. If the goods demand is sufficiently large, the firm could control $e$ to maximize the profit under the prevailing prices, ignoring the quantity constraint of goods demand. This solution is the notional labor demand. On the other hand, if there is a shortage of goods demand for the notional profit maximization, employment must be determined by the goods demand. This labor demand must be called effective.

According to the following proposition, labor demand is a continuous function of $(k_C, w)$.

**Proposition 1.** The solution for eq.(2.13) is written as follows:

$$l^d = \min\{\phi(w)k_C, \psi(k_C, w)\}, \text{ where } \phi' < 0, \psi_1 < 0, \psi_2 > 0, \psi_{12} < 0, \psi_{11}, \psi_{22} > 0. \tag{2.14}$$

*Proof.* See Appendix A. □

Note that $\phi(w)k_C$ is the notional labor demand, and $\phi(k_C, w)$ is the effective labor demand. Each variable works oppositely on the two labor demand functions: high real wage $w$ lowers the notional demand and increases the effective demand since the goods demand increases.

## 2.4 Temporary equilibrium and uniqueness of transactions

In the static model, the prevailing prices and given stock variables determine the goods transactions. We call these transactions temporary equilibrium.

**Definition 1.** The temporary equilibrium is a solution $(e, c, i_I, i_C, \dot{m})$ to the following simultaneous equations:

$$e = \min\{\phi(w)k_C, \psi(k_C, w), 1\}, \tag{2.15}$$

$$c = \min\{c^d, F_C(e, k_C) - g_C\}, \tag{2.16}$$

$$i_j = \frac{i_j^d}{i_I^d + i_C^d} \cdot \min\{i_I^d + i_C^d, \beta k_I\}, \quad j = I, C, \tag{2.17}$$

$$\dot{m} = \dot{m}^s. \tag{2.18}$$

Note that according to eq.(2.17), the investment follows the proportional rationing rule.

This temporary equilibrium is uniquely determined.

**Proposition 2.** Let the exogenous variables $(P_I, g_c, m) \in \mathbb{R}^2_{++} \times \mathbb{R}$ be given. Then, for any set of positive numbers $X = (k_C, k_I, w) \in \mathbb{R}^3_{++}$, a temporary equilibrium $(e, c, i_I, i_C, \dot{m}) \in (0, 1] \times \mathbb{R}^4_{++}$ is uniquely determined.

*Proof.* First, the uniqueness of $\dot{m}$ is obvious from eq.(2.6). In the following, we confirm the uniqueness of $e$. According to Appendix A, $\phi$ and $\psi$ are real-valued functions, and therefore the right-hand side of eq.(2.15) is uniquely determined by $X$. Using this, $e$, $c$, and $i_C^d$ are also determined. The rest are $i_I$ and $i_C$.

If the realized investment goods production $y_I$ is determined, the investment for each sector can be calculated. Note that $i_I^d$ depends on $y_I^d$, which implies that there is some multiplier process. We show that this process has a unique solution. Under Assumption 2, the realized investment combining the two-sectors is written as follows:

$$y_I = \min\{i_I^d(y_I/k_I, k_I) + i_C^d,\ \beta k_I\}, \quad i_I^d(0, k_I) + i_C^d(e) > 0. \tag{2.19}$$

Let $\sigma(y_I)$ denote the right-hand side of eq.(2.19). It is obvious that $\sigma$ is continuous, $\sigma(0) > 0$, and $\lim_{y_I \to \infty} \sigma(y_I) < \infty$, which mean that the solution for eq.(2.19) exists due to the intermediate-value theorem. Besides, $\sigma' \geq 0$ and $\sigma'' \leq 0$ confirm the uniqueness of solution $y_I$. The proportional rationing rule in eq.(2.17) preserves the uniqueness. □

## 2.5 Regimes of temporary equilibria

As we have seen, variables $X = (k_C, k_I, w)$ uniquely determine the transactions of labor and goods. In the following, we should characterize this temporary equilibrium by dividing regimes such as "Keynesian unemployment" and "Equilibrium."

The following items show the regimes and their condition equations for $X$. The names of regimes follow those in Malinvaud (1977).

- **Keynesian unemployment (KU)**

  Keynesian unemployment is caused by the shortage of (consumption) goods demand. The expressed labor demand is effective demand $\psi$, and it is below unity. Thus, the condition equation is

  $$\psi(k_C, w) \leq \phi(w)k_C \text{ and } \psi(k_C, w) < 1.$$

  When the condition equation holds, both the labor market and consumption goods market face excessive supply.

- **Classical unemployment (CU)**

  Too high wage also causes unemployment even if the goods demand is sufficiently large. We call this phenomenon classical unemployment. The firm restricts employment for profit maximization, and therefore the labor demand is notional, $\phi(w)k_C$:

  $$\phi(w)k_C \leq \psi(k_C, w) \text{ and } \phi(w)k_C < 1.$$

  The consumption goods production is confined to notional supply. Therefore, excess demand occurs in the consumption goods market, while the labor market has excess supply.

- **Repressed inflation (RI)**

  In the repressed inflation regime, large goods demand promotes goods' production, but the labor is insufficient for the desired production. Both the consumption goods market and labor market have excess demand. The condition equation is

  $$1 \leq \phi(w)k_C, \psi(k_C, w).$$

- **Equilibrium (EQ)**

  When the demand and supply are in accord in both the markets, the economy is defined as an equilibrium. This regime is the boundary between KU and RI, and the condition equation is

  $$1 = \psi(k_C, w) \leq \phi(w)k_C.$$

  In the EQ regime, the firm does NOT always maximize the profit. We call the EQ with profit maximization the "Walrasian equilibrium" (WE), which means that notional supply and demand are achieved, and the markets are in equilibrium. The following condition corresponds to WE:

  $$1 = \psi(k_C, w) = \phi(w)k_C$$

  .

These regimes cover up $\mathbb{R}^3_{++}$, which is the space of $X$. Therefore, $X$ determines the regime of economy as well as the quantity of goods transactions. Note that the condition equations are independent of $k_I$, which means that we will illustrate the regimes on the $k_C$ - $w$ plane. Figure 1 shows the regime division on the plane. Curves on the plane indicate the borders of employment-determinant variables, and, in particular, the solid line curves are the borders of regimes.

KU is the region with low wage, and the region expands as capital intensity of the consumption sector increases. Low wage decreases the quantity of goods demand (and effective labor demand), and restrained employment shrinks the consumption goods demand. The shortage of consumption goods demand and substitution of labor for capital are the causes of unemployment in this regime.

CU occurs when the wage is too high or the capital intensity of consumption sector is insufficient. The firm determines the employment following the first-order condition for profit maximization, which is decreasing in $w$ and increasing in $k_C$. The cause of unemployment is the high wage cost compared with the marginal production of labor. Note that the production function has a certain degree of complementarity with the production factors. This implies that ample capital equipment complements labor productivity.

