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21 April 2019

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April 21, 2019

Abstract

Using a dynamic general equilibrium model, this paper theoretically analyzes a negative interest rate policy in a permanent liquidity trap. If the natural nominal interest rate is above the lower bound set by the presence of vault cash held by commercial banks, a reduction in the nominal rate of interest on excess bank reserves can get an economy out of the permanent liquidity trap. In contrast, if the natural nominal interest rate is below the lower bound, then it cannot do so, but instead a rise in the rate of tax on vault cash is useful for doing so.

Keywords: aggregate demand, liquidity trap, negative nominal interest rate, unemployment

JEL Classification Codes: E12, E31, E58

*I would like to thank Takayuki Ogawa and Yoshiyasu Ono for helpful comments, discussions, and suggestions.

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1 Introduction

Recently, negative interest rate policies have been implemented in Europe and Japan (see, e.g., Bech and Malkhozov, 2016, and Angrick and Nemoto, 2017). Some economists presented positive views of negative nominal interest rates, but others presented negative views. For example, using Old and New Keynesian models, Buiter and Panigirtzoglou (2003) showed that paying negative interest on currency (imposing a carry tax on currency) eliminates the zero lower bound on nominal interest rates and, hence, is useful for eliminating a liquidity trap.[1] Without developing theoretical models, Goodfriend (2000) suggested a carry tax on bank reserves as a way of overcoming the zero lower bound, and Fukao (2005) proposed a tax on government-backed financial assets as a way to get the Japanese economy out of the stagnation that it has been experiencing since the 1990s. Abo-Zaid and Garín (2016) showed that the optimal nominal interest rate is negative in a New Keynesian model with a borrowing constraint. Rognlie (2016) constructed a money-in-the-utility-function model where the utility of money is saturated and showed that the optimal interest rate is negative under price rigidity. Meanwhile, Eggertsson et al. (2019) argued that lowering the nominal rate of interest on bank reserves to negative values reduces commercial banks' profits and has a contractionary effect on output. They developed a New Keynesian model with a commercial banking sector and examined the effects of a negative nominal interest rate in a short-run slump caused by a preference shock.

[1]In addition to paying negative interest on currency, Buiter (2010) proposed two ways of overcoming the zero lower bound: abolishing currency and separating a numéraire function from currency.

It seems that the Euro zone and Japan, where negative interest rate policies have been implemented, have not been in short-run but long-run liquidity traps. It is well known that Japan has been in a prolonged liquidity trap since the 1990s. Recently, there have been concerns that the Euro zone may also have been in a prolonged liquidity trap. The purpose of this paper is to theoretically analyze a negative interest rate policy in a permanent liquidity trap.[2] For this purpose, I extend the dynamic general equilibrium model of Murota and Ono (2012) in two ways. First, I consider that negative nominal interest is paid on excess bank reserves. In fact, the European Central Bank and the Bank of Japan have imposed negative nominal interest on excess reserves (see, e.g., Angrick and Nemoto, 2017). Second, I assume that a tax is levied on commercial banks' vault cash holdings in order to examine the effectiveness of a Gesell tax discussed by Goodfriend (2000), Buiter and Panigirtzoglou (2003), and Fukao (2005).

As in Murota and Ono (2012), I present a permanent liquidity trap, where nominal interest rates are stuck at their lower bounds, deficient aggregate demand creates unemployment, excess bank reserves arise, and the money multiplier declines. Furthermore, even the price change rate is not in control of the central bank; that is, deflation can arise despite an increase in the monetary base. These are the phenomena observed in Japan's liquidity trap since the 1990s. In this permanent liquidity trap, I investigate the effects of

[2] Recently, economists have proposed several types of permanent stagnation. The causes of permanent stagnation advocated by them are deleveraging shocks (Eggertsson and Mehrotra, 2014; Eggertsson et al., 2016), wealth preferences (Michaillat and Saez, 2014; Michau, 2018; Ono and Yamada, 2018), pessimistic expectations (Benigno and Fornaro, 2018), and liquidity preferences (Ono and Ishida, 2014; Illing et al., 2018; Murota, 2018; Ono, 2018).

a reduction in the nominal rate of interest on excess reserves, which is the policy rate in the present model.

This paper shows that a reduction in the nominal rate of interest on excess reserves boosts an economy falling into the permanent liquidity trap to the extent that it lowers the nominal deposit rate. It increases household consumption (aggregate demand), reduces unemployment, and raises the price change rate. If the natural nominal interest rate is higher than the lower bound set by the presence of vault cash, it can lower the nominal deposit rate to the level of the natural nominal interest rate. Consequently, the economy gets out of the permanent liquidity trap and reaches a normal steady state. However, if the natural nominal interest rate is lower than the lower bound, the economy cannot escape the permanent liquidity trap no matter how negative the nominal rate of interest on excess reserves becomes. This is because the nominal deposit rate reaches the lower bound and does not go down to the level of the natural nominal interest rate. In this situation, where lowering the nominal rate of interest on excess reserves becomes ineffective, instead, a rise in the rate of tax on vault cash is useful for pulling the economy out of the permanent liquidity trap because it allows the nominal deposit rate to fall to the level of the natural nominal interest rate. This is consistent with the suggestions by Goodfriend (2000), Buiter and Panigirtzoglou (2003), and Fukao (2005). In the present model, however, levying a tax on currency held by the public, which is practically difficult, is not required for overcoming the lower bound.

This paper proceeds as follows. Section 2 develops the model of an economy. Section 3 shows the dynamic system of the economy. Section 4 presents a normal steady state as a benchmark. Sections 5 and 6 discuss the effects of a negative interest rate policy in a permanent liquidity trap. Section 7 concludes this paper.

2 Model

This section extends the dynamic general equilibrium model of Murota and Ono (2012).[3] Excess bank reserves bear nominal interest, which can be negative. Commercial banks hold vault cash, and a tax is levied on vault cash holdings. In addition, I provide a microfoundation for nominal wage stickiness by modifying the fair wage setting of Raurich and Sorolla (2014).

2.1 Household

A representative household derives utility not only from cash but also from bank deposits.[4] The lifetime utility of this household is

$$\int_0^\infty [u(c_t) + v(m_t^h, d_t) - n_t f(e_t)] \exp(-\rho t) dt,$$

where ρ (> 0) is the subjective discount rate. $u(c_t)$ denotes the utility of consumption c_t and satisfies

$$u'(c_t) > 0, \quad u''(c_t) < 0, \quad u'(0) = \infty, \quad u'(\infty) = 0. \tag{1}$$

$v(m_t^h, d_t)$ denotes the utility of real cash holdings m_t^h ($\equiv M_t^h/P_t$) and real deposit holdings d_t ($\equiv D_t/P_t$), where M_t^h is nominal cash holdings, D_t

[3]Murota and Ono (2012) did not consider vault cash or interest paid on bank reserves and assumed nominal wage stickiness without microfoundations.

[4]Romer (1985), Jones et al. (2004), and Agénor and Alper (2012) also assumed that both cash and deposits provide utility. See Buiter (2010, p. 222) for a somewhat similar assumption.

is nominal deposit holdings, and P_t is the price level. $v(m_t^h, d_t)$ is linear homogeneous and satisfies

$$\frac{\partial v}{\partial m_t^h} \equiv v_m(m_t^h, d_t) > 0, \quad \frac{\partial^2 v}{\partial {m_t^h}^2} < 0, \quad v_m(0, d_t) = \infty, \quad v_m(\infty, d_t) = 0;$$
$$\frac{\partial v}{\partial d_t} \equiv v_d(m_t^h, d_t) > 0, \quad \frac{\partial^2 v}{\partial {d_t}^2} < 0, \quad v_d(m_t^h, 0) = \infty, \quad v_d(m_t^h, \infty) = 0.$$

The cash–deposit ratio is defined by x_t:

$$x_t \equiv \frac{m_t^h}{d_t}. \tag{2}$$

Then the marginal utility of cash and of deposits are expressed as functions of x_t:

$$v_m(m_t^h, d_t) \equiv v_m(x_t), \quad v_d(m_t^h, d_t) \equiv v_d(x_t),$$

and the above-mentioned properties of $v(m_t^h, d_t)$ are rewritten as follows:

$$\begin{aligned} &v_m'(x_t) < 0, \quad v_m(0) = \infty, \quad v_m(\infty) = 0; \\ &v_d'(x_t) > 0, \quad v_d(0) = 0, \quad v_d(\infty) = \infty. \end{aligned} \tag{3}$$

$-n_t f(e_t)$ denotes the disutility of effort, where n_t is the amount of employed labor, e_t is effort per unit of employed labor, and $-f(e_t)$ is the disutility of effort per unit of employed labor. Following Raurich and Sorolla (2014), I assume a quadratic disutility function:[5]

$$-n_t f(e_t) = -n_t\big[e_t - e(W_t/W_t^R)\big]^2,$$

where W_t is the nominal wage, W_t^R is the nominal reference wage, and $e(W_t/W_t^R)$ is the norm of effort and where the household takes n_t, W_t, and W_t^R as given. Furthermore, following them, I assume that the reference wage

[5] Akerlof (1982), Collard and de la Croix (2000), Danthine and Kurmann (2004), and Vaona (2013) also assumed quadratic disutility functions in efficiency wage models.

is given by the weighted average of past social averages of income. However, unlike them, the reference wage consists of nominal wages, not real wages, as follows:

$$W_t^R \equiv \int_{-\infty}^{t} I_s \alpha \exp(-\alpha(t-s)) ds, \tag{4}$$

where α is a positive constant and where I_s is the social average of nominal income defined such that

$$I_s \equiv \frac{W_s n_s + \beta W_s (n^f - n_s)}{n^f}, \tag{5}$$

where n^f is the labor endowment that the household inelastically supplies, $n^f - n_s$ is unemployment, and $\beta W_s (n^f - n_s)$ is unemployment benefits received by the household (β is the replacement rate satisfying $0 < \beta < 1$).

