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# Harsanyi support levels payoffs and weighted Shapley support levels values

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## Abstract

This paper introduces a new class of values for level structures. The new values, called Harsanyi support levels payoffs, extend the Harsanyi payoffs from the Harsanyi set to level structures and contain the Shapley levels value (Winter, 1989) as a special case. We also look at extensions of the weighted Shapley values to level structures. These values, we call them weighted Shapley support levels values, constitute a subset of the class of Harsanyi support levels payoffs and coincide on a level structure with only two levels with a class of weighted coalition structure values, already mentioned in Levy and McLean (1989) and discussed in McLean (1991). Axiomatizations of the studied classes are provided for both exogenously and endogenously given weights.

**Keywords** Cooperative game · Level structure · (Weighted) Shapley (levels) value · Level sharing system · Harsanyi set · Dividends

## 1 Introduction

Many organizations, companies, governments and so on are organized in hierarchical structures. Typically, there is one unit at the top and in the following levels, each unit of the superior level is split into two or more subordinate units, which usually have a lower rank than the higher ones. A similar organizational structure appears in some respects in supply chains. Effectiveness increases by sharing or pooling of physical objects, resources, and information. Queuing problems or electricity and other networks have a related background. A central characteristic of all these forms of organization is that a cooperating unit itself can be an actor in order to gain advantages of cooperation for the members of the unit. The question is: how should we share the realized benefits and allocate the costs incurred.

To distribute the profits of cooperating coalitions, the application of a cooperative game seems to be a natural approach. Winter (1989) defined a model for cooperative games with a hierarchical structure, called level structure, which consists of a sequence of coalition structures (the levels). In each level, the player set is partitioned into components. Winter's

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value (Winter, 1989) for such a model, we call it Shapley levels value, extends the Owen value (Owen, 1977), by itself an extension of the Shapley value (Shapley, 1953b). Thus, this value satisfies extensions of the symmetry axioms that are satisfied by the Owen value.

To treat symmetric players differently when there are exogenously given weights for the players, Shapley (1953a) introduced the weighted Shapley values. Vidal-Puga (2012) established a value for coalition structures with weights given by the size of the coalitions. With a step by step top-down algorithm, Gómez-Rúa and Vidal-Puga (2011) extended it to level structures. Besner (2019) generalized this value to the class of the weighted Shapley hierarchy levels values for arbitrary exogenously given weights. These values satisfy an extension of the consistency property of the weighted Shapley values in Hart and Mas-Colell (1989) but do not satisfy the null player axiom in general.

The weighted values for coalition structures in Levy and McLean (1989) and McLean (1991) have the opposite behavior, they satisfy the null player property but do not match a consistency property in the above sense. In Levy and McLean (1989) are examined several classes of weighted values for coalition structures which use the same weight system as the weighted Shapley values: either for the players within a component or for the components themselves if the components act as players, representing the players they contain. The combined use of such a weight system, both for players and for components, is only mentioned. This latter class of extensions of the weighted Shapley values and an extension of the class of random order values (Weber, 1988) in general to coalition structures is discussed in McLean (1991). Dragan (1992) called McLean's class of extensions of the weighted Shapley values McLean weighted coalition structure values, presented for them a formula related to that of the Owen value, and showed that these values coincide for a fixed coalition structure with a multiweighted Shapley value (Dragan, 1992).

A new view on coalition functions was introduced by Harsanyi (1959). He used so-called (Harsanyi) dividends, assigned to all feasible coalitions of a player set according to the coalition function: the singletons receive the singleton worth as dividend and the dividend of each larger coalition amounts to the worth of this coalition minus the sum of all dividends of the proper sub-coalitions of this coalition. The weighted Shapley values give each player as a payoff, according to her for all coalitions equal weight, shares in the dividends of the coalitions in which they are contained. Compared to this, the Harsanyi payoffs, the TU-values from the Harsanyi set (Hammer, 1977; Vasil'ev, 1978), are more flexible here. They allow each player to receive a share of a dividend from a coalition containing him, individual for each coalition.

In this article, we introduce the Harsanyi support levels payoffs. To the best of our knowledge, these are the first values for level structures that extend the Harsanyi payoffs. Each value of the new class can be represented by a formula with dividends. The coefficients in the formulas constitute a dividend sharing system, i. e., all coefficients are non-negative and amount to one for each coalition. Thus, by definition of a Harsanyi payoff, the values from this class coincide with a Harsanyi payoff for a given set of players and a fixed level structure and inherit so all properties (adapted to level structures) of these values. Since the Harsanyi payoffs are in general no random order values, we cannot take over the proof procedures for characterizations, e.g., in Winter (1989) or McLean (1991).

All our proofs are based on dividends, whereby two new lemmas are of significant assistance. We give axiomatizations for exogenously given level sharing systems and provide an axiomatization for the entire class of values where weights are derived endogenously through the actual solution. In addition, we present, as a subset of the class of Harsanyi support levels payoffs, the weighted Shapley support levels values, which coincide on a level

structure with only two levels with the McLean weighted coalition structure values. For this new class of values, our axiomatizations extend axiomatizations in [Nowak and Radzik \(1995\)](#).

The outline of the paper is structured as follows. Section 2 contains preliminaries, Section 3 presents the axioms and Section 4 gives a quick look at the Shapley levels value. As the main part, we introduce in Section 5 the Harsanyi support levels payoffs with appropriate axiomatizations and, in Section 6, the weighted Shapley support levels values. Section 7 gives a conclusion and discuss the results. The Appendix (Section 8) provides all the proofs, two related lemmas and a note to the logical independence of the axioms in our axiomatizations.

## 2 Preliminaries

We denote by  $\mathbb{R}$  the real numbers and by  $\mathbb{R}_{++}$  the set of all positive real numbers. A cooperative game with transferable utility (**TU-game**) is a pair  $(N, v)$  consisting of a finite set  $N$  of players and a **coalition function**  $v: 2^N \rightarrow \mathbb{R}$ ,  $v(\emptyset) = 0$ , where  $2^N$  is the power set of  $N$ . The subsets  $S \subseteq N$  are called **coalitions**,  $v(S)$  is the **worth** of coalition  $S$ , the set of all nonempty subsets of  $S$  is denoted by  $\Omega^S$ , and the set of all TU-games with player set  $N$  is denoted by  $\mathbb{V}^N$ . The **null game**  $\mathbf{0}$  is defined by  $\mathbf{0}(T) = 0$  for all  $T \subseteq N$ .

Let  $(N, v) \in \mathbb{V}^N$  and  $S \subseteq N$ . The **dividends**  $\Delta_v(S)$  ([Harsanyi, 1959](#)) are defined inductively by

$$\Delta_v(S) := \begin{cases} v(S) - \sum_{R \subsetneq S} \Delta_v(R), & \text{if } S \in \Omega^N, \text{ and} \\ 0, & \text{if } S = \emptyset. \end{cases} \quad (1)$$

A TU-game  $(N, u_T) \in \mathbb{V}^N$ ,  $T \in \Omega^N$ , with  $u_T(S) := 1$  if  $T \subseteq S$  and  $u_T(S) := 0$  otherwise for all  $S \subseteq N$  is called an **unanimity game** and  $T$  is called an **unanimity coalition**. It is well-known that any coalition function  $v$  on  $N$  has a unique presentation

$$v = \sum_{T \in \Omega^N} \Delta_v(T) u_T. \quad (2)$$

$(N, v)$  is called **totally positive** ([Vasil'ev, 1975](#)) if  $\Delta_v(T) \geq 0$  for all  $T \subseteq N$ . The **marginal contribution**  $MC_i^v(S)$  of player  $i \in N$  to  $S \subseteq N \setminus \{i\}$  is given by  $MC_i^v(S) := v(S \cup \{i\}) - v(S)$ . We call a coalition  $S \subseteq N$  **essential** in  $v$  if  $\Delta_v(S) \neq 0$ . Player  $i \in N$  is called a **null player** in  $v$  if  $v(S \cup \{i\}) = v(S)$  for all  $S \subseteq N \setminus \{i\}$ ; players  $i, j \in N$ ,  $i \neq j$ , are called (mutually) **dependent** ([Nowak and Radzik, 1995](#)) in  $v$  if  $v(S \cup \{i\}) = v(S) = v(S \cup \{j\})$  for all  $S \subseteq N \setminus \{i, j\}$  or, equivalent to it as a well-known fact, if

$$\Delta_v(S \cup \{k\}) = 0, \quad k \in \{i, j\}, \quad \text{for all } S \subseteq N \setminus \{i, j\}. \quad (3)$$

A **coalition structure**  $\mathcal{B}$  on  $N$  is a partition of the player set  $N$ , i.e., a collection of nonempty, pairwise disjoint, and mutually exhaustive subsets of  $N$ . Each  $B \in \mathcal{B}$  is called a **component** and  $\mathcal{B}(i)$  denotes the component that contains a player  $i \in N$ . A **level structure** ([Winter, 1989](#)) on  $N$  is a finite sequence  $\underline{\mathcal{B}} := \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$  of coalition structures  $\mathcal{B}^r$ ,  $0 \leq r \leq h+1$ , on  $N$  such that:

- $\mathcal{B}^0 = \{\{i\}: i \in N\}$ .
- $\mathcal{B}^{h+1} = \{N\}$ .

- For each  $r$ ,  $0 \leq r \leq h$ ,  $\mathcal{B}^r$  is a refinement of  $\mathcal{B}^{r+1}$ , i. e.,  $\mathcal{B}^r(i) \subseteq \mathcal{B}^{r+1}(i)$  for all  $i \in N$ .

$\mathcal{B}^r$  is called the  $r$ -th **level** of  $\underline{\mathcal{B}}$ ;  $\overline{\mathcal{B}}$  is the set of all components  $B \in \mathcal{B}^r$  of all levels  $\mathcal{B}^r \in \underline{\mathcal{B}}$ ,  $0 \leq r \leq h$ ;  $\mathcal{B}^r(B^k)$  is the component of the  $r$ -th level which contains the component  $B^k \in \mathcal{B}^k$ ,  $0 \leq k \leq r \leq h + 1$ .

