Quasiseparable aggregation in games with common local utilities

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Abstract

Strategic games are considered where each player's total utility is an aggregate of local utilities obtained from the use of certain "facilities." All players using a facility obtain the same utility therefrom, which may depend on the identities of users and on their behavior. Individual improvements in such a game are acyclic if a "trimness" condition is satisfied by every facility and all aggregation rules are consistent with a separable ordering. Those conditions are satisfied, for instance, by bottleneck congestion games with an infinite set of facilities. Under appropriate additional assumptions, the existence of a Nash equilibrium is established.
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Key words: Bottleneck congestion game; Game with structured utilities; Potential game; Aggregation; Separable ordering

1 Introduction

The origins of this research can be traced back to two papers from the 1970's: Rosenthal (1973) and Germeier and Vatel' (1974), although nobody noted the similarities between them at the time. In both cases, each player's total utility was an aggregate of local utilities obtained from the use of certain "facilities"; all players using a facility obtained the same (local) utility therefrom. In the first case, that utility only depended on the number of users, while the users had a certain freedom in deciding which facilities to use. In the second case, each player had a fixed set of facilities to use, but was able to decide *how* to use those facilities, and the local utilities depended on that. In the first case, the aggregation of local utilities consisted in summing them up; in the second case, in taking the minimum of them. In the first case, the existence of a Nash equilibrium was shown (actually, even of an "exact potential" as defined later by Monderer and Shapley (1996)); in the second case, even the existence of a strong Nash equilibrium (although this was done in later papers).

Both approaches were considered simultaneously in Kukushkin (2007), where it was shown that only additive aggregation ensures the "universal" existence of Nash equilibrium in both "generalized congestion games" and "games with structured utilities" (supposing that the aggregation rules must be continuous and *strictly* increasing in all local utilities). Le Breton and Weber (2011) introduced a

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class of potential games similar to those from either class, but, generally, belonging to neither, and found conditions for the existence of a Nash equilibrium in such games. Kukushkin (2018) finally united the two approaches by formulating the notion of a "trim" game with common local utilities (henceforth, a *CLU game*); every such game admits an exact potential, which attains its maximum (at a Nash equilibrium) under certain additional assumptions.

In this paper, we pay no attention at all to the concept of an exact potential; we are only interested in aggregation rules that ensure the acyclicity of individual improvements and, under appropriate additional assumptions, the existence of a Nash equilibrium in CLU games. We also do not consider strong Nash equilibrium here. Quite a few papers studied its existence in "bottleneck congestion games," (see, e.g., Epstein, Feldman, and Mansour, 2009; Feldman and Tennenholtz, 2010; Harks, Klimm, and Möhring, 2013). An analysis for general CLU games was given in Kukushkin (2017).

Our basic construction is described in the following section. In Section 3, the necessity of certain properties of aggregation rules for the guaranteed existence of a Nash equilibrium are established. In contrast to previous results of this kind (Kukushkin, 2007, 2017), arbitrary restrictions on the possible values of local utilities are allowed (e.g., they may be assumed to be integer). Unfortunately, the necessity of *separable* aggregation remains elusive.

Section 4 introduces the notions of a "universal separable ordering" and aggregation rules consistent with such an ordering. Theorem 1 asserts the existence of an "order potential," i.e., the acyclicity of individual improvements, in every CLU game with such aggregation rules. Essentially the same result, but restricted to "generalized congestion games," was obtained in Kukushkin (2014).

In Section 5, the question of when the potential attains its maximum is addressed. We formulate a list of assumptions ensuring the "ω-transitivity" of the potential, and hence the existence of a Nash equilibrium (Theorem 2 from Kukushkin (2018) and new Theorems 3–5). The proof of Theorem 3 is in Section 6; a sketch of the proof of Theorem 5, in Section 7.

2 Basic definitions

A *strategic game* Γ is defined by a finite set N of *players*, and, for each $i \in N$, a set X_i of *strategies* and a real-valued *utility function* u_i on the set $X_N := \prod_{i\in N} X_i$ of *strategy profiles*. We denote $\mathcal{N} := 2^N \setminus \{\emptyset\}$ and $X_I := \prod_{i\in I} X_i$ for each $I \in \mathcal{N}$. Given $i, j \in N$, we use notation X_{-i} instead of $X_{N\setminus\{i\}}$ and X_{-ij} instead of $X_{N\setminus\{i,j\}}$.

Being interested in games with *ordinal* preferences here and following Kukushkin (1999), we define a *potential* of Γ as an irreflexive and transitive relation $\ggcurly$ on X_N satisfying

$$\forall x_N, y_N \in X_N \left[\exists i \in N \left[y_{-i} = x_{-i} \;\&\; u_i(y_N) > u_i(x_N)\right] \Rightarrow y_N \ggcurly x_N\right]. \tag{1}$$

When X_N is finite, the existence of a potential in our sense is equivalent to the existence of a generalized ordinal potential (Monderer and Shapley 1996, Lemma 2.5); and it obviously implies the existence of a Nash equilibrium. Generally, the existence of a generalized ordinal potential implies the existence of a potential (1), but not the other way round; the existence of a Nash equilibrium also needs more than (1).

A *game with common local utilities* (a *CLU game*) may have an arbitrary finite set N of players and arbitrary sets of strategies X_i ($i \in N$), whereas the utilities are defined by the following construction.

