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Abstract

We introduce an infinite-horizon endogenous growth framework for studying the effects of foreign aid on the economic growth in a recipient country. Aid is used to partially finance the recipient's public investment. We point out that the same rule of aid may have very different outcomes, depending on the recipient's circumstances in terms of development level, domestic investment, efficiency in the use of aid and in public investment, etc. Foreign aid may promote growth in the recipient country, but the global dynamics of equilibrium are complex (because of the non-monotonicity and steady state multiplicity). The economy may converge to a steady state or grow without bounds. Moreover, there are rooms for the divergence and a two-period cycle. We characterize conditions under which each scenario takes place. Our analysis contributes to the debate on the nexus between aid and economic growth and in particular on the conditionality of aid effects.

Keywords: Aid effectiveness, economic growth, cycle, poverty trap, public investment, threshold.
JEL Classification: H50, O19, O41

1 Introduction

Since the United Nations Summit in September 2000 at which the Millennium Development Goals (MDGs) were agreed, foreign aid, in particular, Official Development Assistance (ODA) has been continually increasing. For example, in 2015, development aid provided by the donors in the OECD Development Assistance Committee (DAC) was 131.6 billion USD, increased by 6.9% in real terms from 2014, and by 83% from 2000.

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At the same time, bilateral aid, provided by one country to another, risen by 4% in real terms.[1] Many issues are under debate regarding the effectiveness of aid in terms of economic growth. Indeed, extensive empirical investigations using different data samples show conflicting results.

On the one hand, some studies show that aid may exert a positive and conditional effect on economic growth. In a seminal paper Burnside and Dollar (2000) use a database on foreign aid developed by the World Bank and find that foreign aid has a positive effect on growth only in recipient countries which have good fiscal, monetary and trade policies. Collier and Dollar (2001, 2002) use the World Bank's Country Policy and Institutional Assessment (CPIA) as a measure of policy quality and show that aid may promote economic growth and reduce the poverty in recipient countries if the quality of their policies is sufficiently high. The findings in Guillaumont and Chauvet (2001), Chauvet and Guillaumont (2003, 2009) indicate that the marginal effect of aid on growth is contingent on the recipient countries' economic vulnerability. While economic vulnerability is negatively associated with growth, the marginal effect of aid on growth is an increasing function of vulnerability.

On the other hand, other empirical studies, not rejecting the conditionality of aid effects, show a certain fragility of results and suggest a non-linear effect of aid on growth (Hansen and Tarp, 2001; Easterly et al., 2004; Islam, 2005; Roodman, 2007; Clemens et al., 2012; Guillaumont and Wagner, 2014). For example, Islam (2005) shows an aid Laffer curve in recipient countries with political stability. The effect of aid on growth may be negative at a high level of aid inflows. Hansen and Tarp (2001) find that the effectiveness of aid is conditional on investment and human capital in recipient countries and aid has no effect on growth when controlling for these variables. Their findings shed light on the link between aid, investment, and human capital and show that aid increases economic growth through its impact on capital accumulation. Using the same empirical specification like that in Burnside and Dollar (2000), but expanding the data set sample, Easterly et al. (2004) nuance the claim from that of these authors. The results on aid effectiveness seem to be fragile when varying the sample and the definition of different variables such as aid, growth and good policy (Easterly, 2003).

The aforementioned conflicting results in the literature raise a concern about the effectiveness of foreign aid. Our paper deals with this concern by investigating the following questions: (1) How the recipient country use foreign aid to enhance economic growth? (2) What are the determinants of the effectiveness of foreign aid? (3) Why are the effects of foreign aid significant for some countries but not for others?

To address these questions, we consider a tractable discrete-time infinite-horizon growth model where public investment, which is financed by foreign aid and capital tax, may improve the total factor productivity (TFP) if it is large enough. Inspired by the empirical literature, we formulate aid flows taking into account the donor's rules and the recipient's need represented by its initial capital stock. In the case of a poor country, we also consider the efficiency in the use of aid and in public investment, then examine their impacts on the aid effectiveness. Our model allows us to find explicitly the dynamics of capital stocks, and provide a full analysis of equilibrium transitional dynamics. We show that if the initial circumstances of the recipient are good enough (high productivity and initial capital), the country does not need foreign aid to achieve its development goals. This

[1] For more information, see http://www.oecd.org/development/development-aid-rises-again-in-2015-spending-on-refugees-doubles.htm

result is in line with the findings in growth models with increasing return to scale. Consequently, our analysis focuses on the case in which the recipient country's initial capital and productivity are not high. The main results can be described as follows:

First, when the foreign aid is very generous and/or the use of aid is efficient, and the recipient country has a high quality of circumstances (high and efficient public investment and/or low fixed cost in public investment), then the economy will grow without bounds for any level of initial capital stock. Consequently, the country will no longer receive aid from some date on.

The second case, corresponding to the richest dynamics of equilibrium, is found when foreign aid is not very generous and the recipient has an intermediate quality of circumstances. In this case, we distinguish 2 regimes: (R1) the recipient country focuses on its domestic investment, characterized by a remarkable level of capital tax financing public investment, and (R2) it focuses on foreign aid, characterized by the fact that the use of foreign aid is sufficiently efficient. In the first regime (domestic investment focus), if the country has sufficient initial capital or/and the foreign aid is quite high, the economy can grow. Otherwise, it would collapse (i.e., the capital level tends to zero) or stay at the unique steady state. In the second regime, the transitional dynamics are complex because of the non-monotonicity of capital dynamics. The non-monotonicity is due to the fact that the country focuses on foreign aid which decreases when the economy gets better. In this regime, there are two steady states: the lower one is interpreted as a middle-income trap while the second one as a poverty trap. Let us present our findings under this regime R2.

R2.1. We prove that any poor country, receiving a middle-level aid flow and using it efficiently, always grows at the first stage of its development process and hence will never collapse. This is intuitive because if the capital stock is very low, the country receives a significant aid flow which improves its investment. Under mild conditions, we show that the economy can increasingly converge to the low steady state.

R2.2. However, the convergence may fail in some cases and there may exist a two-period cycle capital path. The intuition is the following: When a poor country receives aid at an initial date (date 0), its economy may grow at date 1. By the rule of aid, the aid flow for date 1 may decrease, leading to a decrease of total public investment at date 1. Hence, the capital at date 2 may decrease, and so on. Thus, a two-period cycle may arise.

R2.3. Last, we point out that a poor country, having strong dynamics of capital, can take advantage of foreign aid to finance its public investment. This may lead it to overcome the middle-income trap in a finite number of periods and to obtain growth in the long run. In this case, the recipient will no longer receive aid from some date on.[2]

Several empirical studies on the effects of aid and conditionality of aid effectiveness in different recipient countries may illustrate our theoretical analyses. First, South Korea offers an illustration for our results in regime R1 and regime R2.3. Indeed, this country was a recipient country during the period of 1960-1990 (after the Korean War 1950-1953) and experienced a high domestic investment during this period. South Korea is now

[2] Strong dynamics of capital are defined in our paper in the sense that there is some value of capital under the middle-income trap, which produces an output higher than the middle-income trap.

a developed country and has become a member of the OECD-DAC (since 2010). The average aid flows have decreased during the period of 1960-1980, from 6.3% to 0.1% of GDP. It became negative at the beginning of the 1990s.[3] Second, our analysis in the regime R2.1 may be illustrated by the Tunisia case where aid flows have also decreased during 1960-2003, from 8.1% of GDP in the 1960s to 1.5% during 1990-2003.[4]

From a theoretical point of view, our paper is closely related to Dalgaard (2008) who considers that aid flows depend on the recipient's income per capita and on the donor's exogenous degree of inequality aversion. However, there are some differences. First, Dalgaard (2008) considers an OLG growth model while we use an infinite-horizon model à la Ramsey. Second, in Dalgaard (2008), public investments are fully financed by foreign aid while in our paper public investments are financed, not only by foreign aid but also by capital tax. In Dalgaard (2008), the transitional dynamics of capital stock are totally determined by the degree of inequality aversion on the part of the donor. Our contribution is to show that the transitional dynamics depend not only on the aid rule but also on the country's characteristics. In particular, our framework allows us to study whether a poor country can surpass, not only the low steady state but also the high one and then achieve growth in the long run.

Our theoretical results complemented by a number of numerical simulations, indicate that the effects of aid (in the short run and in the long run) are complex, non-linear and conditional on recipient countries' characteristics. By the way, our paper is related to and complements the points in Charterjee et al. (2003), Charteerjee and Tursnovky (2007). Indeed, Charterjee et al. (2003) examine the effects of foreign transfers on the economic growth of the recipient country given that foreign transfers are positively proportional to the recipient's GDP but are not subject to conditions. They show that their effects on growth and welfare are different according to the type of transfers, untied or tied to investment in public infrastructures. Charteerjee and Tursnovky (2007) underly the role of endogeneity of labor supply as a crucial transmission mechanism for foreign aid.

Our paper is likewise related to the literature on optimal growth with increasing returns (Romer, 1986; Jones and Manuelli, 1990; Kamihigashi and Roy, 2007) and with the presence of threshold (Azariadis and Drazen, 1990; Bruno et al., 2009; Le Van et al., 2010, 2016). Different from numerous papers in this literature, we point out the role of aid which can provide investment for the least developed country, this helps the recipient country to evade poverty and potentially obtain positive growth in the long run. Moreover, policy function in our framework may not be monotonic.

The remainder of the paper is organized as follows. Section 2 characterizes the case of a small recipient country. Section 3 presents the dynamics of capital, in particular, the poverty trap without international aid. In Section 4, we emphasize the role of foreign aid by analyzing the conditions for the effectiveness of aid. Section 5 studies effects of aid in a centralized economy. Section 6 concludes. Technical proofs are presented in Appendix A.

[3] See also Marx and Soares (2013) and Guillaumont and Guillaumont Jeanneney (2010).

[4] See also Guillaumont and Guillaumont Jeanneney (2010).

2 A small economy with foreign aid

This section presents our framework. We consider an economy with infinitely-lived identical consumers and a representative firm. The population size is constant over time and normalized to unity. Labor is exogenous and inelastic. The representative firm produces a single good, which can be used for either consumption or investment. The government uses capital tax and foreign aid to finance public investment (including R&D investment) which can improve the total factor productivity. The waste in aid spending is considered by the presence of unproductive aid. The latter has no direct effect on the household's welfare, nor on the production process. The fraction of wasteful aid may reflect the degree of inefficiency (including corruption) in the use of aid.

2.1 Foreign aid and public investment

The literature on aid conditionalities has a large consensus on the recipient's need as a significant criterion of aid allocation: countries with a high need should receive a high amount of aid. This criterion, among others, is used in several bilateral and multilateral aid policies. For instance, the World Bank's International Development Association (IDA) uses a specific rule of aid allocation which gives priority to the poorest countries and also those with the ability to use aid effectively. Compared to the IDA, the Asian Development Bank formula assigns a higher weight to recipient countries' poverty, but a lower weight to the recipient's population size.[5] In a study for the 2008 Development Cooperation Forum at the UN Economic and Social Council, Andersson (2008) (Box 1 and Figure 7) evoked different factors, including initial income, influencing the form of aid allocation. Other studies (Carter, 2014; Guillaumont and Wagner, 2015; McGillivray and Pham, 2017) provided more details on these aid allocation rules and mentioned a formula of aid allocated to a recipient country as a function of its poverty.

In this sense, we can assume that aid per capita a_t takes the following form:

$$a_t = (\bar{a} - \phi k_t)^+ \equiv \max\{\bar{a} - \phi k_t, 0\} \tag{1}$$

where $\bar{a} > 0$, the maximal aid amount that the recipient country can receive, and ϕ, independent of the per capita capital, may be referred to all exogenous rules imposed by the donor.

Equation (1) means that the higher the per capita capital k_t, the lower the country ranks in its need, then the lower the aid received. A similar assumption may be found in Carter (2014) and Dalgaard (2008).[6] The form of equation (1) implies that a decrease in ϕ and/or an increase in $\bar{a}$ lead(s) to a higher aid flow. Moreover, from a threshold $(\bar{a}/\phi)$

[5] Other international institutions such as the Asian Development Bank, the European Development Fund, the UK's Department for International Development, etc. use different variants of this rule.

[6] Carter (2014) (page 135) considers that aid flow received by country i is positively correlated with country performance rating as underlined in Collier and Dollar (2001, 2002) (with index Country Policy and Institutional Assessment) and is negatively associated with income per capita. Dalgaard (2008) assumes that per capita flow of aid at time t is also a reversed function of income of per capita at period $(t-1)$, $a_t = \theta y_{t-1}^{\lambda}$, where $\theta > 0$ and $\lambda < 0$. In this aid function, λ reflects the degree of inequality aversion of the donor. Parameter θ represents exogenous determinants of aid. Although Appendix C.2 in Dalgaard (2008) presents a generalized aid allocation rule, it seems that this rule does not cover the form of (1) because the function $(\bar{a} - \phi x)^+$ is not differentiable. Our analyses of effects of threshold $(\bar{a}/\phi)$ are an added-value of our paper.

of capital, the recipient country no longer receives aid. In Section 4.5, we will work under a general form of aid.

The positive couple $(\bar{a}, \phi)$ is interpreted as the aid rule imposed by the donor. It is taken as given by the recipient country and represents aid conditionalities. Allocation of aid may be conditional on the policy performance as underlined in Burnside and Dollar (2000), Collier and Dollar (2001, 2002). According to these authors, a country with a high policy quality is more able to use aid in an efficient way. Guillaumont and Chauvet (2001) focus on a fairness argument when they focus on the recipient's economic vulnerability: more aid should be provided to countries with a high economic vulnerability since in these countries aid would be more efficient. This argument also fits in a philosophy of fairness which proposes that aid should compensate the recipient country for its vulnerable initial situation (in macroeconomic conditions or lack of human capital), so that all countries can begin at the same initial opportunities. McGillivray and Pham (2017), Guillaumont et al. (2017) consider the lack of human capital as a determinant criterion. Other analyses underline the link between aid and political variables (strategic allies, former colonial status, and the ability to use aid effectively), between aid and macroeconomic conditions (trade openness, commercial allies, etc.) (Alesina and Dollar, 2000; Berthelemy and Tichit, 2004). All these factors are exogenous for the recipient country as they are chosen by donors and may be considered as different interpretations of the parameter ϕ. Equation (1) may reflect a trade-off between needs (low initial capital) and country-selectivity (high ϕ) or a compatibility between needs (low initial capital) and country-selectivity (low ϕ) based on aid performance or other criteria.

