When and how to dismantle nuclear weapons

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Abstract This paper first derives revenue-maximizing auctions with identity-specific externalities among all players (seller and buyers). Our main findings are as follows. Firstly, a modified second-price sealed-bid auction with appropriate entry fees and reserve price is revenue-maximizing. Secondly, seller may physically destroy the auctioned item if the item is unsold or use destroying the item as nonparticipation threat. Thirdly, the revenue-maximizing auction induces full participation of buyers. Fourthly, each losing buyer's payment includes an externality-correcting component that equals the allocative externality to him. These components eliminate the impact of externalities on strategic bidding behavior. The paper further studies revenue-maximizing auctions with financial externalities. One-to-one correspondences between revenue-maximizing auctions for settings with and without financial externalities are established through incorporating externality-correcting payments. This result provides a general method for designing revenue-maximizing auctions in different settings of financial externalities, since revenue-maximizing auctions can be obtained through transforming the revenue-maximizing auctions for the regular settings without externalities.

Keywords: Auctions design; Endogenous participation; Externality.

JEL classifications: D44, D82.

1 Introduction

Auctions design with externalities among buyers has been studied by a number of papers. Jehiel, Moldovanu and Stacchetti (1996, 1999), Varma (2002), and Brocas (2003, 2005) among others consider identity-specific externalities imposed on losers by the winning buyer, while Maasland and Onderstal (2002, 2005) and Goeree, Maasland, Onderstal and Turner (2005) study the cases where financial externalities among buyers are proportional to the total payments of other buyers or all buyers. While the literature largely focuses on externalities among buyers, situations abound where the existence of externalities between the seller and buyers is the major concern. One recent example is the North Korea's nuclear weapon case, where the seller (North Korea) puts great externalities on the buyers (China, Japan, Russia, South Korea, US) if it keeps its nuclear arsenal. In this paper, we will first derive the revenue-maximizing auction while allowing identity-specific externalities among **all** players, including the seller and buyers.[1] We will also develop a unified approach for deriving the revenue-maximizing auctions with financial externalities in a general setting allowing asymmetry across buyers. This method illustrates some common principles for revenue-maximizing auctions design in both settings with identity-specific and financial externalities.

We first study in this paper the revenue-maximizing auction while allowing externalities among all players. One major contribution of this paper lies in that the analysis brings in the option for the seller to physically destroy the item (i.e., dismantle its nuclear arsenal) at a cost. One should note that "destroying the item" differs from "not-selling" in our setting. In previous auctions design literature, destroying the auctioned item has not been formulated as a possible outcome or as a nonparticipation threat. The signif-

[1]Potipiti (2005) also considers a setting of selling retaliation in the WTO where externalities exist between the seller and buyers. However, both the setting and the focus of Potipiti (2005) are quite different from that of this paper.

icance of this option is the following. First, we are particularly interested in addressing when and how to dismantle the nuclear weapons, i.e. under what conditions should the seller destroy the object and what actions should be taken by the seller to maximize his revenue if he destroys the object. Second, allowing this new option enlarges the freedom of auctions design with externalities. Specifically, destroying the item can be an optimal allocation outcome for the seller or be used by the seller as an optimal nonparticipation threat, since it eliminates the externalities imposed on buyers. Specifically, eliminating these externalities has two effects. First, seller's threat to a buyer who refuses to participate can be made more severe. This happens when a buyer enjoys positive externalities whoever else gets the object. In this case, the most severe nonparticipation threat is to destroy the object. Second, the seller may extract higher rent when he destroys the object if the object is unsold. This occurs when the sum of the seller's valuation, the destroying cost of the seller and the total externalities to the buyers is negative, if the seller keeps the item. In this situation, the seller can be better off by destroying the object and collecting a payment from each buyer which equals the externality to the buyer.

We start with a baseline setting where identity-specific externalities among all players are public information.[2] We allow heterogeneity in the externalities among the players. Moreover, our analysis does not require the externalities to be uniformly positive or negative.

The revenue-maximizing auction is fully characterized in terms of the nonparticipation threats, the winning probabilities, the probability of destroying the item, and the payments of buyers. A modified second-price sealed-bid auction with appropriately set (nonnegative) entry fees and reserve price is established as the revenue-maximizing auc-

[2] We adopt this setting because the multi-dimensional setting in Jehiel, Moldovanu and Stacchetti (1999) is too complicated for a thorough characterization of the optimum, and the Jehiel, Moldovanu and Stacchetti (1996) setting is fundamentally one-dimensional.

tion. This auction induces full participation. As a special feature, the revenue-maximizing auction involves externality-correcting transfers between seller and losing buyers. The detailed specifications of the auction rule are as follows. (i) If only buyer i does not participate, the item is then assigned to the one (including the seller) generating buyer i the smallest externality provided that this externality is nonpositive. Otherwise the seller destroys the object. (ii) Every participant pays a nonnegative entry fee, which equals the absolute value of the smallest possible externality to him.[3] (iii) The highest buyer wins if his bid is higher than the reserve price, and he pays the second highest bid or the reserve price whichever is higher. Each buyer pays an externality-correcting payment (positive or negative) that equals the allocative externality to him.[4] (iv) If no buyer bids higher than the reserve price, the seller may keep the item or destroy it. The seller destroys the item if and only if the sum of his own valuation, his dismantling cost and the total externalities on the buyers if he keeps the item is negative. The externality-correcting payments in the revenue-maximizing auction lead to a situation that mimics a setting without externalities to buyers. This explains intuitively why a modified second-price auction with these externality-correcting payments is revenue-maximizing, if the entry fees and reserve price are set appropriately.

What is the intuition behind the optimality of full participation? Note that the sum of the externality-correcting payment and the entry fee for each buyer must be nonnegative in the above auction. Thus, the seller gains (weakly) from the participation of every type of buyers.[5] This explains why the seller wants every type of buyers to participate, although some types have no chance of winning.

[3]This smallest externality must be nonpositive due to the option of destroying the object by the seller.

[4]This externality-correcting payment is a unique feature which is first discovered in the literature to my best knowledge.

[5]Besides the additional payment, each buyer (winner or loser) makes another nonnegative payment as in a standard second-price auction without externalities.

An interesting issue is whether the insights from our baseline setting also apply to more complex settings. Jehiel, Moldovanu and Stacchetti (1996) have shown that when buyers have private information on the externalities they create for others, the revenue-maximizing auction design problem can be transformed into a one-dimensional problem. Therefore, our findings obtained in the public-information externalities setting remain valid for a multi-dimensional setting of Jehiel, Moldovanu and Stacchetti (1996). A slight modification is to replace the public-information externalities in our results by the expectations of the private-information externalities.

Based on these insights from our baseline setting, we further study the auctions design in general settings of financial externalities where the externality to every buyer equals a proportion of the payments of all buyers or all other buyers. Useful linkages between revenue-maximizing auctions for settings with and without externalities are established. Specifically, we establish one-to-one correspondences between revenue-maximizing auctions with and without financial externalities. Therefore, the revenue-maximizing second-price auction for a regular setting without externalities can be properly modified to be revenue-maximizing in various settings of financial externalities. These findings provide a general way of deriving the revenue-maximizing auctions for settings of financial externalities. Being consistent with the insights from our baseline setting, in all these auctions all buyers' payments consist of an externality-correcting component which equals the externalities to them at the outcome of the auction. [6]

This paper is organized as follows. Section 2 derives the revenue-maximizing mechanism in our baseline setting where identity-specific externalities are public information. We further show that for a symmetric setting the mechanism turns to be a modified

[6] Goeree, Maasland, Onderstal and Turner (2005) show that a lowest-price all-pay auction is revenue-maximizing in symmetric settings of financial externalities. Applying our findings to their setting leads to alternative revenue-maximizing second-price auctions. One advantage of this alternative second price auction lies in that the maximal expected revenue is implemented through weakly dominant strategy.

second price auction. This section also extends these findings to a setting where buyers have private information on the externalities they create for others. Section 3 provides a unified approach for deriving revenue-maximizing auctions with financial externalities, based on the insights from our baseline setting with identity-specific externalities. Section 4 concludes.