First of all, there is a set A of *facilities*; we denote $\mathcal{B}$ the set of all (nonempty) finite subsets of A. For each $i \in N$, there is a mapping $\mathrm{B}_i \colon X_i \to \mathcal{B}$ describing what facilities player i uses having chosen x_i. Every strategy profile x_N determines *local utilities* at all facilities $\alpha \in \mathrm{A}$; each player's *total utility* is an aggregate of local utilities over chosen facilities. The exact definitions need plenty of notations.

For every $\alpha \in \mathrm{A}$, we denote $I_\alpha^- := \{i \in N \mid \forall x_i \in X_i\, [\alpha \in \mathrm{B}_i(x_i)]\}$ and $I_\alpha^+ := \{i \in N \mid \exists x_i \in X_i\, [\alpha \in \mathrm{B}_i(x_i)]\}$; without restricting generality, we may assume $I_\alpha^+ \neq \emptyset$. For each $i \in I_\alpha^+$, we denote $X_i^\alpha := \{x_i \in X_i \mid \alpha \in \mathrm{B}_i(x_i)\}$; if $i \in I_\alpha^-$, then $X_i^\alpha = X_i$. Then we set $\mathcal{I}_\alpha := \{I \in \mathcal{N} \mid I_\alpha^- \subseteq I \subseteq I_\alpha^+\}$ and $\Xi_\alpha := \{\langle I, x_I\rangle \mid I \in \mathcal{I}_\alpha \;\&\; x_I \in X_I^\alpha\}$. The *local utility function* at α is $\varphi_\alpha \colon \Xi_\alpha \to \mathbb{R}$.

For every $\alpha \in \mathrm{A}$ and $x_N \in X_N$, we denote $I(\alpha, x_N) := \{i \in N \mid \alpha \in \mathrm{B}_i(x_i)\}$: the set of players using α at x_N. Obviously, $I_\alpha^- \subseteq I(\alpha, x_N) \subseteq I_\alpha^+$. We denote $n^-(\alpha) := \min_{I \in \mathcal{I}_\alpha} \#I = \max\{1, \#I_\alpha^-\}$: the minimal number of players using α provided somebody *is* using it.

For every $i \in N$ and $x_i \in X_i$, there is a mapping $U_i^{x_i} \colon \mathbb{R}^{\mathrm{B}_i(x_i)} \to \mathbb{R}$, an *aggregation rule*. The "ultimate" or total utility functions of the players are aggregates of the local utilities:

$$u_i(x_N) := U_i^{x_i}\big(\langle \varphi_\alpha(I(\alpha, x_N), x_{I(\alpha,x_N)})\rangle_{\alpha \in \mathrm{B}_i(x_i)}\big),$$

for all $i \in N$ and $x_N \in X_N$.

We call a facility $\alpha \in \mathrm{A}$ *trim* if there is a real-valued function $\psi_\alpha(m)$ defined for integer m between $n^-(\alpha)$ and $\#I_\alpha^+ - 1$ for which

$$\varphi_\alpha(I, x_I) = \psi_\alpha(\#I) \tag{2}$$

whenever $I \in \mathcal{I}_\alpha$, $I \neq I_\alpha^+$, and $x_I \in X_I^\alpha$. In other words: whenever a trim facility is not used by all potential users, neither the identities of the users, nor their strategies matter, only the number of users. We call a CLU game *trim* if so is every facility.

The class of trim CLU games includes two important subclasses: "generalized congestion games" and "games with structured utilities." The former are obtained by replacing the sum of local utilities in Rosenthal's (1973) congestion games with arbitrary aggregates; to be more precise, $X_i \subseteq \mathcal{B}$ for all $i \in N$ and (2) holds for all $I \in \mathcal{I}_\alpha$, even for $I = I_\alpha^+$. In the latter class, $\mathrm{B}_i(x_i)$ only depends on i. It follows immediately that $\mathcal{I}_\alpha = \{I_\alpha^+\}$ for each facility α and hence (2) is not required at all.

Note that A is finite in both cases, which is not required generally.

3 Aggregation rules

Given a subset $V \subseteq \mathbb{R}$, an (abstract) *aggregation rule* over V is a mapping from a power of V to $\mathbb{R}$. Given a set $\mathfrak{U}$ of aggregation rules over V and a CLU game Γ, we say that player $i \in N$ *aggregates local utilities with rules from* $\mathfrak{U}$ iff (i) for every $\alpha \in \mathrm{A}$ and $\langle I, x_I\rangle \in \Xi_\alpha$ such that $i \in I$, there holds $\varphi_\alpha(I, x_I) \in V$, and (ii) for every $x_i \in X_i$, there is $U \in \mathfrak{U}$ for which the set $\mathrm{B}_i(x_i)$ can be ordered in such a way that $U_i^{x_i} = U$.

Remark. Interestingly, none of the results of this section needs the monotonicity or continuity (in any sense) of aggregation rules, however natural such assumptions would seem.

Proposition 1 (Kukushkin, 2014, Proposition 6.1)**.** *Let $U \colon V^m \to \mathbb{R}$, where $V \subseteq \mathbb{R}$, have the property that every generalized congestion game where $\#x_i = m$ for each strategy of each player and each player aggregates local utilities with U possesses a Nash equilibrium. Then U is symmetric.*

Remark. The necessity of symmetry could not be derived without the stipulation that a Nash equilibrium must exist when *all* players use the same aggregation rule.