The recipient country uses aid and tax on capital to finance public investment which improves private capital productivity. Since some spending of aid is wasted in most recipient countries, there is a significant part of the unproductive activity, noted as a_t^u. This is potentially explained by corruption, administrative fees, etc. Then, the attribution of aid may be written as:

$$a_t = a_t^i + a_t^u \tag{2}$$

where a_t^i represents the part of aid which contributes to the public investment of the recipient country. If we consider a fixed fraction of aid for each activity, we can rewrite equation (2) as follows:

$$a_t = \alpha_i a_t + \alpha_u a_t$$

with $\alpha_u = 1 - \alpha_i$. Parameter $\alpha_u \in (0, 1)$ reflects the degree of inefficiency (including the corruption) in the use of aid while α_i represents the efficiency in the use of aid.

Let us denote B_t the public investment financed by tax on capital and by aid, B_t may be written as:

$$B_t = T_{t-1} + a_t^i \tag{3}$$

where T_{t-1} is the tax at period $t - 1$. We assume that $T_{t-1} = \tau K_t$ used to finance public investment which will have its effect at date t. Since, all capital tax is used to fund public investment, τ may be interpreted as the government effort in financing public investment.[7]

[7] The amount $T_t = \tau K_{t+1}$ in our model can be viewed as the total expenditure in R&D which may affect the TFP. In this sense, τ should be lower than 1 as the total expenditure in R&D in most countries does not exceed 4% of the GDP. Only the South Korea's R&D expenditure exceeds 4%; in 2016, it was 4.2% of the GDP (according to the World Bank database). However, in our theoretical analyses, we do not need that $\tau \leq 1$.

The positive effect of foreign aid on public investment is an obvious finding in empirical studies (Khan and Hoshino, 1992; Franco-Rodrigez et al., 1998; Ouattara, 2006; Feeny and McGillivray, 2010).

2.2 Production with endogenous productivity

At each date, the representative firm maximizes its profit. The production function at date t is given by $F_t(K_t) = A_t K_t$ where K_t represents the capital while A_t represents the total factor productivity. In the spirit of Barro (1990), we assume that A_t is endogenous and depends on public investment B_t as follows:

$$A_t \equiv A\left[1 + (\sigma B_t - b)^+\right]. \tag{4}$$

Parameter $A \in (0, \infty)$ is interpreted as autonomous productivity. Parameter $\sigma \in (0, \infty)$ measures the extent to which the public investment translates into technology and it reflects the efficiency of public investment. So, σB_t can be viewed as a flow of new technology/innovation generated by the investment B_t. Parameter b is the threshold from which the flow of technology σB_t has a strictly positive impact on the TFP. The threshold b may be viewed as a fixed cost or setup cost (Azariadis and Drazen, 1990).[8] When $B_t \leq b/\sigma$, we have $(\sigma B_t - b)^+ = 0$, and then $A_t = A$. This means that the positive effect of public investment on total productivity is observed only from the level b/σ. In Section 4.5, we will work under a general form of TFP A_t.

According to (3) and (4), foreign aid has a significant effect on the capital productivity and production only if $a_t^i \geq b/\sigma - \tau K_t$. As in Charterjee et al. (2003), Charteerjee and Tursnovky (2007), Dalgaard (2008), we consider that aid is used to finance public expenditures. However, for countries with low capital (in the sense that $\sigma\tau K_t < b$), aid should be higher than a critical level to improve the technology which is necessary for positive growth in the long run.[9] This assumption may be referred to the big push concept of aid supported by Sachs (2005) and discussed in Guillaumont and Guillaumont Jeanneney (2010). Our setup is also supported by Wagner (2014) who uses a database including 89 recipient countries and identified the existence of a critical level above which aid is effective in terms of economic growth.

At each period t, given B_t, r_t, the representative firm maximizes its profit:

$$(P_{ft}): \qquad \pi_t \equiv \max_{K_t \geq 0} \Big(F_t(K_t) - r_t K_t\Big)$$

It is straightforward to obtain r_t and π_t for a competitive economy:

$$r_t = A\big[1 + \big(\sigma B_t - b\big)^+\big] \text{ and } \pi_t = 0. \tag{5}$$

Remark 1. We may introduce a more general setup

$$A'_t \equiv A\left[1 + \left(\sigma(B_t - b_1)^+ - b_2\right)^+\right]. \tag{6}$$

[8]Our setup (4) is related to that in Bruno et al. (2009). The role of b (resp., σB_t)in our framework is similar to that of the parameter X (resp., $Y_{e,t}$) in Bruno et al. (2009).

[9]Our setup is different from that in Dalgaard (2008). Indeed, Dalgaard (2008) considers a production function: $y_t = k_t^\alpha g_t^{1-\alpha}$, where g_t represents government services, entirely financed by international aid a_t. This means that the first dollars received from donors have a positive effect on the recipient's production.

where $\bar{b}_1$ represents the fixed cost due to, for example, bribery and b_2 plays the role of b in (4). Notice that $A'_t = A\left[1 + \left(\sigma B_t - b\right)^+\right]$ where $b \equiv \sigma b_1 + b_2$. So, our main results still hold under this general setup. However, there is a difference in terms of implications which will be discussed in Remark 3.

2.3 Household

Let us consider the representative consumer's optimization problem. She maximizes her intertemporal utility by choosing consumption and capital sequences (c_t, k_t):

$$(P_c): \quad \max_{(c_t, k_{t+1})_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t)$$
$$\text{s.t: } c_t + k_{t+1} + T_t \le (1-\delta)k_t + r_t k_t + \pi_t$$

where $k_0 > 0$ is given, β is the rate of time preference and $u(\cdot)$ is the consumer's instantaneous utility function. T_t is the tax, r_t is the capital return while π_t is the firm's profit at date t.

For the sake of tractability, we assume that the consumer knows that $T_t = \tau k_{t+1}$ and instantaneous utility function is logarithmic, $u(c_t) = \ln(c_t)$. According to Lemma 8 in Appendix A.1, we establish the relationship between k_{t+1} and k_t

$$k_{t+1} = \beta \frac{1 - \delta + r_t}{1 + \tau} k_t. \tag{7}$$

Since the utility function is strictly concave, the solution is unique.

2.4 Intertemporal equilibrium

Definition 1 (intertemporal equilibrium)**.** *Given tax rate τ, a list $(r_t, c_t, k_t, K_t, a_t)$ is an intertemporal equilibrium if:*

1. *(c_t, k_t) is a solution of the problem P_c, given a_t^i, r_t, π_t.*
2. *(K_t) is a solution of the problem P_{ft}, given B_t and r_t.*
3. *(Market clearing conditions) $K_t = k_t$ and $c_t + k_{t+1} + T_t = (1-\delta)k_t + Y_t$.*
4. *The government budget is balanced: $T_t = \tau k_{t+1}$.*
5. *(Rule of Aid) $a_t = \max\{\bar{a} - \phi k_t, 0\}$ and $a_t^i = \alpha_i a_t$.*

Combined with (5), the dynamics of capital stock may be rewritten as follows:[10]

$$k_{t+1} = G(k_t) \equiv f(k_t)k_t \tag{8a}$$

$$\text{where} \quad f(k_t) \equiv \beta \frac{1 - \delta + A\Big[1 + \big(\sigma(\tau k_t + \alpha_i(\bar{a} - \phi k_t)^+) - b\big)^+\Big]}{1 + \tau}. \tag{8b}$$

This dynamic system is non-linear and non-monotonic. The next sections analyze the global dynamics of capital stocks (k_t) and the effects of international aid on the recipient's economic growth. Before doing this, it is useful to introduce some notions of growth and collapse.

[10]We implicitly assume that $\sum_t \beta^t u(G^t(k_0)) < \infty$ where $G^0(k_0) \equiv k_0$ and $G^{t+1}(k_0) = G(G^t(k_0))$ $\forall t \ge 1$. This ensures that the intertemporal utility is finite.

Definition 2 (growth, collapse, and poverty trap)**.**

1. *The economy collapses if* $\lim_{t\to\infty} k_t = 0$*. It grows without bounds if* $\lim_{t\to\infty} k_t = \infty$*.*

2. *A value* $\bar{k}$ *is called a trap if, for any initial capital stock* $k_0 < \bar{k}$*, we have* $k_t < \bar{k}$ *for any* t *high enough.*

Our formal definition of trap means that a poor country ($k_0 \leq \bar{k}$) continues to be poor. It is in line with the notion of poverty trap in Azariadis and Stachurski (2005): A poverty trap is a self-reinforcing mechanism which causes poverty to persist.

3 Equilibrium dynamics without foreign aid

This section considers an economy which does not receive foreign aid. Its public investment B_t is entirely financed by tax revenue. We will analyze the transitional dynamics of capital. From equation (8a), we have:

$$k_{t+1} = f_b(k_t)k_t, \text{ where } f_b(k_t) \equiv \beta \frac{1-\delta+A\Big[1+\big(\sigma\tau k_t - b\big)^+\Big]}{1+\tau} \tag{9}$$

Let us denote:

$$r_a \equiv \beta \frac{1-\delta+A}{1+\tau}. \tag{10}$$

We observe $f_b(k_t) \geq r_a$ for any t. Therefore, we have:

Remark 2 (role of autonomous technology)**.** Consider an economy without aid. If $r_a > 1$, the economy will grow without bounds.[11]

Condition $r_a > 1$ is equivalent to $A > \frac{1+\tau}{\beta} + \delta - 1$. If we define the subjective interest rate ρ by $(1+\rho)\beta = 1$, then $\frac{1+\tau}{\beta} + \delta - 1 = (1+\tau)(1+\rho) + \delta - 1 \approx \tau + \rho + \delta$ which can be interpreted as the investment cost. By consequence, condition $r_a > 1$ means that the autonomous productivity is higher than the investment cost. Under this condition the economy will have growth whatever the levels of other factors such as: initial capital or efficiency of public investment. In this case, the country does not need foreign aid to get economic growth. Since our purpose is to look at the impacts of public investment and foreign aid, from now on, we will work under the following assumption.

Assumption 1 (for the rest of the paper)**.** $r_a \leq 1$ *or equivalently,*
$A \leq \frac{1+\tau}{\beta} + \delta - 1$.

Under this assumption, the economy would never reach economic growth in the long run without public investment B_t (in infrastructure, in R&D program, etc.). Public investment B_t is then required to improve technology, and this is necessary for a positive economic growth in the long run.

According to equation (9) and the fact that $f_b(k_t)$ is an increasing function, we get the following properties concerning the dynamics of capital stock:

[11] In this case, $f_b(k_t) \geq r_a > 1$, then $k_{t+1} > k_t$ for any t.

Proposition 1 (poverty trap and growth: role of public investment)**.** *Consider an economy with a low level of autonomous technology (Assumption 1 holds), and without foreign aid. The public investment in technology is entirely financed by tax revenue. The dynamics of capital, characterized by equation (9), are as follows:*

1. *If $f_b(k_0) > 1$, i.e., $\sigma\tau k_0 > b + D$, then (k_t) increases and the economy grows without bounds.*

2. *If $f_b(k_0) < 1$, i.e., $\sigma\tau k_0 < b + D$, then (k_t) decreases and the economy collapses.*

3. *If $f_b(k_0) = 1$, i.e., $\sigma\tau k_0 = b + D$, then $k_t = k_0$ for any t.*

There exists a unique steady state k^{**}, and $k^{**} = \frac{b+D}{\tau\sigma}$ where

$$D \equiv \frac{1}{A}\Big(\frac{1+\tau}{\beta} + \delta - 1\Big) - 1 \geq 0. \tag{11}$$

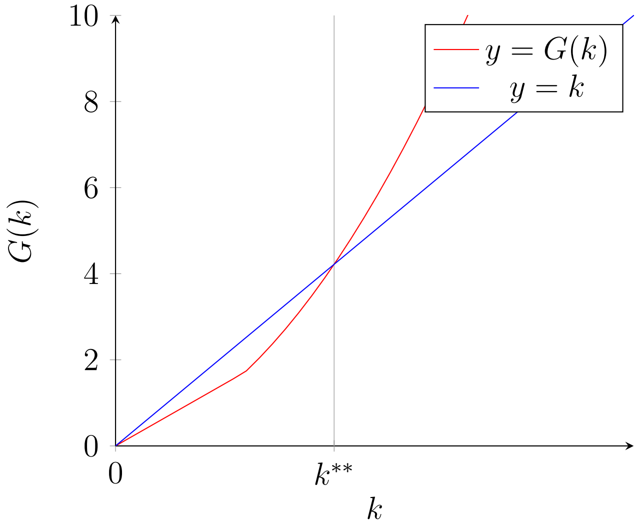

Figure 1: Transitional dynamics of capital without foreign aid and $r_a < 1$.

We may interpret b as a fixed cost of public investment. If the return of public investment ($\sigma B_t \equiv \sigma\tau k_0$), also interpreted as the flow of new technology, is less than $b + D$, public investment τk_0 does not make any change to the total factor productivity. Following this interpretation, $b + D$ can be viewed as the threshold so that if the return of public investment in R&D (σB_t) is less than this level, there is no growth of capital stock, i.e. $k_{t+1} < k_t$ for all t.

Figure 1 illustrates Proposition 1.[12] The point of interaction between the convex curve and the first bisector corresponds to the unstable steady state k^{**} which is considered as a poverty trap for this economy (see Definition 2). For all initial capital k_0 higher than k^{**} (corresponding to $\sigma\tau k_0 > b + D$), the economy will grow without bounds while it will collapse if the initial capital is lower than k^{**}. It should be noticed that k^{**} is decreasing in A, σ but increasing in b. This means that an economy having a high autonomous technology A, high efficiency σ and low fixed cost b in public investment, obtains a higher probability to surpass its poverty trap as the condition $\sigma\tau k_0 > b + D$ is more likely to be satisfied.

[12]In Figure 1, parameters are $\beta = 0.8, \delta = 0.2, A = 0.5, \tau = 0.4, \sigma = 2$, $\bar{a} = 0$, $b = 2$.