The budget constraint in real terms is

$$\dot{a}_t = r_t^D d_t - \pi_t m_t^h + w_t n_t + \beta w_t (n^f - n_t) - c_t - s_t, \tag{6}$$

where a_t is real asset holdings, r_t^D is the real rate of interest on deposits, π_t ($\equiv \dot{P}_t / P_t$) is the rate of price change, w_t ($\equiv W_t / P_t$) is the real wage, and s_t is a lump-sum tax or transfer. The components of a_t are cash and deposits:

$$a_t = m_t^h + d_t. \tag{7}$$

The current-value Hamiltonian $\mathscr{H}_t$ for the utility-maximization problem is

$$\begin{aligned} \mathscr{H}_t = u(c_t) + v(m_t^h, d_t) - n_t \big[e_t - e(W_t / W_t^R) \big]^2 \\ + \lambda_t \big[r_t^D d_t - \pi_t m_t^h + w_t n_t + \beta w_t (n^f - n_t) - c_t - s_t \big] \\ + \gamma_t \big(a_t - m_t^h - d_t \big) , \end{aligned}$$

where λ_t is the costate variable associated with (6) and γ_t is the Lagrange multiplier associated with (7). The first-order conditions with respect to c_t, m_t^h, d_t, a_t, and e_t are

$$\begin{aligned} u'(c_t) &= \lambda_t, \\ v_m(x_t) - \pi_t \lambda_t &= \gamma_t, \\ v_d(x_t) + r_t^D \lambda_t &= \gamma_t, \\ \dot{\lambda}_t - \rho \lambda_t &= -\gamma_t, \\ e_t &= e(W_t/W_t^R). \end{aligned} \tag{8}$$

The transversality condition is

$$\lim_{t \to \infty} \lambda_t a_t \exp(-\rho t) = 0. \tag{9}$$

The last equation of (8) shows that in contrast with Raurich and Sorolla (2014), effort e_t depends on nominal wages (not real wages).[6] I assume that

$$e'(W_t/W_t^R) > 0, \quad e''(W_t/W_t^R) < 0.$$

The assumption that $e'(\cdot) > 0$ implies that as a firm pays a higher nominal wage compared with the nominal reference wage (which is the criterion for judging fairness), the household provides greater effort in return. There are empirical findings consistent with this assumption. Kahneman et al. (1986) and Blinder and Choi (1990) found evidence of money illusion that people tend to judge fairness in terms of nominal wages. Shafir et al. (1997) and Mees and Franses (2014) also found evidence of money illusion that nominal wages tend to influence worker morale. Moreover, Campbell and Kamlani (1997), Bewley (1999), and Kawaguchi and Ohtake (2007) found

[6] Raurich and Sorolla (2014) analyzed the relationship between real wage stickiness and economic growth in a neoclassical growth model where effort depends on real wages.

that reductions in nominal wages decrease worker morale. The assumption that $e''(\cdot) < 0$ is required for the second-order condition for the firm's profit-maximization problem. See Murota (2016) for a somewhat similar effort function where effort depends on nominal wages because of money illusion. From (8) except the last equation, I obtain

$$\rho + \eta(c_t)\frac{\dot{c}_t}{c_t} + \pi_t = \frac{v_m(x_t)}{u'(c_t)} = R_t^D + \frac{v_d(x_t)}{u'(c_t)}, \tag{10}$$

where $\eta(c_t) \equiv -u''(c_t)c_t/u'(c_t)$ and R_t^D $(\equiv r_t^D + \pi_t)$ is the nominal rate of interest on deposits. According to (10), the household decides to consume or save and allocates wealth between cash and deposits. Equation (10) implies that even when the nominal deposit rate R_t^D is negative, the marginal utility of cash $v_m(\cdot)$ can be positive owing to the presence of the marginal utility of deposits $v_d(\cdot)$. This makes equilibrium with negative nominal interest rates feasible.

2.2 Firm

The production function of a representative firm is linear as follows:

$$y_t = e_t n_t = e(W_t/W_t^R)n_t, \tag{11}$$

where y_t is output, effort e_t is labor productivity, and n_t is labor input. The firm chooses n_t and W_t to maximize its profit:

$$P_t e(W_t/W_t^R)n_t - W_t n_t,$$

where the firm takes P_t and W_t^R as given because the goods market is perfectly competitive and because the reference wage consists of the social averages of income. The first-order conditions with respect to n_t and W_t are[7]

$$e(W_t/W_t^R) = \frac{W_t}{P_t}, \tag{12}$$

$$\frac{P_t e'(W_t/W_t^R)}{W_t^R} = 1. \tag{13}$$

Eliminating P_t from (12) and (13) yields a modified Solow condition:

$$\frac{(W_t/W_t^R)e'(W_t/W_t^R)}{e(W_t/W_t^R)} = 1, \tag{14}$$

which gives W_t/W_t^R as a constant. Denoting it by ω:

$$\frac{W_t}{W_t^R} \equiv \omega, \tag{15}$$

I find that effort (labor productivity) is constant:

$$e_t = e(W_t/W_t^R) = e(\omega) \equiv e. \tag{16}$$

2.3 Commercial Bank

A representative commercial bank collects deposits D_t from the household and buys government bonds B_t, which bears nominal interest at the rate R_t^B. In this regard, however, the commercial bank is required to put an amount of money greater than or equal to a portion of the deposits in the central bank as bank reserves:

$$M_t^b \geq \epsilon D_t, \tag{17}$$

[7]Under the linear production technology, the firm chooses labor input and output as follows:

$$\begin{aligned} n_t = \infty, \quad y_t = \infty \quad &\text{if} \quad e(W_t/W_t^R) > W_t/P_t, \\ 0 < n_t < \infty, \quad 0 < y_t < \infty \quad &\text{if} \quad e(W_t/W_t^R) = W_t/P_t, \\ n_t = 0, \quad y_t = 0 \quad &\text{if} \quad e(W_t/W_t^R) < W_t/P_t. \end{aligned}$$

Since W_t is determined by the firm so as to satisfy (13), P_t flexibly falls (rises) so as to eliminate excess supply (excess demand) in the perfectly competitive goods market when $y_t = \infty$ ($y_t = 0$). Consequently, (12) is satisfied.

where M_t^b is bank reserves (commercial bank's deposits with the central bank) and ϵ is the required reserve ratio ($0 < \epsilon < 1$). Unlike Murota and Ono (2012), I consider that excess reserves ($M_t^b - \epsilon D_t$) bear nominal interest at the rate R, which is the policy rate and an exogenous variable controlled by the central bank. In the present model, a negative interest rate policy indicates the case of

$$R < 0.$$

Moreover, unlike them, I take into consideration vault cash. Besides B_t and M_t^b, the commercial bank can hold vault cash Z_t:

$$Z_t \geq 0. \tag{18}$$

Naturally, the nominal rate of interest on vault cash is zero. In sum, the following relationship holds:

$$B_t + M_t^b + Z_t = D_t. \tag{19}$$

The relationship between bank reserves and vault cash varies in different countries. For example, in Japan, vault cash is not included in bank reserves (bank reserves consist only of commercial banks' deposits with the Bank of Japan),[8] which means that (17) holds. In contrast, in the USA, both vault cash and deposits with Federal Reserve Banks are included in bank reserves,[9] which means that the equation $M_t^b + Z_t \geq \epsilon D_t$ holds instead of (17). I adopt (17) because Japan is the country that has implemented a negative interest rate policy.[10]

[8] See http://www.boj.or.jp/en/announcements/education/oshiete/seisaku/b33.htm/.

[9] See https://www.federalreserve.gov/monetarypolicy/reservereq.htm.

[10] To be more precise, the Bank of Japan has imposed negative interest on a fraction of excess reserves (see, e.g., Angrick and Nemoto, 2017).

The commercial bank's profit-maximization problem is as follows:

$$\max R_t^B B_t + R(M_t^b - \epsilon D_t) - \tau Z_t - R_t^D D_t,$$
$$\text{s.t. } M_t^b \geq \epsilon D_t, \quad Z_t \geq 0, \quad B_t + M_t^b + Z_t = D_t,$$

where τ is the rate of tax on vault cash. The presence of the cost of holding vault cash plays an important role in considering negative nominal interest rates. Given R_t^B and R_t^D, the commercial bank chooses B_t, M_t^b, Z_t, and D_t to maximize its profit. The Lagrange function $\mathscr{L}_t$ for this maximization problem is

$$\begin{aligned}\mathscr{L}_t = R_t^B B_t + R(M_t^b - \epsilon D_t) - \tau Z_t - R_t^D D_t \\ + \kappa_t(M_t^b - \epsilon D_t) + \xi_t Z_t + \delta_t(D_t - B_t - M_t^b - Z_t),\end{aligned}$$

where κ_t, ξ_t, and δ_t are the Lagrange multipliers associated with (17), (18), and (19), respectively. The first-order conditions are

$$\begin{gathered} R_t^B = \delta_t, \\ R + \kappa_t = \delta_t, \\ -\tau + \xi_t = \delta_t, \\ R_t^D = \delta_t - \epsilon(R + \kappa_t), \\ \kappa_t \geq 0,\ M_t^b - \epsilon D_t \geq 0,\ \kappa_t(M_t^b - \epsilon D_t) = 0, \\ \xi_t \geq 0,\ Z_t \geq 0,\ \xi_t Z_t = 0. \end{gathered} \tag{20}$$

From (20), I obtain

$$\begin{gathered} R_t^B \geq R,\ M_t^b - \epsilon D_t \geq 0,\ (R_t^B - R)(M_t^b - \epsilon D_t) = 0, \\ R_t^B \geq -\tau,\ Z_t \geq 0,\ (R_t^B + \tau) Z_t = 0. \end{gathered} \tag{21}$$

From (21), the lower bound on R_t^B is given by the high side of R and $-\tau$:[11,12]

$$R_t^B \geq \max\{R, -\tau\}, \tag{22}$$

which implies that the lower bound is created by the presence of excess reserves and vault cash.

In what follows, I consider the case of

$$R > -\tau. \tag{23}$$

The case of $R < -\tau$ is analyzed later in Section 6. In the case of (23), from (22), the lower bound on R_t^B is R:

$$R_t^B \geq R > -\tau,$$

which means that vault cash is less profitable than government bonds and excess reserves. Therefore, as is clear from (21), the commercial bank does not hold vault cash:

$$Z_t = 0, \tag{24}$$

i.e., (18) is binding ($\xi_t > 0$).

In the case of (23), in contrast with (18), (17) is either binding or non-binding. When (17) is binding ($\kappa_t > 0$), i.e., the commercial bank does not hold excess reserves:

$$M_t^b = \epsilon D_t, \tag{25}$$

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[11] If $R_t^B < \max\{R, -\tau\}$ (government bonds are less profitable than excess reserves or vault cash), the commercial bank by no means buys government bonds. In this case, R_t^B rises to a level higher than or equal to $\max\{R, -\tau\}$ so that the government gets the commercial bank to buy bonds. Consequently, (22) holds.

