Should Central Banks Take On Credit-Risk?

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Abstract

Central banks have a long tradition of minimizing their exposure to credit-risk. The Federal Reserve's response to the recent financial crisis, which entailed greater risk-taking, has raised the question of whether such 'unusual' practices are desirable. This paper addresses the vacuum in the literature with a highly simplified model that has nevertheless the characteristics missing in the literature: it is a monetary dynamic stochastic general equilibrium model with an inflation-targeting central bank, aggregate risk, bankruptcy, and it is tractable. The main contribution is showing that, in an economy with bankruptcy rights and considerably indebted households, a Central Bank that operates exclusively with risk-free assets effects important distortions; in particular, it benefits the failure-free industries and punishes the failure-prone ones. Thus, on average, it takes longer for the economy to recover and risk-taking behaviour is pro-cyclical. This is even with complete financial markets, perfect competition, both well-behaved production technologies and preferences, as well as flexible prices. Other results include conditions under which the behavior of the Credit-Spread is pro-cyclical or counter-cyclical (despite of a constant probaility of failure) and proposals of different monetized subsidy schemes that would avoid the distortions and even the Zero Lower Bound problem.

JEL: E02, E13, E42, E43, E44, E51, E52, E58, G12, G13.

Keywords: Central Bank, Monetary Policy, Default, Bankruptcy, Cash-in-Advance Models, Inflation Targeting, Credit Spreads, Zero Lower Bound.

1 Introduction

Should central banks take on credit risk? Up until the last financial crisis, they surely didn't; and some still don't. But the financial crisis changed the landscape, and some big

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central banks (like the Fed and the ECB) employed different "unconventional" policies, most of which involved greater risk-taking in one way or another.[1] Surprisingly though, there is little–if any–theory for answering the question of whether central banks should take on credit-risk (a question already posed by Cecchetti, back in 2009).

This paper aims to provide a starting point for answering such a complex question. It does so with a simple model and some greatly simplifying assumptions on the bankruptcy procedure. In spite of this, the paper yields a far more plausible framework than models that ignore bankruptcy altogether (as has usually been done), and shows that aggregate risk and bankruptcy procedures require central banks to modify their more traditional practices.

To be more specific, the model has four agents: The representative household, the representative risky firm, the representative safe firm, and an inflation-targeting central bank. The risky-industry (i.e., the representative risky firm) has a constant probability of failure. Prices are flexible and there is only one perishable good. There is no physical capital and the time horizon is infinite. Every period, households purchase the consumption good, work at the firms, transact financial assets, and receive both wages and dividends. Wages and dividends are paid out after the markets close though.

The inclusion of a risky industry and an inflation-targeting central bank sets this model apart from other cash-in-advance models. But what really separates this model from all the rest in the literature is the inclusion of bankruptcy rights without the loss of tractability.

Collateral and other credit market 'imperfections' haven't permeated the macro literature, probably because we see most big financing of firms and governments happening through unsecured loans or issuance of unsecured bonds. This paper shows how, in equilibrium, bankruptcy rights / limited liability of agents imply a set of restrictions that are very similar to those of imperfect (collateral-requiring) financial markets.

In the real world, bankruptcy laws are complex and typically have different approaches to different types of agents. Grossly speaking though, both consumers and firms have two similar choices when filing for bankruptcy: propose a credible plan to clean up their act and restructure their debts, or proceed with the liquidation of their assets to pay out the creditors and be discharged of any remaining debts. In the case of consumers, there is a certain amount of wealth that is exempt of liquidation and they can "start afresh" (although a public record of the bankruptcy remains for a couple of years). In the case of firms, there is no amount exempt from liquidation, they stop operations and go completely

[1] In fact, at the time of writing, the ECB announced a purchase program of sovereign bonds from euro area countries; which points to the timeliness and relevance of the opening question. See European Central Bank (2015).

out of business.[2]

In this paper's model, filing for bankruptcy will directly mean liquidation, for both households and firms. But bankruptcy declaration and liquidation of assets don't have any further consequence for households (no public record is kept), so they can start afresh immediately and borrow cash for that same period's consumption. For firms, there are no fixed, sunk, or startup costs, and as soon as they go out of business they are immediately replaced by new identical ones in a mechanical fashion. This is, in a nutshell, *the simplified bankruptcy framework* employed by the paper.

Financial markets are complete and there is a publicly-known time-invariant ordering of agents. Risk-free assets are paid out first; and within each asset class, assets are paid out according to the ordering of agents.[3]

The first result is that, in the absence of bankruptcy rights, there exists a traditional monetary policy (one that involves open market operations exclusively with risk-free assets) that achieves an optimal monetary equilibrium–one where the allocation of real resources is optimal, thus obviating the need for taking on credit-risk; furthermore, it is shown that this optimal monetary equilibrium implies a pro-cyclical credit-spread. On the other hand, the second result is that–in the presence of *the simplified bankruptcy framework*–a traditional monetary policy that achieves the inflation target will—in general—distort the economy. In fact, a third result shows that in an economy with bankruptcy rights and significantly indebted households, a traditional policy effects a counter-cyclical bias on the credit-spread, causing a slowdown of the recovery of output in recessions and greater risk-taking in expansions. Finally, it is shown that–under certain conditions–the credit-spread is entirely counter-cyclical for this last case. These results are summarized in the following table.

Equilibria with Traditional Monetary Policy

	Without Bankruptcy	**With Bankruptcy**
Inflation Target is	Perfectly Enforced	Perfectly Enforced
Allocation of (Real) Resources is	Optimal	Suboptimal
Credit-Spread is	Pro-Cyclical	Possibly Counter-Cyclical

[2]This description is based on the U.S. bankruptcy legislation.

[3]This priority rule is like Fama and Miller's "me-first" rule, except for (1) the priority of risk-free over risky assets and (2) the ordering of agents' loans being exogenous. This is done for simplicity. See Fama and Miller (1972, p.152).

The paper is related to different strands of literature. One strand of literature is that on the counter-cyclicality of credit-spreads. Gilchrist and Zakrajsek (2012) is an example of this empirical literature that has widely documented the aforesaid phenomenon; a phenomenon that is consistent with this model's conclusions on traditional monetary policy. Another strand of related literature is the one on consumer bankruptcy. Chatterjee and Gordon (2012) would be an example of this literature which is not involved with monetary policy. A third strand of related literature is that of Taylor-rules that take credit-spreads into account. Cúrdia, V., and M. Woodford (2010) is an example of this literature, which is concerned with the management of traditional monetary policy when taking an additional piece of information into account. There is another strand of literature concerned with the so-called credit multiplier. Olivella and Roldan-Peña (2013) is an example of this literature, which deals with collateralized loans, and typically omits money. Finally, there is the literature on financial stability/fragility. This literature is likewise concerned with bankruptcy (or the closely related 'default'), but it focuses on regulation of financial intermediaries rather than on monetary policy. An example of this would be Goodhart, C., Sunirand, P., and D. Tsomocos (2006).