4 Equilibrium dynamics with foreign aid

Point 2 of Proposition 1 shows that the economy collapses without international aid if $\sigma\tau k_0 < b + D$. Since we want to investigate the effectiveness of aid, we will work under the following assumption in Section 4:

Assumption 2 (for the whole Section 4)**.**

$$\sigma\tau k_0 < D + b \tag{12}$$

where D is defined by (11)

Given this pessimistic initial situation of the recipient country, we examine how international aid could generate positive perspectives in the short run as well as in the long run. Recall that $k_{t+1} = G(k_t)$. Before exploring the dynamics of capital stock, it is essential to underline properties of function $f(k)$ and $G(k)$. To do so, we introduce some notations

$$x_1 \equiv \bar{a}/\phi, \quad x_2 \equiv \frac{\sigma\alpha_i\bar{a} - b}{\sigma(\alpha_i\phi - \tau)}, \quad x_3 \equiv \frac{1 - \delta + A(1 + \sigma\alpha_i\bar{a} - b)}{2A\sigma(\alpha_i\phi - \tau)}. \tag{13}$$

Let us explain the meaning of x_1, x_2, x_3. First, x_1 is the maximum level of capital stock so that the recipient country does not receive international aid. Second, when the country receives aid, x_2 is the critical threshold from which public investment B_t (financed by aid and tax revenue) has a positive impact on productivity (x_2 is a solution to $\sigma\big(\tau k + \alpha_i(\bar{a} - \phi k)\big) - b) = 0$. Last, when the country receives aid (i.e., $\bar{a} - \phi k > 0$) and public investment has positive impact on productivity (i.e., $\sigma\big(\tau k + \alpha_i(\bar{a} - \phi k)\big) - b > 0$), x_3 is a local-maximum point of function G (because $f_3'(x_3) = 0$) where

$$f_3(x) \equiv \beta\frac{1 - \delta + A\Big[1 + \big(\sigma(\tau x + \alpha_i(\bar{a} - \phi x)) - b\big)\Big]}{1 + \tau}x. \tag{14}$$

Lemma 1 (increasingness of G)**.** *The function G is increasing on $[0, \infty)$ if one of the following conditions is satisfied.*

1. $\tau \geq \alpha_i\phi$.
2. $\tau < \alpha_i\phi$ *and* $\sigma\alpha_i\bar{a} < b$ *(which imply that* $x_2 < 0$*).*
3. $\tau < \alpha_i\phi$, $\sigma\alpha_i\bar{a} > b$ *and* $x_3 > \min(x_1, x_2)$.

Condition $\tau \geq \alpha_i\phi$ means that the government effort is high (τ is high or/and α_i is low). In this case, the policy function is increasing. In point 2, the policy function G is also increasing because the flow of aid $\alpha_i(\bar{a} - \phi x)^+$ plays no role on the endogenous productivity. Conditions in point 3 mean that the government effort is low, the maximum level of aid $\bar{a}$ and/or the efficiency of public investment σ is high with respect to the fixed cost b, and the local-maximum point of output is high enough.

When the local-maximum point is not high enough, the policy function G may not be increasing.

Lemma 2 (non-monotonicity of G)**.** *Assume that* $\tau < \alpha_i\phi$, $\sigma\alpha_i\bar{a} > b$, *and* $x_3 < \min(x_1, x_2)$. *Then G is increasing on* $[0, x_3]$, *decreasing on* $[x_3, \min(x_1, x_2)]$, *and increasing on* $[\min(x_1, x_2), \infty)$.

Proofs of Lemmas 1, and 2 are presented in Appendix A.2.

We can find non-trivial fixed points (capital steady states) by computing strictly positive solutions of the equation $G(k) = k$, or equivalently $f_2(k) \equiv \tau k + \alpha_i(\bar{a} - \phi k)^+ = \frac{D+b}{\sigma}$.

Lemma 3 (steady states)**.**

1. *If $\sigma \bar{a} \min(\alpha_i, \tau/\phi) > D + b$, then there is no fixed point.*

2. *Consider the case where $\sigma \bar{a} \min(\alpha_i, \tau/\phi) \leq D + b$.*

   (a) *If $\tau > \alpha_i \phi$, which implies $\sigma \bar{a} \alpha_i \leq D + b$, then the unique fixed point is*

$$\begin{cases} k^* \equiv \frac{\frac{D+b}{\sigma} - \bar{a}\alpha_i}{\tau - \alpha_i \phi} \in (0, \bar{a}/\phi) & \text{if } \sigma \bar{a} \tau/\phi > D + b \\ k^{**} \equiv \frac{D+b}{\tau \sigma} \in (\bar{a}/\phi, \infty) & \text{if } \sigma \bar{a} \tau/\phi < D + b. \end{cases} \quad (15)$$

   (b) *If $\tau < \alpha_i \phi$, which implies $\sigma \bar{a} \tau/\phi \leq D + b$, then*

      i. *If $\sigma \bar{a} \alpha_i < D + b$, then the unique fixed point is $k^{**} \equiv \frac{D+b}{\tau \sigma} \in (\bar{a}/\phi, \infty)$.*

      ii. *If $\sigma \bar{a} \alpha_i > D + b$, then there are two fixed points $k^* \equiv \frac{\bar{a}\alpha_i - \frac{D+b}{\sigma}}{\alpha_i \phi - \tau} \in (0, \bar{a}/\phi)$ and $k^{**} \equiv \frac{D+b}{\tau \sigma} \in (\bar{a}/\phi, \infty)$.*[13]

Proof. See Appendix A.2.[14] □

4.1 Growth under high-quality circumstances

This section investigates effects of aid on the recipient prospects when the recipient country has high-quality circumstances in terms of efficiency in the use of aid, fixed cost and efficiency in public investment, autonomous technology, etc.

Proposition 2 (growth without bounds thanks to foreign aid)**.** *Considering an aid recipient under a poverty trap without aid, characterized by condition (12). The dynamics of capital with foreign aid are characterized by (8a).*

If

$$r_d \equiv \frac{\beta}{1+\tau}\Big[1 - \delta + A\Big(1 + \big(\sigma \bar{a} \min(\alpha_i, \tau/\phi) - b\big)^+\Big)\Big] > 1$$
$$\text{or equivalently, } \sigma \bar{a} \min(\alpha_i, \tau/\phi) > D + b, \quad (16)$$

then we have that,

1. *the economy will grow without bounds for any level of initial capital k_0,*

2. *international aid $a_t = (\bar{a} - \phi k_t)^+$ decreases in t. Consequently, there exists a time T such that aid flows $a_t = 0$ for any $t \geq T$.*

[13] Condition $\sigma \bar{a} \alpha_i > D + b$ ensures that $k^* > 0$.

[14] In Appendix A.2, we also study the case where $\tau = \alpha_i \phi$ but in the main text we do not focus on this case because it is not generic and the result in this case is similar to that in Proposition 3.

Proposition 2 can be proved by using point 4 of Lemma 1 and point 1 of Lemma 3. Notice that in this case, G is increasing and a steady state does not exist.

Condition (16) in Proposition 2 may be written as follows

$$\sigma\bar{a}\frac{\tau}{\phi} > D + b \quad \text{and} \quad \sigma\alpha_i\bar{a} > D + b, \tag{17}$$

where D is given by equation (11). Two conditions in (17) mean that the foreign aid is generous (high $\bar{a}$ and low ϕ) and/or the recipient country has high-quality circumstances (that is, a high efficiency σ and low fixed cost b in public investment, and/or a high level of autonomous technology A). In particular, the first condition in (17) may be associated with a high government effort (high τ) in financing public investment while the second condition may be associated with a high efficiency in the use of aid (high α_i). In other words, given aid flows and the donor's rules characterized by the couple $(\bar{a}, \phi)$, condition (17) is more likely to be satisfied if the recipient country has high-quality circumstances, decisive for the effectiveness of aid.

Proposition 2 presents the best and ideal scenario since whatever the initial capital, generous aid combined with high-quality circumstances could help the recipient country to grow without bounds in the long run. Figure 2 illustrates this proposition under condition (16).[15] The graph on the left corresponds to the case $\alpha_i < \tau/\phi$ and that on the right corresponds to the case $\alpha_i > \tau/\phi$. We observe that, without foreign aid (corresponding to $\bar{a} = 0$), the dynamics of capital are similar to that in Figure 1 and there is one poverty trap. Thanks to development aid, the dynamics of capital change and they are represented by the curve above the first bisector.

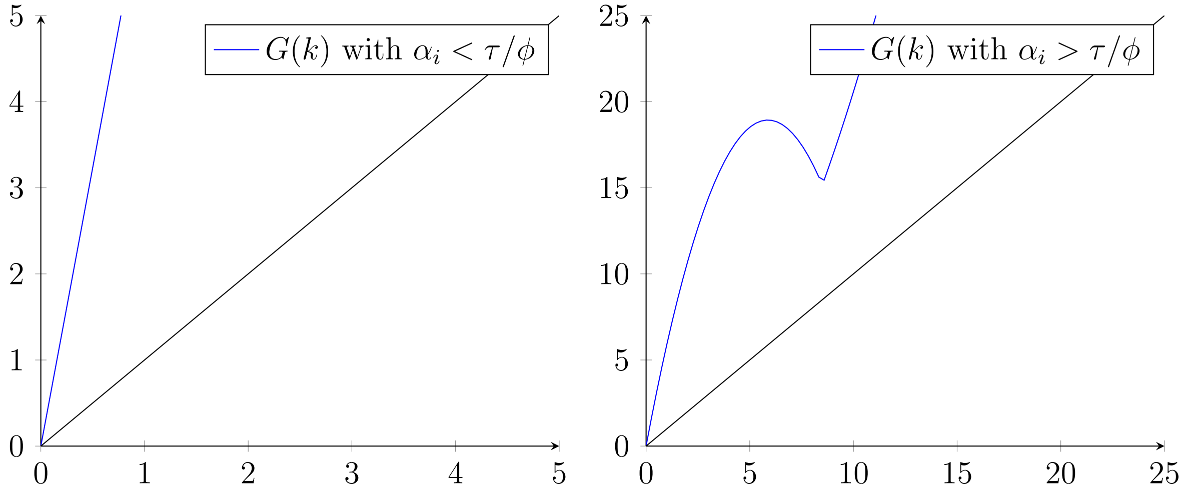

Figure 2: Growth without bounds. Conditions $r_a < 1$ and (16) holds

Remark 3. Assume that the TFP is $A_t' := A\left[1 + \sigma(B_t - b_1)^+\right]$ instead of (4). Assume also that $\alpha_i\bar{a} < b_1$, $\alpha_i\phi > \tau$ and $\beta\frac{1-\delta+A}{1+\tau} < 1$. If $x \in [0, \bar{a}/\phi]$, then we have

$$f(x) = \beta\frac{1 - \delta + A\Big[1 + \sigma\Big(\alpha_i\bar{a} - (\alpha_i\phi - \tau)x - b_1\Big)^+\Big]}{1+\tau} = \beta\frac{1-\delta+A}{1+\tau} < 1.$$

[15] Parameters in Figure 2 are $\beta = 0.8; \tau = 0.4; \delta = 0.2; A = 0.4; \sigma = 2; \alpha_i = 0.8; \bar{a} = 17, b = 2, \phi = 2$ verifying conditions $r_a < 1$ and (16). On the left: $\phi = 0.4$. On the right: $\phi = 2$.

By consequence, we have $\lim_{t\to\infty} k_t = 0$ for any $k_0 \leq \bar{a}/\phi$, whatever the level of efficiency σ. This is different from Proposition 2 where $\lim_{t\to\infty} k_t = 0$ if σ is high enough. This is due to the presence of threshold b_1 which implies that $A'_t = A$ for any $B_t \leq b_1$, whatever the level of σ. We refer to Le Van et al. (2016) for endogenous threshold in an optimal growth model.

4.2 Growth or collapse? The role of aid

We are now interested in the case where condition (16) is not satisfied: recipient countries do not have high-quality circumstances and/or aid flows, subject to conditions, are bounded, due to the budget constraint from the donors. In the next sections, we will work under the following condition:

Assumption 3 (for the rest of the paper)**.**

$$\sigma \bar{a} \min(\alpha_i, \tau/\phi) < D + b. \tag{18}$$

From (18), we can identify three cases:

$$\text{Low circumstances: } \frac{\sigma\tau}{\phi} < \frac{D+b}{\bar{a}} \text{ and } \sigma\alpha_i < \frac{D+b}{\bar{a}} \tag{19a}$$

$$\text{Intermediate circumstances 1 (domestic investment focus): } \sigma\alpha_i < \frac{D+b}{\bar{a}} < \frac{\sigma\tau}{\phi} \tag{19b}$$

$$\text{Intermediate circumstances 2 (aid focus): } \frac{\sigma\tau}{\phi} < \frac{D+b}{\bar{a}} < \sigma\alpha_i \tag{19c}$$

In both (19b) and (19c), we have $\sigma \bar{a} \min(\alpha_i, \tau/\phi) < D + b < \sigma \bar{a} \max(\alpha_i, \tau/\phi)$. However, we distinguish two intermediate circumstances: one with domestic investment focus when $\alpha_i \phi < \tau$, that is the government investment (measured by τ) is quite high with respect to the efficiency degree in the use of aid (measured by α_i); and another with aid focus when $\alpha_i > \tau/\phi$, that is the use of aid is quite efficient.

We firstly consider the cases of low circumstances and the domestic investment focus. According to Lemma 1, we have the following result.

Proposition 3 (growth or collapse? The role of aid)**.** *Consider an aid recipient under poverty trap without aid, characterized by condition (12): $\sigma\tau k_0 < D+b$. Assume that one of three conditions in Lemma 1 holds. Then (k_t) is monotonic in t and the transitional dynamics of (k_t) are characterized as follows.*

1. *If $f(k_0) > 1$, i.e., $\big(\sigma(\tau k_0 + \alpha_i(\bar{a} - \phi k_0)^+) - b\big)^+ > D$, then (k_t) increases and the economy grows without bounds. Consequently, there exists a time T such that aid flows $a_t = 0$ for any $t \geq T$.*

2. *If $f(k_0) < 1$, i.e., $\big(\sigma(\tau k_0 + \alpha_i(\bar{a} - \phi k_0)^+) - b\big)^+ < D$, then (k_t) decreases and the economy collapses. Consequently, there exists a time T_1 such that aid flows $a_t > 0$ for any $t \geq T_1$.*

3. *If $f(k_0) = 1$, then $k_t = k_0$ for any t.*

Moreover, following Lemma 3, we have:

$$\text{The unique steady state } = \begin{cases} k^{**} = \frac{D+b}{\tau\sigma} \in (\bar{a}/\phi, \infty) \text{ if (19a) holds (low circumstances)} \\ k^{*} = \frac{\bar{a}\alpha_i - \frac{D+b}{\sigma}}{\alpha_i\phi - \tau}(0, \bar{a}/\phi) \text{ if (19b) holds (intermediate circumstances 1).} \end{cases}$$

We are considering a country with a low initial capital stock in the sense that $\sigma\tau k_0 < b + D$ (Assumption 2). According to Proposition 3, we observe that: given such an initial capital stock k_0, if the aid rule is generous (in the sense that $\bar{a}$ is high and/or ϕ is low) and the use of aid is efficient (α_i is high) so that $\big(\sigma(\tau k_0 + \alpha_i(\bar{a} - \phi k_0)^+) - b\big)^+ > D$, then the economy will grow without bounds. Otherwise, the economy will collapse or stay at the steady state. In other words, the development aid might help the recipient to surpass its poverty trap while this is impossible without foreign assistance. Our result indicates that low-income and vulnerable countries need not only a large scaling-up of aid but also the efficiency in the use of aid (parameter α_i) to help them to get out of the poverty trap. Our finding may be considered as a theoretical illustration for the argument evoked in Kraay and Raddatz (2007) using a Solow model.[16]

We observe that the poverty trap in the intermediate circumstances 1 (with domestic investment focus) is k^*, which is lower that k^{**}, i.e. the poverty trap in the low circumstances. This means that the intermediate circumstances give a better outcome than the low circumstances as the recipient's possibility of escaping its poverty trap is higher in the intermediate circumstances.