[12] In Murota and Ono (2012), where $R = \tau = 0$, the lower bound on R_t^B is zero.

from (20) I obtain

$$R_t^B > R, \quad R_t^D = (1-\epsilon)R_t^B > (1-\epsilon)R \tag{26}$$

and from (19), (24), and (25) I get

$$D_t = \frac{B_t}{1-\epsilon}, \quad M_t^b = \frac{\epsilon}{1-\epsilon}B_t. \tag{27}$$

Meanwhile, when (17) is not binding ($\kappa_t = 0$), i.e., the commercial bank holds excess reserves:

$$M_t^b - \epsilon D_t > 0,$$

from (20) I obtain

$$R_t^B = R, \quad R_t^D = (1-\epsilon)R_t^B = (1-\epsilon)R. \tag{28}$$

In other words, excess reserves arise when the return on excess reserves equals that on government bonds. From (26) and (28), I find that independently of whether (17) is binding (excess reserves arise), the following relationship holds:

$$R_t^D = (1-\epsilon)R_t^B, \tag{29}$$

where the left-hand side (LHS) and the right-hand side (RHS) denote the marginal cost of and the marginal revenue of collecting deposits, respectively, and that the lower bound on R_t^D is $(1-\epsilon)R$:

$$R_t^D \geq (1-\epsilon)R.$$

Note that from (24), (26), and (27) or from (19), (24), and (28) the profit of the commercial bank is zero:

$$R_t^B B_t + R(M_t^b - \epsilon D_t) - \tau Z_t - R_t^D D_t = 0. \tag{30}$$

2.4 Government and Central Bank

Besides controlling the nominal rate of interest on excess reserves, R, the central bank increases or decreases the nominal monetary base M_t at a constant rate μ:

$$\frac{\dot{M}_t}{M_t} = \mu,$$

which implies that the real monetary base m_t ($\equiv M_t/P_t$) evolves according to

$$\frac{\dot{m}_t}{m_t} = \mu - \pi_t. \tag{31}$$

The budget constraint of the government in nominal terms is

$$\dot{B}_t + \dot{M}_t + P_t s_t + \tau Z_t - R(M_t^b - \epsilon D_t) = R_t^B B_t + \beta W_t(n^f - n_t) + P_t g,$$

where g is government purchases and where $-R(M_t^b - \epsilon D_t)$ denotes the government revenue arising from negative interest on excess reserves when $R < 0$. In real terms, it is

$$\dot{b}_t + \mu m_t + s_t + \tau z_t - R(m_t^b - \epsilon d_t) = r_t^B b_t + \beta w_t(n^f - n_t) + g, \tag{32}$$

where b_t ($\equiv B_t/P_t$) is real government bonds, z_t ($\equiv Z_t/P_t$) is real vault cash holdings, m_t^b ($\equiv M_t^b/P_t$) is real reserve holdings, and r_t^B ($\equiv R_t^B - \pi_t$) is the real rate of interest on government bonds. To prevent b_t from diverging, the government collects the lump-sum tax s_t according to

$$s_t = r_t^B b_t + \beta w_t(n^f - n_t) + g + R(m_t^b - \epsilon d_t) - \tau z_t - \mu m_t + \phi(b_t - \bar{b}),$$

where ϕ is a positive constant and $\bar{b}$ is the target level of real government bonds. Substituting this equation into (32) yields the law of motion for b_t:

$$\dot{b}_t = -\phi(b_t - \bar{b}). \tag{33}$$

3 Dynamics

This section derives the dynamic system of the economy. From (4), (5), and (15), the nominal wage changes according to[13]

$$\frac{\dot{W}_t}{W_t} = \frac{\dot{W}_t^R}{W_t^R} = \alpha\left(\frac{I_t}{W_t^R} - 1\right) = \alpha\left(\frac{\omega(1-\beta)n_t + \beta\omega n^f}{n^f} - 1\right) = \sigma\left(\frac{n_t - \overline{n}}{n^f}\right), \tag{34}$$

where σ and $\overline{n}$ are positive constants defined such that

$$\sigma \equiv \alpha\omega(1-\beta), \quad \overline{n} \equiv \left[\frac{1-\beta\omega}{\omega(1-\beta)}\right] n^f < n^f,$$

where the inequality is established by the assumption that $\omega > 1$.[14] From (34), I find[15]

$$\frac{d(\dot{W}_t/W_t)}{dn_t} = \frac{\sigma}{n^f} > 0,$$

which is produced as follows. An increase in employment n_t leads to an increase in I_t/W_t^R and, hence, to a rise in $\dot{W}_t^R/W_t^R$. This rise in the reference wage puts downward pressure on effort ($\partial e(W_t/W_t^R)/\partial W_t^R < 0$). Since the

[13]Murota (2016, 2018) derived nominal wage stickiness similar to (34) in a discrete time model where worker morale depends on the current and last nominal wages and on the unemployment rate and in a model where labor unions are concerned about nominal wages and employment, respectively.

[14]For example, if the effort function is logarithmic:

$$e(W_t/W_t^R) = \ln(W_t/W_t^R),$$

then from (14) the assumption that $\omega > 1$ is satisfied:

$$\omega = \mathrm{e} = 2.71828\cdots > 1.$$

[15]The rate of change in the nominal wage $\dot{W}_t/W_t$ is related negatively to the unemployment rate $(n^f - n_t)/n^f$ (i.e., a Phillips curve appears):

$$\frac{\dot{W}_t}{W_t} = -\sigma\left(\frac{n^f - n_t}{n^f}\right) + \sigma\left(\frac{n^f - \overline{n}}{n^f}\right).$$

firm raises the nominal wage in order to maintain labor productivity at the optimal level, $\dot{W}_t/W_t$ rises at the same rate as $\dot{W}_t^R/W_t^R$.

In the case of (23), where the commercial bank does not hold vault cash (see (24)), the money market equilibrium condition is

$$m_t^h + m_t^b = m_t. \tag{35}$$

Supply equals demand in the goods market as follows:[16]

$$c_t + g = y_t = e n_t. \tag{36}$$

From (12), (16), (34), and (36), the price change rate is

$$\pi_t = \frac{\dot{W}_t}{W_t} = \sigma\left(\frac{n_t - \overline{n}}{n^f}\right) = \sigma\left(\frac{c_t + g - \overline{y}}{y^f}\right), \tag{37}$$

where

$$\overline{y} \equiv e\overline{n}, \quad y^f \equiv e n^f, \tag{38}$$

where y^f denotes full employment output. From (31) and (37), the law of motion for m_t is

$$\frac{\dot{m}_t}{m_t} = \mu - \pi_t = \mu - \sigma\left(\frac{c_t + g - \overline{y}}{y^f}\right). \tag{39}$$

From the first equality of (10) and (37), the law of motion for c_t is

$$\frac{\dot{c}_t}{c_t} = \eta(c_t)^{-1}\left[-\sigma\left(\frac{c_t + g - \overline{y}}{y^f}\right) + \frac{v_m(x_t)}{u'(c_t)} - \rho\right]. \tag{40}$$

When (17) is binding (the commercial bank does not hold excess reserves), from (2), (27), and (35), x_t in (40) is expressed by b_t and m_t:

$$x_t = \frac{m_t^h}{d_t} = (1 - \epsilon)\frac{m_t}{b_t} - \epsilon. \tag{41}$$

[16]Equation (36) is derived from (6), (7), (11), (12), (16), (19), (24), (30), (31), (32), and (35).

When (17) is not binding (the commercial bank holds excess reserves), from the second equality of (10) into which (28) is substituted:

$$\frac{v_m(x_t)}{u'(c_t)} = (1-\epsilon)R + \frac{v_d(x_t)}{u'(c_t)}, \tag{42}$$

x_t in (40) is given as a function of c_t and R:

$$x_t = x(c_t; R), \tag{43}$$

which satisfies[17]

$$x(0;R) = \infty \text{ if } R < 0, \;\; x(0;R) = \overline{x} \text{ if } R = 0, \;\; x(0;R) = 0 \text{ if } R > 0, \tag{44}$$

where $\overline{x}$ is a value satisfying $v_m(\overline{x}) = v_d(\overline{x})$. In addition, x_t satisfies[18]

$$\frac{\partial x_t}{\partial c_t} < 0 \text{ if } R < 0, \;\; \frac{\partial x_t}{\partial c_t} = 0 \text{ if } R = 0, \;\; \frac{\partial x_t}{\partial c_t} > 0 \text{ if } R > 0; \tag{45}$$

$$\frac{\partial x_t}{\partial R} = \frac{(1-\epsilon)u'(c_t)}{v_m'(x_t) - v_d'(x_t)} < 0, \tag{46}$$

where the inequality of (46) is established by (1) and (3). In sum, the dynamic system consists of (33), (39), and (40) with (41) or (43).

[17] Arranging (42) yields

$$\frac{v_m(x_t) - v_d(x_t)}{u'(c_t)} = (1-\epsilon)R,$$

where the RHS is a finite constant and the denominator of the LHS is infinity when $c_t = 0$ ($u'(0) = \infty$ from (1)). If $R < 0$, for the equality to be satisfied, then the numerator of the LHS must be minus infinity when $c_t = 0$. Therefore, when $c_t = 0$, from (3) I have $x_t = \infty$ (the numerator is $v_m(\infty) - v_d(\infty) = 0 - \infty = -\infty$). If $R > 0$, I have $x_t = 0$ ($v_m(0) - v_d(0) = \infty$) because the numerator of the LHS must be plus infinity.

[18] From (42), I derive

$$\frac{\partial x_t}{\partial c_t} = \frac{u''(c_t)\left[v_m(x_t) - v_d(x_t)\right]}{u'(c_t)\left[v_m'(x_t) - v_d'(x_t)\right]} = \frac{(1-\epsilon)Ru''(c_t)}{v_m'(x_t) - v_d'(x_t)}.$$

Taking (1) and (3) into account, I obtain the signs of $\partial x_t/\partial c_t$ in (45).

4 Normal Steady State

Prior to dealing with a permanent liquidity trap, in this section, I consider a normal steady state, where the nominal interest rates R^B and R^D are above the lower bounds R and $(1-\epsilon)R$, respectively, and where there is no aggregate demand deficiency. From (33), (39), and (40) where $\dot{b}_t = 0$, $\dot{m}_t = 0$, and $\dot{c}_t = 0$, the normal steady state is represented by

$$b^* = \overline{b}, \quad \mu = \pi^* = \sigma\left(\frac{c^* + g - \overline{y}}{y^f}\right), \quad \rho + \mu = \frac{v_m(x^*)}{u'(c^*)}, \tag{47}$$

where the asterisk is attached to endogenous variables. Throughout this paper, I assume that $\mu > -\rho$. The existence of this steady state is easily shown. The real bond b^* and the price change rate π^* are straightforwardly determined by the first and second equations of (47). From (36), (38), and the second equation of (47), consumption and employment are

$$c^* = \frac{\mu}{\sigma}y^f + \overline{y} - g, \quad n^* = \frac{\mu}{\sigma}n^f + \overline{n}. \tag{48}$$

The second equation of (48) implies the existence of unemployment as follows:

$$n^f - n^* > 0 \ \text{ if } \ \mu < \sigma\left(\frac{n^f - \overline{n}}{n^f}\right).$$

Moreover, from (48), I find a crowding-out effect of government purchases g:

$$\frac{dc^*}{dg} = -1, \quad \frac{dn^*}{dg} = 0,$$

which implies that there is no aggregate demand deficiency, i.e., the efficiency wage is the only cause of unemployment in the normal steady state.