The paper proceeds as follows. Section 2 presents the baseline model and some general definitions. Section 3 characterizes the equilibrium with traditional monetary policy in the absense of bankruptcy rights. Section 4 introduces bankruptcy rights through the aforementioned *simplified bankruptcy framwork*; it then shows that the rational expectations equilibrium for the economy with bankruptcy rights can be defined by adding a few new stochastic constraints to the (previous) equilibrium definition for the economy without bankruptcy rights. Having done so, the section proceeds to characterize the suboptimality of the equilibrium with traditional monetary policy in the presence of bankruptcy rights in a couple of important results. Section 5 touches upon a critical semantics issue that separates exogenous from endogenous bankruptcy in Rational Expectations Equilibria and suggests a new practice in economic modeling. Section 6 provides a few options of how to implement the optimal monetary policy in the presence of bankruptcy rights. Section 7 mentions how the Zero Lower Bound problem vanishes under one of the proposed options, and reflects on what the paper's model has to say on Credit-Spreads. Section 8 concludes.

2 Baseline Model

The monetary DSGE model of the paper is (purposely) built upon a very simple real economy (hereafter 'the underlying real-economy') whose equilibrium is unique and optimal in a Pareto sense. This unique, optimal equilibrium is used as a benchmark throughout the paper. The corresponding (unique and optimal) allocation is used as a measuring rod for optimality: monetary equilibria are considered optimal–or suboptimal–if their allocation (of real resources) is the same as–or different from–that of the benchmark.

Comparisons of relative prices in the monetary economy against relative prices in the underlying real-economy are also useful for characterizing the price-distortions or absence thereof. After all, in a flexible price economy money is supposed to be just a veil.

The need for money is exclusively justified by a standard cash-in-advance constraint, so preferences and production possibilities are unaffected. As a result, the use of the 'underlying real-economy' as a benchmark is not only warranted but also called for.

Both the underlying real-economy and the different measures of credit-risk are covered in the appendices A and B in greater detail.

Let us get started with the description of the model. Firms are non-financial, meaning that they do not make loans, and are not allowed to borrow but for paying wages. Extraordinary gains or losses are passed on to shareholders in the form of (potentially negative) dividends. This means shareholders have to "chip in" if the firm is not able to meet its debt payments. Throughout the whole paper, the same production technology and preferences are kept. Information is symmetric, but size is not. All but the central bank are small agents and have therefore no market power. The central bank has market power, but its objective is price-stability (i.e., not profit maximization); it can therefore take all but the consumption-good price as given for making its choices (without further consequence), thus simplifying the analysis.[4]

There are many economy-wide state-variables: the technological state (or total factor productivity), A, which follows a *deterministic* path; the "luck" state, Υ, which is an i.i.d. random variable taking the value one with probability $\upsilon \in (0,1)$ and zero otherwise; the aggregate amount of money at the beginning of the period, M, which depends on previous balances and monetary policy; the previous price level, $\mathfrak{P}$, which is the price level that cleared last period's market for the consumption good; the output of the representative safe firm, $Y^{\mathbb{SF}}$, which is predetermined; the output of the representative risky firm conditional on a "lucky" state, Y^{RF}, which is also predetermined; last period's purchase of safe assets by the representative safe firm, $B_{\mathsf{s}}^{\mathbb{SF}}$, whose return in cash will be paid out this period; last period's purchase of "up" assets by the representative safe firm, $B_{\mathsf{u}}^{\mathbb{SF}}$, whose return in cash will be paid out this period if the state is a "lucky" one; last period's purchase of safe assets by the representative risky firm, $B_{\mathsf{s}}^{\mathrm{RF}}$, whose return in cash will be paid out this period; last period's purchase of "up" assets by the representative risky firm,

[4]It could also take into account its market-power/effect on other prices, but it would be a useless exercise resulting in the same decision-rule.

$B^{\text{RF}}_{\mathsf{u}}$, whose return in cash will be paid out this period if the state is a "lucky" one; and last period's purchase of safe assets by the central bank, $B^{\text{CB}}_{\mathsf{s}}$, whose return in cash will be paid out this period. All but the central bank's "B"s take negative values in equilibrium since–by assumption–firms sell (not buy) financial assets. As for the central bank, the sign of $B^{\text{CB}}_{\mathsf{s}}$ is–in general–ambiguous.

All economy-wide state-variables are found in the vector S. That is,

$$S = (A, \Upsilon, M, \mathfrak{P}, Y^{\text{SF}}, Y^{\text{RF}}, B^{\text{SF}}_{\mathsf{s}}, B^{\text{SF}}_{\mathsf{u}}, B^{\text{RF}}_{\mathsf{s}}, B^{\text{RF}}_{\mathsf{u}}, B^{\text{CB}}_{\mathsf{s}}) \ .$$

Before we begin describing each agent, we need to introduce the concept–and notation–of 'preceding state'.

Definition. *$S^{\backprime}$ or 'S backprime' is said to be a preceding state, or predecessor, of S if and only if* $\Pr(S \mid S^{\backprime}) > 0$.

Notice that if the state contains some variables that are endogenous to the system, then the concept of preceding state will be, in general, equilibrium-dependent.

2.1 Household

At the beginning of each period, households have some cash-holdings m and some net-of-cash financial wealth w (which are financial claims to be redeemed in the current period). Like all agents in the economy, households know the economy-wide state S at the beginning of the period as well. They choose their current consumption c, their future cash-holdings m', their purchased amount of safe bonds $b^{\text{H}'}_{\mathsf{s}}$, and their purchased amount of risky bonds $b^{\text{H}'}_{\mathsf{u}}$ to maximize their total utility.[5] The given price for the consumption good is $\mathcal{P}(S)$; the given price for the safe bond is $\mathcal{I}_{\mathsf{s}}(S)$; the given price for the risky bond is $\mathcal{I}_{\mathsf{u}}(S)$; the given price for labor services is $\mathcal{W}(S)$; and the given value of received dividends is $D(S)$.

The value function is

$$V^{\text{H}}(m, w, S) = \max_{c, m', b^{\text{H}'}_{\mathsf{s}}, b^{\text{H}'}_{\mathsf{u}}} \left\{ u(c) + \beta \, \text{E} \left[V^{\text{H}}(m', w', S') \right] \right\} \tag{1}$$

subject to

$$\mathcal{P}(S)c + \mathcal{I}_{\mathsf{s}}(S) b^{\text{H}'}_{\mathsf{s}} + \mathcal{I}_{\mathsf{u}}(S) b^{\text{H}'}_{\mathsf{u}} \leq m + w \tag{2}$$

$$m' + \mathcal{P}(S)c + \mathcal{I}_{\mathsf{s}}(S) b^{\text{H}'}_{\mathsf{s}} + \mathcal{I}_{\mathsf{u}}(S) b^{\text{H}'}_{\mathsf{u}} \leq m + w + \mathcal{W}(S) + D(S) \tag{3}$$

$$0 \leq m', c \tag{4}$$

[5]Recall that risky bonds pay only if the realization of the Bernoulli random variable Υ' equals one; and also notice that these two bonds are enough for having complete markets.

where

$$w' \equiv b_{\mathsf{s}}^{\mathrm{H}\prime} + \Upsilon' b_{\mathsf{u}}^{\mathrm{H}\prime} \tag{5}$$

$$D(S) \equiv \left(\mathcal{P}(S)Y^{\mathrm{SF}} + B_{\mathsf{s}}^{\mathrm{SF}} + \Upsilon B_{\mathsf{u}}^{\mathrm{SF}}\right) + \left(\mathcal{P}(S)\Upsilon Y^{\mathrm{RF}} + B_{\mathsf{s}}^{\mathrm{RF}} + \Upsilon B_{\mathsf{u}}^{\mathrm{RF}}\right) \tag{6}$$

The instant utility function $u(\cdot)$ is increasing and of the CRRA class, with a positive relative-risk-aversion coefficient.[6] Qualitatively analogous results can be obtained for more general utility functions.