2 The Revenue-Maximizing Mechanism with Identity-Specific Externalities

In this section we derive the revenue-maximizing mechanism when there are identity-specific externalities among all players, including the seller and buyers. Externalities lead to an auctions design problem in which the buyers have mechanism-dependent reservation utilities. We will first establish that full participation is optimal in terms of the seller's expected revenue. For this purpose, we explicitly deal with the revenue-maximizing endogenous participation. Following Stegeman (1996), we define participation of buyers as submitting a signal. Since a mechanism implementing endogenous participation essentially cannot require the buyers who do not participate to submit signals, we consider the mechanisms based on only the signals submitted by the participating buyers.

2.1 The Setting

There is one seller who wants to sell one indivisible object to N potential buyers through an auction. We use $\mathcal{N} = \{1, 2, ..., N\}$ to denote the set of all potential buyers, where $\mathcal{N}$ is public information. The seller's value for the object is v_0, which is public information. Hereafter, we represent the seller as player 0 and bidder i as player i. The ith buyer's private value of the object is v_i, which is his private information. These values v_i, $i \in$

$\mathcal{N}$ are independently distributed on intervals $[\underline{v}_i, \overline{v}_i]$ respectively following cumulative distribution function $F_i(\cdot)$ with density function $f_i(\cdot)(> 0)$. We assume the regularity condition that the virtual valuation functions $J_i(v) = v - (1 - F_i(v))/f_i(v)$ increase on intervals $[\underline{v}_i, \overline{v}_i]$. The density $f_i(\cdot)$ is assumed to be public information. Every buyer observes his private value before his participation decision. The seller and the buyers are assumed to be risk neutral.

Player i enjoys/suffers an externality $e_{i,j}$ when player j keeps the item, $i, j = 0, 1, ..., N$. By definition, $e_{i,i} = 0,\ i = 0, 1, ..., N$. These externalities are public information. The auctioned item can be destroyed by the seller at a cost of $c_0 \geq 0$. If the item is destroyed, no player enjoys/suffers any externality. As a result, buyer i's payoff is $v_i - x_i$ if he wins and pays x_i; his payoff is $e_{i,j} - x_i$ if he pays x_i while another player j (seller or buyer) wins; and his payoff is $-x_i$ if he pays x_i while the item is destroyed.

The timing of the game is as follows.

Time 0: The externalities $e_{i,j}$, the seller's value v_0, the destroying cost c_0 and the distributions of $v_i,\ i \in \mathcal{N}$ are revealed by Nature as public information. Every buyer $i,\ i \in \mathcal{N}$ observes his private value v_i.

Time 1: The seller announces the rule of the selling mechanism. The possibility of destroying the item by the seller is allowed. We assume that the seller has the power of committing to the proposed rule.

Time 2: The buyers simultaneously and confidentially make their participation decisions and announce their bids if they decide to participate.

Time 3: The payoffs of the seller and buyers are determined according to the announced rule at time 1.

In this paper, we derive the revenue-maximizing mechanism that renders the highest seller's expected revenue among all threshold-participation mechanisms. Here, threshold-participation refers to an entry pattern where the buyers only participate if their valuations are equal to or higher than their corresponding thresholds. First, we show that there is no loss of generality to consider only the mechanisms that induce full participation of every type of buyers. Here, **full participation** refers to the entry pattern, where every type of buyers participates. Second, we derive the revenue-maximizing mechanism among the full-participation class.

2.2 The Optimality of Full Participation

Based on the "semirevelation" principle established by Stegeman (1996) that allows no participation, we only need to consider truthful direct semirevelation mechanisms, which require buyers to submit signals if and only if they participate, and reveal truthfully their types if they participate. Following Stegeman (1996), we introduce a null message $\emptyset$ to denote the signal of a nonparticipant.[7] Let $\mathbf{m} = (m_1, m_2, ..., m_N)$, where m_i is the signal of buyer i and it takes values in $\mathcal{M}_i = [\underline{v}_i, \overline{v}_i] \cup \{\emptyset\}, \ \forall i \in \mathcal{N}$. Define $\mathcal{M} = \prod_{i=1}^{N} \mathcal{M}_i$. The seller determines how to allocate the object and how much each buyer pays, using a set of outcome functions that accommodates all participation possibilities. These outcome functions announced by the seller consist of the probability $p_0(\mathbf{m})$ for the seller to keep the item, the winning probability functions $p_i(\mathbf{m})$ and payment functions $x_i(\mathbf{m})$ of buyer $i, \forall i \in \mathcal{N}$. Note that $1 - \sum_{i=0}^{N} p_i(\mathbf{m})$ is the probability of destroying the item by the seller. This set of allocation functions is denoted by $(\mathbf{p}, \mathbf{x})$. Following Jehiel, Moldovanu and Stacchetti (1996), we assume that the buyers who do not participate have no chance to win the object and their payments to the seller are zero, i.e., $p_i(\mathbf{m}) = 0$ and $x_i(\mathbf{m}) = 0$

[7] Unlike the revelation principle whose applicability requires full participation of buyers, the "semi-revelation" principle accommodates all entry patterns including the full participation. Condition (10) in Lemma 2 will further show that in our setting with allocative externalities, the mechanism should accommodate the null signal $\emptyset$ even though full participation should be induced at the optimum.

if $m_i = \emptyset,\ \forall i \in \mathcal{N}$.[8] In addition, clearly the feasibility of mechanism $(\mathbf{p}, \mathbf{x})$ requires $\sum_{i=0}^{N} p_i(\mathbf{m}) \leq 1,\ \forall \mathbf{m} \in \mathcal{M}$.

Define $\mathbf{v^e} = (v_1^e, ..., v_N^e)$, where $v_i^e \in [\underline{v}_i, \overline{v}_i]$ is the entry threshold for buyer i, $i \in \mathcal{N}$. We say $(\mathbf{p}, \mathbf{x})$ is a truthful direct semirevelation mechanism implementing threshold participation $\mathbf{v^e}$ if and only if the following conditions hold:

(a) The buyers with private values lower than their participation thresholds do not participate, i.e., if they participate, they get expected utility which is equal to or lower than their expected utility from nonparticipation. Thus these types of buyers submit the null signal;

(b) The buyers with private values equal to or higher than their participation thresholds participate and reveal truthfully their valuations.

(c) $p_i(\mathbf{m}) \geq 0,\ \forall 0 \leq i \leq N$, with $\sum_{i=0}^{N} p_i(\mathbf{m}) \leq 1,\ \forall \mathbf{m} \in \mathcal{M}$.

(d) $p_i(\mathbf{m}) = 0$ and $x_i(\mathbf{m}) = 0$ if $m_i = \emptyset,\ \forall i \in \mathcal{N},\ \forall \mathbf{m} \in \mathcal{M}$.

When $v_i^e = \underline{v}_i, \forall i \in \mathcal{N}$, we have the case of full participation. The following Lemma shows that we can focus on full-participation mechanisms for revenue maximization.

Lemma 1: *There is no loss of generality to consider only the truthful direct semirevelation mechanisms that induce full participation for the revenue-maximizing mechanism.*

Proof: It is sufficient to show that any allocation outcome implemented by a threshold-participation truthful direct semirevelation mechanism is replicable through a full-participation truthful direct semirevelation mechanism.

Consider any given entry thresholds vector $\mathbf{v^e} = (v_1^e, ..., v_N^e)$ where $v_i^e \in [\underline{v}_i, \overline{v}_i],\ i \in \mathcal{N}$. Without loss of generality, we assume $v_i^e > \underline{v}_i, \forall i \in \mathcal{N}$. Consider a truthful direct semirevelation mechanism $(\mathbf{p}, \mathbf{x})$ that implements $\mathbf{v^e}$. Then conditions (a) to (d) must hold for $(\mathbf{p}, \mathbf{x})$.