Both wage $w$ and capital intensity $k_C$ work in opposite directions in each unemployment regime; that is, a decline in wage is desirable in the CU regime but is harmful in the KU regime. RI, in which the labor demand exceeds the labor supply, lies between the two unemployment regions. In this regime, the shortage of labor supply induces excess demand in the consumption goods market.

Let capital equipment or $k_C$ increase to resolve the shortage of goods supply and excess demand for labor. Then, we could attain the EQ regime as a borderline between the RI and KU regimes. Besides, the WE regime is a center point among the three disequilibrium regimes, and this point is part of the EQ curve. The position of WE is intuitive.

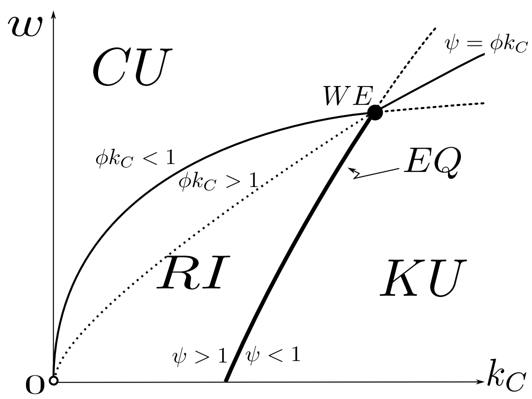


Figure 1: Regime dividing

As the regime dividing is independent of $k_I$, the regions on the plane are invariant regardless of which $k_I \in \mathbb{R}_{++}$ we choose. Note that, however, exogenous variable $g_C$ changes the borders: an increase in $g_C$ (goods demand expansion) increases $\psi$, and then curve $\psi = \phi k_C$ pivots clockwise and curve $\psi = 1$ shifts to the right. Therefore, an increase in government purchases narrows KU but expands the other disequilibrium regions.

# 3 Formulation of Dynamic Equations

As we have seen, the three variables, $X = (k_C, k_I, w) \in \mathbb{R}^3_{++}$, determine the unique temporary equilibrium that is characterized by a(n) (dis)equilibrium regime. In this section, we formulate the dynamics of $X$.

For simplicity, we impose the new assumption as follows.

**Assumption 4.** $(P_I, g_c) \in \mathbb{R}^2_{++}$ is constant.

The constant $P_I$ implies that there is no price adjustment in the investment goods market. Although this is unrealistic, the dynamic property would not be destroyed. The private investment in this model is affected by the profitability, and the price perturbations would not be very influential. Second, the constant $g_C$ means that the fiscal policy is not discretionary and the aggregate government demand expands at the rate of population growth.

Under the above assumption, we set the dynamic system as follows:

$$\dot{k}_C = i_C - (n+\delta)k_C, \tag{3.1}$$

$$\dot{k}_I = i_I - (n+\delta)k_I, \tag{3.2}$$

$$\dot{w} = \dot{w}(l^d), \tag{3.3}$$

where $\delta > 0$ is a constant capital depreciation rate that is common between the two-sectors. Note that capital accumulation is determined by the realized investment, which is sometimes different from the investment demand. As the money growth, $\dot{m}/m$, is independent of the system, hereinafter we ignore it. To exclude a meaningless case, we introduce the following assumption:

**Assumption 5.**

$$\beta > n+\delta \quad \text{and} \quad \beta k_I > i^d_I(\beta, k_I), \quad \forall k_I > 0. \tag{3.4}$$

According to this assumption, the capital in the investment goods sector will always be able to expand, as long as all the capital is used in the production of investment goods. This reproduction condition is a natural assumption.

## 3.1 Investment demand and the realized investment

Capital accumulation is determined by the realized investment, not the investment demand. We analyze the condition for the existence of the quantity constraint on the investment and formulate the realized investment function.

First, the investment demand in each sector depends on the real return rate from the production, which is calculated as follows:

$$r_I = \frac{y_I}{k_I} = u\beta, \tag{3.5}$$

$$r_C = \frac{y_C - we}{k_C} = f_C\left(\frac{e}{k_C}\right) - w\frac{e}{k_C}, \tag{3.6}$$

where $f_C(x) = F_C(x, 1)$. $r_C$ is a function of $(k_C, w)$, and the partial derivatives are calculated as follows:

$$KU: \frac{\partial r_C}{\partial k_C} = (f'_C - w)\frac{\psi_1 k_C - \psi}{k_C^2} < 0, \qquad \frac{\partial r_C}{\partial w} = \frac{(f'_C - w)\psi_2 - \psi}{k_C} < 0, \tag{3.7}$$

$$CU: \frac{\partial r_C}{\partial k_C} = 0, \qquad \frac{\partial r_C}{\partial w} = -\phi < 0, \tag{3.8}$$

$$RI: \frac{\partial r_C}{\partial k_C} = -\frac{f'_C - w}{k_C^2} < 0, \qquad \frac{\partial r_C}{\partial w} = -\frac{1}{k_C} < 0. \tag{3.9}$$

Therefore, the investment demand for consumption sector $i_C^d$ is a function of $(k_C, w)$:

$$i_C^d = i_C^d\left(r_C(k_C, w), k_C\right) = i_C^d(k_C, w). \tag{3.10}$$

As $i_C^d$ is given, the investment demand for investment sector $i_I^d$ determines the combined investment demand $\hat{y}_I^d$. The equation for $\hat{y}_I^d$ is

$$i_I^d(\hat{y}_I^d/k_I, k_I) + i_C^d(k_C, w) = \hat{y}_I^d. \tag{3.11}$$

This investment demand is uniquely determined if it exists,[10] and it is constrained by quantity capacity $\beta k_I$ when the following condition is satisfied:

$$\beta k_I < i_I^d(\beta, k_I) + i_C^d. \tag{3.12}$$

If this inequality holds, the investment demand for each sector is rationed according to the rule in eq.(2.17). The realized investment is smaller than the demand when rationing occurs.

For the convenience of notations, we define the subset $S \subset \mathbb{R}^3_{++}$ as the region in which the investment is not quantity constrained:

$$S \equiv \{X \in \mathbb{R}^3_{++} | i_C^d(k_C, w) \leq \beta k_I - i_I^d(\beta, k_I)\}. \tag{3.13}$$

We summarize the realized investment as follows:

$$X \in S \rightarrow \begin{cases} i_C &= i_C^d, \\ i_I &= i_I^d = \hat{y}_I^d - i_C^d, \end{cases} \tag{3.14}$$

$$X \notin S \rightarrow \begin{cases} i_C &= (i_C^d / y_I^d)\beta k_I \equiv \eta(k_C, k_I, w)\beta k_I, \\ i_I &= (i_I^d / y_I^d)\beta k_I = [1 - \eta(k_C, k_I, w)]\beta k_I. \end{cases} \tag{3.15}$$

Let $\partial S$ denote the boundary of $S$, and by definition,

$$i_C^d \geq \eta\beta k_I \text{ and } i_I^d \geq (1-\eta)\beta k_I \quad \text{if } X \notin S, \tag{3.16}$$

$$\lim_{X \to \partial S} i_C^d = \eta\beta k_I \text{ and } \lim_{X \to \partial S} i_I^d = (1-\eta)\beta k_I. \tag{3.17}$$

According to these equations, the quantity constraint works smoothly in the sense that the constrained investment is equal to the demand on the boundary. This property is the result of the minimum function's continuity.