The collection of all level structures with player set  $N$  is denoted by  $\mathbb{L}^N$ . A TU-game  $(N, v) \in \mathbb{V}^N$  together with a level structure  $\underline{\mathcal{B}} \in \mathbb{L}^N$  is an **LS-game**  $(N, v, \underline{\mathcal{B}})$ . The set of all LS-games on  $N$  is denoted by  $\mathbb{VL}^N$ . To make clear that a level structure  $\underline{\mathcal{B}} := \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$  has a total of  $h + 2$  levels, we also write  $\underline{\mathcal{B}}_h$  instead of  $\underline{\mathcal{B}}$ . Note that each TU-game  $(N, v)$  corresponds to an LS-game  $(N, v, \underline{\mathcal{B}}_0)$  with a **trivial level structure**  $\underline{\mathcal{B}}_0$  and we would like to say that each LS-game  $(N, v, \underline{\mathcal{B}}_1)$  corresponds to a game with coalition structure (Aumann and Drèze, 1974; Owen, 1977).

Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ . For each LS-game, we can also have a look at associated games where components of  $\overline{\mathcal{B}}$  act as players. We define for each level  $r$ ,  $0 \leq r \leq h$ , the level structure  $\underline{\mathcal{B}}^r := \{\mathcal{B}^{r^0}, \dots, \mathcal{B}^{r^{h+1-r}}\} \in \mathbb{L}^{\mathcal{B}^r}$  as the induced  **$r$ -th level structure** from  $\underline{\mathcal{B}}$  by considering the components  $B \in \mathcal{B}^r$  as players. There, all levels below  $r$  are dropped from the original level structure. In the  $k$ -th level  $\mathcal{B}^{r^k}$  of  $\underline{\mathcal{B}}^r$ ,  $0 \leq k \leq h + 1 - r$ , we have for each component  $B^{r+k} \in \mathcal{B}^{r+k}$  of the  $(r + k)$ -th level in the original level structure  $\underline{\mathcal{B}}$  an associated component  $B^{r^k} \in \mathcal{B}^{r^k}$ . This component  $B^{r^k} \in \mathcal{B}^{r^k}$  contains the components  $B \in \mathcal{B}^r$  as players which are subsets of the original component  $B^{r+k} \in \mathcal{B}^{r+k}$  so we have  $\mathcal{B}^{r^k} := \{\{B \in \mathcal{B}^r : B \subseteq B^{r+k}\}$  for all  $B^{r+k} \in \mathcal{B}^{r+k}\}$ <sup>1</sup>.

If a coalition  $T \in \Omega^N$ ,  $T = \bigcup_{B \subseteq T, B \in \mathcal{B}^r} B$ , is the union of components of the  $r$ -th level from  $\underline{\mathcal{B}}$  and we want to stress this property,  $T$  is denoted by  $T^r$ . Each such  $T^r$  is an **associated coalition** to a coalition of all players  $B \in \mathcal{B}^r$ ,  $B \subseteq T^r$ , in the induced  $r$ -th level structure, denoted by  $\mathcal{T}^r := \{B \in \mathcal{B}^r : B \subseteq T^r\}$  and vice versa. The induced  **$r$ -th level game**  $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r) \in \mathbb{VL}^{\mathcal{B}^r}$ , where  $\mathcal{B}^r$  is the player set with  $B \in \mathcal{B}^r$  as players, is given by

$$v^r(\mathcal{T}^r) := v(T^r) \text{ for all } \mathcal{T}^r \in \Omega^{\mathcal{B}^r}. \quad (4)$$

A **TU-value**  $\phi$  is an operator that assigns to any  $(N, v) \in \mathbb{V}^N$  a payoff vector  $\phi(N, v) \in \mathbb{R}^N$ , an **LS-value**  $\varphi$  is an operator that assigns payoff vectors  $\varphi(N, v, \underline{\mathcal{B}}) \in \mathbb{R}^N$  to all LS-games  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ .

We define  $W^N := \{f : N \rightarrow \mathbb{R}_{++}\}$ ,  $w_i := w(i)$  for all  $w \in W^N$  and  $i \in N$ , as the set of all positive **weight systems** on  $N$ . For all  $\underline{\mathcal{B}} \in \mathbb{L}^N$ , we define  $W^{\overline{\mathcal{B}}} := \{f : \overline{\mathcal{B}} \rightarrow \mathbb{R}_{++}\}$ ,  $w_B := w(B)$  for all  $w \in W^{\overline{\mathcal{B}}}$  and  $B \in \overline{\mathcal{B}}$ , as the set of all positive **components weight systems** on  $\underline{\mathcal{B}}$ .

The collection  $\Lambda^N$  on  $N$  of all **sharing systems**  $\lambda \in \Lambda^N$  is defined by

$$\Lambda^N := \left\{ \lambda = (\lambda_{T,i})_{T \in \Omega^N, i \in T} \mid \sum_{i \in T} \lambda_{T,i} = 1 \text{ and } \lambda_{T,i} \geq 0 \text{ for each } T \in \Omega^N \text{ and all } i \in T \right\}.$$

Let  $\underline{\mathcal{B}} \in \mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ , and  $B^{r+1} \in \mathcal{B}^{r+1}$ ,  $0 \leq r \leq h$ . The collection  $\Lambda^{B^{r+1}}$  on  $B^{r+1}$  of all

<sup>1</sup>Loosely speaking, the components in both layer structures are related to each other from the  $r$ -th level upwards in such a way that the same original players from the player set  $N$  are somehow the underlying part of two associated components.

<sup>2</sup>Vaguely formulated, all the coalitions of the  $r$ -th level game and the associated coalitions of the original game, they contain the same players of the original player set  $N$  in some way, have the same worth.

**component sharing systems**  $\lambda \in \Lambda^{B^{r+1}}$  is defined by

$$\Lambda^{B^{r+1}} := \left\{ \lambda = (\lambda_{T^r, B^r})_{T^r \in \Omega^{B^{r+1}}, T^r = \bigcup_{B \subseteq T^r, B \in \mathcal{B}^r} B, B^r \in \mathcal{B}^r, B^r \subseteq T^r} \mid \sum_{B^r \in \mathcal{B}^r, B^r \subseteq T^r} \lambda_{T^r, B^r} = 1 \text{ and} \right. \\ \left. \lambda_{T^r, B^r} \geq 0 \text{ for each } T^r \in \Omega^{B^{r+1}}, T^r = \bigcup_{B \subseteq T^r, B \in \mathcal{B}^r} B, \text{ and all } B^r \in \mathcal{B}^r, B^r \subseteq T^r \right\}.$$

A **level sharing system**  $\lambda = \lambda^{\underline{\mathcal{B}}} := \{\lambda^B \mid B \in \{N\} \cup \overline{\mathcal{B}} \setminus \mathcal{B}^0, \lambda^B \in \Lambda^B\}$  is a set system that contains a component sharing system for each component of  $\overline{\mathcal{B}} \cup \{N\}$  that is not a singleton. The collection of all level sharing systems  $\lambda$  on  $\underline{\mathcal{B}}$  is denoted by  $\Lambda^{\underline{\mathcal{B}}}$ .

For all  $(N, v) \in \mathbb{V}^N$  and  $w \in W^N$ , the (positively) **weighted Shapley value**  $Sh^w$  (Shapley, 1953a) is defined by

$$Sh_i^w(N, v) := \sum_{S \subseteq N, S \ni i} \frac{w_i}{\sum_{j \in S} w_j} \Delta_v(S) \text{ for all } i \in N.$$

As a special case of a weighted Shapley value, all weights are equal, the **Shapley value**  $Sh$  (Shapley, 1953b) is defined by

$$Sh_i(N, v) := \sum_{S \subseteq N, S \ni i} \frac{\Delta_v(S)}{|S|} \text{ for all } i \in N.$$

Hammer (1977) and Vasil'ev (1978) introduced independently a set of TU-values, called **Harsanyi set**, also known as **selectope**, which we designate by  $\mathcal{H}$ . The payoffs are obtained by distributing the Harsanyi dividends with the help of a sharing system. Each TU-value  $H^\lambda \in \mathcal{H}$ ,  $\lambda \in \Lambda^N$ , titled **Harsanyi payoff**, is defined by

$$H_i^\lambda(N, v) := \sum_{S \subseteq N, S \ni i} \lambda_{S,i} \Delta_v(S), \text{ for all } i \in N.$$

Obviously, the weighted Shapley values are a proper subset of the Harsanyi set. The best-known LS-value is the Shapley levels value<sup>3</sup> (Winter, 1989). We introduce this value here with a formula, presented in Calvo, Lasaga, and Winter (1996, Eq. (1)): Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ , and for all  $T \in \Omega^N$ ,  $T \ni i$ , be

$$K_T(i) := \prod_{r=0}^h K_T^r(i), \text{ where } K_T^r(i) := \frac{1}{|\{B \in \mathcal{B}^r : B \subseteq \mathcal{B}^{r+1}(i), B \cap T \neq \emptyset\}|}.$$

The **Shapley levels value**  $Sh^L$  is given by

$$Sh_i^L(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} K_T(i) \Delta_v(T) \text{ for all } i \in N.$$

It is easy to see that  $Sh^L$  coincides with  $Sh$  if  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_0$ .

<sup>3</sup>The value is also known as level(s) structure value or Winter's (Shapley type) value. Our designation is used, e. g., in Álvarez-Mozos et al. (2017).

### 3 Axioms

We refer to the following axioms for LS-values  $\varphi$  which are mostly simple adaptations of standard axioms:

**Efficiency, E.** For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ , we have  $\sum_{i \in N} \varphi_i(N, v, \underline{\mathcal{B}}) = v(N)$ .

**Null player, N.** For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$  and  $i \in N$  such that  $i$  is a null player in  $v$ , we have  $\varphi_i(N, v, \underline{\mathcal{B}}) = 0$ .