[8] This assumption is consistent with the **no passive reassignment** (NPR) assumption adopted by Stegeman (1996).

Define $\eta_i(m_i) = m_i$ if $m_i \geq v_i^e$, and $\eta_i(m_i) = \emptyset$ otherwise. We now construct a new direct semirevelation mechanism $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ as follows.

$$\tilde{\mathbf{p}}(\mathbf{m}) = \mathbf{p}(\eta(\mathbf{m})),\ \forall \mathbf{m} \in \mathcal{M},$$
$$\tilde{\mathbf{x}}(\mathbf{m}) = \mathbf{x}(\eta(\mathbf{m})),\ \forall \mathbf{m} \in \mathcal{M},$$

where $\eta(\mathbf{m}) = (\eta_1(m_1), \eta_2(m_2), ..., \eta_N(m_N)),\ \forall \mathbf{m} \in \mathcal{M}$.

Suppose that all the buyers other than i always participate regardless of their valuations and they reveal truthfully their types when participating. From the construction of $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$, it is clearly the best strategy for buyer i to always participate and reveal truthfully his type regardless of his type, if $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ is adopted. Therefore, $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ is a truthful direct semirevelation mechanism that induces full participation. Furthermore, $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ implements the same allocation (the players' winning probabilities and the buyers' payments for every realization of the buyers' valuations) as $(\mathbf{p}, \mathbf{x})$. Thus $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ must render the same expected revenue for the seller as $(\mathbf{p}, \mathbf{x})$. $\square$

2.3 The Revenue-Maximizing Mechanism

For any truthful direct semirevelation mechanism $(\mathbf{p}, \mathbf{x})$ implementing full participation, the seller's expected revenue is given by:

$$\begin{aligned}
&R(\mathbf{p}, \mathbf{x}) \\
&= E_{\mathbf{v}}\Big\{(v_0 + e_{0,0})p_0(\mathbf{v}) + \sum_{i=1}^{N} e_{0,i}\ p_i(\mathbf{v}) - c_0\Big(1 - \sum_{i=0}^{N} p_i(\mathbf{v})\Big) + \sum_{i=1}^{N} x_i(\mathbf{v})\Big\} \\
&= E_{\mathbf{v}}\Big\{(v_0 + c_0 + e_{0,0})p_0(\mathbf{v}) + \sum_{i=1}^{N}(e_{0,i} + c_0)\ p_i(\mathbf{v}) + \sum_{i=1}^{N} x_i(\mathbf{v})\Big\} - c_0, \qquad (1)
\end{aligned}$$

where $\mathbf{v} = (v_1, v_2, ..., v_N)$. The support of $\mathbf{v}$ is $\mathcal{V} = \prod_{i=1}^{N}[\underline{v}_i,\ \overline{v}_i]$.

For buyer i with private value v_i, if he submits signal $m_i \in \mathcal{M}_i$, his interim expected payoff is given by:

$$U_i(v_i, m_i; \mathbf{p}, \mathbf{x}) = E_{\mathbf{v}_{-i}}\Bigg(v_i\ p_i(m_i, \mathbf{v_{-i}}) + \sum_{j\geq 0} e_{i,j}\ p_j(m_i, \mathbf{v_{-i}}) - x_i(m_i, \mathbf{v_{-i}})\Bigg), \qquad (2)$$

where $\mathbf{v_{-i}} = (v_1, ..., v_{i-1}, v_{i+1}, ..., v_N)$. The support of $\mathbf{v}_{-i}$ is $\mathcal{V}_{-i} = \prod_{j=1, j\neq i}^{N}[\underline{v}_j,\ \overline{v}_j]$.

The seller's optimization problem is to find the revenue-maximizing truthful direct semirevelation mechanism $(\mathbf{p}^*, \mathbf{x}^*)$ that implements full participation, i.e.,

$$\max_{(\mathbf{p},\mathbf{x})}\ R(\mathbf{p}, \mathbf{x}) \qquad (3)$$

Subject to:

$$(i)\ U_i(v_i, v_i; \mathbf{p}, \mathbf{x}) \geq U_i(v_i, \emptyset; \mathbf{p}, \mathbf{x});\ \forall i \in \mathcal{N},\ \forall v_i \in [\underline{v}_i, \overline{v}_i], \qquad (4)$$

$$(ii)\ U_i(v_i, v_i; \mathbf{p}, \mathbf{x}) \geq U_i(v_i, v_i'; \mathbf{p}, \mathbf{x});\ \forall i \in \mathcal{N},\ \forall v_i \in [\underline{v}_i,\ \overline{v}_i],\ v_i' \in [\underline{v}_i,\ \overline{v}_i], \qquad (5)$$

$$(iii)\ p_i(\mathbf{m}) = x_i(\mathbf{m}) = 0\ if\ m_i = \emptyset,\ p_i(\mathbf{m}) \geq 0,\ \forall i \in \mathcal{N},\ \sum_{i=0}^{N} p_i(\mathbf{m}) \leq 1,\ \forall \mathbf{m} \in \mathcal{M}. \qquad (6)$$

Restrictions (4)-(6) come from conditions (a)-(d).

For any direct semirevelation mechanism $(\mathbf{p}, \mathbf{x})$ implementing full participation, we define

$$Q_i(v_i; \mathbf{p}) = E_{\mathbf{v}_{-i}} p_i(\mathbf{v}). \qquad (7)$$

If $(\mathbf{p}, \mathbf{x})$ is a truthful direct semirevelation mechanism implementing full participation, then $Q_i(v_i; \mathbf{p})$ is the conditional expected probability that buyer i wins the object if his private value is v_i.

As in Myerson (1981), we have the following necessary and sufficient conditions for a direct semirevelation mechanism to be a **truthful** direct semirevelation mechanism that implements full participation.[9]

[9]The proof is omitted as it follows closely Myerson (1981). For the same reason, the proof for Lemma 3 is also omitted.

Lemma 2: *Direct semirevelation mechanism* $(\mathbf{p}, \mathbf{x})$ *is a* **truthful** *direct semirevelation mechanism that implements full participation, if and only if* $\forall i \in \mathcal{N}$ *the following conditions and (6) hold:*

$$Q_i(s_i; \mathbf{p}) \leq Q_i(v_i; \mathbf{p}),\ \ \forall \underline{v}_i \leq s_i \leq v_i \leq \overline{v}_i, \tag{8}$$

$$U_i(v_i, v_i; \mathbf{p}, \mathbf{x}) = U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}) + \int_{\underline{v}_i}^{v_i} Q_i(s_i; \mathbf{p}) ds_i,\ \ \forall v_i \in [\underline{v}_i, \overline{v}_i], \tag{9}$$

$$U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}) \geq U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x}). \tag{10}$$

Note that (10) differs from its counterpart in Lemma 2 of Myerson (1981). In Myerson (1981), the outside utility level $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x})$ that buyer i obtains if he does not participate is not mechanism dependent. In particular, $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x})$ is fixed at zero in Myerson (1981). However, in our setting with allocative externalities, $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x})$ must be determined by the mechanism adopted and thus can differ from zero.

Based on Lemma 2, we can replace (4) and (5) by (8), (9) and (10) in the seller's optimization problem. As a result, the expected revenue of the seller from a mechanism $(\mathbf{p}, \mathbf{x})$ satisfying conditions (4)-(6) is given in the following Lemma.