[10] If the solution for eq.(3.11) does not exist, we define $\hat{y}_I^d$ as $+\infty$. This means that the realized investment is determined by the capital capacity, $\beta k_I$.

### 3.2 Real wage dynamics

In this study, we adopt the price adjustment process (Walrasian adjustment) for the real wage dynamics.[11] We use a linear equation:

$$\dot{w} = \omega(l^d - \gamma) = \omega(\min\{\phi(w)k_C, \psi(k_C, w)\} - \gamma), \quad \omega > 0. \tag{3.18}$$

$\gamma \in (0, 1]$ is interpreted as a natural employment rate.

### 3.3 The dynamical system

We formulate the autonomous system for $X$:

$$\dot{k}_C = \begin{cases} i^d_C(k_C, w) - (n+\delta)k_C & \text{if } X \in S, \\ \eta(k_C, k_I, w)\beta k_I - (n+\delta)k_C & \text{otherwise,} \end{cases} \tag{3.19}$$

$$\dot{k}_I = \begin{cases} i^d_I(k_C, k_I, w) - (n+\delta)k_I & \text{if } X \in S, \\ [1 - \eta(k_C, k_I, w)]\beta k_I - (n+\delta)k_I & \text{otherwise,} \end{cases} \tag{3.20}$$

$$\dot{w} = \omega(\min\{\phi(w)k_C, \psi(k_C, w)\} - \gamma). \tag{3.21}$$

The steady state of the system is the state in which the time derivatives of the three variables become zero.

**Definition 2.** The steady state $X^*$ is defined as follows:

$$X^* = \{X \in \mathbb{R}^3_{++} | \dot{k}_C(X^*) = \dot{k}_I(X^*) = \dot{w}(X^*) = 0\}.$$

Before the dynamic analysis, we should note that the reproduction of $k_I$ is crucial for our system: the shape of function $y^d_I(r_I, k_I)$ plays an important role in the dynamic property. In this study, we utilize a simple function: $i^d_j$ depends only on the rate of return $r_j$, which means the investment determination simply responds to capital utilization $u$, or a simple disequilibrium signal in the $I$-sector. Although we have another choice, this function is the best way to derive our conclusion since it has the simplest form to retain some *nonlinearity*, which is needed to illustrate the spillover effect between the two-sectors. This problem will be discussed after the dynamic analysis.

# 4 Dynamic Analysis with the Specified Function

The return on capital is affected by the goods market disequilibrium: if the goods supply is constrained, profitability of the sector decreases. In this section, we specify the investment function as follows:

$$i^d_j = i^d_j(r_j), \quad (i^d_j)' > 0, (i^d_j)'' < 0, \lim_{r_j \to \infty} (i^d_j)''(r_j) = 0, i^d_I < \bar{i}_I, \quad j = I, C. \tag{4.1}$$

The investment demand simply responds to the rate of return, which means that the investment is increased to adjust the rate of return to some desired level.[12] According to the last inequality, the investment demand for $I$ is bounded by some positive $\bar{i}_I$.

[11] We ignore the consumption goods price dynamics in this study. If both the price and nominal wage are adjusted, the difference in adjustment speeds changes the dynamic property; see Picard (1983). Our setting follows Ginsburgh et al. (1985), who explore the Ramsey model with disequilibrium.

[12] Our formulation is a simplified version of Malinvaud (1980) and Picard (1983).

Under the formulation, the investment demand is calculated as follows:

$$i_I^d(\hat{y}_I^d/k_I) + i_C^d(k_C, w) = \hat{y}_I^d \tag{4.2}$$

The effective demand $\hat{y}_I^d$ is a function of $(k_I, i_C^d)$, and the partial derivatives of $i_I^d(k_I, i_C^d) = \hat{y}_I^d - i_C^d$ are

$$\frac{\partial i_I^d}{\partial k_I} < 0, \quad \frac{\partial i_I^d}{\partial i_C^d} > 0. \tag{4.3}$$

The realized investment under the quantity constraint is determined by the proportion function $\eta(k_I, i_C^d)$, as shown in eq.(3.15). $\eta$ is an increasing function of each variable:

$$\frac{\partial \eta}{\partial k_I} = -\frac{i_C^d}{(i_I^d + i_C^d)^2}\frac{\partial i_I^d}{\partial k_I} > 0, \tag{4.4}$$

$$\frac{\partial \eta}{\partial i_C^d} = \frac{i_I^d}{(i_I^d + i_C^d)^2}\left[1 - \frac{\partial i_I^d}{\partial i_C^d}\frac{i_C^d}{i_I^d}\right] > 0. \tag{4.5}$$

First, an increase in $k_I$ reduces $i_I^d$, and then the proportion for investment of the $C$-sector expands. An increase in the investment demand for the $C$-sector increases the $I$-sector's investment, but the expansion of $i_I^d$ is not as large as that of $i_C^d$. This impeding effect of capital accumulation results from the concavity of $i_I^d(r_I)$.

## 4.1 Graphical analysis

For convenience of intuition, we visualize the dynamical system using $\mathbb{R}^3_{++}$. First, we analyze the shape of nullclines, or curves that satisfy the time derivative of each variable that is equal to zero.

$w$ - nullcline is equal to the isocline of the employment rate $e$, which is independent of $S$ or $S^c$.

The shape of $k_C$ - nullcline is affected by the quantity constraint of investment. In $S$, the investment is independent of $k_I$.

Analogous with Figure 1, we illustrate the two nullclines on the $k_C$ - $w$ plane. The slopes of the nullclines are as follows:

$$k_C : \frac{dw}{dk_C} = \begin{cases} \frac{dw}{dk_C}|_{r_C = \text{const}} + \frac{n+\delta}{i_C^{d\,\prime}\frac{\partial r_C}{\partial w}} < 0 & \text{if } X \in S, \\ \frac{dw}{dk_C}|_{r_C = \text{const}} + \frac{n+\delta}{i_C^{d\,\prime}\frac{\partial r_C}{\partial w}}\left(\frac{\beta k_I}{\hat{y}_I^d}\frac{i_I^d}{\hat{y}_I^d}\left[1 - \frac{\partial i_I^d}{\partial i_C^d}\frac{i_C^d}{i_I^d}\right]\right)^{-1} < 0 & \text{if } X \notin S. \end{cases} \tag{4.6}$$

$$w : \frac{dw}{dk_C} = \frac{\partial e}{\partial k_C} > 0. \tag{4.7}$$

These equations indicate that the steady state is locally determinate if it exists. Besides, $k_C$ - nullcline becomes steeper when it is in $S^c$.