The first restriction (the liquidity constraint) requires housholds to hold liquid wealth in advance to pay for consumption and asset purchases. In a sense, net-of-cash wealth w can be seen as inside money. The second restriction is the usual one, and the third restriction is just for the non-negativity of consumption and cash holdings.

The definition of future net-of-cash financial wealth w' is straightforward. Dividends $D(S)$ are defined as the sum of dividends from the safe industry and dividends from the risky industry; each of them being simply the profit of the corresponding industry, given the state S and the price $\mathcal{P}(S)$.

Recall that for the purpose of this paper, filing for bankruptcy can only mean liquidation, for both household and firms. So a household will file for bankruptcy at the beginning of the period whenever his wealth in case of paying is less than the exempted amount whenever he declares bankrupt, that is whenever

$$m + w < e(S^{\backprime})$$

where $e(S^{\backprime})$ is the exempted amount as a function of the preceding state. Of course, in the absence of bankruptcy rights this whole paragraph is vacuous.

2.2 Safe Firm

It is assumed that firms (including risky ones) maximize the present value of their whole profits-stream, given the prices of the complete set of assets. The main result of the paper (the theorem in section 4) prevails exactly if we have households own firms (in fact, a latter assumption is not even required if we do so).[7] Still, we have chosen the present approach for the sake of exposition and simplicity.[8]

[6]The utility function $u(\cdot)$ is of the CRRA class if and only if $u_c(x) \equiv \frac{\mathrm{d}u(x)}{\mathrm{d}x} = \alpha_1 x^{-\alpha_2}$, with α_1 and α_2 real numbers. The number α_2 is the (constant) relative risk aversion coefficient, which is defined as $-\frac{u_{cc}(x)x}{u_c(x)}$. The paper assumes $\alpha_1, \alpha_2 > 0$.

[7]This assumption is that, when indifferent, firms prefer to finance themselves by issuing safe, rather than risky, debt.

[8]To be clear, *this* section's results would not be exactly the same because of liquidity premia; but taking this other path would only distract us from the–far more pressing–topic of this paper.

The paper employs the easier decision-problem of maximizing the value of next period's profits, instead of maximizing the value of the whole stream of profits. Both approaches are equivalent for the present paper.[9] At the beginning of the period, the safe firm knows the economy-wide state S. It chooses how much labor l^{SF} to hire and how to finance this through the purchase[10] of safe bonds $b_{\mathsf{s}}^{\mathrm{SF}\prime}$ and risky bonds $b_{\mathsf{u}}^{\mathrm{SF}\prime}$. $\mathcal{M}(S',S)$ is the monetary stochastic discount factor and $\Psi^{\mathrm{SF}}(l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime}, S', S)$ is next period's profit for the safe firm (which depends on next period's state S'); they are both defined below.

The value function for the safe firm is therefore

$$V^{\mathrm{SF}}(S) = \max_{l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime}} \left\{ \mathrm{E}\left[\mathcal{M}(S',S)\Psi^{\mathrm{SF}}(l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime}, S', S) \right] \right\} \tag{7}$$

subject to

$$\mathcal{W}(S)l^{\mathrm{SF}} + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{SF}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{SF}\prime} = 0 \tag{8}$$

$$b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime} \geq 0 \tag{9}$$

where

$$\Psi^{\mathrm{SF}}(l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime}, S', S) \equiv \mathcal{P}(S')Af(l^{\mathrm{SF}}) + b_{\mathsf{s}}^{\mathrm{SF}\prime} + \Upsilon' b_{\mathsf{u}}^{\mathrm{SF}\prime} \tag{10}$$

$$\mathcal{M}(S',S) \equiv \Upsilon' \frac{\mathcal{I}_{\mathsf{u}}(S)}{\upsilon} + (1-\Upsilon')\frac{(\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))}{1-\upsilon} \tag{11}$$

The function $f(\cdot)$ employed for both the risky and the safe firm is twice continuously differentiable with $f(0) = 0$, $f_l(\cdot) > 0 > f_{ll}(\cdot)$, $\lim_{x\to 0^+} f_l(x) = \infty$, and a Relative Risk Aversion coefficient $-\frac{f_{ll}(x)x}{f_l(x)}$ smaller than $1 - \alpha_2 x \frac{[\gamma f_l(x) - f_l(1-x)]}{[\gamma f(x) + f(1-x)]}$. This last requirement is a sufficient condition to guarantee uniqueness for some final results of the paper.

Recall that for the purpose of this paper, filing for bankruptcy can only mean liquidation, for both household and firms. So firms will do so if they have losses. Now, bankruptcy must be declared at the very beginning of each period, so a representative safe firm will file for bankruptcy at the beginning of the period if

$$\mathcal{P}(S)Y^{\mathrm{SF}} + B_{\mathsf{s}}^{\mathrm{SF}} + \Upsilon B_{\mathsf{u}}^{\mathrm{SF}} < 0$$

In the absence of bankruptcy rights this last paragraph is vacuous.

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[9]Basically, today's choices don't affect the future decision problem because shareholders are assumed to to take profits or pay residual claims such that the firm begins next period with a clean slate.

[10]The firm will actually sell bonds (borrow), and the sign of this variables will be non-positive; but we have modeled all bond transactions of agents as purchases for notational ease.

2.3 Risky Firm

At the beginning of the period, the risky firm knows the economy-wide state S. It chooses how much labor l^{RF} to hire and how to finance this through the purchase[11] of safe bonds $b_{\mathsf{s}}^{\mathrm{RF}\prime}$ and risky bonds $b_{\mathsf{u}}^{\mathrm{RF}\prime}$. $\mathcal{M}(S',S)$ is the monetary stochastic discount factor and $\Psi^{\mathrm{RF}}(l^{\mathrm{RF}}, b_{\mathsf{s}}^{\mathrm{RF}\prime}, b_{\mathsf{u}}^{\mathrm{RF}\prime}, S', S)$ is next period's profit for the risky firm (which depends on next period's state S'); they are both defined below.