4.3 Stability, fluctuations or take-off? The complexity of aid's effects

We have so far analyzed three circumstances (high, low and intermediate circumstances 1 with domestic investment focus) in which the capital path (k_t) is monotonic. In these cases, the recipient country may or may not fully exploit the same flow of aid following its initial situation. This section focuses on the remaining cases characterized by the following assumption:

Assumption 4 (for the whole Section 4.3)**.**

1. *$\sigma\tau/\phi < \frac{D+b}{\bar{a}} < \sigma\alpha_i$ (condition (19c) - intermediate circumstances 2 with aid focus)*
2. *$0 < x_3 < \min(x_1, x_2)$ where x_1, x_2, x_3 are given by (13).*

Assumption 4 means that: (1) the government investment is low (i.e. τ is low) but the use of aid is quite efficient (i.e. α_i is quite hight); (2) the maximum level of aid $\bar{a}$ and/or the efficiency of public investment σ are quite high with respect to the fixed cost b, but the local-maximum point x_3 of function G is not high enough (i.e. dynamics of capital are not very strong) to surpass thresholds x_1, x_2. Notice that if Assumption 4 is violated, we recover analyses in the previous sections.

[16] In a Solow model with two exogenous saving rates, there are two steady states which are locally stable. Kraay and Raddatz (2007) indicate that in such a model, if the saving rate is low, foreign aid could help the recipient to accumulate capital. Saving rate might jump to the higher level, and then, the economy would converge to the high steady state.

Under Assumption 4, G is not monotonic. It is increasing on $[0, x_3]$, decreasing on $[x_3, \min(x_1, x_2)]$, and increasing on $[\min(x_1, x_2), \infty)$. By combining Assumption 4, Lemma 2 and point $(2.b.i)$ of Lemma 3, there exist two steady states :

$$\text{low steady state } k^* = \frac{\bar{a}\alpha_i - \frac{D+b}{\sigma}}{\alpha_i\phi - \tau} \in (0, \bar{a}/\phi), \text{ and high steady state } k^{**} = \frac{D+b}{\tau\sigma} \in (\bar{a}/\phi, \infty).$$

It is easy to see that the high steady state k^{**} is unstable. The main question in this section is whether the recipient country can encompass the high steady state and attain an economic take-off. It is also about to investigate whether the capital stock converges to the low steady state or fluctuates around it.

Let us start by considering a poor country (i.e., k_0 is low).

Proposition 4. *Assume that $\sigma\alpha_i\bar{a} > D + b$. When the initial capital stock k_0 is low enough, the capital stock at the next period will be higher than k_0: $k_1 > k_0$.*

Proof. Condition $\sigma\alpha_i\bar{a} > D + b$ ensures that $f(0) > 1$. Since the function f is continuous, $f(k_0) > 1$ for any k_0 low enough. By consequence, $k_1 = G(k_0) = f(k_0)k_0 > k_0$. □

Proposition 4 leads to an important implication: any poor country (characterized by Assumption 4) receiving foreign aid and using it efficiently always grows at the first stage of its development process (see Figure 3 for an illustration, with k_0 sufficiently far from the low steady state). In this case, aid may not promote growth but the economy never collapses: this is an important difference between the case of intermediate circumstances with aid focus and the cases of low circumstances or intermediate circumstances with domestic investment focus (which may rise a collapse). It follows that we should provide development aid for such poor countries.

However, our result does not mean that we should provide more development aid for any country at any stage of its development. A natural question arises: What happens to poor or developing countries (having a low or middle value of k_0)? We will address this question in next subsections.

4.3.1 Stability and fluctuations

We start this section by considering the stability of capital path.

Proposition 5 (stability of low steady state)**.** *Let Assumption 4 be satisfied.*

1. *Considering the case where $\sigma\bar{a}\alpha_i < D + b + \frac{1}{A}\left(\frac{1+\tau}{\beta}\right)$, or equivalently $x_3 > k^*$. We have that: if $k_0 \in (0, k^*)$, then $k_t \in (0, k^*)$ for any t and $\lim_{t\to\infty} k_t = k^*$.*

2. *Considering the case where $\sigma\bar{a}\alpha_i > D + b + \frac{1}{A}\left(\frac{1+\tau}{\beta}\right)$, or equivalently $x_3 < k^*$. The steady state k^* is locally stable[17] if and only if*

$$\sigma\bar{a}\alpha_i < D + b + \frac{2}{A}\left(\frac{1+\tau}{\beta}\right) \tag{20}$$

Proof. See Appendix A.3. □

[17] It means that there exists $\epsilon > 0$ such that $\lim_{t\to\infty} k_t = k^*$ for any $k_0 \in (k^* - \epsilon, k^* + \epsilon)$.

Recall that we are considering $\sigma\tau k_0 < D + b$, i.e., $k_0 < k^{**}$ the country is in a situation sufficiently vulnerable to have a possibility of collapse if there is no aid (according to point 1 of Proposition 1). Point 1 of Proposition 5 shows the role of aid: a country receiving development aid may converge to some point. This may happen under Assumption 4 and $x_3 > k^*$, that is the low steady state is lower than the local-maximum point (x_3) of output. This finding complements Proposition 2 and Proposition 3: foreign aid may promote growth in the recipient country. It should be noticed that Propositions 2, 3 and 5 consider different circumstances (high, low, intermediate 1 and intermediate 2 circumstances) which are not overlapped.

Figure 3 illustrates Proposition 5.[18] On the left we have $x_3 > k^*$, and $\lim_{t\to\infty} k_t = k^*$ for any $k_0 \in (0, k^*)$. However the convergence of capital stock may fail when $x_3 < k^*$. Indeed, point 2 of Proposition 5 shows that there may be room for local instability when k_0 is around the low steady state.

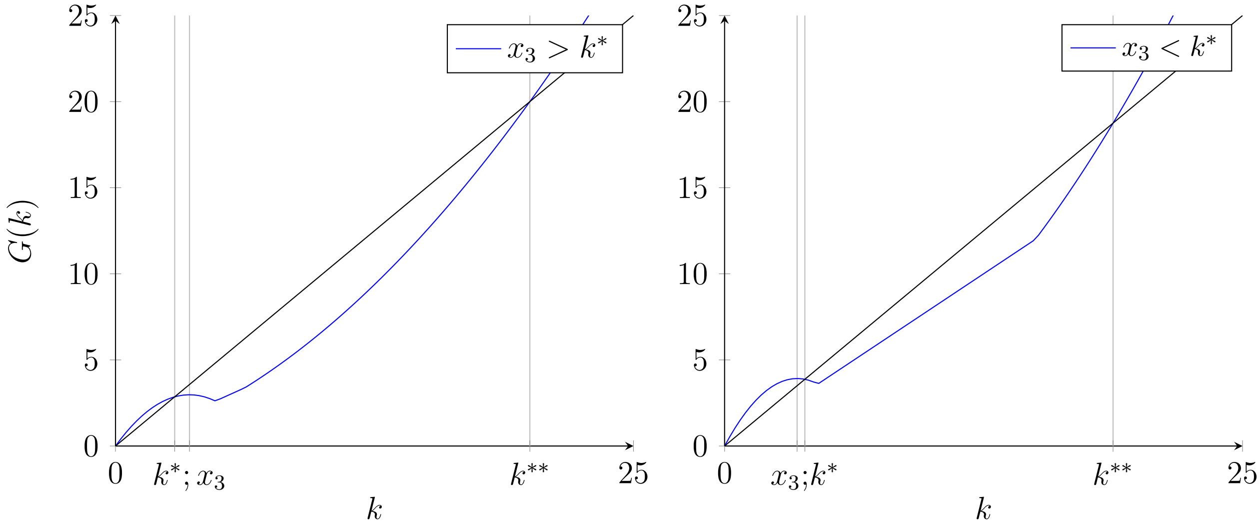

Figure 3: Assumption 4 is satisfied. On the left: $x_3 > k^*$. On the right: $x_3 < k^*$ and (20) holds.

Another question arises: is there fluctuation of capital paths or cycle around the low steady state? Our analysis is based on the following intermediate result.

Lemma 4. *Assume conditions in Assumption 4 hold and $x_3 < k^*$. Assume also that*

$$\sigma\bar{a}\alpha_i > D + b + \frac{2}{A}\left(\frac{1+\tau}{\beta}\right). \tag{21}$$

Then, there exists $y_1 \in (x_3, k^*)$ and $y_2 > 0$ in $(0, x_2)$ such that

$$y_1 \neq y_2, \quad f_3(y_1) = y_2, \quad f_3(y_2) = y_1. \tag{22}$$

Moreover, if we add assumption that $G(y_1) < x_2$, then such values y_1, y_2 satisfy

$$y_1 \neq y_2, \quad G(y_1) = y_2, \quad G(y_2) = y_1. \tag{23}$$

[18]Parameters in Figure 3. On the left: $\beta = 0.5; \tau = 0.2; \delta = 0.2; A = 0.5; \sigma = 0,8; \alpha_i = 0.8; \bar{a} = 10, \phi = 2, b = 1$. On the right: $\beta = 0.8; \tau = 0.2; \delta = 0.2; A = 0.4; \sigma = 1; \alpha_i = 0.7; \bar{a} = 12, \phi = 2, b = 3$.

Proof. See Appendix A.4. □

Considering y_1, y_2 determined by (23) of Lemma 4, let us denote

$$\mathcal{F}_0 \equiv \{y_1, y_2\}, \quad \mathcal{F}_{t+1} \equiv G^{-1}(\mathcal{F}_t) \quad \forall t \geq 0, \quad \mathcal{F} \equiv \cup_{t\geq 0}\mathcal{F}_t.$$

The following result is a direct consequence of Lemma 4 and definition of $\mathcal{F}$.

Proposition 6 (a two-period cycle around the low steady state)**.** *Under Assumption 4 and conditions in Lemma 4, we have: if $k_0 \in \mathcal{F}$, then there exists t_0 such that $k_{2t} = y_1$, $k_{2t+1} = y_2$ for any $t \geq t_0$.*

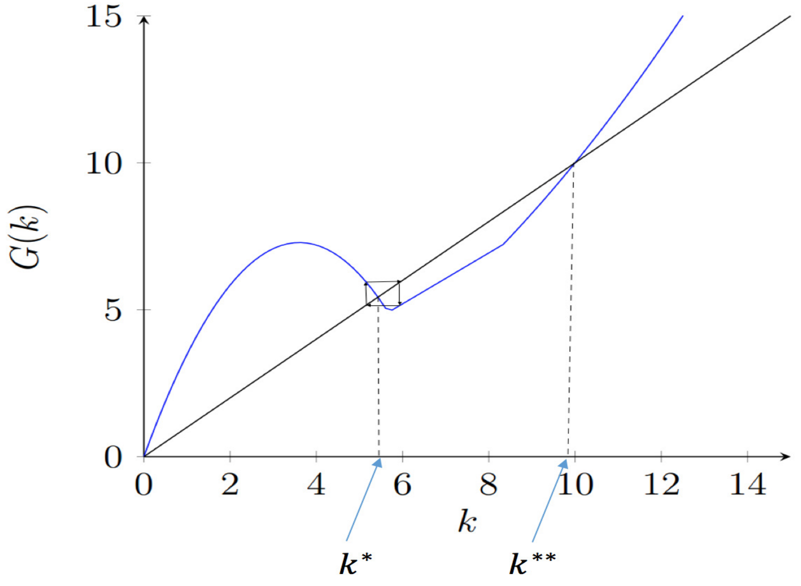

Figure 4: *Fluctuation around the low steady state.* Condition (21) holds and $x_3 < k^*$.

Proposition 6 indicates that if the initial capital belongs to $\mathcal{F}$ of $\mathbb{R}^+$, there is neither possibility for the recipient country to converge to the low steady state, nor the possibility of reaching an economic take-off.[19] The key for obtaining Proposition 6 is condition (21) which is equivalent to $3\frac{1+\tau}{\beta} - (1-\delta) < A(1 + \sigma\alpha_i\bar{a} - b)$. This holds if and only if

$$1 + \sigma\alpha_i\bar{a} > b \text{ and } A > \frac{3\frac{1+\tau}{\beta} - (1-\delta)}{1 + \sigma\alpha_i\bar{a} - b} \tag{24}$$

It means that the maximum of aid $\bar{a}$ and the efficiency in the use of aid and the TFP are quite high.

The intuition of Proposition 6 is the following: consider a country having a middle-level of initial capital and satisfying condition (24), when it receives aid at the initial date, its economy may grow at date 1 (according to Proposition 4). When the economy grows, its capital at date 1 increases. By the rule of aid, the aid flow for date 1 may decrease, leading to a decrease of total investment at date 1. Hence, the capital at date 2 may decrease, and so on. It follows that a two-period cycle may arise. Figure 6 illustrates this cycle.[20]

[19]However, it should be noticed that the fluctuation around the low steady state is not necessarily worse than the convergence towards this level.

[20]Parameters in Figure 6 are $\beta = 0.8, \tau = 0.2; \delta = 0.2, A = 0.5, \sigma = 1.2, \alpha_i = 0.8; \bar{a} = 12, b = 2, \phi = 2$.

4.3.2 Lucky growth

As shown above, a country having intermediate circumstances 2 with aid focus can converge to the low stead state or fluctuate around it. In this section, we wonder whether such a country can achieve growth in the long run. Notice that we continue to consider Assumption 4 under which we have $x_3 < k^{**}$.

Since x_3 is a local-maximum point of function G, we distinguish two sub-cases: (1) $G(x_3) \leq k^{**} = G(k^{**})$ corresponding to low dynamics of capital; and (2) $G(x_3) > k^{**} = G(k^{**})$ strong dynamics of capital, meaning that with some value in $(0, k^{**})$ (here it is x_3), the output can overcome the critical threshold k^{**}. Condition $G(x_3) > k^{**}$ is equivalent to

$$\frac{\beta}{1+\tau}\frac{\Big(1-\delta+A(1+\sigma\alpha_i\bar{a}-b)\Big)^2}{4A(\alpha_i\phi-\tau)} > \frac{D+b}{\tau}. \tag{25}$$

Under Assumption 4, the left hand side of (25) increases in $\bar{a}, \sigma, \alpha_i$ but decreasing in ϕ.[21] The right hand side depends neither on $(\bar{a}, \phi)$ nor on (σ, α_i). Hence, condition $G(x_3) > k^{**}$ is more likely to hold if $\bar{a}, \sigma, \alpha_i$ are high and/or ϕ is low.