$k_I$ - nullcline is a graph of $(i_C^d, k_I)$, as implied by eq.(4.2). Eq.(3.10) shows that the investment demand for $C$-sector is uniquely determined by $(k_C, w)$, which means that the variable $i_C^d$ contains the information of $(k_C, w)$.

In Figure 2, $k_I$ - nullcline is illustrated on the $i_C^d$ - $k_I$ plane. Each level of $i_C^d$ corresponds with the level of $r_C$, which implies that $k_I$ - nullcline is an isocline of $r_C$ on the $k_C$ - $w$ plane. The figure shows that the dynamics of $k_I$ are stable, and capital accumulation is impeded in $S^c$ as the increase in $i_C^d$ shrinks $k_I$.

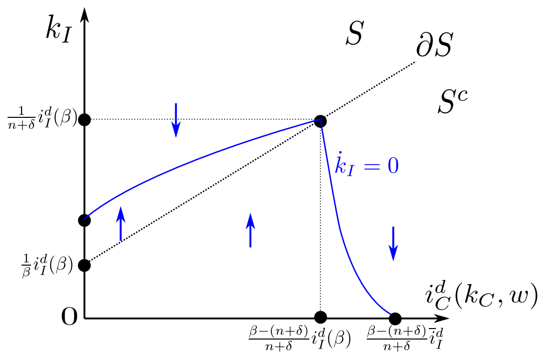


Figure 2: $k_I$ - nullcline

We are now ready to illustrate the dynamic system on the $\mathbb{R}^3_{++}$ space. First, we analyze the existence of a steady state. Before the analysis, we define the regions for convenience.

**Definition 3.** The region in which $l^d = \phi(k_C, w)$ is *Keynesian*, and the region in which $l^d = \psi(w)k_C$ is *Classical*.

This region dividing is about the labor demand. The border of the regions is the set $\psi = \phi k_C$, which is independent of $k_I$. Both regions include the RI regime.

Let us see the dynamics in $S$. Since $w$ - nullcline is the isocline of $e$, at most one steady state might exist in each region, as shown in Figures 3 and 4. As $k_I$ becomes smaller, $k_I$ - nullcline and $\partial S$ on the $k_C$ - $w$ plane expand into the northeast direction. Let ① and ② denote the indexes of steady state in the *Keynesian* and *Classical* regions, respectively. Figure 4 shows that ① corresponds with the larger $k_I$, which implies that if $k_I$ at ② (we call it $k_I^{(2)}$) is smaller than $i_I^d(\beta)/(n+\delta)$, at least one steady state exists in $S$.

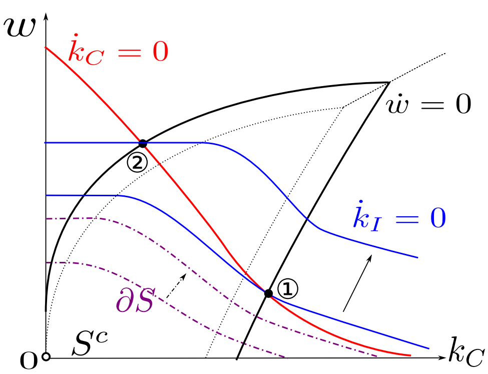


Figure 3: Steady states in $S$: $k_C$ - $w$ plane (the arrow indicates the decrease in $k_I$)

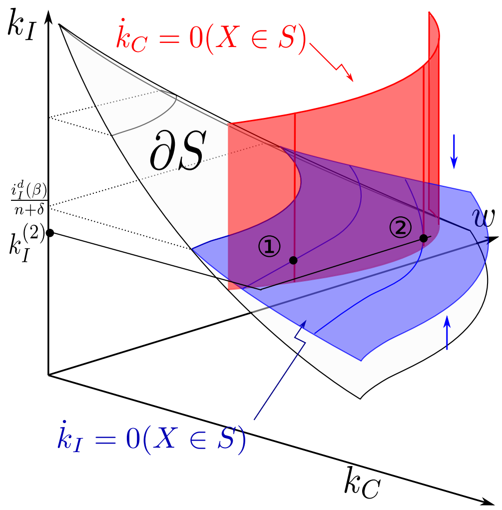


Figure 4: Steady states in $S$: $\mathbb{R}^3_{++}$ space

Similarly, $S^c$ also has at most two steady states; see Figures 5 and 6. As $k_I$ decreases, the $k_C$ - and $k_I$ - nullclines shift to the southwest on the $k_C$ - $w$ plane. Therefore, the steady state in the *Keynesian* region indexed as ③ corresponds more with the larger

$k_I$ than steady state ④ in the *Classical* region. This implies that $k_I^{(4)} < i_I^d(\beta)/(n+\delta)$ is a sufficient condition for the existence of the steady state in $S^c$. When we calculate the steady state using specified functions, the existence condition is derived. Generally, sufficiently large $\beta$ (reproduction is not difficult) guarantees the existence.

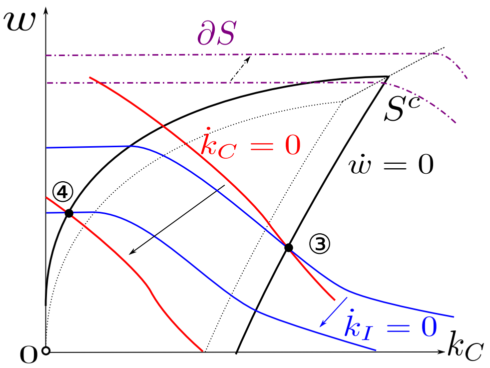


Figure 5: Steady states in $S^c$: $k_C$ - $w$ plane (the arrow indicates the decrease in $k_I$)

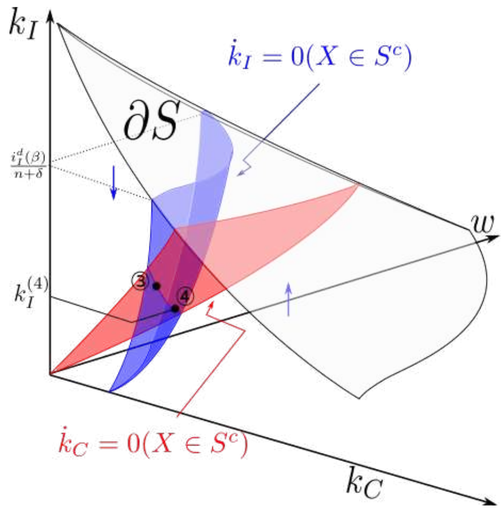


Figure 6: Steady states in $S^c$: $\mathbb{R}^3_{++}$ space

## 4.2 Stability

Hereinafter, we suppose that all possible steady states (①, ..., ④) exist.[13] We check the stability of each steady state.

Note that the dynamics of $(k_C, w)$ in $S$ are autonomous, which means that we will consider the "sub"system of these variables. For this subsystem, the following lemma is derived.