The value function is

$$V^{\mathrm{RF}}(S) = \max_{l^{\mathrm{RF}}, b_{\mathsf{s}}^{\mathrm{RF}\prime}, b_{\mathsf{u}}^{\mathrm{RF}\prime}} \left\{ \mathrm{E}\left[\mathcal{M}(S',S)\Psi^{\mathrm{RF}}(l^{\mathrm{RF}}, b_{\mathsf{s}}^{\mathrm{RF}\prime}, b_{\mathsf{u}}^{\mathrm{RF}\prime}, S', S) \right] \right\} \tag{12}$$

subject to

$$\mathcal{W}(S)l^{\mathrm{RF}} + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{RF}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{RF}\prime} = 0 \tag{13}$$

$$b_{\mathsf{s}}^{\mathrm{RF}\prime}, b_{\mathsf{u}}^{\mathrm{RF}\prime} \geq 0 \tag{14}$$

where

$$\Psi^{\mathrm{RF}}(l^{\mathrm{RF}}, b_{\mathsf{s}}^{\mathrm{RF}\prime}, b_{\mathsf{u}}^{\mathrm{RF}\prime}, S', S) \equiv \mathcal{P}(S')\Upsilon'\gamma A f(l^{\mathrm{RF}}) + b_{\mathsf{s}}^{\mathrm{RF}\prime} + \Upsilon' b_{\mathsf{u}}^{\mathrm{RF}\prime} \tag{15}$$

$$\mathcal{M}(S',S) \equiv \Upsilon' \frac{\mathcal{I}_{\mathsf{u}}(S)}{\upsilon} + (1-\Upsilon')\frac{(\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))}{1-\upsilon} \tag{16}$$

Recall that for the purpose of this paper, filing for bankruptcy can only mean liquidation, for both household and firms. So firms will do so if they have losses. Now, bankruptcy must be declared at the very beginning of each period, so a representative risky firm will file for bankruptcy at the beginning of the period if

$$\mathcal{P}(S)\Upsilon Y^{\mathrm{RF}} + B_{\mathsf{s}}^{\mathrm{RF}} + \Upsilon B_{\mathsf{u}}^{\mathrm{RF}} < 0$$

Again, in the absence of bankruptcy rights this last paragraph is vacuous.

2.4 Inflation-Targeting Central Bank

It is assumed that the central bank cares only about currently reaching the inflation target π^{T} or, equivalently, having current prices $\mathcal{P}$ equal past prices times one plus the inflation target $\mathfrak{P}(1+\pi^{\mathrm{T}})$. Since we are limiting ourselves to *traditional monetary policy*, the only instrument of the central bank is the purchase/sale of risk-free bonds $b_{\mathsf{s}}^{\mathrm{CB}\prime}$. The central

[11] The firm will actually sell bonds (borrow), and the sign of this variables will be non-positive; but we have modeled all bond transactions of agents as purchases for notational ease.

bank's value function is parameterized for simplicity with an indicator function $I(\cdot)$. So it will reach the value 1 if the inflation target is perfectly enforced for the current period ($\pi = \pi^{\mathrm{T}}$) and 0 if not ($\pi \neq \pi^{\mathrm{T}}$):

$$V^{\mathrm{CB}}(S) = \max_{b_{\mathsf{s}}^{\mathrm{CB}\prime}} \left\{ I\Big(\mathcal{P}(b_{\mathsf{s}}^{\mathrm{CB}\prime}, S) = (1+\pi^{\mathrm{T}})\mathfrak{P}\Big) \right\} \tag{17}$$

where

$$\mathcal{P}(b_{\mathsf{s}}^{\mathrm{CB}\prime}, S) = \frac{M - \Upsilon(B_{\mathsf{u}}^{\mathrm{SF}} + B_{\mathsf{u}}^{\mathrm{RF}}) - (B_{\mathsf{s}}^{\mathrm{SF}} + B_{\mathsf{s}}^{\mathrm{RF}}) - B_{\mathsf{s}}^{\mathrm{CB}} - \mathcal{W}(S) + \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\mathrm{CB}\prime}}{Y^{\mathrm{SF}} + \Upsilon Y^{\mathrm{RF}}} \tag{18}$$

The last function is derived 'by the central bank' assuming the household's liquidity constraint is binding and markets clear (conditions that will be covered in the next section).[12] The reader will find the detailed derivation in the appendix. Recall that, because it only cares about currently reaching its inflation-target, it doesn't matter if the central bank takes wages and interest rates as given. This greatly simplifies the analysis though.[13]

2.5 Credit-Risk and Cyclicality

We now introduce two measures of credit-risk that will be widely used in our results.

Definition. *The 'nominal risk-ratio', denoted by* $\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}}$*, is the ratio between the safe bond's price and the risky bond's price.*

The nominal risk-ratio, although never used in the financial world, is the most important of both measures from an economic point of view: it is a relative price that should be equal to the corresponding relative price of the 'underlying real-economy'. All the bonds in the main text are one-period bonds. So the interest rate i_j of the bond j with price $\mathcal{I}_j$ is simply given by the relation $\mathcal{I}_j = \frac{1}{1+i_j}$ for $j = \mathsf{u}, \mathsf{s}$.

Definition. *The 'nominal risk-spread', denoted by* ξ*, is the difference between the nominal interest rate of the risky bond and the nominal interest rate of the safe bond:*

$$\xi \equiv i_{\mathsf{u}} - i_{\mathsf{s}} \ .$$

[12]It can be shown that, in this paper's model, a money-growth target is weakly dominated by a price-growth target in terms of welfare. We only mention this in passing, since it is a topic outside of the paper's scope.

[13]Notice that, from this last equation (multiplying it by Y you get $PY = M -$ other terms, but by definition $PY \equiv MV$ so $V = 1 -$ other terms$/M$), the velocity of money is not constant. This is because $-\Upsilon(B_{\mathsf{u}}^{\mathrm{SF}} + B_{\mathsf{u}}^{\mathrm{RF}}) - (B_{\mathsf{s}}^{\mathrm{SF}} + B_{\mathsf{s}}^{\mathrm{RF}}) - B_{\mathsf{s}}^{\mathrm{CB}} - \mathcal{W}(S) + \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\mathrm{CB}\prime}$ would have to be the same for both values of Υ (one of the variables included in the state-vector S).

The nominal risk-spread is the widestly used measure of Credit-Risk. And the paper has results with regards to it as well.

Since most of the results are intimately related to the cyclicality of certain variables, we shall introduce a formal definition now. Naturally, we will use the growth factor of output, or $\frac{Y(S')}{Y(S)}$, for this.

Definition. *A variable $X(S)$ is said to be pro-cyclical if and only if its conditional-on-the-preceding-state correlation with $\frac{Y(S')}{Y(S)}$ is positive, for every preceding state $S^{\backprime}$:*

$$corr\left(X(S), \frac{Y(S')}{Y(S)} \;\middle|\; S^{\backprime} \right) > 0 \qquad \forall S^{\backprime}$$

For the sequential case, a stochastic process $\{X_t\}_{t=0}^{\infty}$ is said to be pro-cyclical if and only if $corr\left(X_t, \frac{Y_{t+1}}{Y_t}\right) > 0$ for all t, given the stochastic process $\{\frac{Y_{t+1}}{Y_t}\}_{t=0}^{\infty}$. A counter-cyclical variable is defined by using the opposite strict-inequality signs, and an acyclical variable is defined by using equality signs.