Henceforth, we restrict ourselves to symmetric aggregation rules and name (or order) their arguments in whatever way is more convenient at a particular moment.

Proposition 2. *Let $\mathfrak{U}$ be a set of aggregation rules over $V \subseteq \mathbb{R}$ such that every generalized congestion game where each player aggregates local utilities with rules from $\mathfrak{U}$ possesses a Nash equilibrium; let $U, U' \in \mathfrak{U}$; let the (finite) set of arguments of U' be M and the (finite) set of arguments of U be $M \cup K$ with $M \cap K = \emptyset$; let $U'(v'_M) > U'(v_M)$ for some $v_M, v'_M \in V^M$; let $w_K \in V^K$. Then $U(v'_M, w_K) \geq U(v_M, w_K)$.*

Proof. Supposing the contrary, $U(v'_M, w_K) < U(v_M, w_K)$, we define the following generalized congestion game. $N := \{1, 2\}$; the facilities $\mathrm{A} := A \cup B \cup C$ with $A := \{a_s\}_{s \in M}$, $B := \{b_s\}_{s \in M}$, $C := \{c_s\}_{s \in K}$, and $a_k \neq b_h \neq c_s \neq a_k$ for all relevant k, h, s; $X_1 := \{A \cup C, B \cup C\}$; $X_2 := \{A, B\}$; $\psi_{a_s}(1) := \psi_{b_s}(1) := v'_s$ and $\psi_{a_s}(2) := \psi_{b_s}(2) := v_s$ for each $s \in M$; $\psi_{c_s}(1) := w_s$ for each $s \in K$; $U_1^{x_1} := U$ for both $x_1 \in X_1$ and $U_2^{x_2} := U'$ for both $x_2 \in X_2$.

The 2×2 matrix of the game looks as follows:

$$\begin{array}{ccc} & A & B \\ A \cup C & (U(v_M, w_K), U'(v_M)) & (U(v'_M, w_K), U'(v'_M)) \\ B \cup C & (U(v'_M, w_K), U'(v'_M)) & (U(v_M, w_K), U'(v_M)). \end{array}$$

There is no Nash equilibrium in the game. ☐

Corollary 2.1. *Let $\mathfrak{U}$ be a set of aggregation rules over $V \subseteq \mathbb{R}$ such that every generalized congestion game where each player aggregates local utilities with rules from $\mathfrak{U}$ possesses a Nash equilibrium; let $U, U' \in \mathfrak{U}$ have the same set of arguments M; let $U(v'_M) > U(v_M)$ for some $v_M, v'_M \in V^M$. Then $U'(v'_M) \geq U'(v_M)$.*

Proof. Set $K := \emptyset$ and drop w_K and C in Proposition 2 and its proof. ☐

Proposition 3. *Let $\mathfrak{U}$ be a set of aggregation rules over $V \subseteq \mathbb{R}$ such that every generalized congestion game where each player aggregates local utilities with rules from $\mathfrak{U}$ possesses a Nash equilibrium; let $U, U' \in \mathfrak{U}$ have the same set of arguments M; let $U'(v_M) > U(v_M)$ for some $v_M \in V^M$. Then $U'(v'_M) \geq U(v'_M)$ for every $v'_M \in V^M$.*

Proof. Supposing the contrary, $U'(v'_M) < U(v'_M)$, we define the following generalized congestion game. $N := \{1, 2\}$; the facilities $\mathrm{A} := A \cup B$ with $A := \{a_k\}_{k \in M}$, $B := \{b_h\}_{h \in M}$, and $a_k \neq b_h$ for all relevant k, h; $X_1 := X_2 := \{A, B\}$; $\psi_{a_s}(1) := \psi_{b_s}(2) := v'_s$ and $\psi_{a_s}(2) := \psi_{b_s}(1) := v_s$ for each $s \in M$; $U_1^A := U_2^B := U$ and $U_1^B := U_2^A := U'$.

The 2×2 matrix of the game looks as follows:

$$\begin{array}{ccc} & A & B \\ A & (U(v_M), U'(v_M)) & (U(v'_M), U(v'_M)) \\ B & (U'(v_M), U'(v'_M)) & (U'(v'_M), U(v'_M)). \end{array}$$

There is no Nash equilibrium in the game. ☐

Proposition 4. *Let $\mathfrak{U}$ be a set of aggregation rules over $V \subseteq \mathbb{R}$ such that every generalized congestion game where each player aggregates local utilities with rules from $\mathfrak{U}$ possesses a Nash equilibrium; let $U, U' \in \mathfrak{U}$; let the (finite) set of arguments of U' be M and the (finite) set of arguments of U be $M \cup K$ with $M \cap K = \emptyset$; let $U(v_M, w_K) > U'(v_M)$ for some $v_M \in V^M$ and $w_K \in V^K$. Then $U(v'_M, w_K) \geq U'(v'_M)$ for every $v'_M \in V^M$.*

Proof. Supposing the contrary, $U(v'_M, w_K) < U'(v'_M)$, we define the following generalized congestion game. $N := \{1, 2\}$; the facilities $\mathrm{A} := A \cup B \cup C$ with $A := \{a_s\}_{s \in M}$, $B := \{b_s\}_{s \in M}$, $C := \{c_s\}_{s \in K}$, and $a_k \neq b_h \neq c_s \neq a_k$ for all relevant k, h, s; $X_1 := \{A, B \cup C\}$; $X_2 := \{B, A \cup C\}$; $\psi_{a_s}(1) := \psi_{b_s}(2) := v_s$ and $\psi_{a_s}(2) := \psi_{b_s}(1) := v'_s$ for each $s \in M$; $\psi_{c_s}(1) := \psi_{c_s}(2) := w_s$ for each $s \in K$; $U_1^A := U_2^B := U'$ and $U_1^{B \cup C} := U_2^{A \cup C} := U$.