**Lemma 1.** Consider the dynamics of $(k_C, w)$ in $S$, which is an autonomous system. If the steady states exist, the steady state in the *Keynesian* region is a saddle point, and that in the *Classical* region is locally asymptotically stable.

*Proof.* Consider a linear approximation of the subsystem at the steady state. The Jacobian matrix $\mathbf{J}_{\text{sub}}$ is

$$\mathbf{J}_{\text{sub}} = \begin{pmatrix} i_C^{d\,\prime}\frac{\partial r_C}{\partial k_C} - (n+\delta) & i_C^{d\,\prime}\frac{\partial r_C}{\partial w} \\ \omega\frac{\partial l^d}{\partial k_C} & \omega\frac{\partial l^d}{\partial w} \end{pmatrix} = \begin{pmatrix} \ominus & \ominus \\ \omega\frac{\partial l^d}{\partial k_C} & \omega\frac{\partial l^d}{\partial w} \end{pmatrix} \tag{4.8}$$

The sign of the second row of $\mathbf{J}_{\text{sub}}$ is $(\ominus\ \oplus)$ in the *Keynesian* region, and it is $(\oplus\ \ominus)$ in the *Classical* region. Therefore, we obtain

$$\textit{Keynesian}\text{ region}: \det\mathbf{J}_{\text{sub}} < 0, \tag{4.9}$$

$$\textit{Classical}\text{ region}: \det\mathbf{J}_{\text{sub}} > 0, \quad \mathrm{tr}\mathbf{J}_{\text{sub}} < 0. \tag{4.10}$$

□

[13] When steady state ② does not exist, an odd dynamic such as a cycle emerges; see Appendix B.

Figure 7 shows the subsystem with $\gamma < 1$, and the paths in the KU regime are saddle paths. Note that in the case of $\gamma = 1$, the steady states lie on the borderlines of regimes, and the derivative equations are different between the regimes. Analysis of this "patch worked" system is usually complicated.[14] However, the stability property in this study is not affected by the regime-switching, since the dynamic system is continuous among regimes. Figure 8 shows the unique steady state in $S$, which is the WE. In this case, the two steady states merge. The stability of this steady state on WE is complicated: the saddle paths separate the space into two, wherein one space is locally stable but the other is unstable.

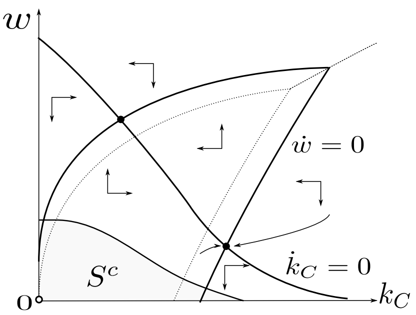


Figure 7: Subsystem: $\gamma < 1$

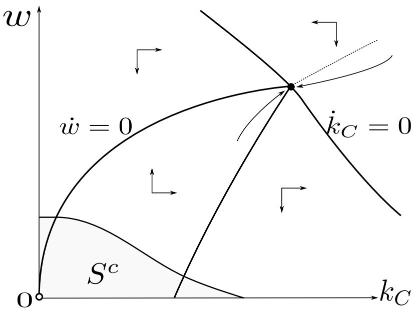


Figure 8: Subsystem: The Walrasian equilibrium is the steady state

Using the subsystem, we derive the following proposition.

**Proposition 3.** When the steady states exist in $S$, the steady state in the *Keynesian* region is a saddle point with a two-dimensional locally stable manifold and a one-dimensional unstable manifold. The other steady state, which is in the *Classical* region, is locally asymptotically stable.

*Proof.* Let $\mathbf{J}$ denote the Jacobian matrix of the linearized system at the steady states. We use the notation $P_J(\lambda)$ for the characteristic polynomial of $\mathbf{J}$, and $P_{\text{sub}}(\lambda)$ for the characteristic polynomial of $\mathbf{J}_{\text{sub}}$:

$$\begin{aligned} P_J(\lambda) &= \lambda^3 - (\text{tr}\mathbf{J})\lambda^2 + (\det\mathbf{J}_{11} + \det\mathbf{J}_{22} + \det\mathbf{J}_{33})\lambda - \det\mathbf{J} \\ &= \lambda^3 - \left(\text{tr}\mathbf{J}_{\text{sub}} + \frac{\partial \dot{k}_I}{\partial k_I}\right)\lambda^2 + \left(\frac{\partial \dot{k}_I}{\partial k_I}\text{tr}\mathbf{J}_{\text{sub}} + \det\mathbf{J}_{\text{sub}}\right)\lambda - \frac{\partial \dot{k}_I}{\partial k_I}\det\mathbf{J}_{\text{sub}} \\ &= \left(\lambda - \frac{\partial \dot{k}_I}{\partial k_I}\right)P_{\text{sub}}(\lambda), \end{aligned} \tag{4.11}$$

where $\det\mathbf{J}_{ii}$ is $(i,i)$ minor determinant for $\mathbf{J}$. As $\partial \dot{k}_I/\partial k_I < 0$, the solutions for equation $P_J(\lambda) = 0$ are composed of the solutions for $P_{\text{sub}}(\lambda) = 0$ and another negative root. Using Lemma 1, the proof is completed. □

As we only add the stable dynamics of $k_I$ to the subsystem, the dynamic property is not so different from the subsystem. If the steady state in $S$ is WE, paths often diverge since the steady state is not stable.

[14]See Honkapohja and Ito (1983) and Eckwert and Schittko (1988).

The stability of the steady states in $S^c$ is not obvious: it depends on the forms of functions and the parameters.

Let $\mathbf{J}_c$ denote the Jacobian matrix in $S^c$. Using eqs. (3.7)–(3.9), the signs of the elements of $\mathbf{J}_c$ are as follows:

$$\mathbf{J}_c = \begin{pmatrix} \varepsilon\frac{\partial r_C}{\partial k_C} - (n+\delta) & \tau + \eta\beta & \varepsilon\frac{\partial r_C}{\partial w} \\ -\varepsilon\frac{\partial r_C}{\partial k_C} & -\tau & -\frac{\partial r_C}{\partial w} \\ \omega\frac{\partial l^d}{\partial k_C} & 0 & \omega\frac{\partial l^d}{\partial w} \end{pmatrix} = \begin{cases} \begin{pmatrix} \ominus & \oplus & \ominus \\ \oplus & \ominus & \oplus \\ \ominus & 0 & \oplus \end{pmatrix}, & \textit{Keynesian} \text{ region } ③ \\ \begin{pmatrix} \ominus & \oplus & \ominus \\ 0 & \ominus & \oplus \\ \oplus & 0 & \ominus \end{pmatrix}, & \textit{Classical} \text{ region } ④ \end{cases} \tag{4.12}$$

$$\text{where } \varepsilon = \frac{\partial \eta}{\partial i_C^d}(i_C^d)'\beta k_I > 0, \quad \tau = \frac{\partial \eta}{\partial k_I}\beta k_I > 0. \tag{4.13}$$

As the trace of $\mathbf{J}_c$ is negative when the wage dynamics are not intensive ($\omega$ is not large), a stable manifold of steady states exists easily: this means that the steady states should not be completely unstable. However, we are not able to guarantee the stability of the steady states in $S^c$, since the signs of coefficients of the characteristic polynomial $P_c(\lambda)$ are indeterminate in this system. Exceptionally, the sufficient condition for the saddle stability of ④ is attainable.