Lemma 3: *For a truthful direct semirevelation mechanism* $(\mathbf{p}, \mathbf{x})$ *that implements full participation, the seller's expected revenue can be written as*

$$R(\mathbf{p}, \mathbf{x}) = E_{\mathbf{v}}\Big\{\sum_{i=0}^{N} p_i(\mathbf{v})\tilde{J}_i(v_i)\Big\} - c_0 - \sum_{i=1}^{N} U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}), \tag{11}$$

where $\tilde{J}_i(v_i) = J_i(v_i) + c_0 + \sum_{j\geq 0} e_{j,i},\ \ i = 0, 1, ..., N$ *are defined as the augmented virtual value functions.*[10]

Based on Lemma 3, we are then able to characterize the revenue-maximizing mechanism as in the following proposition.

[10]Note that $\tilde{J}_0(v_0) = v_0 + c_0 + \sum_{j\geq 0} e_{j,0}$. We use $\tilde{J}_i^{-1}(\cdot)$ to denote the inverse function of $\tilde{J}_i(\cdot)$. If $x < \underline{v}_i - \frac{1}{f_i(\underline{v}_i)} + c_0 + \sum_{j\geq 0} e_{j,i}$, $\tilde{J}_i^{-1}(x)$ is defined as $\underline{v}_i$; if $x > \overline{v}_i + c_0 + \sum_{j\geq 0} e_{j,i}$, $\tilde{J}_i^{-1}(x)$ is defined as $\overline{v}_i$.

Proposition 1: *The following mechanism is a truthful direct semirevelation mechanism that maximizes the seller's expected revenue. The mechanism induces full participation.*
(i) If only buyer i submits signal $\emptyset$, the item is assigned to the one (including the seller) who brings him the smallest externality provided that it is nonpositive, otherwise the seller destroys the item.
(ii) Every participating buyer pays a nonnegative entry fee, which equals the absolute value of the smallest possible externality to him.[11]
(iii) If all buyers participate and buyer i, $\forall i \in \mathcal{N}$ submits signal $m_i \in [\underline{v}_i, \overline{v}_i]$, the object is assigned to the player (including the seller) whose signal renders the highest "augmented virtual value", provided this value is nonnegative.[12] *Ties are broken randomly. If this value is negative, the object is destroyed by the seller.*
(iv) The winning buyer i pays $\tilde{J}_i^{-1}(\max\{0, \max_{j=0, j\neq i}^{N} \tilde{J}_j(v_j)\})$ besides the entry fee. On top of the entry fee, each losing buyer pays an externality-correcting payment (positive or negative) that equals the allocative externality to him.

Proof: From (11), a truthful direct semirevelation mechanism that induces full participation must be revenue-maximizing if it satisfies the following two conditions. First, it minimizes $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x})$, $\forall i \in \mathcal{N}$. Second, it also maximizes $\sum_{i=0}^{N} p_i(\mathbf{v})\tilde{J}_i(v_i)$, $\forall \mathbf{v} \in \mathcal{V}$. We next put forward a direct semirevelation mechanism $(\mathbf{p}^*, \mathbf{x}^*)$ that satisfies the above criterion and thus maximizes the seller's expected revenue. We then verify that $(\mathbf{p}^*, \mathbf{x}^*)$ is truthful.

First, $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x})$ can be pushed to take the lowest possible value $\min_{j\geq 0} e_{i,j}$. This can be achieved by the following specification. If $e_{i,j_0} \leq 0$ where $j_0 = argmin_{j\geq 0, j\neq i} e_{i,j}$, then set $p_{j_0}^*(m_i, \mathbf{v}_{-i}) = 1$ where $m_i = \emptyset$. If $e_{i,j_0} > 0$, then set $p_j^*(m_i, \mathbf{v}_{-i}) = 0$ for $j \geq 0$. The above specifications mean that when $\min_{j\geq 0, j\neq i} e_{i,j} \leq 0$, the item is assigned to the

[11] This smallest externality must be nonpositive due to the option of destroying the object by the seller.
[12] We treat the seller's signal as v_0.

player who brings buyer i the smallest externality, if buyer i does not participate; when $\min_{j\geq 0, j\neq i} e_{i,j} > 0$, the item is destroyed by the seller. When the item is destroyed, externalities cease to exist. As $e_{i,i} = 0,\ \forall i \in \mathcal{N}$, we have that $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x}) = \min_{j\geq 0} e_{i,j}$ always holds for the above specifications of the optimal nonparticipation threats. Clearly, $\min_{j\geq 0} e_{i,j}\ (\leq 0)$ is the lowest utility possible for buyer i if he does not participate.

Second, $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x})$ can be driven down to exactly equal $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x})$, which in turn equals $\min_{j\geq 0} e_{i,j}$. Note that $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x})$ cannot be lower than $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x})$ from (10). The full participation payment $x_i^*(\mathbf{v})$ pushing $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x})$ to $\min_{j\geq 0} e_{i,j}$ is defined as below for any given set of full participation winning probabilities $p_i(\mathbf{v}),\ \forall 0 \leq i \leq N$. From Lemma 2, we have $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}) = U_i(v_i, v_i; \mathbf{p}, \mathbf{x}) - \int_{\underline{v}_i}^{v_i} Q_i(s_i; \mathbf{p}) ds_i$. Thus from (2) and (7), $x_i^*(\mathbf{v})$ must satisfy

$$\min_{j\geq 0} e_{i,j} = E_{\mathbf{v}_{-i}}\left(v_i\ p_i(\mathbf{v}) + \sum_{j\geq 0} e_{i,j}\ p_j(\mathbf{v}) - x_i^*(\mathbf{v}) - \int_{\underline{v}_i}^{v_i} p_i(s_i, \mathbf{v}_{-i}) ds_i \right), \forall i \in \mathcal{N}. \qquad (12)$$

Naturally, we define

$$x_i^*(\mathbf{v}) = v_i\ p_i(\mathbf{v}) + \sum_{j\geq 0} e_{i,j}\ p_j(\mathbf{v}) - \min_{j\geq 0} e_{i,j} - \int_{\underline{v}_i}^{v_i} p_i(s_i, \mathbf{v}_{-i}) ds_i,\ \forall i \in \mathcal{N}. \qquad (13)$$

Clearly, the above defined nonparticipation threats and full-participation payments $\mathbf{x}^*(\mathbf{v}) = (x_i^*(\mathbf{v}))$ minimize $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}), \forall i \in \mathcal{N}$, for any given set of full participation winning probabilities $p_i(\mathbf{v}),\ \forall 0 \leq i \leq N$.

Third, we define the set of full-participation winning probabilities $\mathbf{p}^*(\mathbf{v}) = (p_i^*(\mathbf{v}))$ that maximizes $\sum_{i=0}^{N} p_i(\mathbf{v}) \tilde{J}_i(v_i), \forall \mathbf{v} \in \mathcal{V}$. Clearly, the winning probability of player $i,\ i = 0, 1, ..., N$ should be defined as follows.

$$p_i^*(\mathbf{v}) = \begin{cases} 1, & if\ \tilde{J}_i(v_i) > \max_{j=0, j\neq i}^{N} \tilde{J}_j(v_j)\ and\ \tilde{J}_i(v_i) \geq 0, \\ 0, & otherwise. \end{cases} \qquad (14)$$

As usual, ties are broken randomly.

The corresponding set of full-participation payments $\mathbf{x}^*(\mathbf{v})$ is then defined as follows according to (13). For buyer $i,\ i \in \mathcal{N}$,

$$x_i^*(\mathbf{v}) = \begin{cases} \tilde{J}_i^{-1}(\max\{0, \max_{j=0,j\neq i}^N \tilde{J}_j(v_j)\}) + E_i, & if\ i\ wins, \\ e_{i,j} + E_i, & if\ \ j(\geq 0)\ wins, \\ E_i, & if\ the\ object\ is\ destroyed, \end{cases} \quad (15)$$

where $E_i = -\min_{j\geq 0} e_{i,j}$.