The 2×2 matrix of the game looks as follows:

$$\begin{array}{ccc} & B & A \cup C \\ A & (U'(v_M), U'(v'_M)) & (U'(v'_M), U(v'_M, w_K)) \\ B \cup C & (U(v_M, w_K), U'(v_M)) & (U(v'_M, w_K), U(v_M, w_K)). \end{array}$$

There is no Nash equilibrium in the game. □

Remark. From a technical viewpoint, it is interesting to note that "generalized congestion games" in the formulations of Propositions 1–4 could be replaced with "games with structured utilities" without much change in the proofs.

4 Quasiseparable aggregation

Given a subset $V \subseteq \mathbb{R}$, we define V^∞ as the disjoint union of V^m for $m = 1, 2, \ldots$ A *universal separable ordering on* V is an ordering, i.e., reflexive, transitive, and total binary relation $\succeq^*$, on V^∞ (we denote $\succ^*$ and $\sim^*$, respectively, its asymmetric and symmetric components) such that:

1) $\succeq^*$ on V is the standard order $\geq$ induced from $\mathbb{R}$;

2) for every permutation σ of $\{1, \ldots, m\}$,

$$\langle v_1, \ldots, v_m \rangle \sim^* \langle v_{\sigma(1)}, \ldots, v_{\sigma(m)} \rangle$$

(symmetry); by this condition, $\succeq^*$ can be perceived as defined on the set of unordered lists of $\langle v_s \in V \rangle_{s \in M}$;

3) for every $\langle v_1, \ldots, v_m \rangle \in V^m$, every $\langle v'_1, \ldots, v'_{m'} \rangle \in V^{m'}$, and every $\langle v''_1, \ldots, v''_{m''} \rangle \in V^{m''}$,

$$\langle v_1, \ldots, v_m, v''_1, \ldots, v''_{m''} \rangle \succeq^* \langle v'_1, \ldots, v'_{m'}, v''_1, \ldots, v''_{m''} \rangle \iff \langle v_1, \ldots, v_m \rangle \succeq^* \langle v'_1, \ldots, v'_{m'} \rangle$$

(separability).

A set $\mathfrak{U}$ of aggregation rules over $V \subseteq \mathbb{R}$ is *consistent* with a universal separable ordering $\succeq^*$ if, whenever functions U with m_U arguments and U' with $m_{U'}$ arguments belong to $\mathfrak{U}$, while $\langle v_1, \dots, v_{m_U} \rangle \in V^{m_U}$ and $\langle v'_1, \dots, v'_{m_{U'}} \rangle \in V^{m_{U'}}$, there holds

$$U'(v'_1, \dots, v'_{m_{U'}}) > U(v_1, \dots, v_{m_U}) \Rightarrow \langle v'_1, \dots, v'_{m_{U'}} \rangle \succ^* \langle v_1, \dots, v_{m_U} \rangle. \tag{3}$$

It seems reasonable to call such aggregation rules *quasiseparable.*

Theorem 1. *Let $\succeq^*$ be a universal separable ordering on $V \subseteq \mathbb{R}$; let N be a finite set and $\mathfrak{U}_i$ $(i \in N)$ be sets of aggregation rules over V, each consistent with $\succeq^*$; let Γ be a trim CLU game where each player i aggregates local utilities with rules from $\mathfrak{U}_i$. Then Γ admits a potential in the sense of (1).*

Proof. The proof is a combination of those for Proposition 3.1 from Kukushkin (2014) and Theorem 1 from Kukushkin (2018). We restrict ourselves here to an explicit definition of a potential.

Given $x_N \in X_N$, we denote $\mathrm{A}(x_N) := \{\alpha \in \mathrm{A} \mid I(\alpha, x_N) \neq \emptyset\}$ and $\mathrm{A}^+(x_N) := \{\alpha \in \mathrm{A} \mid \# I(\alpha, x_N) > n^-(\alpha)\}$ $[\subseteq \mathrm{A}(x_N)]$; since N and each $\mathrm{B}_i(x_i)$ are finite, $\mathrm{A}(x_N)$ is finite too. Then we define an unordered list:

$$\varkappa(x_N) := \Big\langle \langle \varphi_\alpha(I(\alpha, x_N), x_{I(\alpha,x_N)}) \rangle_{\alpha \in \mathrm{A}(x_N)}, \langle \psi_\alpha(h) \rangle_{\alpha \in \mathrm{A}^+(x_N), h=n^-(\alpha),\dots,\#I(\alpha,x_N)-1} \Big\rangle. \tag{4}$$

Now we define our potential $\ggcurly$ in this way:

$$y_N \succ\!\!\succ x_N \rightleftharpoons \varkappa(y_N) \succ^* \varkappa(x_N). \tag{5}$$

Claim 1.1. *Whenever $x_N, y_N \in X_N$ and $i \in N$ are such that $y_{-i} = x_{-i}$ and $u_i(y_N) > u_i(x_N)$, there holds $y_N \succ\!\!\succ x_N$.*

The proof, quite similar to that of Proposition 3.1 from Kukushkin (2014) and based on the separability of $\succeq^*$ is omitted. Theorem 1 is proven. □

The simplest and most important example of quasiseparable (actually, just separable) aggregation is given by addition:

$$U^{(m)}(v_1, \dots, v_m) := \sum_{s=1}^{m} v_s; \tag{6}$$

$$v'_{M'} \succeq^\Sigma v_M \rightleftharpoons \sum_{s \in M'} v'_s \geq \sum_{s \in M} v_s.$$

As was noted in Kukushkin (2014), Rosenthal's (1973) congestion games are covered by Theorem 1 with this ordering; moreover, the construction described by (4) and (5) generates just Rosenthal's potential in this case.