**Proposition 4.** If the determinant of $\mathbf{J}_c$ is positive in the *Classical* region, steady state ④ is a saddle point with a two-dimensional locally stable manifold and a one-dimensional unstable manifold.

*Proof.* In the *Classical* region, the characteristic polynomial $P_c(\lambda)$ is written as follows:

$$\begin{aligned} P_c(\lambda) &= \lambda^3 - (\mathrm{tr}\mathbf{J}_c)\lambda^2 + (\det\mathbf{J}_{c11} + \det\mathbf{J}_{c22} + \det\mathbf{J}_{c33})\lambda - \det\mathbf{J}_c \\ &= \lambda^3 - q_1\lambda^2 + q_2\lambda - q_3, \quad q_1 < 0,\ q_2, q_3 > 0. \end{aligned} \tag{4.14}$$

According to Theorem 1 in Benhabib and Perli (1994),[15] the number of roots of $P_c(\lambda)$ with positive real pairs is equal to the number of variations of signs in the following scheme:

$$-1, \quad q_1, \quad -q_2 + \frac{q_3}{q_1}, \quad q_3. \tag{4.15}$$

Under the condition $\det\mathbf{J}_c > 0$, one root with positive real pairs exists. □

Although the local stability of steady states in $S^c$ is ambiguous, we have arrived at an important conclusion: our system has at least one local stable steady state (steady state ②). This stable steady state belongs to the *Classical* region, in which the wage is determined by the first-order condition of profit maximization. The conclusion seems incompatible with the persistence of Keynesian unemployment, which is supported by Varian (1975) and Malinvaud (1980). We should note that this contradiction is derived from our primitive assumptions: consumption goods demand is monotonically increasing

[15] For the mathematical background of this theorem, see Gantmacher (1959, Chapter 15).

in wage, money plays no role in determining goods demands, and wage dynamics are not affected by the goods market disequilibrium. Revising these assumptions is beyond the scope of this study, since we consider a more important problem here: ample consumption demand sometimes ruins capital accumulation.

### 4.3 Recovery path and shrinking path

As we have seen, the stability of each steady state is not obvious. Using graphical analysis, however, we see the dynamic property from a broader viewpoint: will the economy in $S^c$ recover and converge into $S$? If the economy converges into $S$ soon, the quantity constraint in the investment is not an important issue.

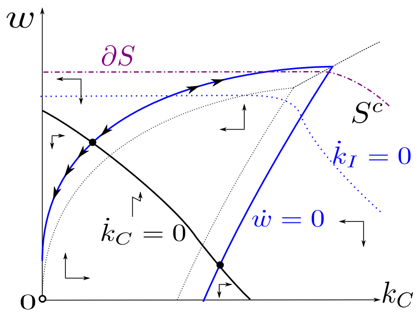


Figure 9: Dynamics in $S^c$

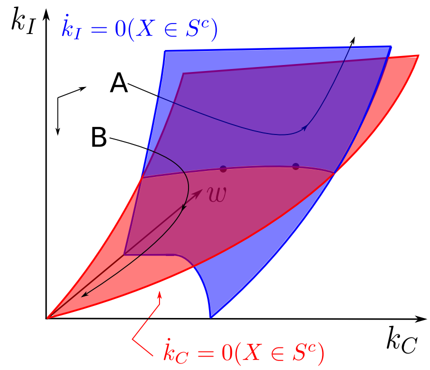


Figure 10: Recovery path and shrinking path

Figure 9 shows the dynamics of $(k_C, w)$, with $k_I$ being constant. Ignoring $k_I$, the point $\dot{k}_C = \dot{w} = 0$ in the *Classical* region is stable, which means the neighboring economy is absorbed into this point. Consider the case in which this point exists in the area of $\dot{k}_I < 0$. As the economy approaches this absorbing point, $k_I$ shrinks, and the point shifts to the southwest on the $k_C$ - $w$ plane. This shift is along the $w$ - nullcline, and it induces further shrinkage of $k_I$. This "trapped" path is shown as Path B in Figure 10. Path A, on the other hand, is the recovering path, wherein the absorbing point lies in the area of $\dot{k}_I > 0$.

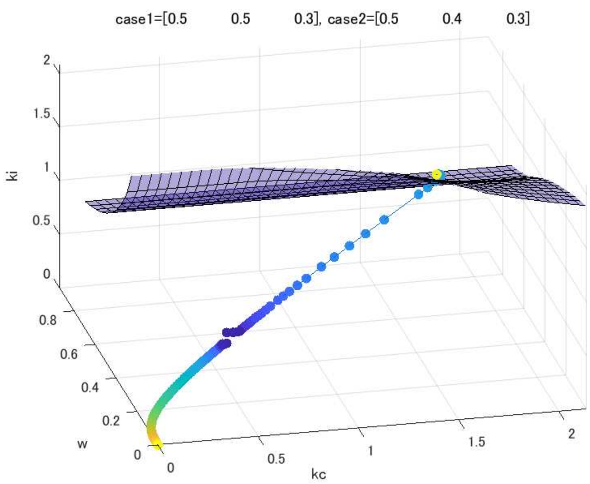


Figure 11: Simulation of Recovery path and shrinking path

In Figure 11, we examine the existence of the recovery path and shrinking path using a numerical experiment.[16] The two initial points (blue points in the figure) are around the *Classical* steady state ④, but the levels of $k_I$ are slightly different. As time passes, the color of the points vary into yellow, and we can easily see that the difference from the initial $k_I$ determines the future of capital accumulation. The path with the larger $k_I$ is the recovery path, which converges into a steady state in $S$ (above the meshed surface $\partial S$), and the path with the smaller $k_I$ shrinks into $X = 0$.

We should note how the shrinking path emerges, because the investment demand for the $C$-sector impedes the capital accumulation for $k_I$, and the reproduction of investment goods does not keep up against the depreciation.[17] Note that the shrinking path emerges in the *Classical* region, in which the consumption goods demand exceeds supply. The ample goods demand increases the rate of return in the $C$-sector, which induces a large investment demand for the $C$-sector. This pessimistic future seems peculiar to the two-sector quantity constrained growth model.

## 4.4 Discussions about the investment function

In this study, we specify the investment demand function as shown in eq.(4.1). We utilized this form since it is the simplest function that retains the nonlinearity of our system. We determine the importance of the nonlinearity to induce the shrinking path as compared with another form.