Let us briefly touch upon the evolution of S.

The economy's state $S = (A, \Upsilon, M, \mathfrak{P}, Y^{\text{SF}}, Y^{\text{RF}}, B^{\text{SF}}_{\mathsf{s}}, B^{\text{SF}}_{\mathsf{u}}, B^{\text{RF}}_{\mathsf{s}}, B^{\text{RF}}_{\mathsf{u}}, B^{\text{CB}}_{\mathsf{s}})$ evolves according to the following laws of motion

$$A' = h_A(A), \quad \text{for some suitable function } h_A(\cdot) \; ; \tag{19}$$

$$\Upsilon' = \begin{cases} 1 & \text{with invariant probability } \upsilon, \\ 0 & \text{otherwise;} \end{cases} \tag{20}$$

$$M' = \int m'(S); \tag{21}$$

$$\mathfrak{P}' = \mathcal{P}(S) \; ; \tag{22}$$

$$Y^{\text{SF}\prime} = \int Af(l^{\text{SF}}(S)) \; ; \tag{23}$$

$$Y^{\text{RF}\prime} = \int \gamma Af(l^{\text{RF}}(S)) \; ; \tag{24}$$

$$B^{\text{SF}\prime}_{\mathsf{s}} = \int b^{\text{SF}\prime}_{\mathsf{s}}(S) \; ; \tag{25}$$

$$B^{\text{SF}\prime}_{\mathsf{u}} = \int b^{\text{SF}\prime}_{\mathsf{u}}(S) \; ; \tag{26}$$

$$B^{\text{RF}\prime}_{\mathsf{s}} = \int b^{\text{RF}\prime}_{\mathsf{s}}(S) \; ; \tag{27}$$

$$B^{\text{RF}\prime}_{\mathsf{u}} = \int b^{\text{RF}\prime}_{\mathsf{u}}(S) \; ; \text{ and} \tag{28}$$

$$B^{\text{CB}\prime}_{\mathsf{s}} = b^{\text{CB}\prime}_{\mathsf{s}}(S) \; . \tag{29}$$

An integral sign $\int$ has been set to emphasize the aggregation of the small agents/the representative character of all but the Central Bank. It is timely to remark that $M' = M - B^{\text{CB}}_{\mathsf{s}} + \mathcal{I}_{\mathsf{s}}(S) b^{\text{CB}\prime}_{\mathsf{s}}$ in any equilibrium. That is, tomorrow's cash equals today's cash after today's central-bank operations.

Remark 2.1. *Issues of nominal indeterminacy are out of this paper's scope. The paper's interest lies in the credit-risk distortions caused by the central bank's implementation of monetary policy. Therefore, we will focus on equilibria where all nominal risk-free interest rates are positive (as in Lucas (1982)), and write "equilibria with positive rates" for short.*[14]

[14] There are many assumptions (or relaxations thereof) that ensure determinacy. 'Positive rates' is a sufficient, but not necessary, condition. We avoid making these assumptions or relaxations thereof, for the sake of simplicity.

3 Equilibrium with unlimited liability (absence of bankruptcy rights)

Definition. *An equilibrium without bankruptcy rights consists of decision rules,* $c(S), m'(S), b_{\mathsf{s}}^{\mathrm{H}\prime}(S), b_{\mathsf{u}}^{\mathrm{H}\prime}(S)$ *for the representative household,* $l^{\mathrm{SF}}(S), b_{\mathsf{s}}^{\mathrm{SF}\prime}(S), b_{\mathsf{u}}^{\mathrm{SF}\prime}(S)$ *for the representative safe firm,* $l^{\mathrm{RF}}(S), b_{\mathsf{s}}^{\mathrm{RF}\prime}(S), b_{\mathsf{u}}^{\mathrm{RF}\prime}(S)$ *for the representative risky firm,* $b_{\mathsf{s}}^{\mathrm{CB}\prime}(S)$ *for the central bank, and a state-dependent price-vector* $\langle \mathcal{P}(S), \mathcal{I}_{\mathsf{s}}(S), \mathcal{I}_{\mathsf{u}}(S), \mathcal{W}(S) \rangle$ *such that:*

1. *the decision rules for the household are a solution to problem (1), given the price-vector* $\langle \mathcal{P}(S), \mathcal{I}_{\mathsf{s}}(S), \mathcal{I}_{\mathsf{u}}(S), \mathcal{W}(S) \rangle$ *and the laws of motion;*
2. *the decision rules for the safe firm are a solution to problem (7), given the price-vector* $\langle \mathcal{P}(S), \mathcal{I}_{\mathsf{s}}(S), \mathcal{I}_{\mathsf{u}}(S), \mathcal{W}(S) \rangle$ *and the laws of motion;*
3. *the decision rules for the risky firm are a solution to problem (12), given the price-vector* $\langle \mathcal{P}(S), \mathcal{I}_{\mathsf{s}}(S), \mathcal{I}_{\mathsf{u}}(S), \mathcal{W}(S) \rangle$ *and the laws of motion;*
4. *the decision rule for the central bank is a solution to problem (17), given the price-vector* $\langle \mathcal{I}_{\mathsf{s}}(S), \mathcal{I}_{\mathsf{u}}(S), \mathcal{W}(S) \rangle$ *and the laws of motion; and*
5. *all markets clear:*

$$\int c(S) = Y^{\mathrm{SF}} + \Upsilon Y^{\mathrm{RF}} \equiv Y \qquad \forall S; \qquad (30)$$

$$\int l^{\mathrm{SF}}(S) + \int l^{\mathrm{RF}}(S) = 1 \qquad \forall S; \qquad (31)$$

$$b_{\mathsf{s}}^{\mathrm{CB}\prime}(S) + \int b_{\mathsf{s}}^{\mathrm{H}\prime}(S) + \int b_{\mathsf{s}}^{\mathrm{SF}\prime}(S) + \int b_{\mathsf{s}}^{\mathrm{RF}\prime}(S) = 0 \qquad \forall S; \qquad (32)$$

$$\int b_{\mathsf{u}}^{\mathrm{H}\prime}(S) + \int b_{\mathsf{u}}^{\mathrm{SF}\prime}(S) + \int b_{\mathsf{u}}^{\mathrm{RF}\prime}(S) = 0 \qquad \forall S. \qquad (33)$$

Proposition 1. *In the absence of bankruptcy rights, every equilibrium with positive rates:*

- *is optimal (i.e., the allocation of real resources is the same as that of the underlying real economy);*
- *has a perfectly enforced inflation target (i.e.,* $\mathcal{P}(S') = (1 + \pi^{\mathrm{T}})P(S)$ *for all* S' *given* S*, for all* S*);*

- *has the same price-vector* $\langle \mathcal{P}(S), \mathcal{I}_{\mathsf{s}}(S), \mathcal{I}_{\mathsf{u}}(S), \mathcal{W}(S)\rangle$ *(i.e., there is price-vector determinacy);*
- *has a constant nominal risk-ratio* $\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}}$.