As shown in Figure 1, when

$$\rho + \mu > \frac{v_m(x(c^*; R))}{u'(c^*)}, \tag{49}$$

where from (42) and (43) $x(c^*; R)$ is the cash–deposit ratio when $c = c^*$ and $R^D = (1-\epsilon)R$, the cash–deposit ratio x^* in the last equation of (47) is determined so as to satisfy

$$x^* < x(c^*; R), \tag{50}$$

and the nominal deposit rate R^D is determined so as to be higher than its lower bound $(1-\epsilon)R$:

$$\left(\frac{v_m(x^*)}{u'(c^*)} - \frac{v_d(x^*)}{u'(c^*)} =\right) R^D > (1-\epsilon)R \left(= \frac{v_m(x(c^*; R))}{u'(c^*)} - \frac{v_d(x(c^*; R))}{u'(c^*)}\right), \tag{51}$$

which straightforwardly results from $-R^D < -(1-\epsilon)R$ in Figure 1.[19] Note that the equalities of (51) are obtained from the second equality of (10) and that the properties of $v_m(x)$ and $v_d(x)$ in (3) yield (50) and (51) under (49). From (29) and (51), the government bond rate R^B is determined so as to be higher than its lower bound R and is defined by R^*:

$$R < \frac{R^D}{1-\epsilon} = R^B \equiv R^*. \tag{52}$$

Then R^D is expressed as

$$R^D = (1-\epsilon)R^*.$$

Thus, in the normal steady state, the optimality condition of the household, (10), holds as follows:

$$\rho + \mu = \frac{v_m(x^*)}{u'(c^*)} = (1-\epsilon)R^* + \frac{v_d(x^*)}{u'(c^*)}. \tag{53}$$

Since R^* and $(1-\epsilon)R^*$ are the nominal interest rates that hold in the normal steady state where R^B and R^D are higher than their lower bounds

[19] Figure 1 shows the case where R^D is negative, but R^D can be negative or positive.

and where aggregate demand is not deficient, I regard R^* and $(1-\epsilon)R^*$ as natural nominal interest rates. In the present model, nominal interest rates rather than real interest rates are important because the cash–deposit ratio, which is affected by the nominal deposit rate (not the real deposit rate), plays a crucial role in determining whether the economy falls into a permanent liquidity trap (see Section 5). From (48) and (53), R^* is given by

$$R^* = \frac{v_m(x^*) - v_d(x^*)}{(1-\epsilon)u'\left(\frac{\mu}{\sigma}y^f + \overline{y} - g\right)}, \tag{54}$$

where x^* is

$$x^* = v_m^{-1}\left((\rho+\mu)u'\left(\frac{\mu}{\sigma}y^f + \overline{y} - g\right)\right). \tag{55}$$

From (54) and (55), I find that R^* is independent of the nominal rate of interest on excess reserves, R, and the rate of tax on vault cash, τ, and that if ρ, σ, and g are sufficiently small and y^f and $\overline{y}$ are sufficiently large, then the natural nominal interest rate is negative:[20]

$$R^* < 0.$$

Moreover, R^* has the following property.

Lemma 1. *Equation (49) is necessary and sufficient for (52).*

[20]From (55), I obtain

$$\frac{\partial x^*}{\partial \rho} = \frac{u'}{v_m'} < 0, \quad \frac{\partial x^*}{\partial \sigma} = -\frac{\mu y^f(\rho+\mu)u''}{\sigma^2 v_m'} < 0, \quad \frac{\partial x^*}{\partial g} = -\frac{(\rho+\mu)u''}{v_m'} < 0,$$
$$\frac{\partial x^*}{\partial y^f} = \frac{\mu(\rho+\mu)u''}{\sigma v_m'} > 0, \quad \frac{\partial x^*}{\partial \overline{y}} = \frac{(\rho+\mu)u''}{v_m'} > 0.$$

Hence, from (3) and (54), I have $R^* < 0$ if ρ, σ, and g are sufficiently small and y^f and $\overline{y}$ are sufficiently large. Note that the influence of μ on R^* is unclear because the sign of $\partial x^*/\partial \mu$ is ambiguous: $\partial x^*/\partial \mu = [\sigma u' + (\rho+\mu)y^f u'']/(\sigma v_m')$.

Proof. Since from (3) $v_m(x)$ is a monotonically decreasing function of x, using the last equation of (47), I find

$$(49) \iff (50).$$

Taking into account that from (3) $v_m(x) - v_d(x)$ is a monotonically decreasing function of x, I have

$$(50) \iff (51).$$

Thus, (49) is necessary and sufficient for (51). Since (51) is equivalent to (52), I obtain

$$(49) \iff (52).$$

□

I formally state the existence of the normal steady state in the following proposition.

Proposition 1. *When (23) and (52) hold:*

$$R > -\tau, \quad R < R^*,$$

there exists the normal steady state represented by (47).

In the normal steady state, where $R^B = R^* > R$, from (21) excess reserves do not arise ($M^B = \epsilon D$), which implies that the money multiplier is larger than one:

$$\frac{M^h + D}{M} = \frac{(m^h/d) + 1}{(m^h/d) + (m^b/d)} = \frac{x^* + 1}{x^* + \epsilon} > 1.$$

Now, I investigate the effects of monetary policies in the normal steady state. From (47) and (48), a rise in the money growth rate μ increases consumption, employment, and the price change rate:

$$\frac{dc^*}{d\mu} = \frac{y^f}{\sigma} > 0, \quad \frac{dn^*}{d\mu} = \frac{n^f}{\sigma} > 0, \quad \frac{d\pi^*}{d\mu} = 1 > 0,$$

where the effects of a rise in μ on c^* and n^* become stronger as the nominal wage becomes more sticky (i.e., σ decreases). This implies that the cause of the increases in c^* and n^* is nominal wage stickiness. Meanwhile, from (47) and (48), I obtain the following proposition.

Proposition 2. *A change in the nominal rate of interest on excess reserves, R, does not affect consumption, employment, or the price change rate:*

$$\frac{dc^*}{dR} = 0, \quad \frac{dn^*}{dR} = 0, \quad \frac{d\pi^*}{dR} = 0.$$

The reason for this ineffectiveness is that the nominal deposit rate R^D is not affected by a change in R (R^D is not stuck at the lower bound $(1-\epsilon)R$).

5 Permanent Liquidity Trap

This section considers the case where the household's desire for savings is so excessive that (49) is not true:

$$\rho + \mu < \frac{v_m(x(c^*; R))}{u'(c^*)} \left(= (1-\epsilon)R + \frac{v_d(x(c^*; R))}{u'(c^*)} \right), \tag{56}$$

i.e., the natural nominal interest rate is so low that (52) is not true:

$$R > R^*. \tag{57}$$

Note that as inferred from Lemma 1, (56) is necessary and sufficient for (57). In this case, as shown in Figure 2, for the normal steady state to exist (for

Point A in Figure 2 to be attained), the equation $-R^D > -(1-\epsilon)R$ must hold, i.e., the nominal deposit rate must be below its lower bound:

$$R^D < (1-\epsilon)R.$$

Naturally, this is infeasible. Hence, it turns out that under (56) the normal steady state does not exist. Then, what is the state that the economy reaches if (56) is true?

Equation (56) implies that the household prefers saving cash and deposits to consuming when $c = c^*$, $R^D = (1-\epsilon)R$, and $x = x(c^*; R)$.[21] This desire for savings is not suppressed by a decline in the nominal deposit rate (the consequent rise in the cash–deposit ratio) because the nominal deposit rate already reaches the lower bound $(1-\epsilon)R$ (the cash–deposit ratio already reaches the upper bound $x(c^*; R)$).[22] Thus, in contrast with (53) in the normal steady state, the optimality condition of the household, (10), is not satisfied by the adjustment of the nominal deposit rate and the cash–deposit ratio. A reduction in consumption is required for satisfying (10). That is, the ungratified desire to save cash and deposits causes the household to decrease consumption to less than c^*. This consumption deficiency creates unemployment. Consequently, the economy reaches a stagnation steady state where the nominal interest rates R^B and R^D are stuck at the respective lower bounds R and $(1-\epsilon)R$, consumption (aggregate demand) is deficient, and unemployment worsens. In addition, as described below, an increase in the

[21]In (56), $\rho + \mu$ intuitively denotes the degree of preference for consumption. Naturally, higher ρ causes the household to save less and consume more. Also higher μ, which means higher π^* when $c = c^*$, urges the household to consume more because it implies a fall in the price of the present good relative to the price of the future good.

[22]From (46), a decline in the nominal deposit rate R^D $(= (1-\epsilon)R)$ leads to a rise in the cash–deposit ratio $x(c^*; R)$.

monetary base is ineffective (even the price change rate is not affected), excess reserves arise, and the money multiplier decreases to one. In short, the economy falls into a permanent liquidity trap. From (33), (39), and (40) with (43), this permanent liquidity trap is represented by

$$b = \overline{b}, \ \frac{\dot{m}}{m} = \mu - \pi = \mu - \sigma\left(\frac{c + g - \overline{y}}{y^f}\right) > 0, \ \rho + \sigma\left(\frac{c + g - \overline{y}}{y^f}\right) = \frac{v_m(x(c; R))}{u'(c)}, \tag{58}$$

where

$$c < c^*, \quad R^B = R, \quad R^D = (1 - \epsilon)R.$$

Recall that from (28), (42), and (43) R^B and R^D are R and $(1-\epsilon)R$, respectively, when the cash–deposit ratio is $x(c; R)$.