The full-participation winning probabilities and payments functions $\mathbf{x}^*(\mathbf{v})$ and $\mathbf{p}^*(\mathbf{v})$ together with the optimal nonparticipation threats lead to a Nash equilibrium in which every type of buyers participates and reveals truthfully their types, because the conditions in lemma 2 are satisfied. We thus have that the full-participation winning probabilities and payments functions $\mathbf{x}^*(\mathbf{v})$ and $\mathbf{p}^*(\mathbf{v})$ together with the optimal nonparticipation threats constitute a **truthful** direct semirevelation mechanism that maximizes the seller's expected revenue. In the same spirit of Jehiel, Moldovanu and Stacchetti (1996), there is no need to consider the joint deviation from the Nash equilibrium.[13] Thus all the other winning probabilities and payments functions which are not relevant to the equilibrium path can be specified in any way. □

Proposition 1 answers the questions of when the auctioned object is destroyed by the seller and how the seller should proceed to maximize his expected revenue if the auctioned item is to be destroyed. From Proposition 1, we have the following results regarding the possibility of destroying the object.

Corollary 1: *If* $\tilde{J}_0(v_0) = v_0 + c_0 + \sum_{j\geq 0} e_{j,0} \geq 0$, *the object is never destroyed by the seller. If instead* $\tilde{J}_0(v_0) = v_0 + c_0 + \sum_{j\geq 0} e_{j,0} < 0$, *the object is destroyed by the seller with probability* $\prod_{i=1}^N Pr(p_i^*(\mathbf{v}) = 0)$, *which equals* $\prod_{i=1}^N F_i(J_i^{-1}(-c_0 - \sum_{j\geq 0} e_{j,i}))$.[14]

[13]Footnote 11 in Jehiel, Moldovanu and Stacchetti (1996) points out that joint deviations of buyers are irrelevant since full-participation Nash equilibrium is studied.

[14]We use $J_i^{-1}(\cdot)$ to denote the inverse function of $J_i(\cdot)$. If $x < \underline{v}_i - \frac{1}{f_i(\underline{v}_i)}$, $J_i^{-1}(x)$ is defined as $\underline{v}_i$; if

We use $(\mathbf{p^0}, \mathbf{x^0})$ to denote the revenue-maximizing mechanism when the externalities are $e_{i,j},\ i,j \in \{0,1,...,N\}$. When $e_{i,0}, \forall i \in \mathcal{N}$ are negative enough, we have $\min_{j\geq 0} e_{i,j} = e_{i,0}, \forall i \in \mathcal{N}$. Thus $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p^0}, \mathbf{x^0}) = e_{i,0}, \forall i \in \mathcal{N}$. From (11), the optimal expected revenue for the seller is

$$R(\mathbf{p^0}, \mathbf{x^0}) = -c_0 - \sum_{j\geq 0} e_{j,0} + \int_{\mathcal{V}} \Big\{ p_0^0(\mathbf{v})(v_0 + c_0 + \sum_{j\geq 0} e_{j,0}) + \sum_{i=1}^{N} p_i^0(\mathbf{v})(v_i + c_0 + \sum_{j\geq 0} e_{j,i} - \frac{1 - F_i(v_i)}{f_i(v_i)}) \Big\} \mathbf{f}(\mathbf{v}) d\mathbf{v}.$$

Let $R'(\mathbf{p^0}, \mathbf{x^0})$ denote the value of the right-hand-side of $R(\mathbf{p^0}, \mathbf{x^0})$ when $e_{i,0}$ decreases to $e'_{i,0}\ i \in \mathcal{N}$. Clearly $R'(\mathbf{p^0}, \mathbf{x^0}) \geq R(\mathbf{p^0}, \mathbf{x^0})$ as $p_0^0(\mathbf{v}) \in [0,1]$. Suppose when $e_{i,0}$ decreases to $e'_{i,0}\ i \in \mathcal{N}$, the corresponding revenue-maximizing auction rule changes to $(\mathbf{p'}, \mathbf{x'})$. Denote the optimal expected revenue by $R(\mathbf{p'}, \mathbf{x'})$ when externalities $e_{i,0}$ decrease to $e'_{i,0},\ i \in \mathcal{N}$. We must have $R(\mathbf{p'}, \mathbf{x'}) \geq R'(\mathbf{p^0}, \mathbf{x^0})$. Therefore, $R(\mathbf{p'}, \mathbf{x'}) \geq R(\mathbf{p^0}, \mathbf{x^0})$, i.e., the seller's optimal expected revenue increase as $e_{i,0}$ decreases. This helps to explain why North Korea tries to convince the relevant countries that it owns very powerful nuclear weapon.

2.4 The Revenue-Maximizing Auction in Symmetric Setting

We now show that the revenue-maximizing mechanism derived above reduces to a modified second price auction in a symmetric setting. In this symmetric setting, $F_i(\cdot) = F(\cdot),\ f_i(\cdot) = f(\cdot)$ on support $[\underline{v}, \overline{v}]$, $\forall i \in \mathcal{N}$. In addition, $e_{i,0} = e_{j,0} = e_{a,0},\ e_{0,i} = e_{0,j} = e_{0,a},\ e_{i,j} = e, \forall i,j \in \mathcal{N}$. As usual, we assume the regularity condition that the virtual valuation $J(v) = v - \frac{1-F(v)}{f(v)}$ is an increasing function. The augmented virtual value function of a representative buyer is defined as $\tilde{J}(\cdot) = J(\cdot) + c_0 + \sum_{j\geq 0} e_{j,i}$. The inverse function of $\tilde{J}(\cdot)$ is denoted by $\tilde{J}^{-1}(\cdot)$.

$x > \overline{v}_i$, $J_i^{-1}(x)$ is defined as $\overline{v}_i$.

Based on (14) and (15), the winning probability of buyer i, $\forall i \in \mathcal{N}$ is defined as

$$p_i^*(\mathbf{v}) = \begin{cases} 1 & \text{if} \quad v_i \geq z_i(\mathbf{v}_{-i}), \\ 0 & \text{if} \quad v_i < z_i(\mathbf{v}_{-i}), \end{cases} \tag{16}$$

and his payment is defined as

$$x_i^*(\mathbf{v}) = \begin{cases} z_i(\mathbf{v}_{-i}) - \min_{j\geq 0} e_{i,j}, & \textit{if } i \textit{ wins}, \\ e_{i,j} - \min_{j\geq 0} e_{i,j}, & \textit{if } \ j(\geq 0) \textit{ wins}, \\ -\min_{j\geq 0} e_{i,j}, & \textit{if the object is destroyed}, \end{cases} \tag{17}$$

where $z_i(\mathbf{v}_{-i}) = \max\{\max_{j\neq i, j\in\mathcal{N}} v_j, \tilde{J}^{-1}(\max\{0, v_0 + c_0 + \sum_{j\geq 0} e_{j,0}\})\}$. In addition,

$$p_0^*(\mathbf{v}) = \begin{cases} 1, & \textit{if } \tilde{J}(\max_{j=1}^N v_j) < v_0 + c_0 + \sum_{j\geq 0} e_{j,0} \geq 0, \\ 0, & \textit{otherwise}. \end{cases} \tag{18}$$

Eq. (17) means that every buyer i pays an entry fee of $-\min_{j\geq 0} e_{i,j}$. Moreover, if buyer i wins, he pays an additional $z_i(\mathbf{v}_{-i})$. If he loses, he pays an externality-correcting payment that equals the externality he enjoys or suffers. From (18), if $v_0 + c_0 + \sum_{j\geq 0} e_{j,0} \geq 0$, the seller keeps it when the object is not sold out, while if $v_0 + c_0 + \sum_{j\geq 0} e_{j,0} < 0$, it is optimal for the seller to destroy the unsold object. In case that the sum of the seller's valuation, the destroying cost of the seller and the total externalities to the buyers is negative when the seller keeps the item, intuitively the seller is better off by destroying the object and collecting a payment from each buyer that equals the externality to him.

Based on the above results, we have the following proposition that describes the revenue-maximizing auction.