**Nullgame, NG.**  $\varphi_i(\mathbf{0}, v, \underline{\mathcal{B}}) = 0$  for all  $i \in N$ .

**Additivity, A.** For all  $(N, v, \underline{\mathcal{B}}), (N, v', \underline{\mathcal{B}}) \in \mathbb{VL}^N$ , we have  $\varphi(N, v, \underline{\mathcal{B}}) + \varphi(N, v', \underline{\mathcal{B}}) = \varphi(N, v + v', \underline{\mathcal{B}})$ .

**Positivity, Po (Vasil'ev, 1975).** For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$  such that  $(N, v)$  is totally positive, we have  $\varphi_i(N, v, \underline{\mathcal{B}}) \geq 0$  for all  $i \in N$ .

**Strict aggregate monotonicity, SAMo (Megiddo, 1974).** For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$  and  $\alpha \in \mathbb{R}_{++}$ , we have  $\varphi_i(N, v + \alpha \cdot u_N, \underline{\mathcal{B}}) > \varphi_i(N, v, \underline{\mathcal{B}})$  for all  $i \in N$ .

**Marginality, M (Young, 1985).** For all  $(N, v, \underline{\mathcal{B}}), (N, v', \underline{\mathcal{B}}) \in \mathbb{VL}^N$  and  $i \in N$  such that  $MC_i^v(S) = MC_i^{v'}(S)$  for all  $S \subseteq N \setminus \{i\}$ , we have  $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i(N, v', \underline{\mathcal{B}})$ .

**Coalitional strategic equivalence, CSE (Chun, 1989).** For all  $\alpha \in \mathbb{R}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ , and  $S \in \Omega^N$ , we have  $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i(N, v + \alpha \cdot u_S, \underline{\mathcal{B}})$  for all  $i \in N \setminus S$ .

A dependent player behaves like a dummy player in all coalitions that do not contain all dependent players: her marginal contribution to such coalitions is always zero. In the next axiom for level structures, the ratio of two player's payoffs is equal to the ratio of the weights of the player's singletons if both players are dependent and are members of the same component of the first level.

**$w$ -weighted dependence,  $\mathbf{D}_0^w$  (Nowak and Radzik, 1995).** For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $w \in W^{\bar{\mathcal{B}}}$ ,  $i, j \in N$  such that  $j \in \mathcal{B}^1(i)$ , and  $i, j$  are dependent in  $v$ , we have

$$\frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{\{i\}}} = \frac{\varphi_j(N, v, \underline{\mathcal{B}})}{w_{\{j\}}}.$$

**Mutual dependence,  $\mathbf{MD}_0$  (Nowak and Radzik, 1995).** For all  $(N, v, \underline{\mathcal{B}}), (N, v', \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $i, j \in N$  such that  $j \in \mathcal{B}^1(i)$ , and  $i, j$  are dependent in  $v$  and  $v'$ , we have

$$\varphi_i(N, v, \underline{\mathcal{B}})\varphi_j(N, v', \underline{\mathcal{B}}) = \varphi_j(N, v, \underline{\mathcal{B}})\varphi_i(N, v', \underline{\mathcal{B}}).$$

This axiom states that the ratios of two player's payoffs in two different games are equal if both players are members of the same component of the first level and both players are dependent in both games if no payoff is zero. The following axiom weakens  $\mathbf{D}_0^w$ .

**Coalitional differential  $\lambda$ -dependence,  $\mathbf{CDD}_0^\lambda$ .** For all  $\alpha \in \mathbb{R}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $\lambda \in \Lambda^{\bar{\mathcal{B}}}$ ,  $S \in \Omega^N$ ,  $S = S^0$ ,  $i, j \in S$  such that  $j \in \mathcal{B}^1(i)$ , and  $i, j$  are dependent in  $v$ , we have

$$\lambda_{S, \{j\}} [\varphi_i(N, v + \alpha \cdot u_S, \underline{\mathcal{B}}) - \varphi_i(N, v, \underline{\mathcal{B}})] = \lambda_{S, \{i\}} [\varphi_j(N, v + \alpha \cdot u_S, \underline{\mathcal{B}}) - \varphi_j(N, v, \underline{\mathcal{B}})].$$

Our new axiom means that if all players of a coalition  $S$  change their cooperation and take this change with them into all supersets of  $S$ , the ratio of the payoff differences of two dependent players from  $S$  who are in the same component of the first level is equal to the

ratio of their sharing weights for coalition  $S$  (if we have no null-weights). The next axiom is typical for LS-values and plays an important role in our examinations. It claims that the sum of all players' payoffs of a component coincides with this component's payoff in an induced level game where the component is regarded as a player.

**Level game property, LG** (Winter, 1989). For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $B \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ , we have

$$\sum_{i \in B} \varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_B(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r). \quad (5)$$

Winter (1989) used the following axiom to characterize the Shapley levels value: the sum of the payoffs to all players of a component is equal to the sum of the payoffs to all players of another component of the same level if both components are subsets of the same component of the next higher level and both components are symmetric players in the  $r$ -th level game.

**Symmetry between components, Sym<sup>4</sup>** (Winter, 1989). For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $B_k, B_\ell \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ , such that  $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$  and  $B_k, B_\ell$  are symmetric in  $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r) \in \mathbb{VL}^{\mathcal{B}^r}$ , we have

$$\sum_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}).$$

This axiom contains, so to speak, an adaptation of the well-known symmetry axiom to LS-values in combination with the level game property. Similarly, we extend all previously introduced axioms, marked by an index zero, into new axioms which use, so to speak, the level game property for their purposes. We could also have done without the previously introduced axioms, but we think that the statement of the individual axioms is easier to follow and that the somewhat inherent level game property becomes clearer. This also makes the message of the axiomatizations in the main part more understandable, whereby each of the following axioms can conceptually be replaced by the associated axiom with subscript 0 and the level game property. For this reason, we do not explain the following axioms in more detail.

**Remark 3.1.** *Each of the following axioms coincides with the associated axiom, marked by an index 0, if  $h = 0$ , and is implied, along with **LG**, by the associated axiom.*

**$w$ -weighted dependence between components, D<sup>w</sup>.** For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $w \in W^{\overline{\mathcal{B}}}$ ,  $B_k, B_\ell \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ , such that  $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ , and  $B_k, B_\ell$  are dependent in  $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r) \in \mathbb{VL}^{\mathcal{B}^r}$ , we have

$$\sum_{i \in B_k} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_k}} = \sum_{i \in B_\ell} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_\ell}}.$$

**Mutual dependence between components, MD.** For all  $(N, v, \underline{\mathcal{B}}), (N, v', \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $B_k, B_\ell \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ , such that  $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ , and  $B_k, B_\ell$  are dependent in  $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r) \in \mathbb{VL}^{\mathcal{B}^r}$  and  $(\mathcal{B}^r, v'^r, \underline{\mathcal{B}}^r) \in \mathbb{VL}^{\mathcal{B}^r}$ , we have

$$\sum_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) \sum_{i \in B_\ell} \varphi_i(N, v', \underline{\mathcal{B}}) = \sum_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}) \sum_{i \in B_k} \varphi_i(N, v', \underline{\mathcal{B}}).$$

<sup>4</sup>This axiom is called coalitional symmetry in Winter (1989).



**Coalitional differential  $\lambda$ -dependence between components,  $\mathbf{CDD}^\lambda$ .** For all  $\alpha \in \mathbb{R}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $\lambda \in \Lambda^{\underline{\mathcal{B}}}$ ,  $S \in \Omega^N$ ,  $B_k, B_\ell \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ , such that  $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ ,  $B_k, B_\ell \cap S \neq \emptyset$ , and  $B_k, B_\ell$  are dependent in  $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r) \in \mathbb{VL}^{\mathcal{B}^r}$  and  $C_S^r := \bigcup_{B \in \mathcal{B}^r: B \subseteq \mathcal{B}^{r+1}(B_k), B \cap S \neq \emptyset} B$ , we have

$$\lambda_{C_S^r, B_\ell} \sum_{i \in B_k} [\varphi_i(N, v + \alpha \cdot u_S, \underline{\mathcal{B}}) - \varphi_i(N, v, \underline{\mathcal{B}})] = \lambda_{C_S^r, B_k} \sum_{i \in B_\ell} [\varphi_i(N, v + \alpha \cdot u_S, \underline{\mathcal{B}}) - \varphi_i(N, v, \underline{\mathcal{B}})]. \quad (6)$$

Finally, we present an axiom for TU-values  $\phi$  that coincides with  $\mathbf{CDD}^\lambda$  when we have a level structure  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_0$ .

**Coalitional differential  $\lambda$ -dependence between players,  $\mathbf{CDDP}^\lambda$ .** For all  $\alpha \in \mathbb{R}$ ,  $(N, v) \in \mathbb{V}^N$ ,  $\lambda \in \Lambda^N$ ,  $S \in \Omega^N$ ,  $i, j \in S$  such that  $i, j$  are dependent in  $v$ , we have

$$\lambda_{S,j} [\phi_i(N, v + \alpha \cdot u_S, \cdot) - \phi_i(N, v)] = \lambda_{S,i} [\phi_j(N, v + \alpha \cdot u_S) - \phi_j(N, v)].$$

## 4 The Shapley levels value

Winter (1989) used the Owen value (Owen, 1977) as a starting point for his LS-value. Therefore, Winter has extended the efficiency, null player, symmetry and additivity axioms to axioms for level structures where symmetry splits into symmetry between components and an individual symmetry axiom. If we define a level structure as above, i. e., the singletons are the elements of the lowest level, Winter (1989, remark 1.6) pointed out that we can omit the individual symmetry axiom. In this sense, we present Winter's first axiomatization of the Shapley levels value<sup>5</sup>.

**Theorem 4.1.** (Winter, 1989) *Sh<sup>L</sup> is the unique LS-value that satisfies  $\mathbf{E}$ ,  $\mathbf{N}$ ,  $\mathbf{Sym}$ , and  $\mathbf{A}$ .*

It should be noted that there exist some further axiomatizations of the Shapley levels value (see Calvo, Lasaga and Winter 1996; Khmelnitskaya and Yanovskaya 2007; Casajus 2010; Besner, 2019).