Let us denote

$$U_0(k^{**}) \equiv \{x \in [0, k^{**}] : G(x) > k^{**}\}, \quad U_{t+1}(k^{**}) \equiv G^{-1}(U_t(k^{**})), \quad \forall t \geq 0 \tag{26a}$$

$$U(k^{**}) \equiv \cup_{t\geq 0} U_t(k^{**}). \tag{26b}$$

Note that $k^* \notin U(k^{**})$ and $k^* > x_3$. Here, k^{**} is the high steady state. It is easy to see that k_t tends to infinity if $k_0 > k^{**}$. The following result shows the asymptotic property of equilibrium capital path (k_t) for the case $k_0 < k^{**}$.

Proposition 7 (lucky growth)**.** *Let Assumption 4 be satisfied.*

1. *If $G(x_3) \leq k^{**}$, then $k_t \leq k^{**}$ for any $k_0 \leq k^{**}$.*
2. *If $G(x_3) > k^{**}$, then we have: $U(k^{**}) \neq \emptyset$. In this case, $\lim_{t\to\infty} k_t = \infty$ if and only if $k_0 \in U(k^{**})$.*

   Moreover, if $k_0 \in U_T(k^{**})$, then $k_t < k^{**}$ for any $t < T$, $k_t > k^{**}$ for any $t > T$.

Proof. See Appendix A.5. □

The first point in Proposition 7 indicates that when the dynamics of capital are weak, then the economy never surpasses the middle-income trap.

Point 2 of Proposition 7 suggests that a poor country, receiving development aid and having strong dynamics of capital, may surpass the middle-income in a finite period and achieve growth in the long run. To understand better this point, let us consider $k_0 \in U_0(k^{**})$. When the dynamics of capital are strong ($G(x_3) > k^{**}$), the stock of capital at the next period will be high (thanks to development aid) and surpass the middle-income trap k^{**} (i.e., $k_1 = G(k_0) > k^{**}$), and then the recipient economy may reach growth. However, in some cases, the economy needs more than one period to surpass the middle-income trap (for example, when $k_0 \in U_T(k^{**})$, the economy only surpasses k^{**} after T periods).

[21] It is easy to see that the left hand side increases in $\bar{a}, \sigma$ but decreasing in ϕ. It is increasing in α_i because $x_3 < x_1$.

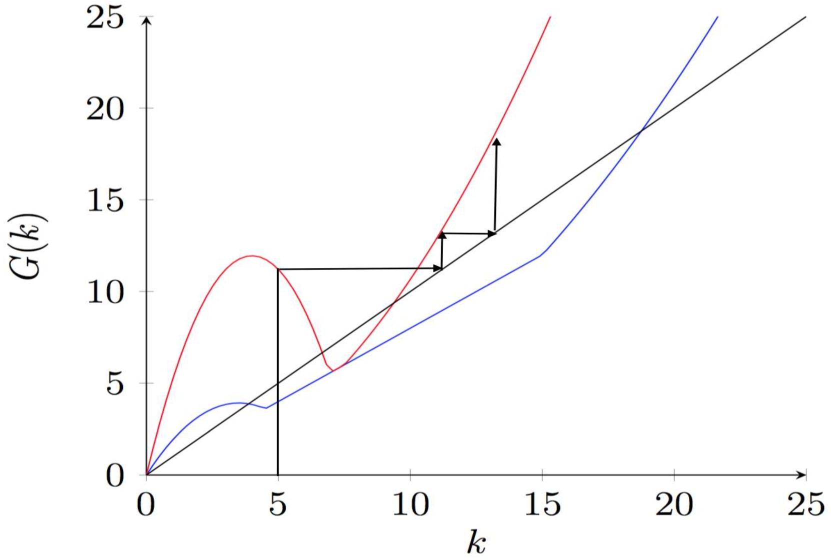

Figure 5: *Lucky growth vs. middle-income trap.* Low curve corresponds to $G(x_3) < G(k^{**})$, with parameters $\beta = 0.8, \tau = 0.2; \delta = 0.2, A = 0.4, \sigma = 1, \alpha_i = 0.7; \bar{a} = 12, b = 3, \phi = 2$. High curve corresponds to $G(x_3) > G(k^{**})$, with $\bar{a} = 14$, $\alpha_i = 0.8, \sigma = 2$, other parameters unchanged.

Corollary 1. *Let Assumption 4 be satisfied and assume that $G(x_3) > k^{**}$. If the initial capital k_0 is close enough to x_3, then $\lim_{t\to\infty} k_t = \infty$. However, if the initial capital k_0 equals k^* which is higher than x_3, we have $k_t = k^*$, $\forall t$.*

Figure 5 illustrates Corollary 1. If $k_0 = 5$, then the economy grows in the long run; however, if $k_0 = k^* > 5$, then $k_t = k^*$ for any t. Corollary 1 suggests that a higher initial capital does not necessarily help the economy to have more growth. Having growth without bounds, k_0 must belong to $U(k^{**})$. For that reason, we use the term "lucky growth", meaning that, with the same rule of aid, a poorer country may have growth but a richer country may not.

Pointing out this scenario is also a contribution of our paper to the literature on economic growth.

4.4 Discussion

We have seen in previous sections that the same rule of aid $(\bar{a}, \phi)$ may generate very different outcomes in the recipient country, following its circumstances. Focusing on autonomous technology, government effort, efficiency in the use of aid, efficiency in public investment and its fixed cost, we can distinguish 4 levels of circumstances, ranked from low to high quality: low circumstances, intermediate circumstances with government effort focus, intermediate circumstances with foreign aid focus, and high circumstances. If the recipient has a relatively high-quality of circumstances, the development aid may help it to reach economic growth whatever the initial capital. Consequently, there will exist a period when this economy no longer needs international aid to stimulate its economic development. In the opposite circumstances with low-quality circumstances, our analysis shows that the recipient country would obtain an economic take-off only if aid flows are sufficiently high. This result might justify a scaling-up of aid for countries suffering initial disadvantages which are not in favor of generating economic growth.

Concerning two intermediate circumstances (focusing on foreign aid or not), as we have shown in Section 4.2 and Section 4.3, their equilibrium outcomes are very different. On the one hand, under the intermediate circumstances 1 with domestic investment focus, k^* is the only steady state and can be viewed as a poverty trap of the economy. The economy will collapse in the long run if and only if the initial capital of the country is lower than this trap. In this case, development aid may promote growth in the recipient country, but under the condition that the use of aid is efficient enough. On the other hand, under the intermediate circumstances 2 with foreign aid focus, there exists two steady states. The lower one k^* can be interpreted as a middle-income trap. In this case, with foreign aid, the economy never collapses, even if its initial capital is very low. However, it does not necessarily mean that the economy will grow in the long run. Instead, the outcomes are fragile. Indeed, it may converge to the middle-income trap or fluctuate around it. With some luck (strong dynamics of capital), the economy may benefit development aid to improve its public investment (including R&D) and thanks to this, it can surpass the poverty trap (i.e. the higher steady-state k^{**}) and get grow after a finite period. To sum up, focusing on foreign aid would make the development process of the recipient country more complicated to predict.

4.5 Extensions

We now consider extensions of our framework to show the robustness of our results and insights. Assume now that aid flow is $a_t = a(k_t)$ instead of equation (1). The flow of new technology depends on the tax revenue and the aid flow in the following way: $H_t = H\big(\tau k_t, a(k_t)\big)$ (instead of σB_t). Assume that the TFP depends on new technologies: $A_t = P(H_t)$. Consequently, the TFP has the following form instead of (4): $A_t = P\Big(H\big(\tau k_t, a(k_t)\big)\Big)$. The dynamics of capital (8a) becomes

$$k_{t+1} = G(k_t) \equiv f(k_t)k_t, \text{ where } \quad f(k_t) \equiv \beta \frac{1-\delta+P\Big(H\big(\tau k_t, a(k_t)\big)\Big)}{1+\tau} \tag{27}$$

We introduce natural assumptions on the functions a, P and H.

Assumption (a). *The function $a(\cdot) : \mathbb{R}_+ \to \mathbb{R}^+$ is continuous, concave, strictly decreasing on $[0, \bar{k}]$ and differentiable on $(0, \bar{k})$. $a(0) = \bar{a} > 0$, $a(k) = 0\ \forall k \geq \bar{k}$*

Assumption (P). *The function $P(\cdot) : \mathbb{R}_+ \to \mathbb{R}^+$ is continous, strictly increasing on $[b, \infty)$ and differentiable on (b, ∞). $P(h) \geq A > 0\ \forall h \geq 0$. $P(h) = A$ if and only if $h \leq b$ (A represents the autonomous productivity).*

Assumption (H). *The function $H(\cdot) : \mathbb{R}^2_+ \to \mathbb{R}^+$ is differentiable, strictly concave, strictly increasing in each component. The aid is not essential: $H(x_1, 0) > 0\ \forall x_1 > 0$. Assume also that $\frac{1+\tau}{\beta} + \delta - 1 > A$ and $H(\infty, 0) > h_p$ where*

$$h_p \equiv P^{-1}\Big(\frac{1+\tau}{\beta} + \delta - 1\Big). \tag{28}$$

First, we study steady states. A steady state $k > 0$ is determined by $f(k) = 1$, i.e.,

$$P\Big(H\big(\tau k, a(k)\big)\Big) = \frac{1+\tau}{\beta} + \delta - 1. \tag{29}$$

Lemma 5. *Let Assumptions (a), (P) and (H) be satisfied. There are at most 3 steady states.*

In the proof of Lemma 5 (see Appendix A.6), we provide a necessary and sufficient condition under which there are i (i=1,2,3) steady states. In particular, if $H\big(\tau k, a(k)\big) = \sigma(\tau k)^m\big(1+\alpha_i(\bar{a}-\phi k)^+\big)^n$ and $P(h) = A(1+(h-b)^+)$, then there may be 3 steady states as illustrated by the following figure.[22]

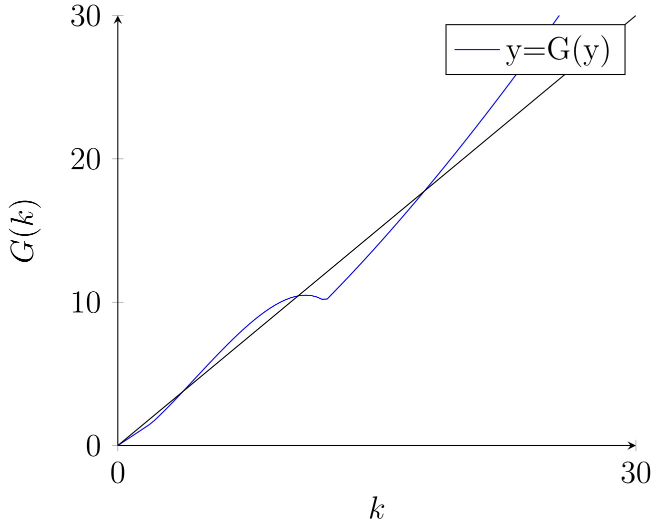

Figure 6: Existence of 3 steady-states.

Second, we look at the monotonicity of the policy function $G(\cdot)$.

Lemma 6. *Let Assumptions (a), (P) and (H) be satisfied. If* $I'(x) \geq 0$ $\forall x \in [0, \bar{k})$*, then* $G'(x) > 0$ $\forall x \in [0, \bar{k})$*. By consequence, the function G is increasing on* $[0, \infty)$*.*

Proof. See Appendix A.6. □

Condition $I'(x) \geq 0$, or equivalently $\tau H_1(\tau x, a(x)) + a'(x)H_2(\tau x, a(x)) > 0$, means that the government always focuses on the domestic investment. With the specification (8b), condition $I'(x) \geq 0$ becomes $\tau \geq \alpha_i\phi$.

We now present our findings concerning the dynamics of capital (27). The following result is a generalized version of Proposition 2.

Proposition 8 (growth without bounds thanks to foreign aid)**.** *Let Assumptions (a), (P) and (H) be satisfied. If* $\min\big(H(\tau\bar{k}, 0), H(0, \bar{a})\big) > h_p$*, then two statements in Proposition 2 hold.*

Proof. See Appendix A.6. □

Notice that condition in Proposition 8 is more likely to be satisfied if : (1) the fixed cost b is low and/or the autonomous productivity A is high (because h_p is increasing in b and decreasing in A); or/and (2) the aid rule is generous in the sense that $\bar{a}$ is high and/or ϕ is low (because $\min\big(H(\tau\bar{k}, 0), H(0, \bar{a})\big)$ is increasing in $\bar{a}$ and decreasing in ϕ).

We now extend Proposition 3 as follows:

[22] Parameters are: $\beta = 0.8; \tau = 0.2; \delta = 0.2; A = 0.4; \sigma = 2.5; \alpha_i = 0.45; \bar{a} = 12; \phi = 1; m = 0.4; n = 0.4; b = 3.4$.

Proposition 9. *(growth or collapse? The role of aid.) Let Assumptions (a), (P) and (H) be satisfied. If* $x\big(1-\delta+P(I(x))\big)$ *is increasing on* $[0,\bar{k}]$*, then* G *is increasing on* $[0,\infty)$*. In this case, three statements in Proposition 3 hold.*

According to this result and Lemma 6, we have that: if $I'(x) = \tau H_1(\tau x, a(x)) + a'(x)H_2(\tau x, a(x)) > 0$, meaning that the government always focuses on the domestic investment, then three statements in Proposition 3 hold.

We now present results in the case where the policy function may not be increasing. Denote k^{**} the unique solution (if it exists) of the equation $H(\tau k, 0) = h_p$. We firstly look at the economic growth in an economy having a high initial capital.

Proposition 10 (high initial capital)**.** *Let Assumptions (a), (P) and (H) be satisfied and* $H(\tau\bar{k}, 0) < h_p$*. We have that: (1)* k^{**} *is the highest steady state; and (2) for any* $k_0 > k^{**}$*, the sequence* (k_t) *increases and the economy grows without bounds.*

Proof. See Appendix A.6. □

We next look at countries having a low initial capital level. The following result generalizes Proposition 4.

Proposition 11 (low initial capital)**.** *Let Assumptions (a), (P) and (H) be satisfied.*

1. *Assume that* $\beta\frac{1-\delta+P(H(0,\bar{a}))}{1+\tau} > 1$*. If the initial capital stock* k_0 *is low enough, then the capital stock at the next period will be higher than* k_0*:* $k_1 > k_0$*.*

2. *Assume that* $\beta\frac{1-\delta+P(H(0,\bar{a}))}{1+\tau} < 1$*. If the initial capital stock* k_0 *is low enough, then* $\lim_{t\to\infty} k_t = 0$*.*

Proof. See Appendix A.6. □

Different from Proposition 4, we also provide a condition under which the economy collapses. This is illustrated by Figure 6 where we see that, if the initial capital k_0 is less than the lowest steady state, then k_t converges to zero.

As Proposition 7, we can prove the following result concerning the dynamics of economies having middle initial capital.