Proposition 2: *In a symmetric setting with identity-specific externalities among the seller and buyers, a modified second-price sealed-bid auction with the following features is the revenue-maximizing auction.*

(a) The nonparticipation threats take the same form as in Proposition 1(i). Every participant pays an entry fee defined as in Proposition 1(ii).

(b) If all buyers participate, the highest buyer wins if his bid is higher than the reserve price, and he pays the second highest bid or the reserve price, whichever is higher. The reserve price is set at $\tilde{J}^{-1}(\max\{0, v_0 + c_0 + \sum_{j\geq 0} e_{j,0}\})$*. If no buyer bids higher than the reserve price, then the seller destroys the item if* $v_0 + c_0 + \sum_{j\geq 0} e_{j,0} < 0$*, otherwise he keeps the item.*
(c) Each losing buyer pays an externality-correcting payment equal to the allocative externality to him.

Each buyer's payment is adjusted by the amount of allocative externality to him, while he suffers or enjoys this externality at the same time. This creates a situation where buyers bid as if there is no externality on them. Based on similar arguments for the standard second-price auction, bidding his true valuation is a weakly dominant strategy for every buyer in the second price auction specified in Proposition 2. This is why a modified second-price auction with the externality-correcting payments is revenue-maximizing, provided that the reserve price and entry fee are properly set. In Proposition 2, the entry fee is set at the highest possible level which can be supported by the optimal nonparticipation threats, and the optimal reserve price is determined by the seller's augmented value $v_0 + c_0 + \sum_{j\geq 0} e_{j,0}$.

2.5 When buyers Have Private Information on the Externalities They Create on Others

An interesting issue is the extent to which the results obtained in a public-information externalities setting apply to a multi-dimensional setting where buyers have private information on the externalities they create on others. Specifically, we consider the setting where $e_{j,i}, \forall 0 \leq j \leq N$, are buyer i's private information. We assume that all v_k and $e_{j,i}$, $\forall 0 \leq k, i, j \leq N$ are mutually independent and their distributions are public information.

Jehiel, Moldovanu and Stacchetti (1996) look at a 2-dimensional setting where winning buyer impose the same externality on other buyers. They show that the winning probability of any buyer must not depend on his externality signal because of the rationality condition. Therefore the private information on externality is a redundant signal, and thus the auction design problem is a one-dimensional program in nature. Although Jehiel, Moldovanu and Stacchetti (1996) show these results while assuming losing buyers suffer the same externality, it is clear that all these results still hold when players experience heterogenous externalities, as long as players' private information is the externalities they create on other players. Specifically, the revenue-maximizing auction problem can be transformed into a one-dimensional program by first integrating over the externalities dimensions. The one-dimensional program obtained is fundamentally the same as the problem assuming public-information externalities. The only difference is that the public-information externalities in the baseline setting are replaced by the expected externalities. Note that these expected externalities are also public information as the distributions of the externalities are public information.

Based on the above arguments, clearly the revenue-maximizing mechanism for the multi-dimensional setting considered can be obtained through replacing the public-information externalities by the expectations of the private-information externalities in our results for the public-information externalities setting.

3 A Unified Approach for Auctions Design with Financial Externalities

In Section 2, we studied the revenue-maximizing auction when the externalities among the players are exogenous. As a special feature, the revenue-maximizing auction requires that every buyer's payment consists of an externality-correcting component which equals the allocative externality to him. In this section, we conduct further studies on revenue-maximizing auctions for general settings with financial externalities where the externalities among buyers are endogenously determined by their payments and there is no externalities between seller and buyers. Useful linkages between revenue-maximizing auctions for settings with and without externalities are established based on the insights from Section 2. Specifically, we will establish one-to-one correspondences between revenue-maximizing auctions with and without externalities. Therefore, revenue-maximizing auctions for various settings of financial externalities can be obtained through transforming the revenue-maximizing second-price auction for a regular setting without externalities. In other words, our findings provide a general approach for deriving the revenue-maximizing auctions with financial externalities.

Goeree, Maasland, Onderstal and Turner (2005) show that a lowest-price all-pay auction is revenue-maximizing in a symmetric setting of financial externalities. Our general result leads to alternative revenue-maximizing second-price auction for their setting, which implements the maximal expected revenues through weakly dominant strategy. Being consistent with the insights from our baseline setting, in this alternative auction all buyers' payments consist of an externality-correcting component which equals the allocative externalities to them.

3.1 When Externalities Are Proportional to the Total Payments of All buyers

We first consider a setting where each buyer enjoys a positive externality that equals a proportion of the total payments of **all** buyers. Specifically, we assume the buyers' private values $v_i,\ i \in \mathcal{N}$ are independently distributed on interval $[\underline{v}_i, \overline{v}_i]$ following respectively cumulative distribution function $F_i(\cdot)$ with density function $f_i(\cdot)(> 0)$. We assume the regularity condition that the virtual valuation functions $J_i(v) = v - (1 - F_i(v))/f_i(v)$

increase on interval $[\underline{v}_i, \overline{v}_i]$. The seller's valuation is zero. Buyer i, $i \in \mathcal{N}$ enjoys a positive externality which equals a proportion α_i of the total payments of all buyers. Denote the payment of buyer i by x_i, $i \in \mathcal{N}$. Buyer i's payoff is then $v_i - x_i + \alpha \sum_{j=1}^{N} x_j$ if he wins; his payoff is $(\alpha \sum_{j=1}^{N} x_j) - x_i$ if he does not win. We assume $\sum_{i=1}^{N} \alpha_i \in [0, 1)$. This setting is a generalization from Goeree, Maasland, Onderstal and Turner (2005).[15] If $\alpha_i = 0$, $\forall i \in \mathcal{N}$, then we have a usual setting without externalities.

Similar to Lemma 1 for the case of identity-specific externalities, for the above setting of financial externalities there is no loss of generality to consider only the truthful direct semirevelation mechanisms that induce full participation for the revenue-maximizing mechanism.[16] We thus focus on the truthful direct semirevelation mechanisms that induce full participation in this section.[17]

Suppose $(\mathbf{p}, \mathbf{x})$ is a truthful direct semirevelation mechanism that induces full participation in the usual setting without externalities. We define $\tilde{\mathbf{p}} = \mathbf{p}$, and define a new full-participation payment $\tilde{x}_i(\mathbf{v})$ of buyer i by $x_i(\mathbf{v}) + \frac{\alpha_i}{1-\sum_{j=1}^{N} \alpha_j} \sum_{j=1}^{N} x_j(\mathbf{v})$, $\forall i \in \mathcal{N}$. Here, $x_i(\cdot)$, $\forall i \in \mathcal{N}$ is buyer i's payment in $(\mathbf{p}, \mathbf{x})$. In addition, buyer i's payment $\tilde{x}_i(\cdot)$ is set to be zero if he does not participate, and his payment $\tilde{x}_i(\cdot)$ for all other cases can be defined in any way. The above defined $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ must be a truthful direct semirevelation mechanism that induces full participation for our setting with financial externalities. This is true as $-\tilde{x}_i(\mathbf{v}) + \alpha_i \sum_{j=1}^{N} \tilde{x}_j(\mathbf{v}) = -x_i(\mathbf{v})$, $\forall i \in \mathcal{N}$. For the same reason, if $(\tilde{\mathbf{p}}', \tilde{\mathbf{x}}')$ is a truthful direct revelation mechanism that induces full participation for our setting with financial externalities, then $(\mathbf{p}', \mathbf{x}')$ must be a truthful direct revelation mechanism that induces full participation for the setting without externalities, where $\mathbf{p}' = \tilde{\mathbf{p}}'$ and

[15]Goeree, Maasland, Onderstal and Turner (2005) instead assumes symmetry across buyers.

[16]The proof is omitted as it is similar to the proof of Lemma 1.

[17]Note that for a regular setting without externalities, a truthful direct semirevelation mechanism inducing full participation is well defined, although the utility of a nonparticipant from a truthful direct semirevelation mechanism is no longer mechanism-dependent.