Let me examine the existence of this permanent liquidity trap. As in the normal steady state, from the first equation of (58), the real bond b is given by $\overline{b}$. Meanwhile, consumption c is determined by the last equation of (58) so as to be lower than the level of the normal steady state c^* as follows. Using the first equation of (48) and (56), I find that in the last equation of (58) the LHS is smaller than the RHS at $c = c^*$:

$$\rho + \sigma\left(\frac{c^* + g - \overline{y}}{y^f}\right) = \rho + \mu < \frac{v_m(x(c^*; R))}{u'(c^*)}.$$

Therefore, if the LHS is larger than the RHS at $c = 0$:[23]

$$\rho + \sigma\left(\frac{g-\overline{y}}{y^f}\right) > 0 \quad \text{if} \quad R \leq 0, \quad \rho + \sigma\left(\frac{g-\overline{y}}{y^f}\right) > (1-\epsilon)R \quad \text{if} \quad R > 0, \tag{59}$$

at least one value of c satisfying the last equation of (58), denoted by $\tilde{c}$, exists between 0 and c^*:

$$0 < \tilde{c} < c^*.$$

Furthermore, if the slope of the LHS is smaller than that of the RHS at $c = \tilde{c}$:[24]

$$\frac{\sigma}{y^f} < \frac{v_m'(x(\tilde{c};R))}{u'(\tilde{c})} \cdot \frac{\partial x(\tilde{c};R)}{\partial \tilde{c}} - \frac{v_m(x(\tilde{c};R))u''(\tilde{c})}{[u'(\tilde{c})]^2} \equiv f(\tilde{c};R), \tag{60}$$

then $\tilde{c}$ is unique (Figure 3 illustrates the unique existence of $\tilde{c}$ in the case of $R \leq 0$).[25] Hence, in contrast with (53) in the normal steady state, the optimality condition of the household, (10), holds as follows:

$$\rho + \sigma\left(\frac{\tilde{c}+g-\overline{y}}{y^f}\right) = \frac{v_m(x(\tilde{c};R))}{u'(\tilde{c})} = (1-\epsilon)R + \frac{v_d(x(\tilde{c};R))}{u'(\tilde{c})}. \tag{61}$$

Using (36) and (37), I find that the consumption deficiency ($\tilde{c} < c^*$) makes employment $\tilde{n}$ and the price change rate $\tilde{\pi}$ in the permanent liquidity trap

[23]When $c = 0$, from (1), (3), (42), (43), and (44), the RHS of (58) is

$$\frac{v_m(x(0;R))}{u'(0)} = 0 \quad \text{if} \quad R \leq 0, \quad \frac{v_m(x(0;R))}{u'(0)} = (1-\epsilon)R \quad \text{if} \quad R > 0,$$

where

$$\frac{v_d(x(0;R))}{u'(0)} = -(1-\epsilon)R > 0 \quad \text{if} \quad R \leq 0, \quad \frac{v_d(x(0;R))}{u'(0)} = 0 \quad \text{if} \quad R > 0.$$

[24]From (1), (3), and (45), whereas the first term of $f(\tilde{c};R)$ is positive if $R < 0$, it is negative if $R > 0$ and vanishes if $R = 0$. However, the second term is always positive, which allows $f(\tilde{c};R)$ to satisfy the inequality of (60) even if $R \geq 0$.

[25]In the case of $R > 0$, the only difference from Figure 3 is that the intercept of the RHS is not zero but $(1-\epsilon)R$.

lower than the levels of the normal steady state:

$$\tilde{n} = \frac{\tilde{c}+g}{e} < \frac{c^*+g}{e} = n^*, \quad \tilde{\pi} = \sigma\left(\frac{\tilde{c}+g-\overline{y}}{y^f}\right) < \sigma\left(\frac{c^*+g-\overline{y}}{y^f}\right) = \pi^* = \mu. \tag{62}$$

The first equation of (62) implies that unemployment in the permanent liquidity trap is $n^f - \tilde{n}$, which is the sum of unemployment created by the consumption deficiency, $n^* - \tilde{n}$, and unemployment created by the efficiency wage, $n^f - n^*$. The second equation of (62) implies that $\tilde{\pi}$ can be positive or negative and that the real monetary base permanently increases ($m = \infty$), as shown by the second equation of (58).[26] Taking $m = \infty$ into account, from (2), (19), (24), and (35), I find that real cash holdings m^h, real deposit holdings d, and real bank reserves m^b also increase to infinity:

$$m^h = \frac{x(\tilde{c}; R)(m+\overline{b})}{1+x(\tilde{c}; R)} = \infty, \;\; d = \frac{m+\overline{b}}{1+x(\tilde{c}; R)} = \infty, \;\; m^b = \frac{m - x(\tilde{c}; R)\overline{b}}{1+x(\tilde{c}; R)} = \infty. \tag{63}$$

Although household's wealth holdings increase to infinity ($a = m^h + d = \infty$), household consumption remains insufficient ($c = \tilde{c} < c^*$). This is why the liquidity trap is permanent. I summarize the above discussion in the following proposition.

[26] From (7), the first equation of (8), (19), (24), (35), and (58), I obtain

$$a_t = m_t + b_t, \quad \lim_{t\to\infty} \lambda_t = u'(\tilde{c}), \quad \lim_{t\to\infty} b_t = \overline{b}, \quad \lim_{t\to\infty}\left(\frac{\dot{m}_t}{m_t} - \rho\right) = \mu - \frac{v_m(x(\tilde{c}; R))}{u'(\tilde{c})}.$$

Therefore, when μ is so low as to satisfy

$$\mu < \frac{v_m(x(\tilde{c}; R))}{u'(\tilde{c})},$$

the rate of growth in m_t is lower than ρ, and the transversality condition (9) is satisfied:

$$\lim_{t\to\infty} \lambda_t a_t \exp(-\rho t) = u'(\tilde{c}) \left[\lim_{t\to\infty} m_t \exp(-\rho t) + \lim_{t\to\infty} b_t \exp(-\rho t)\right] = 0.$$

Proposition 3. *When (23) and (57) hold:*

$$R > -\tau, \quad R > R^*,$$

there exists the permanent liquidity trap represented by (58).

In this liquidity trap, from (21), excess reserves appear ($M^b - \epsilon D > 0$) because the government bond rate equals the nominal rate of interest on excess reserves ($R^B = R$). The presence of excess reserves decreases the money multiplier to one:

$$\frac{M^h + D}{M} = \frac{(m^h/d) + 1}{(m^h/d) + (m^b/d)} = 1,$$

where from (63) the reserve–deposit ratio is

$$\frac{m^b}{d} = \lim_{m \to \infty} \frac{1 - (x(\tilde{c}; R)\bar{b}/m)}{1 + (\bar{b}/m)} = 1.$$

The effects of fiscal and monetary policies in the permanent liquidity trap are in contrast to those in the normal steady state. From (60), the first equality of (61), and (62), an increase in government purchases g raises consumption, employment, and the price change rate:

$$\frac{d\tilde{c}}{dg} = \frac{\sigma/y^f}{f(\tilde{c}; R) - (\sigma/y^f)} > 0, \quad \frac{d\tilde{n}}{dg} = \frac{1}{e}\left(\frac{d\tilde{c}}{dg} + 1\right) > 0, \quad \frac{d\tilde{\pi}}{dg} = \frac{\sigma}{y^f}\left(\frac{d\tilde{c}}{dg} + 1\right) > 0.$$

The reason why an increase in government purchases boosts consumption is that it raises the price change rate (it lowers the price of the present good relative to the price of the future good). Hence, if the price is fixed ($\sigma = 0$), the effect of g vanishes ($d\tilde{c}/dg = 0$). In contrast with an increase in g, from the first equality of (61) and (62), a rise in the money growth rate μ has no effect:

$$\frac{d\tilde{c}}{d\mu} = 0, \quad \frac{d\tilde{n}}{d\mu} = 0, \quad \frac{d\tilde{\pi}}{d\mu} = 0.$$

It is noteworthy that even the price change rate is not affected, which implies that deflation ($\tilde{\pi} < 0$) can arise despite a monetary expansion ($\mu > 0$). See Ono and Ishida (2014) and Murota (2016, 2018) for similar effects of fiscal and monetary expansions in stagnation steady states.

Whereas a rise in μ is ineffective, a change in the nominal rate of interest on excess reserves, R, affects the economy. Totally differentiating the first equality of (61) yields

$$\frac{d\tilde{c}}{dR} = -\frac{v_m'(x(\tilde{c};R))}{u'(\tilde{c})} \cdot \frac{\partial x(\tilde{c};R)}{\partial R}\left[f(\tilde{c};R) - \frac{\sigma}{y^f}\right]^{-1} < 0,$$

where the inequality is established by (1), (3), (46), and (60). Hence, from (62), I obtain

$$\frac{d\tilde{n}}{dR} = \frac{1}{e} \cdot \frac{d\tilde{c}}{dR} < 0, \quad \frac{d\tilde{\pi}}{dR} = \frac{\sigma}{y^f} \cdot \frac{d\tilde{c}}{dR} < 0.$$

I restate this result in the following proposition.

Proposition 4. *In the permanent liquidity trap, where R^D is stuck at $(1-\epsilon)R$, a reduction in the nominal rate of interest on excess reserves, R, increases consumption, employment, and the price change rate.*

This proposition is produced through the following mechanism. Since a reduction in R lowers R^D $(= (1-\epsilon)R)$, the household is encouraged to shift its portfolio from deposits to cash (from (46) a reduction in R raises the cash–deposit ratio $x(\tilde{c};R)$). The rise in $x(\tilde{c};R)$ works to lower the marginal utility of cash $v_m(x(\tilde{c};R))$ (i.e., it works to gratify the household's desire to hold cash). This causes the household to increase consumption, and the increase in consumption (aggregate demand) leads to increases in employment and the price change rate.

Note that naturally, in this permanent liquidity trap, a rise in the rate of tax on vault cash, τ, does not have any effects on consumption, employment, or the price change rate. From (61) and (62), I have

$$\frac{d\tilde{c}}{d\tau} = 0, \quad \frac{d\tilde{n}}{d\tau} = 0, \quad \frac{d\tilde{\pi}}{d\tau} = 0.$$

Before going on to the next section, I summarize Propositions 1 and 3 in Figure 4. In Region (A) consisting of $R > -\tau$ and $R < R^*$, the normal steady state exists. In Region (B) consisting of $R > -\tau$ and $R > R^*$, the permanent liquidity trap appears. If R is lowered from Point A to Point B in Figure 4, the economy moves from the permanent liquidity trap to the normal steady state. Then, if R is lowered from Point C to Point D in Figure 4, what state does the economy reach? To answer this question, in the next section, I analyze the case of

$$R < -\tau.$$

6 Ineffectiveness of Negative Interest Rate Policy

This section first derives the dynamic system in the case of $R < -\tau$. It then shows that the normal steady state also exists in Region (A) composed of $R < -\tau$ and $R^* > -\tau$ in Figure 5 and that there exists the permanent liquidity trap, where R^D is stuck not at $(1-\epsilon)R$ but at $-(1-\epsilon)\tau$, in Region (C) composed of $R < -\tau$ and $R^* < -\tau$ in Figure 5. Moreover, it investigates the effects of a fall in R and a rise in τ.