The investment demand function is written as follows:

$$i_j^d(r_j, k_j) = \chi_j(r_j)k_j, \quad \chi_j' > 0, \quad \chi_j'' < 0, \ \lim_{r_j \to \infty} \chi_j' = 0, \ \chi_I(\beta) < \beta, \quad j = I, C. \tag{4.16}$$

This separated form corresponds with Tobin's q and micro-founded (Murakami, 2016), and therefore it seems more desirable than the specified one. When we utilize it, the equilibrium of the multiplier process in $I$-sector is

$$\hat{r}_I = \chi_I(\hat{r}_I) - i_C^d/k_I, \tag{4.17}$$

where $\hat{r}_I = \hat{y}_I/k_I$. As $i_I^d/k_I = \chi_I(\hat{r}_I)$, eq.(4.17) indicates that the demanded accumulation rate of the $I$-sector is determined by the ratio of $i_C^d/k_I$. Considering that $k_I$ - nullcline is the set that satisfies $\chi_I(r_I) = n + \delta$, the nullcline is depicted as a straight line; see Figure 12. The border of the quantity constraint $\partial S$ corresponds with $\hat{r}_I = \beta$, and $S$ is the area of $\hat{r}_I < \beta$. $k_I$ - nullcline is the isocline of $\hat{r}_I$, which satisfies $\chi(\hat{r}_I) = n + \delta$, and therefore the steady states could not simultaneously exist in two areas, such as $S$ and $S^c$.

The $k_C$ - and $w$ - nullclines are not much different from the former formulations, and one steady state could exist in each region: the *Keynesian* and *Classical* regions. We summarize the above analyses.

**Proposition 5.** Under the investment demand function in eq.(4.17), at most two steady states exist. They belong to $S$ if $\beta > \chi_I^{-1}(n + \delta)$, $\partial S$ if $\beta = \chi_I^{-1}(n + \delta)$, and $S^c$ if $\beta < \chi_I^{-1}(n + \delta)$.

The analyses above show that there is no shrinking path, and the presence of the quantity constraint at the steady states is determined by the parameters $\beta$, $n$, and $\delta$ and

[16]The settings of this simulation are presented in Appendix B.

[17]Obviously, irreversible capital accumulation is also one cause of the shrinking path.

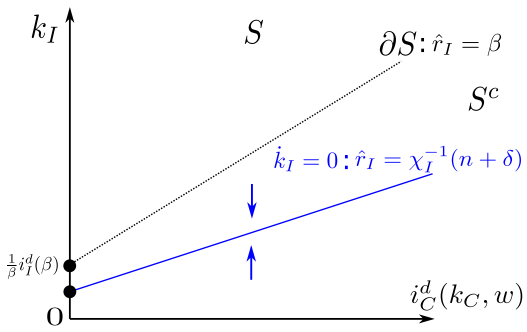


Figure 12: $k_I$ - nullcline: the case of $i_I^d = \chi_I k_I$

the form of $\chi_I$. Figure 12 shows that nonexistence of the shrinking path is caused by the linearity of $k_I$ - nullcline.

Note that this linearity is due to not only the form of $i_j^d$ but also the assumption of the production function and proportional rationing rule in eq.(2.17). This implies that the utilization of $i_j^d = \chi_j(r_j)k_j$ itself would be suitable if we extend our basic model.

# 5 Conclusion

In this study, we built a simple two-sector quantity constrained growth model using the disequilibrium approach. Our main contribution is in identifying the existence of the shrinking path: the large consumption goods demand impedes capital accumulation when the capital is in shortage. Therefore, we should pay attention to the term "demand-led growth," since consumption goods demand does NOT always enhance capital accumulation.

The important problem is determining whether the existence of the shrinking path is common to the quantity constrained growth model. As we intend to build one starting point for disequilibrium growth, our analyses are graphical and intuitive but too simple: employment in the $I$-sector is omitted. As von Hayek (1939) points out, capital intensity in each sector plays an important role in its dynamics; see Benhabib and Nishimura (1985) for more details on equilibrium dynamics. Besides, Takahashi, Mashiyama, and Sakagami (2012) empirically show that the difference in capital intensities between the sectors is not large. Further theoretical analyses on capital accumulation with market disequilibrium are needed, and our primitive model would help them.

# Acknowledgment

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# A The calculation of labor demand

In this appendix, we induce eq.(2.14) from eq.(2.13) and prove proposition 1, by calculating two labor demand functions; notional and effective.

The notional labor demand $l^{d*}$ is a first order condition for profit maximization under no quantity constraint:

$$w = \frac{\partial F_C(l^{d*}, k_C)}{\partial e} \tag{A.1}$$

As $F_C$ is homogeneous of degree 1, the partial derivative of $F_C$ is homogeneous of degree zero. It means there exists a function $\phi(w)$ which satisfies

$$l^{d*} = \phi(w)k_C, \phi' < 0. \tag{A.2}$$

The effective labor demand $\tilde{l}^d$ is derived from the quantity constraint $F_C = y_C^d$:

$$\begin{aligned} F_C(\tilde{l}^d, k_C) &= c^d(w\tilde{l}^d, r_C k_C) + g_C \\ &= c^d(w\tilde{l}^d, F_C(\tilde{l}^d, k_C) - w\tilde{l}^d) + g_C \end{aligned} \tag{A.3}$$

We should prove that there exists a positive $\tilde{l}^d$ which satisfies this equation for all $(k_C, w)$. Let us define a function of $e$, $\Theta : [0, +\infty) \to \mathbb{R}$ as follows:

$$\Theta(e) = c^d(we, F_C(e, k_C) - we) + g_C - F_C(e, k_C) \tag{A.4}$$

We could easily check $\Theta$ is continuous, $\Theta(0) > 0$, $\Theta' < 0$, and $\lim_{e\to+\infty} \Theta(e) < 0$. From intermediate value theorem and the monotonicity of $\Theta$, there exists a unique $e = \tilde{l}^d$ which makes $\Theta(\tilde{l}^d) = 0$. Denoting $\tilde{l}^d = \psi$, we attain partial derivatives by totally differentials for the equation $F_C(\tilde{l}^d, k_C) = c^d(w\tilde{l}^d, F_C(\tilde{l}^d, k_C) - w\tilde{l}^d) + g_C$,

$$\frac{\partial \psi}{\partial k_C} = \frac{(c_2^d - 1)F_{C2}}{(1 - c_2^d)F_{C1} - (c_1^d - c_2^d)w} < 0, \tag{A.5}$$

$$\frac{\partial \psi}{\partial w} = \frac{(c_1^d - c_2^d)e}{(1 - c_2^d)F_{C1} - (c_1^d - c_2^d)w} > 0, \tag{A.6}$$

and $\psi_{12} = \psi_{21} < 0$, $\psi_{11}, \psi_{22} > 0$ for $\psi(k_C, w)$.

# B Simulation of paths and an example of a cyclical growth

We introduce the settings of our numerical experiment in section 4 and an odd result like cyclical growth.