$x_i'(\mathbf{v}) = \tilde{x}_i'(\mathbf{v}) - \alpha_i \sum_{j=1}^N \tilde{x}_j'(\mathbf{v}),\ \forall i \in \mathcal{N}$.

Note that $E_{\mathbf{v}}(\sum_{j=1}^N x_j(\mathbf{v}))$ is the expected revenue from $(\mathbf{p}, \mathbf{x})$ in the setting without externalities, and $E_{\mathbf{v}}(\sum_{j=1}^N \tilde{x}_j(\mathbf{v}))$ is the expected revenue from $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ in the setting with financial externalities.[18] Since $\sum_{j=1}^N x_j(\mathbf{v}) = (1 - \sum_{i=1}^N \alpha_i) \sum_{j=1}^N \tilde{x}_j(\mathbf{v})$, we have $E_{\mathbf{v}}(\sum_{j=1}^N x_j(\mathbf{v})) = (1 - \sum_{i=1}^N \alpha_i) E_{\mathbf{v}}(\sum_{j=1}^N \tilde{x}_j(\mathbf{v}))$. Clearly, the same relation holds between mechanisms $(\tilde{\mathbf{p}}', \tilde{\mathbf{x}}')$ and $(\mathbf{p}', \mathbf{x}')$. Based on these observations, we must have the following results.

Proposition 3: (i) *If* $(\mathbf{p}, \mathbf{x})$ *is a revenue-maximizing truthful direct semirevelation mechanism that induces full participation in the setting without externalities, then the corresponding* $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ *defined above must be a revenue-maximizing truthful direct semirevelation mechanism that induces full participation in the setting with financial externalities.* (ii) *If* $(\tilde{\mathbf{p}}', \tilde{\mathbf{x}}')$ *is a revenue-maximizing truthful direct semirevelation mechanism that induces full participation in the setting with financial externalities, then* $(\mathbf{p}', \mathbf{x}')$ *defined above must be a revenue-maximizing truthful direct semirevelation mechanism that induces full participation in the setting without externalities.* (iii) *The seller's expected revenue from any revenue-maximizing auction with financial externalities equals* $\frac{1}{1-\sum_{i=1}^N \alpha_i}$ *times of that from any revenue-maximizing auction without externalities.*

Proposition 3(i) means that if we know the revenue-maximizing auction without externalities, then the revenue-maximizing auction with externalities can easily be obtained. Proposition 3(iii) means that the seller's optimal expected revenue does not depend on the distribution of the externalities across the buyers. For our general setting with financial externalities, we define the following mechanism.

(a.1) There is no entry fee, the reserve price for buyer i, $\forall i \in \mathcal{N}$ is $\hat{v}_i$ $(\geq \underline{v}_i)$, which is the unique solution of $J_i(\hat{v}_i) = 0$;

(a.2) If at least one buyer does not participate, the seller keeps the item to create zero

[18] Note that destroying the item is never desired in the case of financial externalities.

externality for all buyers;

(a.3) If all participate, we denote buyer i's bid by b_i, $\forall i \in \mathcal{N}$. Buyer i wins if $J_i(b_i)$ is the highest among all $J_j(b_j)$, $\forall j \in \mathcal{N}$ and $b_i \geq \hat{v}_i$. Ties are broken randomly. Suppose buyer i, $\forall i \in \mathcal{N}$ is the winner. The payments are the following. First, buyer i pays z_1, which is $J_i^{-1}(\max_{j=1,j\neq i}^{N} J_j(b_j))$ or the reserve price $\hat{v}_i$ ($\geq \underline{v}_i$), whichever is higher. Second, every buyer $j \in \mathcal{N}$ pays $z_2 = \frac{\alpha_j z_1}{1-\sum_{i=1}^{N}\alpha_i}$. If no one wins, the seller keeps the item, and no one pays.

Corollary 2: *The mechanism defined by* $(a.1)$, $(a.2)$ *and* $(a.3)$ *is revenue-maximizing for our setting with financial externalities.*

Proof: The result follows immediately from Proposition 3(i). From Myerson (1981), we have that in the setting without externalities, the auction defined in (a.1) to (a.3) with $z_2 = 0$ is a revenue-maximizing truthful direct semirevelation mechanism. It follows from Proposition 3(i) that the mechanism defined in (a.1) to (a.3) is a revenue-maximizing truthful direct semirevelation mechanism for our setting with financial externalities. □

Goeree, Maasland, Onderstal and Turner (2005) study a symmetric independent private value setting where buyers' values follow cumulative distribution function $F(\cdot)$ on $[\underline{v}, \overline{v}]$ and the seller's valuation is zero. They assume that each buyer enjoys a positive externality which equals a common proportion (denoted by $\alpha < \frac{1}{N}$ where N is the number of buyers) of the total payments of all buyers. They show that a two-stage lowest-price all-pay auction with proper entry fee Φ and reserve price R is revenue-maximizing. In the first stage, buyers make the decision whether or not to pay the entry fee and participate. All types of buyers participate, however, there exists a bidding threshold $\hat{v}$ ($\geq \underline{v}$) which is also the threshold of winning type. The bidding threshold $\hat{v}$ is the unique solution of $\hat{v} - \frac{1-F(\hat{v})}{F'(\hat{v})} = 0$, the reserve price R equals $\frac{\hat{v}F(\hat{v})^{N-1}}{1-\alpha}$ and the entry fee Φ equals $\frac{\alpha R(N-1)(1-F(\hat{v}))}{1-N\alpha}$.[19]

[19]Please refer to Proposition 5 in Goeree, Maasland, Onderstal and Turner (2005) for details.

For this symmetric setting, the revenue-maximizing auction defined by $(a.1)$ to $(a.3)$ can be described as follows.

(b.1) There is no entry fee, the reserve price is $\hat{v}$ ($\geq \underline{v}$);

(b.2) If at least one buyer does not participate, the seller keeps the item to create zero externality for all buyers;

(b.3) If all participate, the highest buyer wins if his bid is no less than the reserve price $\hat{v}$, and his payment consists of two components. First, he pays z_1, which is the second highest bid or the reserve price $\hat{v}$ ($> \underline{v}$), whichever is higher. Second, every buyer pays z_2, where $z_2 = \frac{\alpha z_1}{1-\alpha N} > 0$. If the highest bid is less than $\hat{v}$, the seller keeps the item, and no one pays.

From Corollary 2, the auction defined by $(b.1)$, $(b.2)$ and $(b.3)$ is also revenue-maximizing. We thus have the following result.

Corollary 3: *The modified second price auction defined by* $(b.1)$, $(b.2)$ *and* $(b.3)$ *is revenue equivalent to the revenue-maximizing two-stage lowest-price all-pay auction established by Goeree, Maasland, Onderstal and Turner (2005).*

3.2 When Externalities Are Proportional to the Other Buyer's Total Payments

We now turn to a setting of financial externalities where each buyer instead enjoys a positive externality which equals a proportion (denoted by $\varphi < \frac{1}{N-1}$) of the total payments of the **other** buyers. The seller's valuation is zero. The buyers' private values $v_i,\ i \in \mathcal{N}$ are independently distributed on interval $[\underline{v}_i, \overline{v}_i]$ following respectively cumulative distribution function $F_i(\cdot)$ with density function $f_i(\cdot)(> 0)$. The virtual valuation functions $J_i(v) = v - (1 - F_i(v))/f_i(v)$ increase on $[\underline{v}_i, \overline{v}_i]$. Denote the payment of buyer i by $x_i,\ i \in \mathcal{N}$. Buyer i's payoff is then $v_i - x_i + \varphi \sum_{j=1,j\neq i}^{N} x_j$ if he wins; his payoff is $(\varphi \sum_{j=1,j\neq i}^{N} x_j) - x_i$ if he does not win. If $\varphi = 0$, then we have a usual setting without externalities.