In other words, money is (super) neutral under the "traditional implementation" of monetary policy by an inflation-targeting central bank.[15] Importantly, this neutrality/freedom from distortions will not be the case when we move on to economies with bankruptcy rights.

An important, and perhaps surprising, feature of these monetary equilibria is the pro-cyclicality of the nominal risk-spread (a.k.a. credit-spread). We touch upon this feature now.

Claim 1. *Every equilibrium from Proposition 1 has a pro-cyclical nominal risk-spread.*

The logic behind the result is as follows. First of all, the nominal risk-spread $i_{\mathsf{u}} - i_{\mathsf{s}}$ satisfies $i_{\mathsf{u}} - i_{\mathsf{s}} = (\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}} - 1)(1 + i_{\mathsf{s}})$ whenever an inflation target is perfectly enforced.[16] Second, the previous proposition stated that every equilibria has a constant nominal risk-ratio $\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}}$. Thus the only variation in the nominal risk-spread comes from the variation in the nominal risk-free rate i_{s}. In fact, since inflation is constant at equilibrium, the only source of variation is the risk-free real interest rate. And, as in typical consumption-based asset-pricing models, the risk-free rate is positively correlated with expected consumption growth (which in this model equals expected output growth).

4 Equilibrium with limited liability (presence of bankruptcy rights)

In the real world, as previously said, typically both consumers and firms have the following two choices when filing for bankruptcy: propose a credible plan to clean up their act and restructure their debts, or proceed with the liquidation of their assets to pay out the creditors and be discharged of any remaining debts. In the case of consumers, there is a certain amount that is exempt from liquidation and they can "start afresh" (although a public record of the bankruptcy event remains for a couple of years). In the case of firms, there is no amount exempt from liquidation, they stop operations and go completely out of business.

[15] In the sense that not only the level but also the growth rate of the price level is innocuous for the allocation of (real) resources.

[16] See Appendix B.2 .

In this paper's model, on the other hand, we assume *a simplified bankruptcy framework* where: (1) filing for bankruptcy necessarily entails liquidation, for both households and firms, although households still have the right to an exempted amount; (2) bankruptcy filing and liquidation of assets don't have any further consequence for households (no public record is kept), so they can start afresh immediately and borrow cash for that same period's consumption; and (3) firms have no fixed, sunk, or startup costs, and as soon as they go out of business they are immediately replaced by new identical ones in a mechanical fashion.

It is assumed that there is a publicly-known time-invariant ordering of agents, and that redemption of bonds follows two rules: (1) Risk-free assets are paid out first; and (2) within each asset class, assets are paid out according to the ordering of agents.[17]

Now, recall that:

- The representative safe firm will file for bankruptcy at the beginning of the period if

$$\mathcal{P}(S)Y^{\mathrm{SF}} + B^{\mathrm{SF}}_{\mathsf{s}} + \Upsilon B^{\mathrm{SF}}_{\mathsf{u}} < 0$$

- The representative risky firm will file for bankruptcy at the beginning of the period if

$$\mathcal{P}(S)\Upsilon Y^{\mathrm{RF}} + B^{\mathrm{RF}}_{\mathsf{s}} + \Upsilon B^{\mathrm{RF}}_{\mathsf{u}} < 0$$

- And the representative household will file for bankruptcy at the beginning of the period if

$$m + w < e(S^{`})$$

If any of these three bankruptcy cases occur (safe firm, risky firm, or household), *it means that at least one agent in the economy is not being repaid according to the return-vector.*[18] Hence, at least one agent is not rational and this cannot be a Rational Expectations Equilibrium.[19] Indeed, these are out-of-equilibrium bankruptcies. We will talk about in-equilibrium bankruptcies in the next section.

[17]As previously noted, this priority rule is like Fama and Miller's "me-first" rule except for (1) the priority of risk-free over risky assets and (2) the ordering of agents' loans being exogenous. This last assumption is done for simplicity.

[18]Formally, it should read "a positive measure of agents is not being repaid..."

[19]Remember, in a Rational Expectations Equilibrium everyone knows the model and–hence–everyone else's strategies. Since assets are paid out according to a public ordering of agents, every agent knows in advance whether he will be on the "short list" or not.

4.1 Immediate Implications of Bankruptcy Rights for the Rational Expectations Equilibrium (REE)

Recall our assumptions that (1) there is a publicly-known time-invariant ordering of agents; (2) Risk-free assets are paid out first; and (3) within each asset class, assets are paid out according to the ordering of agents. A rational expectations equilibrium cannot have agents lending at the same rate as everyone else, if they know they are not getting paid according to the return-vector of the asset (like everyone else is).

It can be shown that this implies a new set of restrictions for a rational expectations equilibrium. Namely:

$$m' + b_{\mathsf{s}}^{\mathrm{H}\prime} + \Upsilon' b_{\mathsf{u}}^{\mathrm{H}\prime} \geq e(S) \qquad \text{a.s.} \tag{34}$$

$$\mathcal{P}(S')Af(l^{\mathrm{SF}}) + b_{\mathsf{s}}^{\mathrm{SF}\prime} + \Upsilon' b_{\mathsf{u}}^{\mathrm{SF}\prime} \geq 0 \qquad \text{a.s.} \tag{35}$$

$$\mathcal{P}(S')\Upsilon'\gamma Af(l^{\mathrm{RF}}) + b_{\mathsf{s}}^{\mathrm{RF}\prime} + \Upsilon' b_{\mathsf{u}}^{\mathrm{RF}\prime} \geq 0 \qquad \text{a.s.} \tag{36}$$

where 'a.s.' stands for 'almost surely' (i.e., with probability one).

Thus, despite the completeness and perfection of financial markets, bankruptcy considerations imply that agents are 'endogenously constrained' in their borrowings. This is because in a rational equilibrium no one will lend an amount beyond the "repayment-is-optimal boundary." *Borrowing constraints are by no means new, but this model's constraints are considerably different from the typical constraints.* First, typical constraints are exogenously set; here, on the other hand, agents are in principle unconstrained in their choice problems but *it is in equilibrium that these constraints have to be satisfied.* In other words, it is not that firms and/or households cannot borrow any further,[20] it's that the households and/or the central bank would not be rational in lending them any further because they wouldn't be repaid as promised.[21] This equilibrium-quality of the constraints does not require a completely different mathematical method of solution, but it does make an important difference on the understanding, policy implications, and interpretation of the economy. Second, typical borrowing constraints depend on the value of the agent's collateral at the time of borrowing; here, on the other hand, loans are unsecured but courts force debtors to pay with any available means, so the constraints depend on the *future* value of the borrower's means. Third, typical borrowing constraints depend exclusively on deterministic variables; here, on the other hand, constraints depend on the distribution of random variables as well.

[20]In fact, one could set all the constraints into the household's and central bank's choice problems, and leave the firms' problems intact. This is because, if counterparties are unwilling to lend, the effect is–in equilibrium–that of a borrowing constraint.