Another example of quasiseparable aggregation is the minimum ("weakest-link")

$$U^{(m)}(v_1, \dots, v_m) := \min\{v_1, \dots, v_m\}, \tag{7a}$$

which is consistent with the leximin universal separable ordering:

$$\min\{v'_1, \dots, v'_{m'}\} > \min\{v_1, \dots, v_m\} \Rightarrow \langle v'_1, \dots, v'_{m'} \rangle >^{\mathrm{Lmin}} \langle v_1, \dots, v_m \rangle. \tag{7b}$$

The exact definition of the leximin ordering is assumed commonly known and omitted: when comparing two lists of local utility values, we start with the worst in either list; in the case of equality, we move to the second worst, etc. The only point needing special mentioning is this: when comparing two lists of different lengths, and when all possible comparisons resulted in ties (i.e., equalities), we assume that the shorter list dominates the longer one; one might say that we supplement the shorter list with an appropriate number of $+\infty$ values. (Obviously, this stipulation is not needed for (7b) to hold, but it is convenient in the following.)

A similar connection exists between the maximum ("best-shot") aggregation,

$$U^{(m)}(v_1, \ldots, v_m) := \max\{v_1, \ldots, v_m\}, \tag{8}$$

and the leximax ordering:

$$\max\{v'_1, \ldots, v'_{m'}\} > \max\{v_1, \ldots, v_m\} \Rightarrow \langle v'_1, \ldots, v'_{m'}\rangle >^{\mathrm{Lmax}} \langle v_1, \ldots, v_m\rangle.$$

The "weakest-link" and additive (disguised as multiplicative) aggregation rules can be combined together in a sense. Consider this family of aggregation rules ($m \in \mathbb{N}$, $v_s \in V \subseteq \mathbb{R}$):

$$U^{(m)}(v_1, \ldots, v_m) := \begin{cases} \prod_{s=1,\ldots,m} v_s, & \text{if } \forall s = 1, \ldots, m\, [v_s > 0], \\ \min_{s=1,\ldots,m} v_s, & \text{otherwise.} \end{cases} \tag{9a}$$

To describe the universal separable ordering this family is consistent with, we need some notations. Given $v_M \in V^M$, we define $Z(v_M) := \{s \in M \mid v_s \leq 0\}$, $P(v_M) := \{s \in M \mid v_s > 0\}$, and $p(v_M) := \prod_{s \in P(v_M)} v_s$. Now we are ready to define the promised universal separable ordering $\succeq^{\Pi}$:

$$v'_{M'} \succeq^{\Pi} v_M \rightleftharpoons \Big[v'_{Z(M')} >^{\mathrm{Lmin}} v_{Z(v_M))} \ \text{ or } \ \big[v'_{Z(v'_{M'})} \sim^{\mathrm{Lmin}} v_{Z(v_M)} \ \&\ p(v'_{M'}) \geq p(v_M)\big]\Big]. \tag{9b}$$

5 The existence of Nash equilibrium

Assumption 1. *The set of facilities* A *and each strategy set* X_i *are metric spaces; each* X_i *is compact; each mapping* B_i *is continuous in the Hausdorff metric on the target.*

Henceforth, we assume each set X_I ($I \in \mathcal{N}$) to be endowed with the maximum metrics. We denote the distances in A, as well as in each X_I, with the same letter d. For each $i \in N$ and $m \in \mathbb{N}$, we denote $X_i^m := \{x_i \in X_i \mid \#\mathrm{B}_i(x_i) = m\}$.

Assumption 2. *For each* $i \in N$ *and* $m \in \mathbb{N}$, *either* $X_i^m = \emptyset$ *or* X_i^m *is a compact subset of* X_i.

Assumption 3. *For each* $i \in N$, $X_i^m \neq \emptyset$ *only for a finite number of* $m \in \mathbb{N}$.

Assumption 4. *Every function* $\varphi_\alpha(I, \cdot)\colon X_I \to \mathbb{R}$ *is upper semicontinuous in* x_I, *i.e., for every* $\alpha \in \mathrm{A}$, $I \in \mathcal{I}_\alpha$, $x_I \in X_I^\alpha$, *and* $\varepsilon > 0$, *there is* $\delta > 0$ *such that:*

$$\varphi_\alpha(I, x_I) > \varphi_\alpha(I, y_I) - \varepsilon \tag{10}$$

whenever $y_I \in X_I^\alpha$ *and* $d(x_I, y_I) < \delta$.

For $\alpha \neq \beta$, a stronger version of (10), with a tint of the monotonicity of φ_α in I, is assumed.