For the same reason as in Section 3.1, we can focus on the truthful direct semirevelation mechanisms that induce full participation for the revenue-maximization. Similar to Proposition 3, we have the following results.[20]

Proposition 4: *(i) If $(\mathbf{p}, \mathbf{x})$ is a revenue-maximizing truthful direct semirevelation mechanism that induces full participation in the setting without externalities, then $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ must be a revenue-maximizing truthful direct semirevelation mechanism that induces full participation in the setting with externalities, where $\tilde{\mathbf{p}} = \mathbf{p}$ and $\tilde{x}_i(\mathbf{v}) = \frac{1}{1+\varphi}[x_i(\mathbf{v}) + \frac{\varphi}{1-(N-1)\varphi}\sum_{j=1}^{N} x_j(\mathbf{v})]$, $\forall i \in \mathcal{N}$.[21] (ii) If $(\tilde{\mathbf{p}}', \tilde{\mathbf{x}}')$ is a revenue-maximizing truthful direct semirevelation mechanism that induces full participation in the setting with financial externalities, then $(\mathbf{p}', \mathbf{x}')$ must be a revenue-maximizing truthful direct semirevelation mechanism that induces full participation in the setting without externalities, where $\mathbf{p}' = \tilde{\mathbf{p}}'$ and $x_i'(\mathbf{v}) = \tilde{x}_i'(\mathbf{v}) - \varphi\sum_{j=1, j\neq i}^{N} \tilde{x}_j'(\mathbf{v})$, $\forall i \in \mathcal{N}$. (iii) The seller's expected revenue from any revenue-maximizing auction with financial externalities equals $\frac{1}{1-(N-1)\varphi}$ times of that from any revenue-maximizing auction without externalities.*

For our setting where the externality to each buyer depends on the other buyers' total payments, we define the following mechanism.

(c.1) There is no entry fee, the reserve price for buyer i, $\forall i \in \mathcal{N}$ is $\hat{v}_i$ ($\geq \underline{v}_i$), which is the unique solution of $J_i(\hat{v}_i) = 0$;

(c.2) If at least one buyer does not participate, the seller keeps the item to create zero externality for all buyers;

(c.3) If all participate, we denote buyer i's bid by b_i, $\forall i \in \mathcal{N}$. buyer i wins if $J_i(b_i)$ is the highest among all $J_j(b_j)$, $\forall j \in \mathcal{N}$ and $b_i \geq \hat{v}_i$. Ties are broken randomly. Suppose buyer i, $\forall i \in \mathcal{N}$ is the winner. The payments are the following. First, buyer i pays z_1, which

[20] The proof is omitted as it is similar to that of Proposition 3.

[21] Buyer i's payment $\tilde{x}_i(\cdot)$ is set to be zero if he does not participate, and his payment $\tilde{x}_i(\cdot)$ for all other cases can be defined in any way.

is $J_i^{-1}(\max_{j\neq i} J_j(b_j))$ or the reserve price $\hat{v}_i$ ($\geq \underline{v}_i$), whichever is higher. Second, he pays $\varphi(N-1)z_2$, where $z_2 = \frac{\varphi z_1}{1-\varphi(N-2)-\varphi^2(N-1)} > 0$. Every losing buyer pays z_2. If no one wins, the seller keeps the item, and no one pays.

Corollary 4: *The mechanism defined by* (c.1), (c.2) *and* (c.3) *is revenue-maximizing for the setting where each buyer enjoys an externality equal to a proportion* φ *of the other buyers' total payments.*

Proof: The result follows immediately from Proposition 4(i). From Myerson (1981), we have that in the setting without externalities, the auction defined in (c.1) to (c.3) with $z_2 = 0$ is a revenue-maximizing truthful direct semirevelation mechanism. It follows from Proposition 4(i) that the mechanism defined by (c.1) to (c.3) is a revenue-maximizing truthful direct semirevelation mechanism for the setting considered. □

In Maasland and Onderstal (2002), buyers' private signals t_i, $\forall i \in \mathcal{N}$ follow the same distribution on $[\underline{t}, \overline{t}]$. buyers' valuations v_i, $\forall i \in \mathcal{N}$ are functions of the buyers' signals $\mathbf{t} = (t_1, t_2, ..., t_N)$. The seller's valuation is zero. Each buyer enjoys a positive externality which equals a proportion of the other buyers' total payments. Although Proposition 4 does not directly apply to the setting of Maasland and Onderstal (2002), clearly the essence of our results extends to their setting. As long as we know the revenue-maximizing auction for the corresponding setting without externalities, the revenue-maximizing auction with financial externalities can be obtained through modifying the payment functions following Proposition 4(i).

3.3 Some Remarks

Consistent with the findings of Sections 2.2 and 2.3, every buyer's payment consists of a component that equals the allocative externalities to them in the revenue-maximizing auctions proposed in Sections 3.2 and 3.3. Moreover, the spirit of the revenue-maximizing auctions of Section 3 also catches the essence of constructing the optimal nonparticipation threats and optimal entry fees in Sections 2.2 and 2.3. All these results suggest that the findings from our baseline identity-specific setting extend to the case of financial externalities. Interestingly, in the Section 3 auctions, the optimal reserve prices are set in the same way as in Myerson (1981) where no externalities are involved. This result is quite intuitive as the Sections 3 auctions create an environment that mimics one without externalities. Thus the optimal reserve price should be set in the same way as in Myerson (1981).

4 Conclusion

This paper first derives the revenue-maximizing mechanism when identity-specific externalities among all players (seller and buyers) are allowed. These externalities are not restricted to be uniformly negative or positive. For a symmetric setting, the revenue-maximizing mechanism derived reduces to be a modified second price auction. We show that introducing the possibility for the seller to destroy the item enlarges the freedom of revenue-maximizing mechanism design. At the optimum, the seller may keep or destroy the item if it is unsold. The seller destroys the item if and only if the sum of his own valuation, his dismantling cost and the total externalities on the buyers if he keeps the item is negative. When buyers suffer highly negative externalities if the seller holds the item, we find that the seller's optimal expected revenue increases as these externalities become more negative. This provides an alternative explanation to why North Korea tries to convince relevant countries that its nuclear weapons are powerful.

Jehiel, Moldovanu and Stacchetti (1996) point out that the seller is better off by not selling at all if the total externalities generated by a sale is larger than total valuations. Our analysis reveals that the seller can be further better off by physically destroying the item while extracting payments from all buyers, if the sum of his valuation, his dismantling cost and the total externalities to the buyers is negative when he keeps the item. This reveals that the crucial force driving the dismantling result is the externalities on the buyers imposed by the seller instead of those among the buyers.

A unique feature of the revenue-maximizing mechanism established is that buyer's payments consist of externality-correcting components that equal the allocative externality to him. These components eliminate the impact of the externalities on the strategic bidding behavior. Introducing these externality-correcting payments leads to a situation that mimics a standard auctions design problem with no externalities to buyers. This is why a modified second-price auction with these externality-correcting payments is revenue-maximizing in a symmetric setting, provided that the entry fees and reserve price are appropriately set. Since the sum of the externality-correcting payment and the entry fee for every buyer is always nonnegative, there is no loss of generality to consider only the full-participation mechanisms for the revenue-maximizing auction with externalities.[22]

Although the above feature of the revenue-maximizing auction and other findings are established in a baseline setting where the identity-specific externalities are public information, these are still valid in an environment where players have private information on the externalities they create for others. Moreover, these insights lead to interesting results on auctions design in general settings with financial externalities, where the externality to every buyer depends on other buyers' total payments or those of all buyers. We establish one-to-one correspondences between revenue-maximizing auctions with and without externalities. As a result, the revenue-maximizing second-price auction for a regular setting without externalities can be properly modified to be revenue-maximizing in various settings of financial externalities. These findings provide a general approach for deriving revenue-maximizing auctions for settings with financial externalities.

[22] Please refer to footnote 5.

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