Proposition 12 (middle initial capital)**.** *Let Assumptions (a), (P), (H) be satisfied and* $H(\tau\bar{k}, 0) < h_p$*.*

1. *If* $\max_{x\in[0,k^{**}]} G(x) \leq k^{**}$*, then* $k_t \leq k^{**}$ *for any* $k_0 \leq k^{**}$*.*

2. *If* $\max_{x\in[0,k^{**}]} G(x) > k^{**}$*, then we have:* $U(k^{**})$ *defined by (26a, 26b) is not empty, and* $\lim_{t\to\infty} k_t = \infty \ \forall k_0 \in U(k^{**})$*. Consequently, aid flow* $a_t = 0$ *for* t *high enough.*

We end this section by mentioning two remarks.

Remark 4 (on the essentiality of aid)**.** Let Assumption 1 be satisfied. We can prove that, if the aid is essential in the sense that $H(x_1, 0) \leq b \ \forall x_1 > 0$, then k_t is bounded from above.[23] This leads to an interesting implication: if the foreign aid is essential for the realization of the public investment in the recipient country whose autonomous TFP is not high, then this country never grows.

[23]It suffices to prove that $k_{t+1} < k_t$ if $k_t > \bar{k}$. Indeed, if $k > \bar{k}$, then $a(k) = 0$ and hence $H(\tau k, a(k)) = H(\tau k, 0) \leq b$. This implies that $P(H(\tau k, a(k)) = A$. By consequence, $f(k) = \frac{\beta(1-\delta+A)}{1+\tau} < 1$. So, $k_{t+1} = G(k_t) = f(k_t)k_t < k_t$ if $k_t > \bar{k}$.

Remark 5 (link to Dalgaard (2008))**.** If we consider a particular case with a full depreciation of capital ($\delta = 1$), no fixed cost ($b = 0$), no capital tax ($\tau = 0$), and the rule of aid flows is given by $a_t = \theta k_t^{\lambda}$ where $\theta > 0, \lambda < 0$ as in Dalgaard (2008), the dynamics of capital will be

$$k_{t+1} = \beta A \sigma \alpha_i \theta k_t^{\lambda+1} \tag{30}$$

Then, we recover a dynamic system similar to that in Dalgaard (2008). The transitional dynamics of capital stock in (30) are much simpler than (8a) or (27) in our framework. In Dalgaard (2008) or in (30), the characteristics of the transitional path are determined by λ (the degree of inequality aversion on the part of the donor) while in our model they depend on all parameters. In particular, the model in Dalgaard (2008) has at most one steady state while ours may have two, even three.

5 Foreign aid in a centralized economy

We have so far focused on the outcomes in competitive equilibrium. In this section, we investigate the effects of foreign aid in a centralized economy. The social planner maximizes the intertemporal utility $\sum_{t=0}^{+\infty} \beta^t u(c_t)$ by choosing consumption (c_t), physical capital (k_t) and tax (T_t) subject to sequential constraints: $c_t + k_{t+1} + T_t \leq F(k_t, T_{t-1}, a_t)$, $\forall t \geq 1$ where $w_0 \equiv F(k_0, T_{-1}, a_0)$ is given and the aid flow $a_t = a(k_t)$ is a decreasing function of k_t. We assume that $F(k_t, T_{t-1}, a_t) = A_t F_0(k_t)$ where F_0 is the autonomous production function and $A_t = P\Big(H\big(T_{t-1}, a(k_t)\big)\Big)$ is the TFP at date t depending on new technologies as in Section 4.5.

Denote $S_t \equiv k_{t+1} + T_t$, $k_{t+1} = \theta S_t, T_t = (1-\theta)S_t$ where $\theta \in [\theta_1, \theta_2] \subseteq [0,1]$, where parameters θ_1, θ_2 represent other potential constraints of the government, which we do not model here. The problem of the social planner can be rewritten as follows

$$(CP1) : \max_{(c_t,S_t)_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t) \tag{31a}$$

$$\text{s.t: } c_t + S_t \leq q(S_{t-1}) \tag{31b}$$

$$\text{where } q(x) \equiv \max_{\theta \in [\theta_1,\theta_2]} Q(x,\theta), \text{ and } Q(x,\theta) \equiv F_0(\theta x) P\Big(H\big((1-\theta)x, a(\theta x)\big)\Big) \tag{31c}$$

and $q(s_{-1}) \equiv w_0 > 0$ is given and the utility function u satisfies standard conditions as required in Le Van and Dana (2003), Kamihigashi and Roy (2007).[24]

In particular, if there is no aid and $F_0(x) = x^{\alpha_d}$, $H(x) = A_e x^{\alpha_e}$, $P(x) = A + a(x - \bar{x})^+$, we recover the model in Section 3.1 in Bruno et al. (2009).

Observe that the outcomes (consumption, physical capital, and production) of the social planner's problem are different from those in the decentralized economy. There are two reasons: (1) the presence of externalities in the production function, and (2) the tax rate T_t/S_t is endogenous in the central planner's problem $(CP1)$ while it is exogenous in the maximization problem of the household (P_c).

If the function $Q(x, \theta)$ is increasing in x, then so is the function $q(x)$. However, theses two functions may not be increasing. Moreover, they may not be concave and there are

[24]Precisely, we assume that (1) u is in C^1, strictly increasing and concave and $u'(0) = \infty$, (2) for every $S > 0$, there exists a feasible path (c_t, S_t) from S such that $\sum_{t=0}^{\infty} \beta^t u(c_t) > -\infty$ and (3) for every $S > 0$, we have $\sum_{t=0}^{\infty} \beta^t u(q^t(S)) < \infty$, where q^t is defined by $q^1 = f, q^{t+1} = q(q^t)$.

two thresholds in the function $Q(x,\theta)$. By consequence, providing a full global analysis of the solution of the optimal growth problem $(CP1)$ is a challenge. However, some clear-cut points can be obtained. As in Section 2 and for the sake of tractability, we assume that

$$Q(x,\theta) \equiv \theta x\Big(1-\delta+A\Big[1+\big[\sigma(1-\theta)x+\sigma\alpha_i(\bar{a}-\phi\theta x)^+-b\big]^+\Big]\Big). \tag{32}$$

Even under this specification, the solution of the problem (CP1) is not explicit as that in Section 2.4. In order to study the properties of optimal paths, we have to understand when $q(x)$ and $Q(x,\theta)$ are increasing or decreasing. Similar to Lemmas 1 and 2, we can identify conditions under which the function $Q(x,\theta)$ is increasing in x or not.

Lemma 7 (monotonicity of $Q(x,\theta)$ in x)**.** *Denote*

$$y_1 \equiv \frac{\bar{a}}{\phi\theta}, \quad y_2 = \frac{\sigma\alpha_i\bar{a}-b}{\sigma(\theta(1+\alpha_i\phi)-1)}, \quad y_3 \equiv \frac{1-\delta+A(1+\sigma\alpha_i\bar{a}-b)}{2A\sigma\big(\theta(1+\alpha_i\phi)-1\big)} \tag{33}$$

where θ is given such that y_1, y_2, y_3 are well defined.

We have that $Q(x,\theta)$ is increasing in x on $[y_1,\infty)$. Moreover,

1. *If $1-\theta(1+\alpha_i\phi)\geq 0$, then $Q(x,\theta)$ is increasing in x.*
2. *If $\theta(1+\alpha_i\phi)-1>0$ and $\sigma\alpha_i\bar{a}-b\leq 0$, then $Q(x,\theta)$ is increasing in x.*
3. *If $\theta(1+\alpha_i\phi)-1>0$ and $\sigma\alpha_i\bar{a}-b>0$. In this case, $y_2,y_3>0$, and $Q(x,\theta)$ is increasing in x on $[y_2,\infty)$*
   - *(a) If $y_3\geq min(y_1,y_2)$, then $Q(x,\theta)$ is increasing in x.*
   - *(b) If $y_3< min(y_1,y_2)$, then $Q(x,\theta)$ is increasing on $[0,y_3]$, decreasing on $[y_3,\min(y_1,y_2)]$, and increasing on $[\min(y_1,y_2),\infty)$.*

When $\theta(1+\alpha_i\phi)-1>0$ and $\sigma\alpha_i\bar{a}-b>0$, we observe that

$$y_3\geq y_1 \Leftrightarrow \frac{1}{\theta} > 1+\alpha_i\phi-\phi\frac{1-\delta+A(1+\sigma\alpha_i\bar{a}-b)}{2A\sigma\bar{a}} \text{ while } y_3\geq y_2 \Leftrightarrow 1-\delta+A\geq A(\sigma\alpha_i\bar{a}-b).$$

By consequence, we have the following result.

Corollary 2. *Given $\theta\in[\theta_1,\theta_2]$. The function $Q(\cdot,\theta)$ is increasing on $[0,\infty)$ if one of the following conditions holds.*

1. *$\theta_2(1+\alpha_i\phi)\leq 1$.*
2. *$\theta_1(1+\alpha_i\phi)>1$ and $\sigma\alpha_i\bar{a}\leq b$.*
3. *$\theta_1(1+\alpha_i\phi)>1$, $\sigma\alpha_i\bar{a}>b$, and $1-\delta+A\geq A(\sigma\alpha_i\bar{a}-b)$.*
4. *$\theta_1(1+\alpha_i\phi)>1$, $\sigma\alpha_i\bar{a}>b$, and $\frac{1}{\theta_2}>1+\alpha_i\phi-\phi\frac{1-\delta+A(1+\sigma\alpha_i\bar{a}-b)}{2A\sigma\bar{a}}$.*

Interpretations: condition $\theta_2(1+\alpha_i\phi)\leq 1$ is equivalent to $1-\theta_2\geq\frac{\alpha_i\phi}{1+\alpha_i\phi}$ which ensures that the government focuses on the investment in new technology/innovation (because $T_t/S_t=1-\theta\geq 1-\theta_2$). Conditions 2 and 3 mean that the government focuses on the physical capital (θ_1 is high) and the aid is not very generous (ϕ is high). Under condition 4, the government takes care of both the physical capital (θ_1 is high) and the investment in new technology/innovation (θ_2 is not high).

According to Corollary 2, we obtain the following result.

Proposition 13. *(1) Under conditions in Corollary 2, the function $q(x)$ is increasing, and then the optimal capital path (k_t) is monotonic and converges.*

(2) If $\theta_1 = \theta_2 = \theta$, $\theta(1 + \alpha_i\phi) \leq 1$, $\sigma\alpha_i\bar{a} > b$ and $\beta\theta\big(1 - \delta + A\big(1 + \sigma\alpha_i\bar{a} - b\big)\big) > 1$, then every optimal capital path motonically goes to infinity and aid flow a_t becomes zero when t is high enough.

Proof. See Appendix A.7. □

In Proposition 13, since the function $q(x)$ is increasing, we can apply the optimal growth theory (see Le Van and Dana (2003), Kamihigashi and Roy (2007) among others) to study properties (convergence, boundedness, growth, ...) of optimal capital paths. Conditions in the second statement of Proposition 13 mean that the government focuses on the investment in new technology/innovation, the level of efficiency σ is high, and the aid is generous enough. Under these conditions, the economy obtains growth in the long run whatever the level of initial output. The insight is similar to that of Proposition 2 even though the approches and proofs are different. Proposition 2's second part is also in line with Proposition 3 of Bruno et al. (2009). Our added value is to introduce foreign aid and study its effects.

However, the functions $q(x)$ and $Q(x, \theta)$ may be decreasing in x. According to Lemma 7, the function $Q(x, \theta)$ is not increasing in x only if $\sigma\alpha_i\bar{a} > b$ and $\theta_2(1 + \alpha_i\phi) - 1 > 0$. In such a case, we obtain the following result showing the role of the TFP A and of the efficiency σ as well as of the aid rule $(\bar{a}, \phi)$.

Proposition 14. *We now assume that $\sigma\alpha_i\bar{a} > b$ and $\theta_2(1 + \alpha_i\phi) - 1 > 0$.*

Assume that the initial output of the economy w_0 is low in the sense that $2\sigma(1 + \alpha_i\phi)w_0 < \sigma\alpha_i\bar{a} - b$ and $4\alpha_i\phi\sigma w_0 < \sigma\alpha_i\bar{a} - b$.

1. *If $\beta\theta_2\Big(1 - \delta + A + \frac{A}{2}(\sigma\alpha_i\bar{a} - b)\Big) > 1$, then $c_1 > c_0$ for any optimal path, and by consequence no optimal path converges to zero.*

2. *If $1 - \delta + A(1 + \sigma\alpha_i\bar{a} - b) < 1$, then the economy collapses (S_t and c_t converge to zero).*

Proof. See Appendix A.7. □

The insight of point 1 of Proposition 14 is similar to those of Proposition 4 and of point 1 of Proposition 11: if foreign aid is quite high and the recipient country uses it efficiently, the economy never collapses. However, point 2 shows that the economy will collapse if the autonomous TFP A and the efficiency (σ) of public investment are low. This is in line with point 2 of Proposition 11.

6 Concluding remarks

Our paper presents a tractable model to investigate the effectiveness of foreign aid given the donors' rules and by the way, we contribute to the debate regarding the effectiveness of aid in terms of economic growth, comprising numerous empirical investigations. We have characterized the transitional dynamics of capital in all scenarios. The effectiveness of foreign aid depends strongly on the manners in which aid is used in recipient countries and on the absorptive capacity of these countries as well as the initial development level of the recipient countries.

Our model suggests that some countries with high circumstances may not need aid to grow. Some others with intermediate circumstances need aid for the first stages of their development process. Foreign aid may, in some cases, help a poor country to avoid collapse, to converge towards its low steady state, or to get an economic take-off. But focusing on foreign aid may also make the country dependent on aid and hence economic fluctuations may arise. Our analyses show that the recipient's TFP, the efficiency of public investment and in the use of aid play major roles in the recipient country's development.

In our framework, the recipient country receives foreign aid with exogenous rules $(\bar{a}, \phi)$ (although the aid flow $(\bar{a} - \phi k_t)^+$ is endogenous). For future research, it would be interesting to endogenize the aid rules as well as the efficiency in the use of aid. By doing this, we can investigate the optimal design of development aid and the reaction of the recipient country's government (especially when corruption may happen).

A Appendix

A.1 The solution of the consumer's problem in Section 2

Lemma 8. *Consider the optimal growth problem*

$$\max_{(c_t,s_t)_t} \sum_{t=0}^{\infty} \beta^t \ln(c_t) \tag{A.1}$$

$$c_t + s_{t+1} \leq A_t s_t, \quad c_t, s_t \geq 0. \tag{A.2}$$

The unique solution of this problem is given by $s_{t+1} = \beta A_t s_t$ for any $t \geq 0$.