Assumption 5. *For every $\alpha \in \mathrm{A}$, $I \in \mathcal{I}_\alpha$, and $\varepsilon > 0$, there is $\delta > 0$ such that:*

$$\varphi_\alpha(I, x_I) > \varphi_\beta(J, y_J) - \varepsilon \tag{11}$$

whenever $\beta \in \mathrm{A} \setminus \{\alpha\}$, $J \in \mathcal{I}_\beta$, $x_I \in X_I^\alpha$, $y_J \in X_J^\beta$, $J \subseteq I$, $d(\alpha, \beta) < \delta$, and $d(x_J, y_J) < \delta$.

If A is finite as, e.g., in a game with structured utilities or in a (generalized) congestion game, then Assumption 5 holds vacuously since a $\delta > 0$ smaller than the minimal distance between $\alpha \neq \beta$ can be chosen.

Theorem 2 (Kukushkin, 2018, Theorem 3)**.** *Every trim CLU game with additive aggregation (6) satisfying Assumptions 1–5 possesses a (pure strategy) Nash equilibrium.*

Theorem 3. *Every trim CLU game with the minimum aggregation (7a) satisfying Assumptions 1, 4, and 5 possesses a (pure strategy) Nash equilibrium.*

The proof is deferred to Section 6.

Theorem 4. *Every trim CLU game with the maximum aggregation (8) satisfying Assumptions 1, 4, and 5 possesses a (pure strategy) Nash equilibrium.*

The proof is "dual" to that of Theorem 3, and is omitted.

Theorem 5. *Every trim CLU game with the aggregation (9a) satisfying Assumptions 1–5 possesses a (pure strategy) Nash equilibrium.*

A sketch of the proof, combining those of Theorem 2 and Theorem 3, is given in Section 7.

6 Proof of Theorem 3

As easily understood, the order $\ggcurly$ defined by (5) with $>^{\mathrm{Lmin}}$ as $\succ^*$ need not be continuous in any sense. Nonetheless, it can be shown to satisfy the following condition ("ω-transitivity"): $x_N^\omega \ggcurly x_N^0$ whenever $\langle x_N^k \rangle_{k \in \mathbb{N}}$ converges to x_N^ω and $x_N^{k+1} \ggcurly x_N^k$ for all k. As proven by Gillies (1959) and Smith (1974), such a strict ordering on a compact set always admits a maximizer, and X_N is compact by Assumption 1. In its turn, every maximizer of $\ggcurly$ must be a Nash equilibrium of the game because of (1).

Thus, let $x_N^k \to x_N^\omega \in X_N$ and $x_N^{k+1} \ggcurly x_N^k$, i.e., $\varkappa(x_N^{k+1}) >^{\mathrm{Lmin}} \varkappa(x_N^k)$, for all k. We have to show that $\varkappa(x_N^\omega) >^{\mathrm{Lmin}} \varkappa(x_N^0)$. Since $\mathrm{A}(x_N^\omega)$ is finite and each B_i is continuous in the Hausdorff metric, there is $\bar{\delta} > 0$ so small that, first, if $\alpha, \beta \in \mathrm{A}(x_N^\omega)$ and $\alpha \neq \beta$, then $d(\alpha, \beta) > \bar{\delta}$, and, second, $I(\beta, y_N) \subseteq I(\alpha, x_N^\omega)$ whenever $d(\alpha, \beta) < \bar{\delta}$ and $d(x_N^\omega, y_N) < \bar{\delta}$.

Let $v_1^\omega \leq v_2^\omega \leq \cdots \leq v_m^\omega$ denote all different components of $\varkappa(x_N^\omega)$ and let each v_s^ω enter $\varkappa(x_N^\omega)$ just ν_s^ω times. Let $v_1^k \leq v_2^k \leq \ldots$ and $\nu_1^k, \nu_2^k, \ldots$ have the same meaning for each $k = 0, 1 \ldots$

Claim 6.1. $v_1^\omega \geq v_1^0$.

Proof. Supposing the contrary, we define $\varepsilon := v_1^0 - v_1^\omega > 0$; then we pick $\alpha \in \mathrm{A}(x_N^\omega)$ and $I \subseteq I(\alpha, x_N^\omega)$ such that $v_1^\omega = \varphi_\alpha(I, x_I)$; then, relying on Assumptions 4 and 5, pick a $\delta > 0$ for which both (10) and (11) hold, set $\delta^* := \min\{\delta, \bar{\delta}\}$ with $\bar{\delta}$ defined at the start of the proof, and only consider k for which $d(x_N^\omega, x_N^k) < \delta^*$.

If there is k such that $\alpha \in \mathrm{B}_i(x_i^k)$ for all $i \in I$, then $\varphi_\alpha(I, x_I^k)$ is present in $\varkappa(x_N^k)$ by definition (4), and hence (10) applies. Otherwise, for each $i \in I$ and k large enough, there is $\beta_i \in \mathrm{B}_i(x_i^k)$ such that $d(\alpha, \beta_i) < \delta^*$; picking such a k and $i \in I \setminus I(\alpha, x_N^k)$, and denoting $J := I \cap I(\beta_i, x_N^k)\,[\ni i]$, we can apply (11). In either case, we have $v_1^k \le v_1^\omega + \varepsilon < v_1^0$, and hence $\varkappa(x_N^0) >^{\mathrm{Lmin}} \varkappa(x_N^k)$, contradicting our initial assumption. □

Claim 6.2. *$v_1^\omega \ge v_1^k$ for each k.*

Proof. Since the sequence $x_N^k, x_N^{k+1}, \ldots$ also converges to x_N^ω, Claim 6.1 applies. □

If $v_1^\omega > v_1^k$ for some $k \ge 0$, then $\varkappa(x_N^\omega) >^{\mathrm{Lmin}} \varkappa(x_N^k) \ge^{\mathrm{Lmin}} \varkappa(x_N^0)$ and we are home. Let $v_1^\omega = v_1^k$ for each k. Since $\varkappa(x_N^{k+1}) >^{\mathrm{Lmin}} \varkappa(x_N^k)$, we have $\nu_1^{k+1} \le \nu_1^0$ for each $k \ge 0$; without restricting generality, $\nu_1^k = \nu_1^0$ for all k.