## 5 Harsanyi support levels payoffs

If there are convincing reasons, not contained in the coalition function itself, to treat symmetric players differently, then symmetric players (acting components) should also not get the same payoff, as it is the case by the Shapley levels value. We can have fixed metrics for each player assigned to the players for all the coalitions containing them. For example, sometimes a player's influence on other partners is stronger when another player is not on the team. Or a player can only really exert his influence if specific other players have joined the coalition to support him. The following value gives the possibility to deal with such situations.

First, we assume that a level sharing system is exogenously given. Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ , and  $\lambda \in \Lambda^{\underline{\mathcal{B}}}$ . By (2),  $v$  can be expressed as a linear combination of unanimity games. For each unanimity game (with a multiplicative factor in the form of the unanimity

<sup>5</sup>Winter (1989) introduced his value axiomatically and used this axiomatization as a definition.

coalition dividend), we can describe the payoff of our LS-value as follows. Each  $h$ -th level component that contains at least one player from the unanimity coalition receives a share of the unanimity coalition dividend proportional to its sharing weight for the coalition that contains all such  $h$ -th level components. Then the share of each  $h$ -th level component involved is distributed among all their  $(h - 1)$ -th level subcomponents that contain at least one unanimity coalition player, proportional to their sharing weights from the coalition that contains all these subcomponents, and so on. In the end, each player of the unanimity coalition gets her share, “supported” by the sharing weights of all her supercomponents and these payoffs will be added up over all unanimity games in which she is not a null player.

**Definition 5.1.** Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $\lambda \in \Lambda^{\bar{\mathcal{B}}}$ , and for all  $T^r \in \Omega^N$ ,  $T^r = \bigcup_{B \subseteq T^r, B \in \mathcal{B}^r} B$ ,  $0 \leq r \leq \ell \leq h$ ,  $i \in T^r$ , be

$$K_{\lambda, T^r}(i) := \prod_{\ell=r}^h \lambda_{C_{T^r}^\ell(i), \mathcal{B}^\ell(i)}, \text{ where} \quad (7)$$

$$C_{T^r}^\ell(i) := \bigcup_{\substack{B \in \mathcal{B}^\ell: B \subseteq \mathcal{B}^{\ell+1}(i), \\ B \cap T^r \neq \emptyset}} B. \quad (8)$$

The **Harsanyi support levels payoff**  $H^{\lambda SL}$  is given by

$$H_{\mathcal{B}^r(i)}^{wSL}(\underline{\mathcal{B}}^r, v^r, \underline{\mathcal{B}}^r) := \sum_{T^r \in \Omega^{\mathcal{B}^r}, T^r \ni \mathcal{B}^r(i)} K_{\lambda, T^r}(i) \Delta_{v^r}(T^r) \text{ for all } i \in N, \quad (9)$$

where  $T^r \in \Omega^N$  is the associated coalition to  $T^r$  and  $H_i^{wSL}(N, v, \underline{\mathcal{B}}) = H_{\{i\}}^{wSL}(\mathcal{B}^0, v^0, \underline{\mathcal{B}}^0)$ . The class of all Harsanyi support levels payoffs is called **Harsanyi support levels set** and is denoted by  $\mathcal{H}^{SL}$ .

**Remark 5.2.** We see that the Shapley levels value is a Harsanyi support levels payoff where all components of the same level have the same sharing weights for each coalition to which they belong and add up to one.

**Remark 5.3.** For each  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\lambda \in \Lambda^{\bar{\mathcal{B}}}$ , and for all  $T = T^0 \in \Omega^N$ ,  $T \ni i$ , we have  $\sum_{i \in T} K_{\lambda, T}(i) = 1$  and  $K_{\lambda, T}(i) \geq 0$ . Therefore, for fixed  $N$  and  $\underline{\mathcal{B}} \in \mathbb{L}^N$ , each  $H^{\lambda SL} \in \mathcal{H}^{SL}$  on  $(N, v, \underline{\mathcal{B}})$  coincides with a  $H^{\lambda'} \in \mathcal{H}$ ,  $\lambda' \in \Lambda^N$ , on  $(N, v) \in \mathbb{V}^N$  where  $\lambda'_{T, i} = K_{\lambda, T}(i)$  for each  $T \in \Omega^N$  and all  $i \in T$  and the level structure is disregarded.

The Harsanyi support levels payoffs match a number of axioms, especially those used in our axiomatizations below.

**Proposition 5.4.** Let  $\underline{\mathcal{B}} \in \mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ , and  $\lambda \in \Lambda^{\bar{\mathcal{B}}}$ .  $H^{\lambda SL}$  satisfies **E**, **N**, **NG**, **A**, **P**, **M**, **CDD** $^\lambda$ , and **LG**.

We present a first axiomatization that replaces the symmetry property in Theorem 4.1 by the coalitional differential  $\lambda$ -dependence property.

**Theorem 5.5.** Let  $\underline{\mathcal{B}} \in \mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ , and  $\lambda \in \Lambda^{\bar{\mathcal{B}}}$ .  $H^{\lambda SL}$  is the unique LS-value that satisfies **E**, **N**, **CDD** $^\lambda$ , and **A**.

As the proof shows, in the axiomatization with **CDD** $^\lambda$ , the requirement that two components must be dependent can be omitted. However, then the axiom is no more weaker than the corresponding **D** $^w$ .

**Remark 5.6.** *If we use the coinciding axioms for TU-values, we have a new axiomatization of the Harsanyi payoffs with exogenously given sharing systems.*

The Harsanyi support levels payoffs have an exceptional status among extensions of Harsanyi payoffs to LS-values.

**Theorem 5.7.** *An LS-value  $\varphi$  coincides for fixed  $N$  and  $\underline{\mathcal{B}} \in \mathbb{L}^N$  with a Harsanyi payoff and satisfies **LG** if and only if  $\varphi \in \mathcal{H}^{SL}$ .*

We get an extension of probably the most famous characterization of the Harsanyi set (Vasil'ev, 1981; Derks, Haller, and Peters, 2000) to LS-games.

**Corollary 5.8.** *An LS-value  $\varphi$  satisfies **E**, **N**, **Po**, **A**, and **LG** if and only if  $\varphi \in \mathcal{H}^{SL}$ .*

## 6 Weighted Shapley support levels values

For applications, the class of weighted Shapley values is an important subset of the Harsanyi set. Therefore, we would like to highlight the following subset of the Harsanyi support levels set that extends the class of weighted Shapley values for level structures.

**Definition 6.1.** *Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $w \in W^{\bar{\mathcal{B}}}$ , and for all  $T^r \in \Omega^N$ ,  $T^r = \bigcup_{B \subseteq T^r, B \in \mathcal{B}^r} B$ ,  $0 \leq r \leq \ell \leq h$ ,  $i \in T^r$ , be*

$$K_{w, T^r}(i) := \prod_{\ell=r}^h K_{w, T^r}^{\ell}(i), \text{ where } K_{w, T^r}^{\ell}(i) := \frac{w_{\mathcal{B}^{\ell}(i)}}{\sum_{\substack{B \in \mathcal{B}^{\ell}: B \subseteq \mathcal{B}^{\ell+1}(i), \\ B \cap T^r \neq \emptyset}} w_B}.$$

The *weighted Shapley support levels value*  $Sh^{wSL}$  is given by

$$Sh_{\mathcal{B}^r(i)}^{wSL}(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r) := \sum_{T^r \in \Omega^{\mathcal{B}^r}, T^r \ni \mathcal{B}^r(i)} K_{w, T^r}(i) \Delta_{v^r}(T^r) \text{ for all } i \in N, \quad (10)$$

where  $T^r \in \Omega^N$  is the associated coalition to  $\mathcal{B}^r$  and  $Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) = Sh_{\{i\}}^{wSL}(\mathcal{B}^0, v^0, \underline{\mathcal{B}}^0)$ . The class of all weighted Shapley support levels values is denoted by  $\mathcal{WS}^{SL}$ .

**Remark 6.2.** *For a level structure  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ , a weighted Shapley levels value  $Sh^{wSL}$  coincides with a Harsanyi support levels payoff  $H^{\lambda^{SL}}$  where for each  $T^r \in \Omega^N$ ,  $0 \leq r \leq h$ , we have  $\lambda_{C^{T^r}, B} = w_B$  for all  $B \in \mathcal{B}^r$ .  $Sh^{wSL}$  coincides with  $Sh^w$  if  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_0$  and, if  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_1$ , the  $K_{w, T^0}(i)$  coincide with the " $\lambda_i^S$ " given in Dragan (1992, Sec. 2(e)). Therefore, in this case, the  $Sh^{wSL}$  coincide with the McLean weighted coalition structure values (Dragan, 1992; Levy and McLean, 1989; McLean, 1991).*

Also the weighted Shapley support levels values match a number of axioms.

**Proposition 6.3.** *Let  $w \in W^{\bar{\mathcal{B}}}$ .  $Sh^{wSL}$  satisfies, beside the axioms presented in Proposition 5.4,<sup>6</sup> **SAMo**, **D<sup>w</sup>**, and **MD**.*

The following two theorems are extensions of two characterizations of the weighted Shapley values with exogenously given weights in Nowak and Radzik (1995). The proof from our first theorem shows that additivity can replace the linearity axiom in their first axiomatization.

<sup>6</sup>For **CDD<sup>λ</sup>**, we have to use a  $\lambda$  as stated in Remark 6.2.

**Theorem 6.4.** *Let  $w \in \mathcal{W}^{\bar{B}}$ .  $Sh^{wSL}$  is the unique LS-value that satisfies  $\mathbf{E}$ ,  $\mathbf{N}$ ,  $\mathbf{D}^w$ , and  $\mathbf{A}$ .*

**Theorem 6.5.** *Let  $w \in \mathcal{W}^{\bar{B}}$ .  $Sh^{wSL}$  is the unique LS-value that satisfies  $\mathbf{E}$ ,  $\mathbf{D}^w$ , and  $\mathbf{M}$ .*

Our last theorem axiomatizes the class of weighted Shapley support levels values in general and is closely related to an axiomatization of the weighted Shapley values in [Nowak and Radzik \(1995, Theorem 2.4, Remark 2.3\)](#).