If $R < -\tau$, from (22), the lower bound on R_t^B is $-\tau$ (not R):

$$R_t^B \geq -\tau > R.$$

Since the return on excess reserves is lower than that on government bonds ($R < R_t^B$), from (21), the commercial bank does not hold excess reserves:

$$M_t^b - \epsilon D_t = 0.$$

Moreover, from (21), the following two cases are possible:

$$R_t^B > -\tau \quad \text{and} \quad Z_t = 0,$$
$$R_t^B = -\tau \quad \text{and} \quad Z_t > 0.$$

In the case where the government bond rate is higher than its lower bound ($R_t^B > -\tau$) and where the commercial bank does not hold vault cash ($Z_t = 0$), from (20) where $\kappa_t > 0$ and $\xi_t > 0$, the nominal deposit rate is higher than $-(1-\epsilon)\tau$:

$$R_t^D = (1-\epsilon)R_t^B > -(1-\epsilon)\tau. \tag{64}$$

In this case, because of $Z_t = 0$ and $M_t^b - \epsilon D_t = 0$, the dynamic system is given by (33), (39), and (40) with (41).

In the case where the government bond rate is stuck at its lower bound ($R_t^B = -\tau$) and where the commercial bank holds vault cash ($Z_t > 0$), the money market equilibrium condition (35) is modified as follows:

$$m_t^h + m_t^b + z_t = m_t, \tag{65}$$

where z_t is real vault cash holdings. However, the law of motion of m_t remains (39). Meanwhile, the law of motion of c_t, (40), is modified; the cash–deposit ratio x_t is no longer (41) or (43) as follows. In the case of $R_t^B = -\tau$ and $Z_t > 0$, from (20) where $\kappa_t > 0$ and $\xi_t = 0$, the nominal deposit rate equals $-(1-\epsilon)\tau$:

$$R_t^D = (1-\epsilon)R_t^B = -(1-\epsilon)\tau. \tag{66}$$

From the second equality of (10) with (66):

$$\frac{v_m(x_t)}{u'(c_t)} = -(1-\epsilon)\tau + \frac{v_d(x_t)}{u'(c_t)},$$

x_t in (40) is given by a function of c_t and $-\tau$:

$$x_t = x(c_t; -\tau), \tag{67}$$

where $x(c_t; -\tau)$ is the same as obtained by replacing R of $x(c_t; R)$ in (43) with $-\tau$. Thus, when $R_t^B = -\tau$ and $Z_t > 0$, the dynamic system consists of (33), (39), and (40) with (67). Note from (64) and (66) that regardless of whether $Z_t = 0$ or $Z_t > 0$, the following equation holds:

$$R_t^D = (1-\epsilon)R_t^B$$

and that the lower bound on R^D is $-(1-\epsilon)\tau$ (not $(1-\epsilon)R$):

$$R_t^D \geq -(1-\epsilon)\tau > (1-\epsilon)R.$$

Now, I examine what steady states exist if $R < -\tau$. When

$$\rho + \mu > \frac{v_m(x(c^*; -\tau))}{u'(c^*)}, \tag{68}$$

from (33), (39), and (40) with (41), there exists the same normal steady state as the one represented by (47) in Section 4:

$$b^* = \bar{b}, \quad \mu = \pi^* = \sigma\left(\frac{c^* + g - \bar{y}}{y^f}\right), \quad \rho + \mu = \frac{v_m(x^*)}{u'(c^*)}.$$

As in Section 4, there exists the cash–deposit ratio x^* that satisfies $\rho + \mu = v_m(x^*)/u'(c^*)$. However, x^* satisfies

$$x^* < x(c^*; -\tau)$$

instead of (50). Therefore, R^D and R^B equal the natural nominal interest rates and satisfy

$$R^D = \frac{v_m(x^*)}{u'(c^*)} - \frac{v_d(x^*)}{u'(c^*)} = (1-\epsilon)R^* > -(1-\epsilon)\tau = \frac{v_m(x(c^*;-\tau))}{u'(c^*)} - \frac{v_d(x(c^*;-\tau))}{u'(c^*)},$$

$$R^B = \frac{R^D}{1-\epsilon} = R^* > -\tau$$

instead of (51) and (52). Since (68) is necessary and sufficient for $R^* > -\tau$ as (49) is necessary and sufficient for (52) (see Lemma 1), I obtain the following proposition.

Proposition 5. *In Region (A) in Figure 5:*

$$R < -\tau, \quad R^* > -\tau,$$

there exists the normal steady state represented by (47).

Next, I consider the case of

$$\rho + \mu < \frac{v_m(x(c^*;-\tau))}{u'(c^*)}. \tag{69}$$

In this case, for the normal steady state to be attained, R^D and R^B must fall below the respective lower bounds: $-(1-\epsilon)\tau$ and $-\tau$, as inferred from the discussion at the outset of Section 5. Since this is not feasible, the normal steady state does not exist, and a permanent liquidity trap appears. However, R^B and R^D are stuck not at R and $(1-\epsilon)R$ but at $-\tau$ and $-(1-\epsilon)\tau$. From (33), (39), and (40) with (67), this liquidity trap is characterized by

$$b = \bar{b}, \ \ \frac{\dot{m}}{m} = \mu - \pi = \mu - \sigma\left(\frac{c+g-\bar{y}}{y^f}\right) > 0, \ \ \rho + \sigma\left(\frac{c+g-\bar{y}}{y^f}\right) = \frac{v_m(x(c;-\tau))}{u'(c)}, \tag{70}$$

which is the same as obtained by replacing R of (58) in Section 5 with $-\tau$.

As in Section 5, the value of c satisfying the last equation of (70), denoted by $\hat{c}$, uniquely exists so as to satisfy

$$0 < \hat{c} < c^*$$

when in addition to (69) the following holds:[27]

$$\rho + \sigma\left(\frac{g - \overline{y}}{y^f}\right) > \frac{v_m(x(0; -\tau))}{u'(0)} = 0, \quad \frac{\sigma}{y^f} < f(\hat{c}; -\tau) \tag{71}$$

instead of (59) and (60). Therefore, taking into account that (69) is necessary and sufficient for $R^* < -\tau$ as (56) is necessary and sufficient for (57), I obtain the following proposition.

Proposition 6. *In Region (C) in Figure 5:*

$$R < -\tau, \quad R^* < -\tau,$$

there exists the permanent liquidity trap represented by (70).

In this liquidity trap, because of $R^B = -\tau > R$, the commercial bank does not hold excess reserves but holds vault cash. Hence, from (2), (19), (25), and (65), the money multiplier also decreases to one:[28]

$$\frac{M^h + D}{M} = \frac{(m^h/d) + 1}{(m^h/d) + (m^b/d) + (z/d)} = \frac{x(\hat{c}; -\tau) + 1}{x(\hat{c}; -\tau) + \epsilon + 1 - \epsilon} = 1.$$

[27] $x(c_t; -\tau)$, as well as $x(c_t; R)$ with $R < 0$ in (44), satisfies

$$x(0; -\tau) = \infty.$$

[28] From (2), (19), (25), and (65), z and d are

$$z = \frac{(1-\epsilon)m - [x(\hat{c}; -\tau) + \epsilon]\overline{b}}{1 + x(\hat{c}; -\tau)} = \infty, \quad d = \frac{m + \overline{b}}{1 + x(\hat{c}; -\tau)} = \infty,$$

which implies that

$$\frac{z}{d} = \lim_{m \to \infty} \frac{1 - \epsilon - [x(\hat{c}; -\tau) + \epsilon](\overline{b}/m)}{1 + (\overline{b}/m)} = 1 - \epsilon.$$

From (36) and (37), as in Section 5, the consumption deficiency ($\hat{c} < c^*$) reduces employment $\hat{n}$ and the price change rate $\hat{\pi}$ to less than the levels of the normal steady state:

$$\hat{n} = \frac{\hat{c} + g}{e} < \frac{c^* + g}{e} = n^*, \quad \hat{\pi} = \sigma\left(\frac{\hat{c} + g - \overline{y}}{y^f}\right) < \sigma\left(\frac{c^* + g - \overline{y}}{y^f}\right) = \pi^* = \mu. \tag{72}$$

From (70) and (72), the effects of increases in g and μ are the same as those in Section 5.[29] Moreover, as shown below, although the effects of a reduction in R and a rise in τ are in contrast to those in Section 5, lowering the nominal deposit rate remains important for boosting the economy.

Proposition 7. *When R^D is stuck at $-(1-\epsilon)\tau$, a change in the nominal rate of interest on excess reserves, R, has no effect:*

$$\frac{d\hat{c}}{dR} = 0, \quad \frac{d\hat{n}}{dR} = 0, \quad \frac{d\hat{\pi}}{dR} = 0.$$

This is because a change in R does not affect R^D (the lower bound on R^D is no longer $(1-\epsilon)R$). By contrast, a rise in τ lowers R^D $(= -(1-\epsilon)\tau)$, which raises the cash–deposit ratio $(\partial x(\hat{c}; -\tau)/\partial \tau > 0)$ and reduces the marginal utility of cash $(v'_m(x(\hat{c}; -\tau)) < 0)$.[30] This stimulates consumption, which reduces unemployment and raises the price change rate, as stated in the following proposition.

[29]From (70), (71), and (72), I obtain

$$\frac{d\hat{c}}{dg} = \frac{\sigma/y^f}{f(\hat{c}; R) - (\sigma/y^f)} > 0, \quad \frac{d\hat{n}}{dg} = \frac{1}{e}\left(\frac{d\hat{c}}{dg} + 1\right) > 0, \quad \frac{d\hat{\pi}}{dg} = \frac{\sigma}{y^f}\left(\frac{d\hat{c}}{dg} + 1\right) > 0,$$
$$\frac{d\hat{c}}{d\mu} = 0, \quad \frac{d\hat{n}}{d\mu} = 0, \quad \frac{d\hat{\pi}}{d\mu} = 0.$$

[30]From (67), I obtain

$$\frac{\partial x_t}{\partial \tau} = \frac{(1-\epsilon)u'(c_t)}{v'_d(x_t) - v'_m(x_t)} > 0.$$

Proposition 8. *When R^D is stuck at $-(1-\epsilon)\tau$, a rise in the rate of tax on vault cash, τ, increases consumption, employment, and the price change rate:*

$$\frac{d\hat{c}}{d\tau} = -\frac{v'_m(x(\hat{c};-\tau))}{u'(\hat{c})} \cdot \frac{\partial x(\hat{c};-\tau)}{\partial \tau} \left[f(\hat{c};-\tau) - \frac{\sigma}{y^f} \right]^{-1} > 0,$$

$$\frac{d\hat{n}}{d\tau} = \frac{1}{e} \cdot \frac{d\hat{c}}{d\tau} > 0, \quad \frac{d\hat{\pi}}{d\tau} = \frac{\sigma}{y^f} \cdot \frac{d\hat{c}}{d\tau} > 0.$$

Finally, I summarize Propositions 1, 3, 5, and 6 in Figure 6 to analyze the effectiveness of a reduction in R and a rise in τ as a way of getting the economy out of the permanent liquidity trap.