Proof. Indeed, the Euler condition $c_{t+1} = \beta A_{t+1} c_t$ jointly with the budget constraint becomes $s_{t+2} - \beta A_{t+1} s_{t+1} = A_{t+1}(s_{t+1} - \beta A_t s_t)$. Thus, a solution is given by $s_{t+1} = \beta A_t s_t$. It is easy to check the transversality condition $\lim_{t\to\infty} \beta^t u'(c_t) s_{t+1} = 0$.

Since the utility function $\ln(c)$ is strictly concave, the solution is unique.

□

A.2 Properties of function f and G

To prove Lemma 1, we need the following claims.

Claim 1 (properties of f)**.**

1. *The function $f_1(k) \equiv (k - a)^+$ is increasing in k.*
2. *The function $f_2(k) \equiv \tau k + \alpha_i(\bar{a} - \phi k)^+$ is increasing on $[0, \infty]$ if $\tau \geq \alpha_i \phi$. When $\tau < \alpha_i \phi$, the function f_2 is decreasing on $[0, \bar{a}/\phi]$ and increasing on $[\bar{a}/\phi, \infty]$.*
3. *$f_2(k) \equiv \tau k + \alpha_i(\bar{a} - \phi k)^+ \geq \bar{a} \min(\alpha_i, \tau/\phi)$.*
4. *$f(k_t) \geq \frac{\beta}{1+\tau}\Big[1 - \delta + A\Big(1 + \big(\sigma \bar{a} \min(\alpha_i, \tau/\phi) - b\big)^+\Big)\Big]$.*

Proof of Claim 1. The two first points are obvious. Point 4 is a direct consequence of point 3. Let us prove point 3. We consider 2 cases.

(i) If $k \geq \bar{a}/\phi$, it is easy to see that $f_2(k) \geq \tau k \geq \tau \bar{a}/\phi \geq \bar{a} \min(\alpha_i, \tau/\phi)$.

(ii) If $k \leq \bar{a}/\phi$, then $f_2(k) = \alpha_i \bar{a} + (\tau - \alpha_i \phi)k$.

When $\tau - \alpha_i \phi \geq 0$, we have $f_2(k) \geq \alpha_i \bar{a}$.

When $\tau - \alpha_i \phi \leq 0$, we have $f_2(k) \geq \alpha_i \bar{a} + (\tau - \alpha_i \phi)\bar{a}/\phi = \alpha_i \bar{a}/\phi$. □

Claim 2. *We have the following properties.*

1. *G is increasing on $[x_1, \infty)$.*
2. *Assume that $x_2 > 0$. We have G increasing on $[x_2, \infty)$.*

Consequently, G is increasing on $[\min(x_1, x_2), \infty)$.

Proof of Claim 2.

1. G is increasing on $[x_1, \infty)$ because when $x \geq x_1$, we have $G(x) = \beta \frac{1-\delta+A\left(1+(\sigma\tau x - b)^+\right)}{1+\tau} x$ which is increasing in in x.
2. If $x_1 < x_2$, the function G is increasing on $[x_2, \infty)$ because it is increasing on $[x_1, \infty)$.

   We now consider the case where $x_1 > x_2$. Let x and y be such that $x \geq y \geq x_2$. We have to prove that $G(x) \geq G(y)$. It is easy to see that $G(x) \geq G(y)$ when $x, y \in [x_2, x_1]$ or $x, y \in [x_1, \infty)$. We now assume that $x \geq x_1 \geq y$. In this case, we have

$$G(x) = \beta \frac{1-\delta+A\Big(1+(\sigma\tau x - b)^+\Big)}{1+\tau} x \geq \beta \frac{1-\delta+A}{1+\tau} x$$

$$G(y) = \beta \frac{1-\delta+A\Big(1+(\sigma\alpha_i\bar{a} - b - \sigma(\alpha_i\phi - \tau)y)^+\Big)}{1+\tau} y = \beta \frac{1-\delta+A}{1+\tau} y$$

   where the last equality is from the fact that $y \geq x_2 \equiv \frac{\sigma\alpha_i\bar{a}-b}{\sigma(\alpha_i\phi-\tau)}$. So, it is clear that $G(x) \geq G(y)$.

□

Proof of Lemma 1.

1. When $\tau \geq \alpha_i\phi$, according to point 2 of Claim 1, the function G is increasing on $[0, \infty)$.
2. When $\tau < \alpha_i\phi$ and $\sigma\alpha_i\bar{a} < b$ (or equivalently, $\tau < \alpha_i\phi$ and $x_2 < 0$). We consider two cases.

   (i) If $x \leq \bar{a}/\phi$, then $\big(\sigma(\tau x + \alpha_i(\bar{a} - \phi x)^+) - b\big)^+ = (\sigma\alpha_i\bar{a} - b - \sigma(\alpha_i\phi - \tau)x)^+ = 0$ (because $\sigma\alpha_i\bar{a} - b < 0$). So, in this case, we have $G(x) = \beta\frac{1-\delta+A}{1+\tau}x$.

   (ii) If $x \geq \bar{a}/\phi$, we have $G(x) = \beta\frac{1-\delta+A\left(1+(\sigma\tau x-b)^+\right)}{1+\tau}x$. It is easy to see that G is increasing on $[0, \infty)$.
3. We now consider the last case where $\tau < \alpha_i\phi$ and $x_2 > 0$, and $x_3 > \min(x_1, x_2)$.

   First, according to Claim 2, we observe that G is increasing on $[\min(x_1, x_2), \infty)$.

   Second, we also see that G is increasing on $(0, x_3)$. Since $x_3 > \min(x_1, x_2)$, we obtain that G is increasing on $[0, \infty)$.

□

Proof of Lemma 2. According to Claim 2, we have that G is increasing on $[\min(x_1, x_2), \infty)$.

We now consider G on $[0, \min(x_1, x_2)]$. Let $x \in [0, \min(x_1, x_2)]$, we have

$$G(x) = f_3(x) = \beta \frac{1 - \delta + A\big[1 + \sigma\alpha_i\bar{a} - b - \sigma(\alpha_i\phi - \tau)x\big]}{1 + \tau} x. \tag{A.3}$$

By definition of x_3 and the fact that $\sigma\alpha_i\bar{a} > b$, we have $x_3 > 0$. Moreover, $f_3'(x_3) \geq 0$ if and only if $x \leq x_3$. Therefore, G is increasing on $[0, x_3]$, decreasing on $[x_3, \min(x_1, x_2)]$. □

Proof of Lemma 3. Points 1 and 2 are clear. We only consider here the case where $\sigma\bar{a}\min(\alpha_i, \tau/\phi) \leq D + b$ and $\tau = \alpha_i\phi$. In such a case, we have

$$G(x) = \begin{cases} \beta \dfrac{1 - \delta + A\big(1 + \big(\sigma\alpha_i\bar{a} - b\big)^+\big)}{1 + \tau} x & \text{if } \bar{a} - \phi x \geq 0 \\ \beta \dfrac{1 - \delta + A\big(1 + \big(\sigma\alpha_i x - b\big)^+\big)}{1 + \tau} x & \text{if } \bar{a} - \phi x < 0 \end{cases} \tag{A.4}$$

If $\beta \frac{1-\delta+A\left(1+\left(\sigma\alpha_i\bar{a}-b\right)^+\right)}{1+\tau} = 1$, i.e., $\sigma\bar{a}\alpha_i = D + b$, then $G(x) = x$ for any $x \leq \bar{a}/\phi$ (multiple steady states).

If $\beta \frac{1-\delta+A\left(1+\left(\sigma\alpha_i\bar{a}-b\right)^+\right)}{1+\tau} < 1$, i.e., $\sigma\bar{a}\alpha_i < D + b$, then there is a unique steady state which is k^{**}.

□

A.3 Proof of Proposition 5

Point 1. First, we need the following result.

Claim 3. *Assume that $\bar{a}\alpha_i > \frac{D+b}{\sigma} > \bar{a}\tau/\phi$ and $x_3 < x_2$.*

If $x_3 > k^*$, then $G(x_3) < x_3$. And therefore, $G(x_3) < x_3 < x_2 < k^* = G(k^*)$. In this case, we have $G(x) < k^*$ for any $x < k^*$.

Proof of Claim 3. It is easy to see that if $x_3 > k^*$, then $G(x_3) < x_3 < x_2 < k^* = G(k^*)$.

If $x < k^*$, then we have $G(x) \leq \max_{x \leq k^*} G(x) \leq G(x_3) < x_3 \leq k^*$. □

We now come back to the proof of Proposition 5. If $k_0 < k^*$, according to Claim 3, we have $k_1 = G(k_0) < k^*$. By induction, we have $k_t < k^*$ for any t.

We now prove that $\lim_{t\to\infty} k_t = k^*$ for any $k_0 \in (0, k^*)$. We consider different cases.

Case 1: $k_0 \in (0, x_3]$. Since G is increasing on $[0, x_3]$, we have $\lim_{t\to\infty} k_t = k^{**}$ for any $k_0 \in (0, x_3]$.

Case 2: $k_0 \in (x_3, x_2]$. We see that $k_1 = G(k_0) \leq \max_{x\in[0,x_2]} G(x) = G(x_3) < x_3$. Therefore $k_1 < x_3$, and so $\lim_{t\to\infty} k_t = k^{**}$.

Case 3: $k_0 \in [x_2, \bar{a}/\phi]$, we have $k_1 = G(k_0) = \frac{\beta(1-\delta+A)}{1+\tau} k_0$. Since $\frac{\beta(1-\delta+A)}{1+\tau} < 1$, there exists t_0 such that $k_{t_0} < x_2$. Thus $\lim_{t\to\infty} k_t = k^{**}$.

Case 4: $k_0 \in [\bar{a}/\phi, k^*]$, we have $G(k_0) < k_0$ which means that $f(k_0) < 1$. Combining with $k_1 = f(k_0)k_0$, there exists t_1 such that $k_1 < \bar{a}/\phi$. This implies that $\lim_{t\to\infty} k_t = k^{**}$.

Point 2. Recall that

$$G(k) = f_3(k) \equiv \frac{\beta}{1+\tau}\Big[1-\delta + A\Big(1+\sigma\alpha_i\bar{a} - \sigma(\alpha_i\phi - \tau)k - b\Big)\Big]k \tag{A.5}$$

$$= \frac{\beta}{1+\tau}\Big[1-\delta + A\big(1+\sigma\alpha_i\bar{a} - b\big) - A\sigma(\alpha_i\phi - \tau)k\Big]k \tag{A.6}$$

$$G'(k) = f_3'(k) = \frac{\beta}{1+\tau}\Big[1-\delta + A\big(1+\sigma\alpha_i\bar{a} - b\big) - 2A\sigma(\alpha_i\phi - \tau)k\Big]. \tag{A.7}$$

It is easy to compute that

$$G'(k^*) = \frac{\beta}{1+\tau}\Big[1-\delta + A(1+b+2B-\sigma\bar{a}\alpha_i)\Big]. \tag{A.8}$$

According to Bosi and Ragot (2011), k^* is locally stable if and only if $\|G'(k^*)\| < 1$. Since $x_3 < k^*$, have have $G'(k) < 0$. So, k^* is locally stable if and only if $G'(k) > -1$ which is equivalent to $3\frac{1+\tau}{\beta} - (1-\delta) + A(b-1-\sigma\alpha_i\bar{a}) > 0$.

A.4 Proof of Lemma 4

We will find $y_1, y_2 > 0$ such that (22). Let us denote $n = 1-\delta + A(1+\sigma\alpha_i\bar{a} - b)$ and $m = A\sigma(\alpha_i\phi - \tau)$. y_1, y_2 must satisfy

$$\frac{\beta}{1+\tau}(n - my_1)y_1 = y_2, \quad \frac{\beta}{1+\tau}(n - my_2)y_2 = y_1. \tag{A.9}$$

This implies that

$$y_2 - y_1 = \frac{\beta}{1+\tau}(n-my_1)y_1 - \frac{\beta}{1+\tau}(n-my_2)y_2 = (y_1 - y_2)\frac{\beta}{1+\tau}(n - m(y_1+y_2)).$$

Since $y_1 \neq y_2$, we have $\frac{\beta}{1+\tau}(n - m(y_1+y_2)) = -1$. So, we obtain

$$H(y_1) \equiv \frac{\beta}{1+\tau}(n-my_1)y_1 + y_1 - \frac{1}{m}\Big(n + \frac{1+\tau}{\beta}\Big) = 0 \tag{A.10}$$

We observe that $H(0) < 0$. We also see that $H(k^*) > 0$ if condition (21) is satisfied.

Under condition (21), there exists y_1 such that $H(y_1) = 0$. Therefore, y_1 and $y_2 = f_3(y_1)$ satisfy (22).

A.5 Proof of Proposition 7

Point (1). Since conditions in Assumption 4 hold, Lemma 2 implies that G is increasing on $[0, x_3]$, decreasing on $[x_3, \min(x_1, x_2)]$, and increasing on $[\min(x_1, x_2), \infty)$. So, $\max_{x \leq k^{**}} G(x) \leq Max(G(x_3), G(k^{**})) \leq k^{**}$. Therefore $k_t = G(k_{t-1}) \leq k^{**}$ for any $k_0 \leq k^{**}$.

Point (2). If $G(x_3) > k^{**}$, then $x_3 \in U_0(k^{**}) \subset U(k^{**})$. So, $U(k^{**}) \neq \emptyset$.

Now, let $k_0 \in U(k^{**})$, then there exists t_0 such that $G^{t_0}(k_0) > k^{**}$, where $G^1 \equiv G$ and $G^{s+1} \equiv G(G^s)$ for any $s \geq 1$. So, $k_{t_0} = G^t(k_0) > k^{**}$. This implies that $(k_t)_{t \geq t_0}$ is an increasing sequence and $\lim_{t\to\infty} k_t = \infty$.

Conversely, take $k_0 < k^{**}$ such that $\lim_{t\to\infty} k_t = \infty$. Since $k_{t+1} = G(k_t)$ and $G(k^{**}) = k^{**}$, we must have $k_t \neq k^{**}$ $\forall t$. Moreover, since $G(k) > k$ for any $k > k^{**}$, there exists

$T_0 \geq 1$ such that $k_t < k^{**}$ $\forall t < T_0$ and $k_t > k^{**}$ $\forall t \geq T_0$. By consequence, $k_{T_0-1} < k^{**}$ and $G(k_{T_0-1}) = k_{T_0} > k^{**}$. It mean that $G(k_{T_0-2}) = k_{T_0-1} \in U_0(k^{**})$. This implies that $k_{T_0-2} \in U_1(k^{**})$. By induction argument, we have $k_0 \in U_{T_0-1}(k^{**}) \subset U(k^{**})$.

To sum up, we have: $\lim_{t\to\infty} k_t = \infty$ if and only if $k_0 \in U(k^{**})$.

A.6 Proof of Section 4.5's results

Proof of Lemma 5. Let $k > 0$ be a steady state.