**Theorem 6.6.** *An LS-value  $\varphi$  satisfies  $\mathbf{E}$ ,  $\mathbf{N}$ ,  $\mathbf{SAMo}$ ,  $\mathbf{A}$ , and  $\mathbf{MD}$  if and only if  $\varphi \in \mathcal{WS}^{SL}$ .*

## 7 Conclusion and discussion

The rapidly increasing volume of collected data and global networking make it possible and necessary to share benefits between cooperating actors, often hierarchically structured. According to the above examinations, for the distribution of the generated surpluses, the presented new classes of LS-values provide an alternative to the Shapley levels value and the weighted Shapley hierarchy levels values. A close examination of the Shapley levels value definition given here shows that in unanimity games it is not advantageous for the individual player to merge into components: each component has only the same weight as a single player.

This is generally not the case for the classes of LS-values presented here and the weighted Shapley hierarchy levels values: the greater the weight of a component, the higher the share of an unanimity coalition player on the unanimity coalition dividend. By the weighted Shapley hierarchy levels values, players who do not belong to the unanimity coalition also receive a share of the unanimity dividend if they contribute to the weight of the components involved.

On the contrary, the values of our new classes always leave null players without benefits. However, here again, it is a great competitive advantage for the players to join forces. First of all, of course, for those who form a cooperating subgroup within an unanimity coalition. And then for all players who belong to coalitions with positive dividends within the whole coalition function. Here, the players of a component always “support” each other in changing unanimity games, even if they do not belong to the unanimity coalition. Nevertheless, the dummy players or null players do not receive assistance, although they can contribute to the total weight of the component.

A disadvantage of the new value classes can be seen in the fact that it is usually not clear what the new component weights are when a player leaves the game. As a result, all our axiomatizations, unlike the axiomatizations of the weighted Shapley hierarchy levels in [Besner \(2019\)](#), require a fixed set of players.

## 8 Appendix

### 8.1 Additional lemmas, used in the proofs

The following lemma states that each non-empty coalition  $S$  for each level is a subset of only one coalition that is a union of components from this level which have a non-empty

intersection with  $S$ .

**Lemma 8.1.** *Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $\mathcal{B}^r \in \underline{\mathcal{B}}$ ,  $0 \leq r \leq h$ . Each  $S \in \Omega^N$  is a subset of exactly one coalition  $T^r \in \Omega^N$ ,  $T^r = \bigcup_{\substack{B \subseteq T^r, B \in \mathcal{B}^r \\ B \cap S \neq \emptyset}} B$ . Thus, we can also uniquely designate each  $S \in \Omega^N$  as  $S_{T^r}$ .*

*Proof.* Each coalition  $T^r \in \Omega^N$  is a union of components  $B \in \mathcal{B}^r$ .  $\mathcal{B}^r$  is a partition, and so each player  $i \in S$ ,  $S \in \Omega^N$ , is contained in only one component  $B \in \mathcal{B}^r$ . Therefore, for each coalition  $S \in \Omega^N$  there is exactly one coalition  $T^r \in \Omega^N$  which is a union of all components  $B \in \mathcal{B}^r$  containing at least one player  $i \in S$ .  $\square$

The next lemma shows that for each coalition  $\mathcal{T}$  in an induced level structure the dividend in the induced level game is equal to the sum of the dividends in the original game from all coalitions  $S$  of the original level structure which are subsets of a coalition  $T$  associated to  $\mathcal{T}$  and have the property of the previous lemma with respect to coalition  $T$ .

**Lemma 8.2.** *Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $\mathcal{B}^r \in \underline{\mathcal{B}}$ ,  $0 \leq r \leq h$ , and  $S_{T^r}$  be the coalitions from lemma 8.1 with associated coalitions  $T^r$ . Then we have in the  $r$ -th level game  $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r)$  for each  $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$ , associated to  $T^r \in \Omega^N$ ,*

$$\Delta_{v^r}(\mathcal{T}^r) = \sum_{S_{T^r} \subseteq \mathcal{T}^r} \Delta_v(S_{T^r}). \quad (11)$$

*Proof.* Let  $t = |\{B \in \mathcal{B}^r : B \subseteq T^r\}|$  the number of components  $B \in \mathcal{B}^r$  which are subsets from a coalition  $T^r \in \Omega^N$  with associated  $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$ . We use induction on the size  $t$ ,  $1 \leq t \leq |\mathcal{B}^r|$ .

*Initialization:* Let  $t = 1$ .  $T^r$  is a component  $B \in \mathcal{B}^r$  and  $\mathcal{T}^r$  is a player in  $v^r$ . We have

$$\Delta_{v^r}(\mathcal{T}^r) \stackrel{(1)}{=} v^r(\mathcal{T}^r) \stackrel{(4)}{=} v(T^r) \stackrel{(1)}{=} \sum_{S \subseteq T^r} \Delta_v(S) \stackrel{\text{Lem. 8.1}}{=} \sum_{S_{T^r} \subseteq \mathcal{T}^r} \Delta_v(S_{T^r}).$$

*Induction step:* Assume that (11) holds for an arbitrary  $\hat{t} \geq 1$  (IH). Let now  $\hat{\mathcal{T}}^r \in \Omega^{\mathcal{B}^r}$  with associated  $\hat{T}^r \in \Omega^N$ ,  $\hat{t} = |\{B \in \mathcal{B}^r : B \subseteq \hat{T}^r\}|$  and  $T^r = \hat{T}^r \cup \hat{B}$ ,  $\hat{B} \in \mathcal{B}^r$ ,  $\hat{B} \not\subseteq \hat{T}^r$ . We have  $t = \hat{t} + 1$  and it follows

$$\begin{aligned} \Delta_{v^r}(\mathcal{T}^r) &\stackrel{(1)}{=} \sum_{\mathcal{Q}^r \subseteq \mathcal{T}^r} \Delta_{v^r}(\mathcal{Q}^r) \stackrel{(1)}{=} v(T^r) - \sum_{\mathcal{Q}^r \not\subseteq \mathcal{T}^r} \Delta_{v^r}(\mathcal{Q}^r) \\ &\stackrel{(1)}{=} \Delta_v(T^r) + \sum_{S \subseteq T^r} \Delta_v(S) - \sum_{\substack{\mathcal{Q}^r \subseteq \mathcal{T}^r, S_{\mathcal{Q}^r} \subseteq \mathcal{Q}^r \\ \mathcal{Q}^r \not\subseteq \mathcal{B}^r}} \sum \Delta_v(S_{\mathcal{Q}^r}) \\ &\stackrel{\text{Lem. 8.1}}{=} \Delta_v(T^r) + \sum_{S \subseteq T^r} \Delta_v(S) - \sum_{\substack{S \subseteq T^r, \\ S \neq S_{T^r}}} \Delta_v(S) \\ &= \Delta_v(T^r) + \sum_{S_{T^r} \not\subseteq \mathcal{T}^r} \Delta_v(S_{T^r}) = \sum_{S_{T^r} \subseteq \mathcal{T}^r} \Delta_v(S_{T^r}). \quad \square \end{aligned}$$

## 8.2 Proofs

**Convention 8.3.** In order to avoid cumbersome case distinctions in the proofs with  $\mathbf{CDD}^\lambda$  or  $\mathbf{D}^w$ , if we consider only one single component isolated as a player, we define the component dependent on itself. Then  $\mathbf{CDD}^\lambda$  or  $\mathbf{D}^w$  is trivially satisfied.

### 8.2.1 Proof of Proposition 5.4

Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $\lambda \in \Lambda^{\underline{\mathcal{B}}}$ , and  $K_{\lambda, T^r}$  be the expressions according to Def. 5.1.

• **E, N, NG, A, P, M:** It is well-known that all  $H^{\lambda'} \in \mathcal{H}$ ,  $\lambda' \in \Lambda^N$ , satisfy the mentioned axioms. Thus, the claim follows by Remark 5.3.

• **LG:** Let  $B^r \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ . If  $r = 0$ , (5) trivially is satisfied.

Let now  $1 \leq r \leq h$ . Obviously, by (8), we have for all  $S^0 \subseteq N$ ,  $S^0 \cap B^r \neq \emptyset$ ,

$$\sum_{j \in B^r, j \in S^0} \prod_{\ell=0}^{r-1} \lambda_{C_{S^0}^\ell(j), \mathcal{B}^\ell(j)} = 1. \quad (12)$$

Let  $i \in B^r$  be fixed and  $S_{T^r} \in \Omega^N$  the coalitions from Lemma 8.1 with related coalitions  $T^r$ . Note, if  $i \in S_{T^r}$ , we have  $B^r \subseteq T^r$ . For all  $S_{T^r}^0 := S_{T^r}$ ,  $S_{T^r}^0 \ni i$ ,  $r \leq \ell \leq h$ , we have

$$\lambda_{C_{S_{T^r}^0}^\ell(i), \mathcal{B}^\ell(i)} \stackrel{\text{Lem. 8.1}}{=} \lambda_{C_{T^r}^\ell(i), \mathcal{B}^\ell(i)}. \quad (13)$$