First, we set the functions as follows:

$$c^d = 0.6we + 0.5r_C k_C \tag{B.1}$$

$$i_I^d = 0.2r_I^{0.2} \tag{B.2}$$

$$i_C^d = 0.5r_C^{0.6} \tag{B.3}$$

$$F_I = 0.195uk_I \tag{B.4}$$

$$F_C = 0.55k_C^{1/3}e^{2/3} \tag{B.5}$$

Besides, we set the parameters as follows: $n = 0.01$, $\delta = 0.08$, $\omega = 0.01$, $\gamma = 0.95$、$g_C = 0.28$.[18] Warlasian Equilibrium is supported by $(k_C, w) = (1.6215, 0.4308)$.

We set the initial point around absorbing path at CU in $S^c$ in figure 9 to illustrate the paths explicitly: $(k_{C0}, w_0) = (0.5, 0.3)$. About $k_{I0}$, we use two different initial value; $k_{I0} = 0.5$ as case 1 (larger value); $k_{I0} = 0.4$ as case 2 (smaller value). As shown in section 4, Case 1 induces the recovering path which converges into steady state ② and case 2 induces the shrinking path. Figure B.1 shows the dynamics of variables in each case. Case 1 converges into the steady state at $t = 500$ but case 2 shrinks gradually.

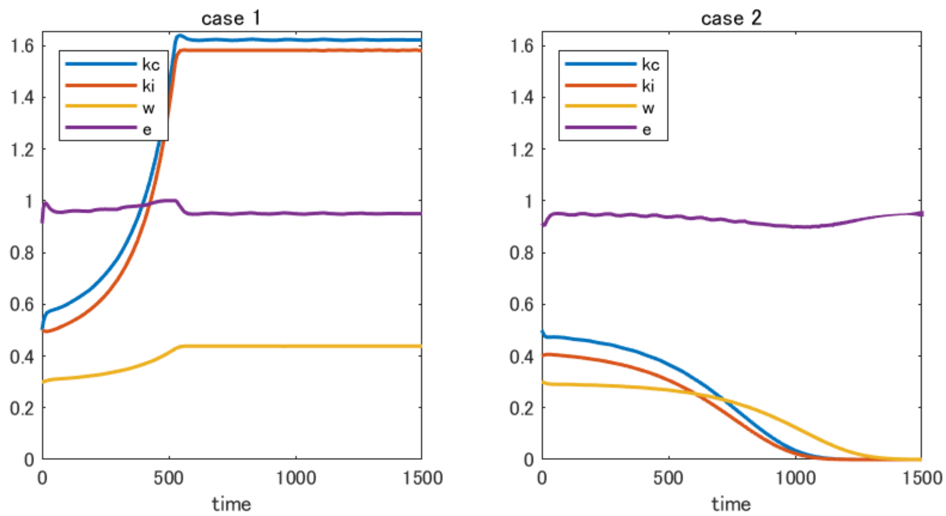


Figure B.1: Dynamics of $X$ and $e$

Figure B.2 shows the paths projected on $(k_C, w)$ plane. The solid lines are the borders of regimes and the dashed lines are the paths.

If we choose a lower $g_C$ or $g_C = 0.27$, the cyclical path emerges; see figure B.3. The initial point is same as case1. This cyclical path crosses $\partial S$ and it seems locally asymptotically stable: even if we set another initial point around the cycle, the path converges to the cycle. From the graphical analysis in section 4, this is the case when

[18] The results are not sensitive to the settings: the characteristic paths are observed under the function form like $c^d = \alpha_1(we)^{\alpha_2} + \alpha_3(r_C k_C)^{\alpha_4}$ or $i_j^d = \beta_1 r_j^{\beta_2} + \beta_3$ where $\alpha_2, \alpha_4 < 1$ and another value of $g_C$. If we set $p_C = p_I$, the government purchase per GDP is 0.2951 at steady state ②.

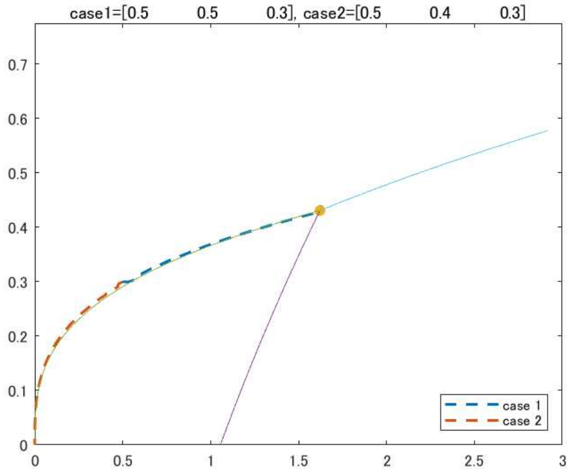


Figure B.2: The paths projected on $(k_C, w)$ plane

steady state ② vanishes. The economy grows along the recovery path, but $k_C$ - and $w$ - nullclines are detached around $k_I = 1.6$ in $S$. The economy then shrinks in *Keynesian* region, and it is absorbed in the recovery path again. However, a rigorous proof of the existence and stability of this cycle is difficult: we need more advanced mathematical tools to analyze differential equations with discontinuous-righthand-sides.

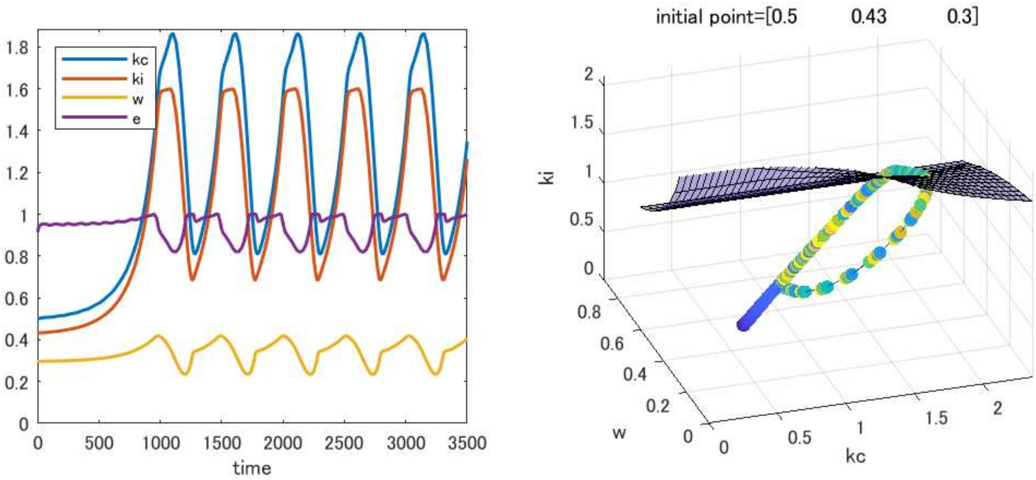


Figure B.3: Cyclical path