Claim 6.3. *If $v_1^\omega = v_1^k$ and $\nu_1^k = \nu_1^0$ for all k, then $\nu_1^\omega \le \nu_1^0$.*

Proof. The proof is quite similar to that of Claim 6.1, but somewhat more complicated. Supposing the contrary, we either have $v_2^0 - v_1^\omega > 0$ or there is no v_2^0 at all, i.e., $\varkappa(x_N^0)$ consists of ν_1^0 entries of v_1^0. In the first case, we define $\varepsilon := v_2^0 - v_1^\omega > 0$; in the second, pick $\varepsilon > 0$ arbitrarily.

Denoting $\mathrm{A}^* := \{\alpha \in \mathrm{A}(x_N^\omega) \mid \exists I \subseteq I(\alpha, x_N^\omega)\,[v_1^\omega = \varphi_\alpha(I, x_I^\omega)]\}$, and relying on the finiteness of both N and A^*, we pick a $\delta > 0$ for which (10) and (11) hold for all $\alpha \in \mathrm{A}^*$ and $I \subseteq I(\alpha, x_N^\omega)$. Then we set $\delta^* := \min\{\delta, \bar{\delta}\}$ with $\bar{\delta}$ defined at the start of the proof, and fix a k large enough that $d(x_N^\omega, x_N^k) < \delta^*$, and hence $I(\alpha, x_N^k) \subseteq I(\alpha, x_N^\omega)$ for each $\alpha \in \mathrm{A}^*$, and that, for each $i \in I(\alpha, x_N^\omega)$, there is $\beta_i \in \mathrm{B}_i(x_i^k)$ such that $d(\alpha, \beta_i) < \delta^*$. Then we consider each $\alpha \in \mathrm{A}^*$ separately.

Let $\alpha \in \mathrm{A}^*$ bring ν_1^α values of $\varphi_\alpha(I, x_I^\omega) = v_1^\omega$ into $\varkappa(x_N^\omega)$, and let $[0{<}]\, m_1 < \cdots < m_{\nu_1^\alpha}\, [\le \# I(\alpha, x_N^\omega)]$ be the cardinalities of those $I \subseteq I(\alpha, x_N^\omega)$. If $\# I(\alpha, x_N^k) \ge m_{\nu_1^\alpha}$, then $I(\alpha, x_N^k)$ contains subsets of all cardinalities $m_1, \ldots, m_{\nu_1^\alpha}$ and (10) applies to each of them, bringing into $\varkappa(x_N^k)$ at least ν_1^α values of $\varphi_\alpha(I, x_I^k) < v_1^\omega + \varepsilon$.

If $m_{\nu_1^\alpha} > \# I(\alpha, x_N^k) \ge m_1$, then (10) applies to subsets $I_1 \subset \cdots \subset I_h \subset I(\alpha, x_N^k)$ with cardinalities $m_1, \ldots, m_h\, [< m_{\nu_1^\alpha}]$. For each $i \in I(\alpha, x_N^\omega) \setminus I(\alpha, x_N^k)$, we have $\alpha \notin \mathrm{B}(x_i^k)$, but there is (at least one) $\beta_i \in \mathrm{B}(x_i^k)$ such that $d(\alpha, \beta_i) < \delta^*$. We fix such a β_i for each i and define an equivalence relation $\sim$ on $I(\alpha, x_N^\omega) \setminus I(\alpha, x_N^k)$ by $i \sim j \rightleftharpoons \beta_i = \beta_j$. Then we linearly order (in an arbitrary way) the equivalence classes, and then linearly order the players within each equivalence class, obtaining a (lexicographic) linear order on $I(\alpha, x_N^\omega) \setminus I(\alpha, x_N^k)$. The order allows us to define a mapping $r\colon I(\alpha, x_N^\omega) \setminus I(\alpha, x_N^k) \to \mathbb{N}$ by $r(i) := \#(I(\alpha, x_N^k) \cup \{j \in I(\alpha, x_N^\omega) \setminus I(\alpha, x_N^k) \mid j \le i\})$. Clearly, $m_h < r(i) \le m_{\nu_1^\alpha}$ for each i. Then we define $i^s := r^{-1}(m_s)$ for $h < s \le \nu_1^\alpha$ and $J^s := \{j \in I(\beta_{i^s}, x_N^k) \mid j \le i^s\}$. Now we have $i^s \in J^s \ne \emptyset$ and $J^s \subseteq I_s$; hence (11) applies. Moreover, $\# J^s \ne \# J^{s'}$ whenever $s \ne s'$ and $i^s \sim i^{s'}$; hence all $\varphi_{\beta_{i^s}}(J^s, x_{J^s}^k)$ separately enter $\varkappa(x_N^k)$.