It applies,  $\mathcal{B}^\ell(i) = \mathcal{B}^\ell(j)$  for all  $j \in B^r$  and  $r \leq \ell \leq h$ . For all  $S_{T^r}^0 \ni i$ , it follows,

$$\begin{aligned} \sum_{\substack{j \in B^r, \\ j \in S_{T^r}^0}} K_{\lambda, S_{T^r}^0}(j) &\stackrel{(7)}{=} \sum_{\substack{j \in B^r, \\ j \in S_{T^r}^0}} \prod_{\ell=0}^h \lambda_{C_{S_{T^r}^0}^\ell(j), \mathcal{B}^\ell(j)} \stackrel{(13)}{=} \sum_{\substack{j \in B^r, \\ j \in S_{T^r}^0}} \prod_{\ell=0}^{r-1} \lambda_{C_{S_{T^r}^0}^\ell(j), \mathcal{B}^\ell(j)} \prod_{\ell=r}^h \lambda_{C_{T^r}^\ell(i), \mathcal{B}^\ell(i)} \\ &\stackrel{(12)}{=} \prod_{\ell=r}^h \lambda_{C_{T^r}^\ell(i), \mathcal{B}^\ell(i)} \stackrel{(7)}{=} K_{\lambda, T^r}(i). \end{aligned} \quad (14)$$

We have  $\Delta_{v^0}(S^0) = \Delta_v(S^0)$  for all  $S^0 \in \Omega^N$  and associated  $\mathcal{S}^0 \in \Omega^{\mathcal{B}^0}$ . Finally, we get the following:

$$\begin{aligned} \sum_{j \in B^r} H_j^{\lambda SL}(N, v, \underline{\mathcal{B}}) &\stackrel{(9)}{=} \sum_{j \in B^r} \sum_{\substack{S^0 \subseteq N, \\ S^0 \ni j}} K_{\lambda, S^0}(j) \Delta_v(S^0) \stackrel{\text{Lem. 8.1}}{=} \sum_{j \in B^r} \sum_{\substack{S_{T^r}^0 \subseteq N, \\ S_{T^r}^0 \ni j}} K_{\lambda, S_{T^r}^0}(j) \Delta_v(S_{T^r}^0) \\ &= \sum_{S_{T^r}^0 \subseteq N} \sum_{\substack{j \in B^r, \\ j \in S_{T^r}^0}} K_{\lambda, S_{T^r}^0}(j) \Delta_v(S_{T^r}^0) \stackrel{(14)}{=} \sum_{\substack{S_{T^r}^0 \subseteq N, \\ T^r \ni B^r}} K_{\lambda, T^r}(i) \Delta_v(S_{T^r}^0) \\ &\stackrel{\text{Lem. 8.1}}{=} \sum_{T^r \subseteq B^r, T^r \ni B^r} K_{\lambda, T^r}(i) \sum_{S_{T^r}^0 \subseteq T^r} \Delta_v(S_{T^r}^0) \\ &\stackrel{\text{Lem. 8.2}}{=} \sum_{T^r \subseteq B^r, T^r \ni B^r} K_{\lambda, T^r}(i) \Delta_{v^r}(T^r) \stackrel{(9)}{=} Sh_{B^r}^{\lambda SL}(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r). \end{aligned}$$

• **CDD<sup>λ</sup>:** Let  $\alpha \in \mathbb{R}$ ,  $S \in \Omega^N$ ,  $k, \ell \in N$ ,  $0 \leq r \leq h$ ,  $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$ ,  $\mathcal{B}^r(k), \mathcal{B}^r(\ell) \cap S \neq \emptyset$ , and  $\mathcal{B}^r(k), \mathcal{B}^r(\ell)$  be dependent in  $v^r$  for the LS-game  $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r) \in \mathbb{V}\mathbb{L}^{\mathcal{B}^r}$ . If  $\lambda_{C_S^r, B_k} = 0$  or  $\lambda_{C_S^r, B_\ell} = 0$ , (6) is satisfied by (9). Otherwise, if  $r = 0$ , then  $k, \ell$  are dependent in  $v$  and we have, with  $S = S^0$ ,

$$\begin{aligned} \frac{H_k^{\lambda SL}(N, v + \alpha \cdot u_S, \underline{\mathcal{B}}) - H_k^{\lambda SL}(N, v, \underline{\mathcal{B}})}{\lambda_{S, \{k\}}} &\stackrel{(9)}{=} \frac{K_{\lambda, S}(k)}{\lambda_{S, \{k\}}} \alpha \\ &\stackrel{(7)}{=} \frac{K_{\lambda, S}(\ell)}{\lambda_{S, \{\ell\}}} \alpha \stackrel{(8)}{=} \frac{H_\ell^{\lambda SL}(N, v + \alpha \cdot u_S, \underline{\mathcal{B}}) - H_\ell^{\lambda SL}(N, v, \underline{\mathcal{B}})}{\lambda_{S, \{\ell\}}}. \end{aligned}$$

Thus, we also have in the  $r$ -th level game,  $0 \leq r \leq h$ ,

$$\begin{aligned} & \frac{H_{\mathcal{B}^r(k)}^{\lambda SL}(\mathcal{B}^r, (v + \alpha \cdot u_S)^r, \underline{\mathcal{B}}^r) - H_{\mathcal{B}^r(k)}^{\lambda SL}(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r)}{\lambda_{C_S^r, B_k}} \\ & \stackrel{\text{Lem. 8.1}}{=} \frac{H_{\mathcal{B}^r(\ell)}^{\lambda SL}(\mathcal{B}^r, (v + \alpha \cdot u_S)^r, \underline{\mathcal{B}}^r) - H_{\mathcal{B}^r(\ell)}^{\lambda SL}(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r)}{\lambda_{C_S^r, B_\ell}} \end{aligned}$$

and the claim follows by **LG**.  $\square$

### 8.2.2 Proof of Theorem 5.5

Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $\lambda \in \Lambda^{\bar{\mathcal{B}}}$ ,  $S \in \Omega^N$ , and  $\varphi$  be an LS-value which satisfies all axioms of Theorem 5.5. Due to Proposition 5.4, property (2), and **A**, it is sufficient to show that  $\varphi$  is uniquely defined on the game  $v_S := \Delta_v(S) \cdot u_S$ .

By Lemma 8.1, for each level  $r$ ,  $0 \leq r \leq h$ , exists exactly one coalition  $T_S^r$ ,  $\mathcal{T}_S^r \subseteq \mathcal{B}^r$ , which is the smallest coalition of all  $R^r$ ,  $R^r \supseteq S$ , with associated  $\mathcal{R}^r \subseteq \mathcal{B}^r$  and so in each game  $(\mathcal{B}^r, v_S^r, \underline{\mathcal{B}}^r) \in \mathbb{V}\mathbb{L}^{\mathcal{B}^r}$  we have  $\Delta_{v_S^r}(\mathcal{T}_S^r) = \Delta_v(S)$  and  $\Delta_{v_S^r}(\mathcal{R}^r) = 0$  for  $\mathcal{R}^r \subseteq \mathcal{B}^r$ ,  $\mathcal{R}^r \neq \mathcal{T}_S^r$ . Therefore, by (3), possibly using Conv. 8.3, all components  $B \in \mathcal{B}^r$ ,  $B \cap S \neq \emptyset$ , are dependent in  $v_S^r$ . If  $B \in \mathcal{B}^r$ ,  $B \cap S = \emptyset$ , we have, by **N**,  $\sum_{i \in B} \varphi_i(N, v_S, \underline{\mathcal{B}}) = 0$ . Also due to **N**, we get

$$\varphi_i(N, \mathbf{0} + \Delta_v(S) \cdot u_S, \underline{\mathcal{B}}) - \varphi_i(N, \mathbf{0}, \underline{\mathcal{B}}) = \varphi_i(N, v_S, \underline{\mathcal{B}}) \text{ for all } i \in N. \quad (15)$$

We use induction on the size  $m$ ,  $0 \leq m \leq h$ , for all levels  $r$ ,  $0 \leq r \leq h$ , with  $m := h - r$ .

*Initialization:* Let  $m = 0$  and so  $r = h$ . It follows for all  $i \in S$  with  $\prod_{\ell=r}^h \lambda_{C_S^\ell(i), \mathcal{B}^r(i)} > 0$ , a such  $i$  always exists, and all  $B \in \mathcal{B}^r$ ,  $B \cap S \neq \emptyset$ ,

$$\sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) \stackrel{\text{(15)}}{=} \frac{\lambda_{C_S^r(i), B}}{\lambda_{C_S^r(i), \mathcal{B}^r(i)}} \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}). \quad (16)$$

$$\Rightarrow \sum_{\substack{B \in \mathcal{B}^r, \\ B \cap S \neq \emptyset}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) = \sum_{\substack{B \in \mathcal{B}^r, \\ B \cap S \neq \emptyset}} \frac{\lambda_{C_S^r(i), B}}{\lambda_{C_S^r(i), \mathcal{B}^r(i)}} \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) \stackrel{\text{(E)}}{=} \Delta_v(S)$$

$$\Rightarrow \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) = \prod_{\ell=r}^h \lambda_{C_S^\ell(i), \mathcal{B}^\ell(i)} \Delta_v(S). \quad (17)$$

By (16), we have for all  $B \in \mathcal{B}^r$ ,  $B \cap S = \emptyset$ , with  $\prod_{\ell=r}^h \lambda_{C_S^\ell(i), B} = 0$ ,

$$\sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) = \prod_{\ell=r}^h \lambda_{C_S^\ell(i), B} \Delta_v(S). \quad (18)$$

*Induction step:* Assume that (17) and (18) hold to  $\varphi$  with an arbitrary  $m-1$ ,  $0 \leq m-1 \leq h-1$  (IH). It follows for all  $i \in S$  with  $\prod_{\ell=r}^h \lambda_{C_S^\ell(i), \mathcal{B}^\ell(i)} > 0$ ,

$$\begin{aligned} & \sum_{\substack{B \in \mathcal{B}^r, B \cap S \neq \emptyset, \\ B \subseteq \mathcal{B}^{r+1}(i)}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) \stackrel{\text{(15)}}{=} \sum_{\substack{B \in \mathcal{B}^r, B \cap S \neq \emptyset, \\ B \subseteq \mathcal{B}^{r+1}(i)}} \frac{\lambda_{C_S^r(i), B}}{\lambda_{C_S^r(i), \mathcal{B}^r(i)}} \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) \\ & \stackrel{\text{(CDD}^\lambda)}{=} \sum_{\substack{B \in \mathcal{B}^r, B \cap S \neq \emptyset, \\ B \subseteq \mathcal{B}^{r+1}(i)}} \frac{\lambda_{C_S^r(i), B}}{\lambda_{C_S^r(i), \mathcal{B}^r(i)}} \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) \\ & \stackrel{\text{(IH)}}{=} \prod_{\ell=r+1}^h \lambda_{C_S^\ell(i), \mathcal{B}^\ell(i)} \Delta_v(S) \\ & \Rightarrow \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) = \prod_{\ell=r}^h \lambda_{C_S^\ell(i), \mathcal{B}^\ell(i)} \Delta_v(S) \end{aligned}$$

and, analogous to before, for all  $B \in \mathcal{B}^r$ ,  $B \cap S \neq \emptyset$ , with  $\prod_{\ell=r}^h \lambda_{C_S^\ell(i), B} = 0$ , we have

$$\sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) = \prod_{\ell=r}^h \lambda_{C_S^\ell(i), B} \Delta_v(S).$$

Therefore,  $\varphi$  is uniquely defined on  $v_S$  (take  $m = h$  and so  $r = 0$ ).  $\square$

### 8.2.3 Proof of Theorem 5.7

Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$  be fixed.