If $\frac{1+\tau}{\beta} + \delta - 1 < A$, then there is no steady state because $P(h) \geq A$ $\forall h \geq 0$.

If $\frac{1+\tau}{\beta} + \delta - 1 = A$, then any $k > 0$ satisfying $H\big(\tau k, a(k)\big) \leq b$ is a steady sate.

We now focus on the case where $\frac{1+\tau}{\beta} + \delta - 1 > A$ as required by Assumption (H). In this case, $k > 0$ is a steady state if and only if

$$H\big(\tau k, a(k)\big) = h_p \equiv P^{-1}\Big(\frac{1+\tau}{\beta} + \delta - 1\Big) \tag{A.11}$$

where P^{-1} is the inverse function of P.

1. If $k \geq \bar{k}$, then $H(\tau k, 0) = h_p$. Since $H(\tau k, 0)$ is increasing in k, the equation $H(\tau k, 0) = h_p$ has a unique solution in $[\bar{k}, \infty)$ if and only if $H(\tau\bar{k}, 0) \leq h_p < H(\infty, 0)$.

2. If $k < \bar{k}$, then $H(\tau k, a(k)) = h_p$. Consider the function $I(x) \equiv H(\tau x, a(x))$.

$$I'(x) = \tau H_1(\tau x, a(x)) + a'(x)H_2(\tau x, a(x)) \tag{A.12}$$
$$I''(x) = \tau^2 H_{11} + 2\tau a'(x)H_{12} + (a'(x))^2 H_{22} + a''(x)H_2(\tau x, a(x)) \tag{A.13}$$

   where $H_i, H_{i,j}$ are the first and second order derivatives. Since $H(x_1, x_2)$ is strictly concave and a is concave, we have $I''(x) < 0$. The function $I'(x)$ is strictly decreasing in x. By consequence, the equation $I'(x) = 0$ has at most one solution. This implies that the equation $I(x) = h_p$ has at most two solutions.

   There are only three cases.

   (a) If $I'(0) \leq 0$, then the function I is strictly decreasing on $(0, \bar{k})$. The equation $I(x) = h_p$ has a solution $k \in (0, \bar{k})$ if and only if $I(0) > h_p > I(\bar{k})$.

   (b) If $I'(\bar{k}) \geq 0$, then the function I is strictly increasing on $(0, \bar{k})$. The equation $I(x) = h_p$ has a solution $k \in (0, \bar{k})$ if and only if $I(0) < h_p < I(\bar{k})$.

   (c) If $I'(0) > 0 > I'(\bar{k})$, then there is a unique, denoted by $x_i \in (0, \bar{k})$, such that $I'(x_i) = 0$. In this case, I is strictly increasing on $(0, x_1)$ and strictly decreasing on $(x_1, \bar{k})$.

      - If $I(x_i) < h_p$, then there is no steady state in $(0, \bar{k})$.
      - If $I(x_i) = h_p$, then x_i is the unique steady state in $(0, \bar{k})$.
      - If $I(x_i) > h_p > \max(I(0), I(\bar{k}))$, then there are 2 steady states in $(0, \bar{k})$, the lower $k_1^* \in (0, x_1)$ and the higher $k_2^* \in (x_1, \bar{k})$.
      - If $\min(I(0), I(\bar{k})) < h_p \leq \max(I(0), I(\bar{k}))$, there is a unique steady state $(0, \bar{k})$.
      - If $h_p \leq \min(I(0), I(\bar{k}))$, then there is no steady state in $(0, \bar{k})$.

□

Proof of Lemma 6. We consider two cases.

1. If $k \geq \bar{k}$, then $G(x) = \beta \frac{1-\delta+P\big(H(\tau x,0)\big)}{1+\tau} x$ is increasing in x. So, the function G is increasing on $[\bar{k}, \infty)$

2. If $k < \bar{k}$, then $G(x) = \beta \frac{1-\delta+P(I(x))}{1+\tau} x$, where we denote $I(x) \equiv H(\tau x, a(x))$. We can compute

$$\frac{1+\tau}{\beta} G'(x) = 1 - \delta + P\big(I(x)\big) + xP'(I(x))I'(x) \quad \text{(A.14)}$$

Since $I'(x) \geq 0 \; \forall x \in [0, \bar{k})$ and $P'(I(x)) > 0 \; \forall x$, we have that $G'(x) > 0 \; \forall x \in [0, \bar{k})$.

So, the function G is strictly increasing on $[0, \infty)$ □

Proof of Proposition 8. It suffices to prove that $\min_{x \geq 0} f(x) > 1$. To do so, we prove that $H\big(\tau x, a(x)\big) > h_p$, $\forall x \geq 0$. We consider two cases.

(1) If $x \geq \bar{k}$, then $H\big(\tau x, a(x)\big) = H(\tau x, 0) \geq H(\tau \bar{k}, 0) > h_p$.

(2) If $0 < x < \bar{k}$, then $H\big(\tau x, a(x)\big) = I(x)$. Since $I(x)$ is concave, then $I(x) \geq \min(I(0), I(\bar{k})) > h_p$, $\forall x \in (0, \bar{k})$.

□

Proof of Proposition 10. Consider the function f on the interval (k^{**}, ∞). Since k^{**} is the highest steady state, we have $f(x) - 1 \neq 0 \; \forall x > k^{**}$. By the continuity of the function $f(x) - 1$, we must have either $f(x) - 1 < 0 \; \forall x > k^{**}$ or $f(x) - 1 > 0 \; \forall x > k^{**}$. According to the fact that $H(\infty, 0) > h_p$, we have $\lim_{x \to \infty}(f(x) - 1) > 0$. By consequence, we get that $f(x) - 1 > 0 \; \forall x > k^{**}$.

Now, let $k_0 > k^{**}$. We have $f(k_0) > 1$ and $k_1 = f(k_0)k_0 > k_0 > k^{**}$. By induction, $k_t > k^{**} \; \forall t$. So, $k_{t+1} = f(k_t)k_t > k_t$. Thus, the sequence the sequence (k_t) increases and hence converges. If $\lim_{t \to \infty} k_t = k < \infty$, we have $k > k^{**}$ and $f(k) = 1$. This is a contradiction because k^{**} is the highest steady state. By consequence, we have $\lim_{t \to \infty} k_t = \infty$ □

Proof of Proposition 11. Point 1 is similar to Proposition 4. Let us prove point 2. Since $\lim_{x \to 0} f(x) = f(0) = \beta \frac{1-\delta+P(H(0,\bar{a}))}{1+\tau} < 1$ and the function f is continuous, there exists $\underline{f} \in (0, 1)$ such that $f(x) < \underline{f}$ for any x low enough. Recall that $k_{t+1} = G(k_t) = f(k_t)k_t$. This implies that $\lim_{t \to \infty} k_t = 0$ for any k_0 low enough, □

A.7 Proof of Section 5's results

Proof of Proposition 13. The first part is clear because the function $Q(x, \theta)$ is increasing in x under conditions in Corollary 2. Let us prove the second part. Since $\theta_1 = \theta_2 = \theta$ (so that the ratio $\theta = K_{t+1}/S_t$ is constant) and $\theta(1 + \alpha_i \phi) \leq 1$, the function $q(x) = Q(x, \theta)$ is increasing in x. So, the optimal path (k_t) monotonically converges.

By using the same argument in proof of point 3 of Claim 1, we have that $(1 - \theta)x + \alpha_i(\bar{a} - \phi\theta x)^+ \geq \min(\bar{a}\alpha_i, \bar{a}\frac{1-\theta}{\theta\phi}) = \bar{a}\alpha_i$ where the last equality is ensured by condition

$\theta(1+\alpha_i\phi) \leq 1$. By consequence, $\sigma\big((1-\theta)x + \alpha_i(\bar{a}-\phi\theta x)^+\big) - b \geq \sigma\alpha_i\bar{a} - b > 0 \ \forall x \geq 0$. This implies that

$$\Big(\sigma\big((1-\theta)x + \alpha_i(\bar{a}-\phi\theta x)^+\big) - b\Big)^+ = \sigma\big((1-\theta)x + \alpha_i(\bar{a}-\phi\theta x)^+\big) - b \quad \forall x \geq 0.$$

We consider two cases.

1. $\bar{a} - \phi\theta x > 0$. In this case, we have

$$q(x) = \theta x\Big(1 - \delta + A\big(1 + \sigma\alpha_i\bar{a} - b\big) + A\sigma x\big(1 - (1+\alpha_i\phi)\theta\big)\Big) \tag{A.15}$$

$$\beta q'(x) = \beta\theta\Big(1 - \delta + A\big(1 + \sigma\alpha_i\bar{a} - b\big) + 2A\sigma x\big(1 - (1+\alpha_i\phi)\theta\big)\Big) \tag{A.16}$$

$$\geq \beta\theta\big(1 - \delta + A\big(1 + \sigma\alpha_i\bar{a} - b\big)\big) > 1. \tag{A.17}$$

2. $\bar{a} - \phi\theta x \leq 0$. In this case, we have

$$q(x) = \theta x\Big(1 - \delta + A\big(1 + \sigma(1-\theta)x - b\big)\Big) \tag{A.18}$$

$$\beta q'_+(x) = \beta\theta\Big(1 - \delta + A\big(1 - b\big) + 2A\sigma x\big(1 - \theta\big)\Big) \tag{A.19}$$

$$\geq \beta\theta\Big(1 - \delta + A\big(1 - b\big) + 2A\sigma\frac{\bar{a}}{\phi\theta}\big(1 - \theta\big)\Big) \tag{A.20}$$

$$\geq \beta\theta\Big(1 - \delta + A\big(1 - b\big) + 2A\sigma\bar{a}\alpha_i\Big) > 1 \tag{A.21}$$

where $q'_+(x) \equiv \limsup_{\epsilon\downarrow 0}\frac{q(x+\epsilon)-q(x)}{\epsilon}$. Since $\beta\theta\big(1 - \delta + A\big(1 + \sigma\alpha_i\bar{a} - b\big)\big) > 1$, we obtain that $\beta q'_+(x) > 1 \ \forall x > 0$. Applying Proposition 4.6 in Kamihigashi and Roy (2007), every optimal capital path goes to infinity. □

Proof of Proposition 14. Step 1. let x be such that $4\alpha_i\phi\sigma x < \sigma\alpha_i\bar{a} - b$ and $2\sigma(1+\alpha_i\phi)x < \sigma\alpha_i\bar{a} - b$ which implies that $\sigma\alpha_i\phi x < \sigma\alpha_i\bar{a} - b$ and $\phi x < \bar{a}$. We have $\phi\theta x \leq \phi x < \bar{a}$ $\forall \theta \leq \theta_2 \leq 1$. Then $(\bar{a}-\phi\theta x)^+ = \bar{a} - \phi\theta x$.

Since $\sigma\alpha_i\phi x < \sigma\alpha_i\bar{a} - b$ and $\theta \leq 1$, we get that

$$\sigma(1-\theta)x + \sigma\alpha_i(\bar{a}-\phi\theta x)^+ \geq \sigma\alpha_i(\bar{a}-\phi\theta x)^+ = (\sigma\alpha_i\bar{a} - \sigma\alpha_i\phi\theta x)^+ > b.$$

Therefore, we have

$$Q(x,\theta) = Q_3(x,\theta) \equiv \theta x\Big(1 - \delta + A\Big[1 + \sigma(1-\theta)x + \sigma\alpha_i(\bar{a}-\phi\theta x) - b\Big]\Big) \tag{A.22}$$

$$= \theta x\Big(1 - \delta + A\big(1 + \sigma x + \sigma\alpha_i\bar{a} - b\big) - (1+\alpha_i\phi)A\sigma x\theta\Big). \tag{A.23}$$

We have $\frac{\partial Q(x,\theta)}{\partial\theta} = x\Big(1 - \delta + A\big(1 + \sigma x + \sigma\alpha_i\bar{a} - b\big) - 2(1+\alpha_i\phi)A\sigma x\theta\Big) > 0$ because $2\sigma(1+\alpha_i\phi)x < \sigma\alpha_i\bar{a} - b$. So, the function $Q(x,\theta)$ is increasing in θ which implies that $q(x) = \max_\theta Q(x,\theta) = Q(x,\theta_2)$ and

$$q(x) = \theta_2 x\Big(1 - \delta + A(1 + \sigma\alpha_i\bar{a} - b) - (\theta_2(1+\alpha_i\phi) - 1)A\sigma x\Big) \tag{A.24}$$

$$q'(x) = \theta_2\Big(1 - \delta + A(1 + \sigma\alpha_i\bar{a} - b) - 2\big(\theta_2(1+\alpha_i\phi) - 1\big)A\sigma x\Big). \tag{A.25}$$

Denote $\bar{q}(\theta_2, \delta, A, \sigma, \alpha_i, \bar{a}, b) \equiv \theta_2\Big(1 - \delta + A + \frac{A}{2}(\sigma\alpha_i\bar{a} - b)\Big)$. Since $4\alpha_i\phi\sigma x < \sigma\alpha_i\bar{a} - b$, we have

$$q'(x) \geq \theta_2\Big(1 - \delta + A(1 + \sigma\alpha_i\bar{a} - b) - 2\alpha_i\phi A\sigma x\Big) \geq \bar{q}(\theta_2, \delta, A, \sigma, \alpha_i, \bar{a}, b) \quad \text{(A.26)}$$

Step 2. Assume that w_0 satisfies $2\sigma(1+\alpha_i\phi)w_0 < \sigma\alpha_i\bar{a} - b$ and $4\alpha_i\phi\sigma w_0 < \sigma\alpha_i\bar{a} - b$. We have $S_0 < w_0$. Step 1 implies that $q(S_0) = Q_3(S_0, \theta_2)$ which is differentiable in S_0. So, we have the Euler equation: $u'(c_0) = \beta u'(c_1)q'(S_0)$.

1. Assume also that $\beta\bar{q}(\theta_2, \delta, A, \sigma, \alpha_i, \bar{a}, b) > 1$. Then $S_0 < w_0$. Again, according to Step 1, we have $q'(S_0) > \bar{q}(\theta_2, \delta, A, \sigma, \alpha_i, \bar{a}, b)$ and hence

$$u'(c_0) = \beta u'(c_1)q'(S_0) \geq u'(c_1)\beta\bar{q}(\theta_2, \delta, A, \sigma, \alpha_i, \bar{a}, b) \geq u'(c_1) \quad \text{(A.27)}$$

   From this, we obtain that $c_1 > c_0$. By consequence, no optimal path (S_t) converges to zero.

2. If $\underline{q} \equiv 1 - \delta + A(1 + \sigma\alpha_i\bar{a} - b) < 1$, then $q(S_0) < \underline{q}S_0 < \underline{q}S_0$. By induction, we have $q(S_t) \leq \underline{q}^t S_0$ $\forall t \geq 0$. This implies that S_t and c_t converge to zero, i.e., the economy collapses.

□

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