If $\# I(\alpha, x_N^k) < m_1$, in particular, if $I(\alpha, x_N^k) = \emptyset$, then we argue exactly as in the preceding paragraph, but with $h + 1 = 1$. We see that $\varkappa(x_N^k)$ contains at least ν_1^α values smaller than $v_1^\omega + \varepsilon$.

Summing up for all $\alpha \in \mathrm{A}^*$, we obtain at least ν_1^ω values smaller than $v_1^\omega + \varepsilon$ in $\varkappa(x_N^k)$; therefore, $\varkappa(x_N^0) >^{\mathrm{Lmin}} \varkappa(x_N^k)$, contradicting our assumption. □

If $\nu_1^\omega < \nu_1^0$, then $\varkappa(x_N^\omega) >^{\mathrm{Lmin}} \varkappa(x_N^0)$ and we are home. Let $\nu_1^\omega = \nu_1^0 = \nu_1^k$ for each k.

Claim 6.4. *If $v_1^\omega = v_1^k$ and $\nu_1^\omega = \nu_1^k$ for all k, then $v_2^\omega \geq v_2^0$.*

Proof. Supposing the contrary, $v_2^\omega < v_2^0$, we argue in essentially the same way as in Claims 6.1 and 6.3. □

If $v_2^\omega > v_2^0$, then we are home. Iterating these arguments further, we come to the conclusion that either $\varkappa(x_N^\omega) >^{\mathrm{Lmin}} \varkappa(x_N^0)$ indeed, or $v_s^\omega = v_s^k$ and $\nu_s^\omega = \nu_s^k$ for all relevant s and all k. The latter alternative, however, is incompatible with our assumptions.

7 Sketch of the proof of Theorem 5

Similarly to the situation of Theorem 3, the order $\ggcurly$ defined by (5) with $\succeq^{\Pi}$ as $\succ^*$ need not be even semi-continuous; however, its "ω-transitivity" can be established, ensuring the existence of a Nash equilibrium of the game.

Assuming that $x_N^k \to x_N^\omega \in X_N$ and $x_N^{k+1} \ggcurly x_N^k$ for all k, we have to show that $x_N^\omega \ggcurly x_N^0$. As long as the first alternative in (9b) is involved, exactly the same argument as in the proof of Theorem 3 works; we do not even need Assumptions 2 and 3. So let $Z(\varkappa(x_N^\omega)) \supseteq Z(\varkappa(x_N^0)) = Z(\varkappa(x_N^k))$ for all k and $\varkappa(x_N^\omega)_{Z(\varkappa(x_N^k))} \sim^{\mathrm{Lmin}} \varkappa(x_N^k)_{Z(\varkappa(x_N^k))}$ for all k. We have to show that $Z(\varkappa(x_N^\omega)) = Z(\varkappa(x_N^0))$ and $p(\varkappa(x_N^\omega)) > p(\varkappa(x_N^0))$.

Exactly as in the proof of Theorem 2 (Theorem 3 of Kukushkin, 2018), Assumptions 2 and 3 imply that $\#\mathrm{B}_i(x_i^\omega) = \#\mathrm{B}_i(x_i^k)$ for all $i \in N$ and k large enough. Supposing, to the contrary, that $Z(\varkappa(x_N^\omega)) \supset Z(\varkappa(x_N^0)) = Z(\varkappa(x_N^k))$, we have at least one "superfluous" pair of $\alpha \in \mathrm{A}$ and $I \in I_\alpha$ such that $\varphi_\alpha(I, x_I^\omega) \leq 0$; but then $\varphi_\alpha(I, x_I^k) > 0$ or $\varphi_\beta(J, x_J^k) > 0$ become arbitrarily close to 0 as $k \to \infty$ by (10) or (11), while all components of $\varkappa(x_N^k)$ remain bounded above. Therefore, $p(\varkappa(x_N^k)) \to 0$ and hence $p(\varkappa(x_N^k)) < p(\varkappa(x_N^0))$ for large enough k, contradicting our initial assumption.

If $Z(\varkappa(x_N^\omega)) = Z(\varkappa(x_N^k))$ for all k, but $p(\varkappa(x_N^\omega)) < p(\varkappa(x_N^0))$, then we would have $p(\varkappa(x_N^k)) < p(\varkappa(x_N^0))$ for large enough k for the same reason as in the preceding paragraph. Finally, the situation of $Z(\varkappa(x_N^\omega)) = Z(\varkappa(x_N^k))$, $\varkappa(x_N^\omega)_{Z(\varkappa(x_N^\omega))} \sim^{\mathrm{Lmin}} \varkappa(x_N^k)_{Z(\varkappa(x_N^k))}$, and $p(\varkappa(x_N^\omega)) = p(\varkappa(x_N^k))$ for all k is also incompatible with our assumption, exactly as at the end of the proof of Theorem 3.

8 Conclusion

To summarize, the main objective of this paper was twofold. First, to advance as far as possible towards establishing that (quasi)separable aggregation of local utilities is necessary and sufficient for the guaranteed acyclicity of individual improvements in trim CLU games. Second, to find out reasonable additional assumptions under which a Nash equilibrium exists. Concerning the first goal, the sufficiency of separability for acyclicity is derived from essentially the same good old construction of Rosenthal's; the necessity of some "separability-style" properties without restrictions on the domain are obtained for the first time ever. The existence of Nash equilibrium for certain quasiseparable aggregation rules is established; in particular, in bottleneck congestion games with an infinite number of facilities.

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