$\Rightarrow$ : By Remark 5.3 and Proposition 5.4, each  $\varphi \in \mathcal{H}^{SL}$  on  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$  coincides with a Harsanyi payoff  $\phi \in \mathcal{H}$  on  $(N, v) \in \mathbb{V}^N$  and satisfies **LG**.

$\Leftarrow$ : By Remark 5.6, each  $\phi \in \mathcal{H}$  satisfies the TU-versions of **E**, **N**, **A**, and **CDDP** $^\lambda$  for some  $\lambda' \in \Lambda^N$ . Therefore, any LS-value  $\varphi$  that coincides with a Harsanyi payoff must satisfy the simply transferred LS-versions of these axioms where we have, as in Remark 5.6,  $K_{\lambda, T}(i) = \lambda'_{T, i}$  for each  $T \in \Omega^N$  and all  $i \in T$ . Note that **CDD** $^\lambda_0$  is implied by the transferred LS-axiom of **CDDP** $^\lambda$ , and **CDD** $^\lambda$  is implied, due to Remark 3.1, by **CDD** $^\lambda_0$  and **LG**. Thus, all the axioms of Theorem 5.5 must be satisfied and we have  $\varphi \in \mathcal{H}^{SL}$ .  $\square$

### 8.2.4 Proof of Corollary 5.8

The claim follows immediately due to the axiomatization of the Harsanyi set (Vasil'ev, 1981) by the TU-versions of **E**, **N**, **Po**, and **A** and Theorem 5.7.  $\square$

### 8.2.5 Proof of Proposition 6.3

Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $w \in W^{\bar{\mathcal{B}}}$ , and  $K_{w, T}^r$  be the expressions according to Def. 6.1. By Remark 6.2,  $Sh^{wSL}$  coincides with  $H^{\lambda SL}$  where for each  $T^r \in \Omega^N$ ,  $0 \leq r \leq h$ , we have for all  $B \in \mathcal{B}^r$ ,  $\lambda_{C^{T^r}, B} = w_B$ . Thus, all axioms from Prop. 5.4 are satisfied (with appropriate modification of any required sharing weights).

- **SAMo**: The claim follows immediately by (10).
- **D<sup>w</sup>**: Let  $k, \ell \in N$ ,  $0 \leq r \leq h$ ,  $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$  and  $\mathcal{B}^r(k), \mathcal{B}^r(\ell)$  be dependent in  $v^r$  on the LS-game  $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r) \in \mathbb{V}\mathbb{L}^{\mathcal{B}^r}$ . If  $r = 0$ , then  $k, \ell$  are dependent in  $v$  and we get

$$\begin{aligned} \frac{Sh_k^{wSL}(N, v, \underline{\mathcal{B}})}{w_{\{k\}}} &\stackrel{(10)}{=} \sum_{T \subseteq N, T \ni k} \frac{K_{w, T}(k)}{w_{\{k\}}} \Delta_v(T) &\stackrel{(3)}{=} \sum_{T \subseteq N, \{k, \ell\} \subseteq T} \frac{K_{w, T}(k)}{w_{\{k\}}} \Delta_v(T) \\ &\stackrel{\text{Def. 6.1}}{=} \sum_{T \subseteq N, \{k, \ell\} \subseteq T} \frac{K_{w, T}(\ell)}{w_{\{\ell\}}} \Delta_v(T) &= \frac{Sh_\ell^{wSL}(N, v, \underline{\mathcal{B}})}{w_{\{\ell\}}}. \end{aligned}$$

Thus we have also in the  $r$ -th level game,  $0 \leq r \leq h$ ,

$$\frac{Sh_{\mathcal{B}^r(k)}^{wSL}(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r)}{w_{\mathcal{B}^r(k)}} = \frac{Sh_{\mathcal{B}^r(\ell)}^{wSL}(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r)}{w_{\mathcal{B}^r(\ell)}}$$

and the claim follows by **LG**.

- **MD**: The claim follows immediately by **D<sup>w</sup>**.  $\square$



### 8.2.6 Proof of Theorem 6.4

Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $w \in \mathcal{W}^{\underline{\mathcal{B}}}$ ,  $S \in \Omega^N$ , and  $\varphi$  be an LS-value that satisfies all axioms of Theorem 6.4. Due to Proposition 6.3, property (2), and **A**, it is sufficient to show that  $\varphi$  is uniquely defined on the game  $v_S := \Delta_v(S) \cdot u_S$ .

By Lemma 8.1, for each level  $r$ ,  $0 \leq r \leq h$ , exists exactly one coalition  $T_S^r$ ,  $\mathcal{T}_S^r \subseteq \mathcal{B}^r$ , which is the smallest coalition of all  $R^r$ ,  $R^r \supseteq S$ , with associated  $\mathcal{R}^r \subseteq \mathcal{B}^r$  and so in each game  $(\mathcal{B}^r, v_S^r, \underline{\mathcal{B}}^r) \in \mathbb{V}\mathbb{L}^{\mathcal{B}^r}$  we have  $\Delta_{v_S^r}(\mathcal{T}_S^r) = \Delta_v(S)$  and  $\Delta_{v_S^r}(\mathcal{R}^r) = 0$  for  $\mathcal{R}^r \subseteq \mathcal{B}^r$ ,  $\mathcal{R}^r \neq \mathcal{T}_S^r$ . Therefore, by (3), possibly using Conv. 8.3, all components  $B \in \mathcal{B}^r$ ,  $B \cap S \neq \emptyset$ , are dependent in  $v_S^r$ . If  $B \in \mathcal{B}^r$ ,  $B \cap S = \emptyset$ , we have, by **N**,  $\sum_{i \in B} \varphi_i(N, v_S, \underline{\mathcal{B}}) = 0$ .

We use induction on the size  $m$ ,  $0 \leq m \leq h$ , for all levels  $r$ ,  $0 \leq r \leq h$ , with  $m := h - r$ .

*Initialization:* Let  $m = 0$  and so  $r = h$ . For an arbitrary  $i \in S$ , we get

$$\begin{aligned} \sum_{\substack{B \in \mathcal{B}^h, \\ B \cap S \neq \emptyset}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) &\stackrel{(\mathbf{D}^w)}{=} \sum_{\substack{B \in \mathcal{B}^h, \\ B \cap S \neq \emptyset}} \frac{w_B}{w_{\mathcal{B}^h(i)}} \sum_{j \in \mathcal{B}^h(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) \stackrel{(\mathbf{E})}{=} \Delta_v(S) \\ \Rightarrow \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) &= \prod_{k=h-m}^h \frac{w_{\mathcal{B}^k(i)}}{\sum_{\substack{B \in \mathcal{B}^k: B \subseteq \mathcal{B}^{k+1}(i), \\ B \cap S \neq \emptyset}} w_B} \Delta_v(S). \end{aligned} \quad (19)$$

*Induction step:* Assume that (19) holds to  $\varphi$  with an arbitrary  $m - 1$ ,  $0 \leq m - 1 \leq h - 1$  (IH). It follows, for an arbitrary  $i \in S$ ,

$$\begin{aligned} \sum_{\substack{B \in \mathcal{B}^r, B \cap S \neq \emptyset, \\ B \subseteq \mathcal{B}^{r+1}(i)}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) &\stackrel{(\mathbf{D}^w)}{=} \sum_{\substack{B \in \mathcal{B}^r, B \cap S \neq \emptyset, \\ B \subseteq \mathcal{B}^{r+1}(i)}} \frac{w_B}{w_{\mathcal{B}^r(i)}} \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) \\ &\stackrel{(\text{IH})}{=} \prod_{k=h-m+1}^h \frac{w_{\mathcal{B}^k(i)}}{\sum_{\substack{B \in \mathcal{B}^k: B \subseteq \mathcal{B}^{k+1}(i), \\ B \cap S \neq \emptyset}} w_B} \Delta_v(S) \\ \Rightarrow \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) &= \prod_{k=h-m}^h \frac{w_{\mathcal{B}^k(i)}}{\sum_{\substack{B \in \mathcal{B}^k: B \subseteq \mathcal{B}^{k+1}(i), \\ B \cap S \neq \emptyset}} w_B} \Delta_v(S). \end{aligned}$$

Therefore,  $\varphi$  is uniquely defined on  $v_S$  (take  $m = h$  and so  $r = 0$ ).  $\square$

### 8.2.7 Proof of Theorem 6.5

Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ ,  $w \in \mathcal{W}^{\underline{\mathcal{B}}}$ , and  $\varphi$  be an LS-value that satisfies all axioms of Theorem 6.5. In Casajus and Huettner (2008), it is shown that **CSE** and **M** are equivalent in TU-games. Obviously, their proof also applies to LS-games. This means that **CSE** is satisfied as well. By Theorem 6.3, we have only to show that  $\varphi$  satisfies (10).