[21]To be precise, a positive measure of them wouldn't be repaid at all.

4.2 The Suboptimality of Traditional Policy in the presence of Bankruptcy Rights

We add the three new REE stochastic restrictions (equations 34–36) to the equilibrium definition and *assume that firms finance themselves exclusively with safe debt if they are indifferent between issuing safe or risky debt* (this assumption is not necessary if households control firms, which would be the formal way to proceed, but the exposition would be more cumbersome). The following lemma sets the stage for our main result.

Lemma 1. *In the model with bankruptcy, equilibria with positive rates*

1. *are–in general–suboptimal, and*

2. *have a perfectly enforced inflation target (i.e., $\mathcal{P}(S') = (1+\pi^{\mathrm{T}})P(S)$ for all S' given S, for all S).*

Grossly speaking, the central bank has to flood/drain the economy with/of money when there are expected contractions/expansions in order to achieve its inflation target. That would not matter if it wasn't for the bankruptcy restrictions, which limit the way some agents can borrow/lend.

The proof of the lemma employs the fact that one of the REE stochastic restrictions can be simplified into the non-stochastic restriction

$$m' + b_{\mathsf{s}}^{\mathrm{H}'} \geq e(S) \tag{37}$$

and that another REE stochastic restriction can be simplified into the non-stochastic restriction

$$b_{\mathsf{s}}^{\mathrm{RF}'} = 0 \; . \tag{38}$$

For some primitives of the model, there are states where households sell risk-free debt until they cannot credibly borrow anymore (at which point restriction (37) starts to bind). From that point onwards, further liquidity distorts the economy because households cannot entirely compensate the increase in the credit-spread by purchasing risky debt and selling risk-free debt (and risky firms can only finance themselves through risky debt due to restriction (38)); yet further liquidity is required to achieve the inflation target. This results in a higher-than-optimal amount of labor at the safe industry, thus lowering the expected output-growth.

The result on the perfect enforcement of the inflation target is extensively used in the upcoming theorem; we state the following definition and assumption before.

Definition. *An equilibrium is said to have significantly indebted households if*

$$m'(S) + b_{\mathsf{s}}^{\mathrm{H}'}(S) \leq e(S)$$

for all S.

In other words, households are significantly indebted if they are willing to file for bankruptcy (or indifferent about doing so) in the unlucky states. Notice that the amount of the household's risky debt is irrelevant for this definition.

Assumption. *Let the exempted amount $e(S)$ be equal or less–but arbitrarily close–to the nominal income $\mathcal{N}(S)$, which is the sum of wage-income and dividends: $\mathcal{N}(S) \equiv \mathcal{W}(S) + D(S)$.*

This assumption is not necessary for the results, but it is sufficient to prove them with ease.

Theorem. *In the presence of bankruptcy rights, equilibria with positive rates and significantly indebted households*

1. *have a counter-cyclical nominal risk-ratio $\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}}$,*

2. *have a counter-cyclical amount of real resources allocated to the safe industry; indeed implying that the economy takes more risk in expansions than in recessions and has slower recoveries on average.*

In short, if the equilibrium has significantly indebted households, then constraint (37) is *always* satisfied with equality (and–in general–binding). Naturally, the distortion is bigger during expected contractions (since the bankruptcy constraint is more binding in such cases): both the nominal risk-ratio $\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}}$ and the amount of labor at the safe industry are greater there. Hence their counter-cyclicality.

Following the previous line of thought, if the nominal risk-ratio $\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}}$ varies even more than the risk-free rate, then the nominal risk spread will become entirely countercyclical.[22] The following claim provides a sufficient–though by no means necessary–condition for this to be true.

Claim 2 (Theorem extension on counter-cyclical Credit-Spreads)**.** *For each set of primitives, there is a real number $\overline{\alpha} > 0$ such that: if the Constant Relative Risk Aversion coefficient is $\alpha_2 < \overline{\alpha}$, then the equilibrium nominal risk-spread (a.k.a. credit-spread) is counter-cyclical as well.*

[22] Recall that the nominal risk-spread $i_{\mathsf{u}} - i_{\mathsf{s}}$ satisfies $i_{\mathsf{u}} - i_{\mathsf{s}} = (\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}} - 1)(1 + i_{\mathsf{s}})$ whenever an inflation target is perfectly enforced.

In fact, like most of the sufficient conditions of this paper, this previous condition is far from necessary; it is quite possible that counter-cyclicality of the credit-spread holds for a much wider range of parameters.

In the next section, we will touch upon an important semantics question regarding bankruptcy in the presence of rational expectations.

5 De-Facto vs De-Jure Returns: the Semantics of Default under Rational Expectations

As awful as this section's title is, it is an incredibly important section that readers are urged to pay attention to.

Suppose we now assumed that the risky firm can only issue debt that is to be repayed in all the states of the world, and label it after the issuer (i.e. the risky firm). Let us employ the subindex RF that we have used so far for this purpose. We would then have that the risky firm can only finance itself as follows:

$$\mathcal{W}(S)l^{\mathrm{RF}} + \mathcal{I}_{\mathrm{RF}}(S)b_{\mathrm{RF}}^{\mathrm{RF}\prime} = 0$$

where $\mathcal{I}_{\mathrm{RF}}(S)$ is the equilibrium price that agents would pay for the risky firm's 'de jure safe' debt.

Now, because this new kind of debt issued by the risky firm is 'de jure safe', the risky firm must take that new repayment schedule (i.e., return vector) into account when computing its (state-dependent) profits:

$$\Psi^{\mathrm{RF}}(l^{\mathrm{RF}}, b_{\mathrm{RF}}^{\mathrm{RF}\prime}, S', S) \equiv \mathcal{P}(S')\Upsilon'\gamma A f(l^{\mathrm{RF}}) + b_{\mathrm{RF}}^{\mathrm{RF}\prime}$$

So the risky firm is selling *debt that, bankruptcy considerations aside, it would be committed to pay in both states of the world.* Of course, given the bankruptcy laws of the economy, the shareholders would rather have the risky firm declare itself bankrupt and close for business than bail it out. Under rational expectations, then, the fact that this new debt is 'de jure safe' is immaterial: if it walks like a dog and talks like a dog, then it's just another dog no matter what its owner calls it. And notice that *it was not only a relabeling* that went on: the contract itself was indeed not state-contingent anymore. What makes this new 'de jure safe' debt a 'de facto risky' debt is that shareholders have limited liability and in the bad state the risky firm will therefore default and go bankrupt. The interpretation of the economy changes considerably: now the risky firm *does* default, *does* go bankrupt, and *does* go out of business. So this economy *does* have bankruptcy in equilibrium! It goes without saying that this modification addresses the valuable observation by Goodhart and

Tsomocos (2009) that "Standard DSGE models do not include the possibility of default". Moreover, it addresses this issue without the hassles of intractability found in other models of endogenous default.