Proposition 9. *If the natural nominal interest rate is higher than the lower bound set by the presence of vault cash ($R^* > -\tau$), a reduction in the nominal rate of interest on excess reserves, R, can move the economy from the permanent liquidity trap to the normal steady state (the economy moves from Region (B) to Region (A) in Figure 6). If the natural nominal interest rate is so low that $R^* < -\tau$, a reduction in R cannot pull the economy out of the permanent liquidity trap (the economy that escapes Region (B) reaches Region (C) in Figure 6).*

If the rate of tax on vault cash is raised from τ to $\overline{\tau}$, then Point A in Region (C) in Figure 6 moves to Region (A) in Figure 7. Meanwhile, Point B in Region (C) in Figure 6 moves to Region (B) in Figure 7, and therefore a reduction in R becomes able to move the economy at Point B to Region (A). I restate this result in the following proposition.

Proposition 10. *A rise in the rate of tax on vault cash, τ, moves the economy from the permanent liquidity trap to the normal steady state or revives* *the ability of a reduction in R to get the economy out of the permanent liquidity trap.*

It turns out from Figure 7 that a high natural nominal interest rate, a low nominal rate of interest on excess reserves, and a high rate of tax on vault cash prevent the economy from falling into the permanent liquidity trap.

7 Conclusion

Using a dynamic general equilibrium model where cash and deposits provide utility, nominal wages are sticky, excess bank reserves bear negative interest, and a tax is levied on vault cash, this paper analyzes the effects of a negative interest rate policy in a permanent liquidity trap where deficient aggregate demand creates unemployment, excess reserves arise, the money multiplier declines to one, and an increase in the monetary base is ineffective. If the natural nominal interest rate is above the lower bound set by the presence of vault cash, a reduction in the nominal rate of interest on excess reserves can reduce the nominal deposit rate to the level of the natural nominal interest rate and can get the economy out of the permanent liquidity trap. By contrast, if the natural nominal interest rate is below the lower bound, the nominal deposit rate does not decline to the level of the natural nominal interest rate and is stuck at the lower bound no matter how negative the nominal rate of interest on excess reserves is. Therefore, in this case, we cannot pull the economy out of the permanent liquidity trap by lowering the nominal rate of interest on excess reserves. Instead, a rise in the rate of tax on vault cash is useful for helping the economy escape the permanent liquidity trap because a decline in the lower bound caused by a rise in the tax rate allows the nominal deposit rate to fall to the level of the natural nominal interest rate.

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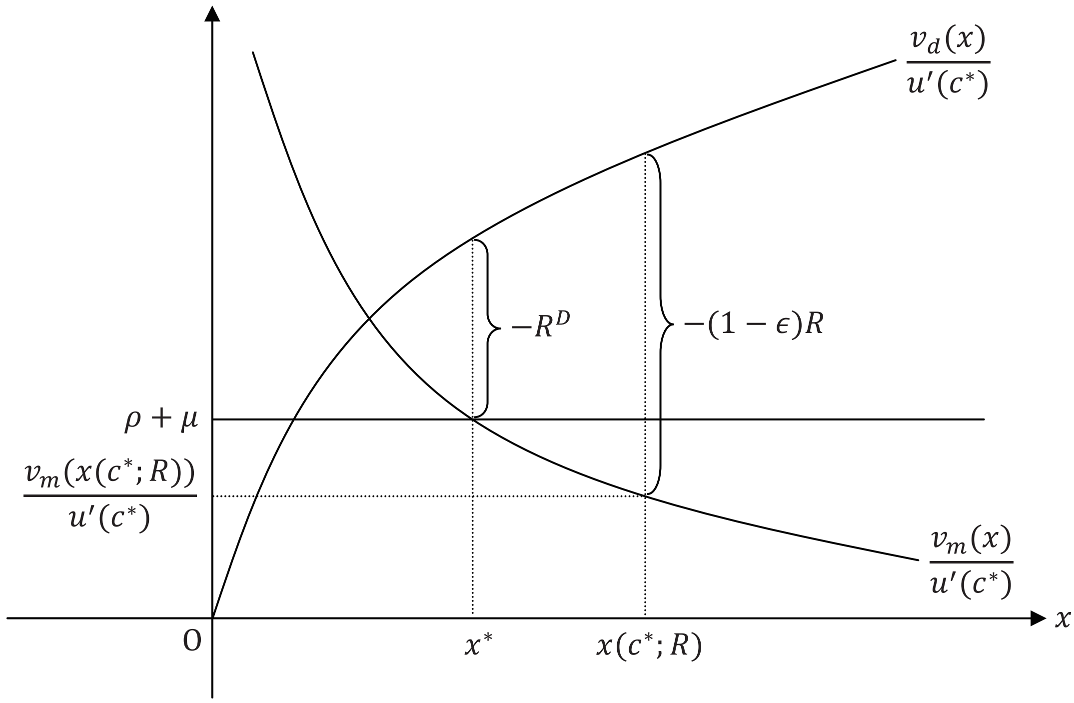

Figure 1: Case of (49)

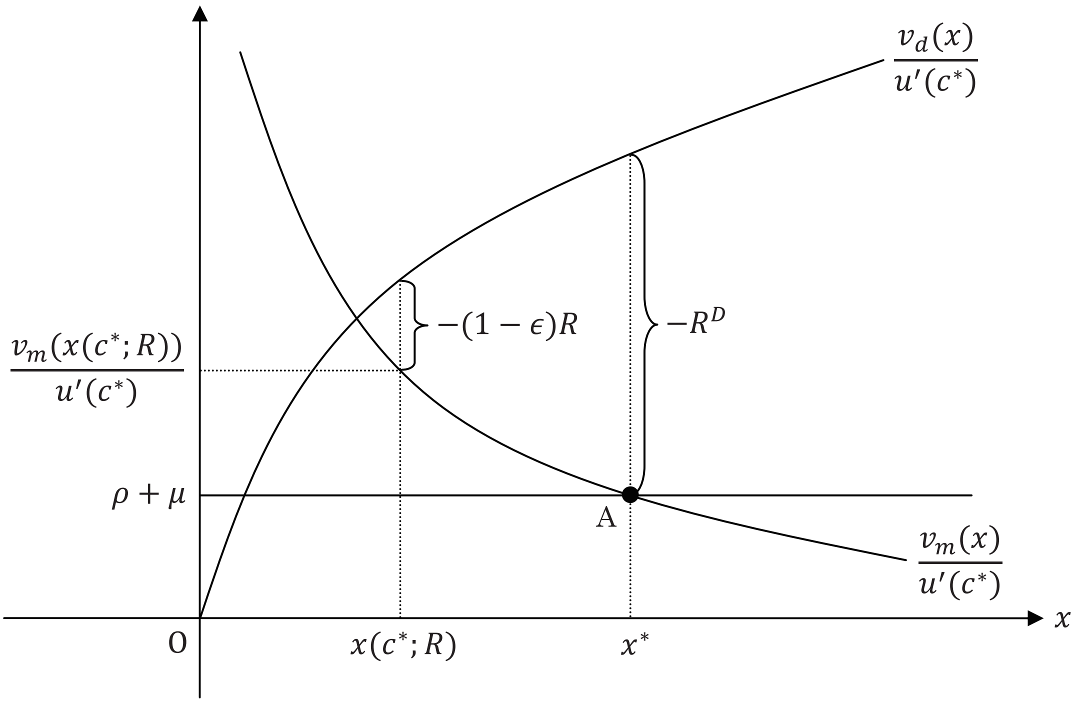

Figure 2: Case of (56)

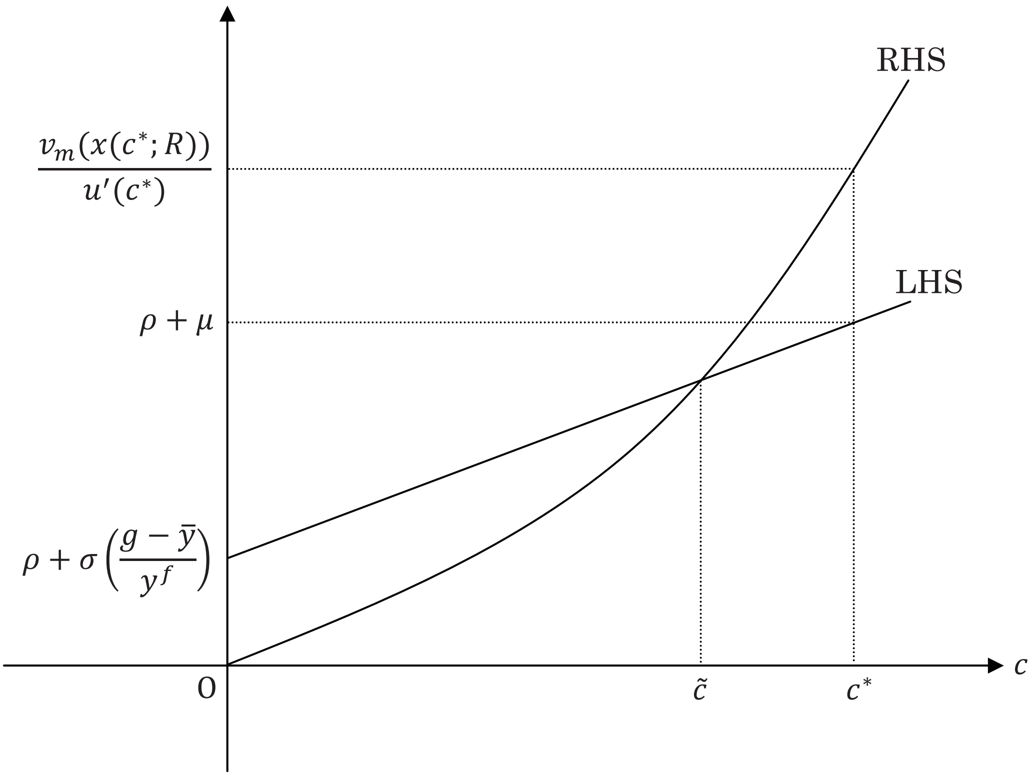

Figure 3: Consumption deficiency $c^* - \tilde{c}$ (> 0) in the case of $R \leq 0$