We use a first induction  $I_1$  on  $t := |\{T \subseteq N : T \text{ is essential in } v\}|$ .

*Initialization  $I_1$ :* Let  $t = 0$ , then for all games  $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r) \in \mathbb{V}\mathbb{L}^{\mathcal{B}^r}$ ,  $0 \leq r \leq h$ ,  $v^r$  is identical to zero on all coalitions. So all players, possibly using Conv. 8.3, are dependent in each game  $v^r$  and for all  $B_k^r, B_\ell^r \in \mathcal{B}^r$ ,  $B_\ell^r \subseteq \mathcal{B}^{r+1}(B_k^r)$ , we have

$$\sum_{i \in B_k^r} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_k^r}} \stackrel{(\mathbf{D}^w)}{=} \sum_{i \in B_\ell^r} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_\ell^r}}.$$

We use a second induction  $I_2$  on the size  $m := h - r$  to show that we have

$$\sum_{i \in B^r} \varphi_i(N, v, \underline{\mathcal{B}}) = 0 \text{ for all } 0 \leq r \leq h \text{ and } B^r \in \mathcal{B}^r. \quad (20)$$

*Initialization  $I_2$ :* Let  $m = 0$  and so  $r = h$ . We get for an arbitrary  $B_k^h \in \mathcal{B}^h$ ,

$$\sum_{B^h \in \mathcal{B}^h} \sum_{i \in B^h} \varphi_i(N, v, \underline{\mathcal{B}}) \stackrel{(\mathbf{D}^w)}{=} \sum_{B^h \in \mathcal{B}^h} \frac{w_{B^h}}{w_{B_k^h}} \sum_{i \in B_k^h} \varphi_i(N, v, \underline{\mathcal{B}}) \stackrel{(\mathbf{E})}{=} 0.$$

It follows that  $\sum_{i \in B^h} \varphi_i(N, v, \underline{\mathcal{B}}) = 0$  for all  $B^h \in \mathcal{B}^h$  because we have  $w_{B^h} > 0$  and  $B_k^h$  is arbitrary.

*Induction step  $I_2$ :* Assume that (20) holds to  $\varphi$  if  $m \geq 0$  ( $IH_2$ ). We get for an arbitrary  $B_k^r \in \mathcal{B}^r$  that

$$\sum_{\substack{B^r \in \mathcal{B}^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \sum_{i \in B^r} \varphi_i(N, v, \underline{\mathcal{B}}) \stackrel{(\mathbf{D}^w)}{=} \sum_{\substack{B^r \in \mathcal{B}^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \frac{w_{B^r}}{w_{B_k^r}} \sum_{i \in B_k^r} \varphi_i(N, v, \underline{\mathcal{B}}) \stackrel{(IH_2)}{=} 0.$$

It follows,  $\sum_{i \in B^r} \varphi_i(N, v, \underline{\mathcal{B}}) = 0$  for all  $0 \leq r \leq h$  and  $B^r \in \mathcal{B}^r$ . Therefore, we have also  $\varphi_i(N, v, \underline{\mathcal{B}}) = 0$  for all  $i \in N$  and (10) is satisfied for  $\varphi$  if  $t = 0$ .

*Induction step  $I_1$ :* Assume that (10) holds to  $\varphi$  if  $t \geq 0$ , ( $IH_1$ ). Let exactly  $t+1$  coalitions  $Q_k \subseteq N$ ,  $1 \leq k \leq t+1$ , be essential in  $v$  and denote

$$Q := \bigcap_{1 \leq k \leq t+1} Q_k.$$

We distinguish two cases: (a)  $i \in N \setminus Q$  and (b)  $i \in Q$ .

(a) Each player  $i \in N \setminus Q$  is a member of at most  $t$  essential coalitions  $Q_k$  and  $v$  has at least one essential coalition  $T_i$ ,  $i \notin T_i$ . Hence, there exists a coalition function  $v_i$  where all coalitions have the same dividend in  $v_i$  as in  $v$ , with the exception of coalition  $T_i$ , that gets the dividend  $\Delta_{v_i}(T_i) = 0$ , and there exists a scalar  $c \in \mathbb{R}$ ,  $c \neq 0$ , where

$$\Delta_{v_i}(S) = \begin{cases} \Delta_{v_i}(T_i) + c, & \text{if } S = T_i, \\ \Delta_{v_i}(S), & \text{else.} \end{cases}$$

By **CSE**, we get  $\varphi_i(v) = \varphi_i(v_i)$  with  $i \in N \setminus T_i$ . Since there is such a  $T_i$  for all  $i \in N \setminus Q$ , it follows that  $\varphi_i(v) = \varphi_i(v_i)$  for all  $i \in N \setminus Q$ . All coalition functions  $v_i$  get at most  $t$  essential coalitions and, by ( $IH_1$ ), we have

$$\varphi_i(v) = Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \text{ for all } i \in N \setminus Q. \quad (21)$$

(b) Each player  $j \in Q$  is a member of all  $t+1$  essential coalitions  $Q_k \subseteq N$ ,  $1 \leq k \leq t+1$ , and therefore, by (3) and Conv. 8.3, all players  $j \in Q$  are dependent in  $v$ . We define for each  $r$ ,  $0 \leq r \leq h$ , a set

$$\mathcal{B}_Q^r := \{B^r \in \mathcal{B}^r : B^r \cap Q \neq \emptyset\}.$$

Note that all components  $B_k^r, B_\ell^r \in \mathcal{B}_Q^r$ ,  $B_\ell^r \subseteq \mathcal{B}^{r+1}(B_k^r)$ , are dependent in  $v^r$ . We use a third induction  $I_3$  on the size  $s := h - r$  to show that we have

$$\sum_{i \in B_k^r} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_k^r} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \text{ for all } B_k^r \in \mathcal{B}_Q^r. \quad (22)$$

*Initialization  $I_3$ :* Let  $s = 0$  and so  $r = h$ . We get for an arbitrary  $B_k^h \in \mathcal{B}_Q^h$

$$\begin{aligned}
& \sum_{B^h \in \mathcal{B}_Q^h} \sum_{i \in B^h} \varphi_i(N, v, \underline{\mathcal{B}}) \stackrel{(\mathbf{D}^w)}{=} \sum_{B^h \in \mathcal{B}_Q^h} \frac{w_{B^h}}{w_{B_k^h}} \sum_{i \in B_k^h} \varphi_i(N, v, \underline{\mathcal{B}}) \\
& \stackrel{\substack{(\mathbf{E}) \\ (21)}}{=} \sum_{B^h \in \mathcal{B}_Q^h} \sum_{i \in B^h} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \stackrel{(\mathbf{D}^w)}{=} \sum_{B^h \in \mathcal{B}_Q^h} \frac{w_{B^h}}{w_{B_k^h}} \sum_{i \in B_k^h} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \\
& \Rightarrow \sum_{i \in B_k^h} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_k^h} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}).
\end{aligned}$$

*Induction step  $I_3$ :* Assume that (22) holds to  $\varphi$  if  $s \geq 0$  ( $IH_3$ ). We get for an arbitrary  $B_k^r \in \mathcal{B}_Q^r$  and, because  $\mathcal{B}^{r+1}(B_k^r) \in \mathcal{B}_Q^{r+1}$ ,

$$\begin{aligned}
& \sum_{\substack{B^r \in \mathcal{B}_Q^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \sum_{i \in B^r} \varphi_i(N, v, \underline{\mathcal{B}}) \stackrel{(\mathbf{D}^w)}{=} \sum_{\substack{B^r \in \mathcal{B}_Q^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \frac{w_{B^r}}{w_{B_k^r}} \sum_{i \in B_k^r} \varphi_i(N, v, \underline{\mathcal{B}}) \\
& \stackrel{\substack{(\mathbf{IH}_3) \\ (21)}}{=} \sum_{\substack{B^r \in \mathcal{B}_Q^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \sum_{i \in B^r} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \stackrel{(\mathbf{D}^w)}{=} \sum_{\substack{B^r \in \mathcal{B}_Q^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \frac{w_{B^r}}{w_{B_k^r}} \sum_{i \in B_k^r} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \\
& \Rightarrow \sum_{i \in B_k^r} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_k^r} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}).
\end{aligned}$$

Finally, we get  $\varphi_i(N, v, \underline{\mathcal{B}}) = Sh_i^{wSL}(N, v, \underline{\mathcal{B}})$  for all  $i \in Q$ .  $\square$

### 8.2.8 Proof of Theorem 6.6

By Proposition 6.3, we only have to show the way back.

Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ , and  $\varphi$  be an LS-value that satisfies  $\mathbf{E}$ ,  $\mathbf{N}$ ,  $\mathbf{SAMo}$ ,  $\mathbf{A}$ , and  $\mathbf{MD}$ . By  $\mathbf{SAMo}$  and  $\mathbf{N}$ , we have  $\varphi_i(N, u_N, \underline{\mathcal{B}}) > 0$  for all  $i \in N$ . Define a  $w \in W^{\bar{\mathcal{B}}}$  by  $w_B := \sum_{i \in B} \varphi_i(N, u_N, \underline{\mathcal{B}})$  for all  $B \in \bar{\mathcal{B}}$ . By  $\mathbf{MD}$ , we have for all  $B_k, B_\ell \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ , such that  $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ , and  $B_k, B_\ell$  are dependent in  $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r) \in \mathbb{V}\mathbb{L}^{\mathcal{B}^r}$ ,

$$\sum_{i \in B_k} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_k}} = \sum_{i \in B_\ell} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_\ell}}$$

and  $\mathbf{D}^w$  is satisfied. The claim follows by Theorem 6.4.  $\square$

### 8.3 Logical independence

All axiomatizations must also be valid for  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_0$ . In this case, the axioms used in this article for axiomatization coincide with axioms for TU-values. These axiomatizations thus correspond in this context to axiomatizations of the Harsanyi payoffs and weighted Shapley values respectively. It is well-known or easy to prove that the axioms there are logically independent. Therefore, all axioms for LS-values in the given axiomatizations must also be logically independent.

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