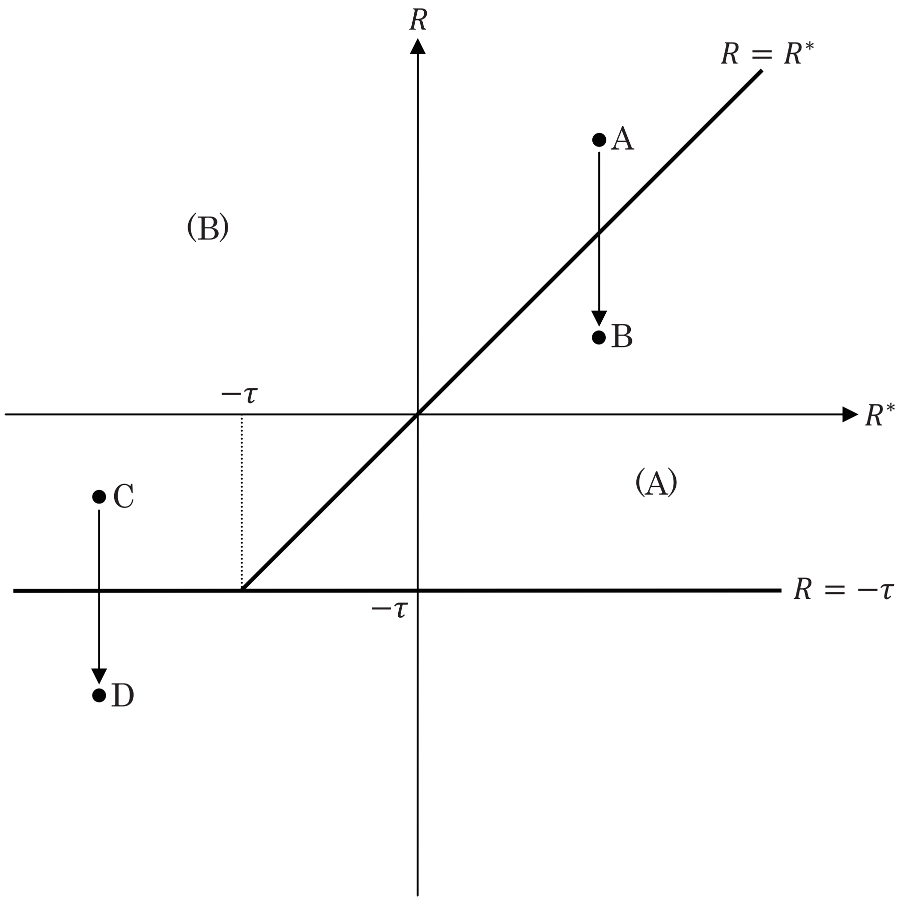

Figure 4: (A) normal steady state and (B) permanent liquidity trap ($R^D = (1-\epsilon)R$)

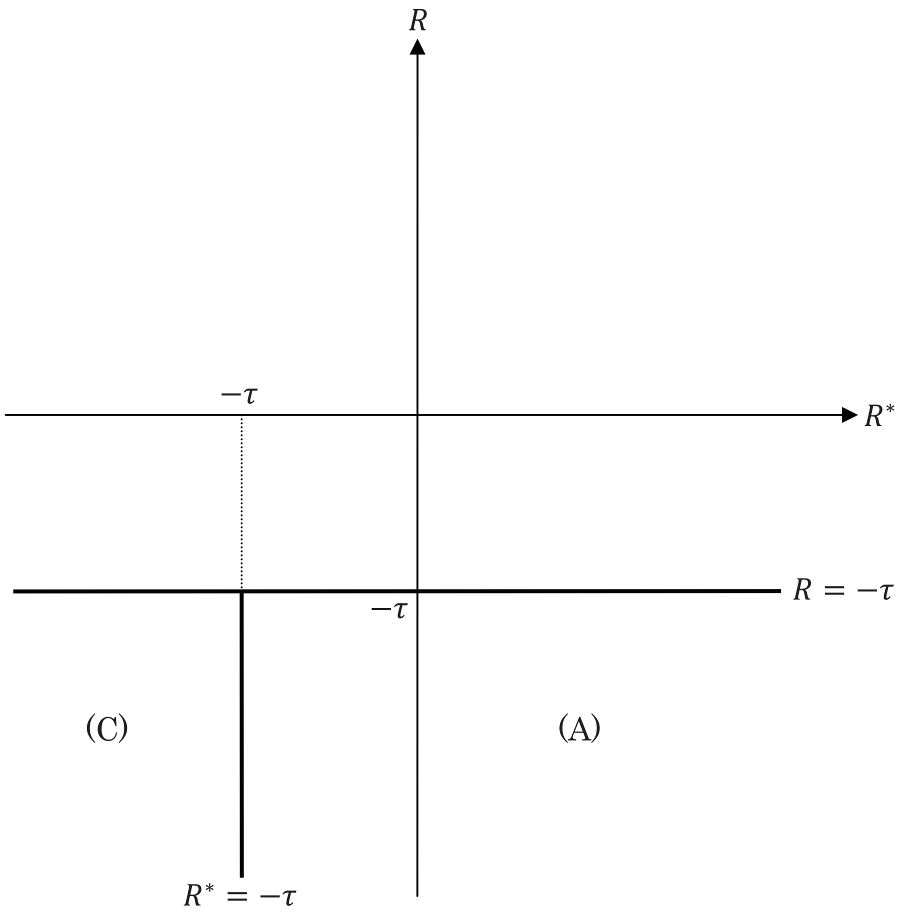

Figure 5: (A) normal steady state and (C) permanent liquidity trap ($R^D = -(1-\epsilon)\tau$)

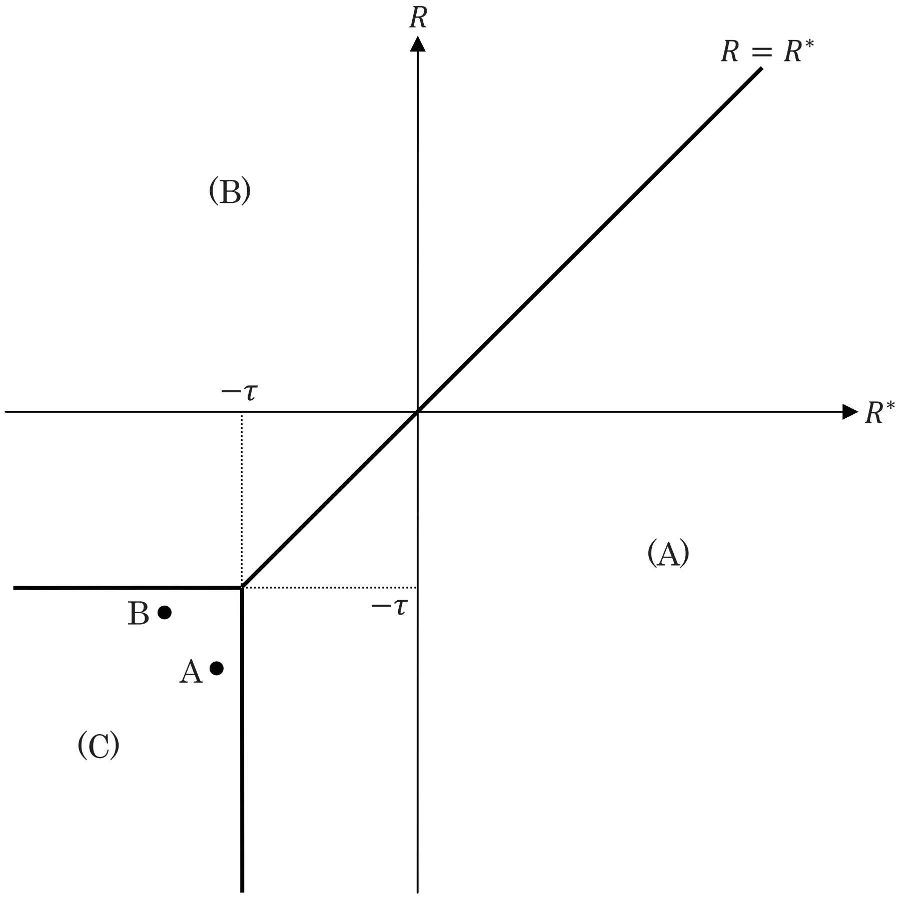

Figure 6: (A) normal steady state, (B) permanent liquidity trap ($R^D = (1-\epsilon)R$), and (C) permanent liquidity trap ($R^D = -(1-\epsilon)\tau$)

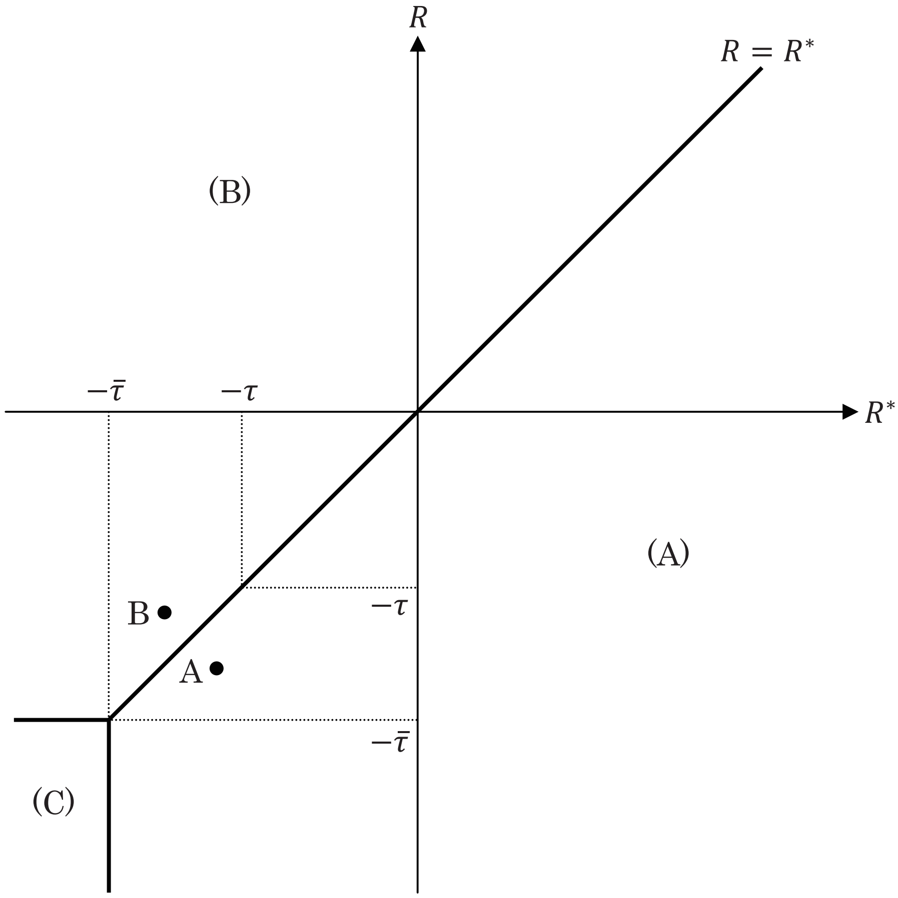

Figure 7: Effect of a rise in the rate of tax on vault cash