We see that the nomenclature of rational-expectations bankruptcy can be confusing. And that is because the following two worlds are apparently equivalent: in one, the risky firm sells 'de jure safe' bonds, defaults, and goes bankrupt; in the other, the risky firm sells 'de jure risky' bonds, does not default, and stays in business. Surely the first world seems closer to reality. And so, before ending this discussion for good, we would like to suggest a new practice in economic models with bankruptcy/default: have assets be labeled after the issuer (and not pool them together if their de-facto return-vectors are different). In the real world firms typically issue only 'de jure' safe debt (with different priorities), and the market takes care of categorizing it in 'de facto' terms (say, as risky or safe debt). This should replace the traditional approach with purely "anonymous" assets if the profession aims to take default/bankruptcy into account. Only then, and in rational expectations equilibrium, should agents decide what is 'safe' and what is 'risky' and what is 'junk' and so on, and put assets issued by different entities in the same bag.

6 But How Should Central Banks Take On Credit-Risk?

The theoretical results from the economy with bankruptcy show that, indeed, "traditional monetary policy" has undesirable consequences, and that having the central bank take on credit risk could alleviate the economy from distortions. Furthermore, empirical studies show that open-market-operations with risky assets do have an effect on credit-spreads (consistent with the present model).[23] But *how* should central banks take on credit risk?

The immediate answer would be "simply by purchasing risky assets," just like the FED has done by purchasing mortgage-backed-securities (MBS) and by selling insurance to distressed institutions, and like both the FED and the ECB have done by giving loans to distressed institutions, and so on.[24] But this route is not a long term solution: choosing one particular set of risky assets / risky lessees is not recommended because of market power / distorted expectations / distorted spot markets, and so on. On the other hand, using all the assets, though optimal, is computationally unrealistic / subject to human error; and as a consequence the central bank could again, though presumably to a lesser

[23]See, for example, Rai(2013).

[24]For the FED, see Federal Reserve of New York (2013), Cecchetti (2009), and Kotlikoff and Mehrling (2008). For the ECB, see Trichet (2009) and Reuters (2014).

extent, distort the economy. Is there an 'easy way' to guarantee the absence of distortions in a way that is not as prone to human error? Possibly. We discuss some options that apply even under a variable probability of risky-industry failure.

6.1 Proportional Monetary Transfers: Theoretically Common, Impossible in Practice

Practically all the monetary models with variable money-supply employ this device. It can be summarized as follows: Let i denote the index of agents in the economy. Then, if i's cash balances at the end of the day are $\overline{m}_i$, his balances at the beginning of the next day will be $m_i' = \overline{m}_i \cdot (1+\theta_m)$, where θ_m is the same for all agents (though it may be stochastic and/or vary over time). As a result of this mechanism, the relative distribution[25] of cash in the economy remains intact with respect to the monetary policy itself. This is an ideal mechanism, but it is also an impossible one: *the central banker would have to be omnipresent in order to pay interest on everyone's cash balances at exactly the same time.*

To illustrate this mechanism with this paper's model, one would have cash transfers be state-dependent and take place right after the state of the world is revealed. As a result, the household's second constraint (equation (3)) would change to

$$\overline{m} + \mathcal{P}(S)c + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}\prime} \leq m + w + \mathcal{W}(S) + D(S) \tag{39}$$

with

$$m' = \overline{m}(1 + \theta_m(S')) \tag{40}$$

And the value function would be

$$V^{\mathrm{H}}(m, w, S) = \max_{c,\overline{m},b_{\mathsf{s}}^{\mathrm{H}\prime},b_{\mathsf{u}}^{\mathrm{H}\prime}} \left\{u(c) + \beta\,\mathrm{E}\left[V^{\mathrm{H}}(m', w', S')\right]\right\} \tag{41}$$

Meanwhile, the Central Bank would have $b_{\mathsf{s}}^{\mathrm{CB}\prime}(S) = 0$ for all S, and instead use θ_m as its instrument:

$$V^{\mathrm{CB}}(S) = \max_{\theta_m}\{I(\mathcal{P}(\theta_m, S) = (1 + \pi^{\mathrm{T}})\mathfrak{P}\} \tag{42}$$

where

$$\mathcal{P}(\theta_m, S) = \frac{\underline{M}(1 + \theta_m) - \Upsilon(B_{\mathsf{u}}^{\mathrm{SF}} + B_{\mathsf{u}}^{\mathrm{RF}}) - (B_{\mathsf{s}}^{\mathrm{SF}} + B_{\mathsf{s}}^{\mathrm{RF}}) - \mathcal{W}(S)}{Y^{\mathrm{SF}} + \Upsilon Y^{\mathrm{RF}}} \tag{43}$$

where $\underline{M}$ is one of the economy's state variables included in vector S and it's

$$\underline{M}' = \int m \tag{44}$$

[25] That is, the distribution of 'the percentage of the total'.

The household's bankruptcy constraint (equation (34)) cannot be binding in this case, because the household will lend positive amounts to both the safe and the risky firm. That is, households do not borrow 'safe' to lend 'risky'. Distortions would be absent.

6.2 Monetized Subsidies for Consumption: Theoretically Obvious, Long Shot in Practice

Since the central bank's aim is to control the price-increase in the consumption-good, one (awful) way to do this would be to have the central bank purchase/sale the consumption good at the desired price (just like when fixing an exchange rate). But in a world with so many different consumption goods this task would be excessively expensive and flawed.

There is, however, the possibility of providing a proportional subsidy (a percentage of the price) for all consumption goods. In fact, the 'infrastructure' for doing so is already available, namely that of Tax Authorities. Furthermore, providing this subsidy may encourage the report of informal transactions and thus help reduce the level of informality in some countries. This could actually be the best mechanism for implementing monetary policy. But right now, it is not.

It has taken a lot of time and trouble to separate monetary from fiscal policy, to have governments become more responsible, and to have people understand (or at least accept) the fact that a low inflation rate is the best for society as a whole. Mixing fiscal and monetary policies again could confuse things and destroy these achievements.

To illustrate this mechanism with this paper's model, one would have consumers receive a proportional subsidy θ_c for their consumption good purchases. That is, households would pay $(1-\theta_c)P(S)$ monetary units for each consumption unit they buy. The size of this subsidy would be state-dependent.

The household's constraints (equations (2) and (3)) would become

$$(1-\theta_c(S))\mathcal{P}(S)c + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}\prime} \leq m + w \tag{45}$$

$$m' + (1-\theta_c(S))\mathcal{P}(S)c + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}\prime} \leq m + w + \mathcal{W}(S) + D(S) \tag{46}$$

Meanwhile, the Central Bank would have $b_{\mathsf{s}}^{\mathrm{CB}\prime}(S) = 0$ for all S, and instead use θ_c as its instrument:

$$V^{\mathrm{CB}}(S) = \max_{\theta_c}\{I(\mathcal{P}(\theta_c, S) = (1+\pi^{\mathrm{T}})\mathfrak{P}\} \tag{47}$$

where

$$\mathcal{P}(\theta_c, S) = \frac{M - \Upsilon(B_{\mathsf{u}}^{\mathrm{SF}} + B_{\mathsf{u}}^{\mathrm{RF}}) - (B_{\mathsf{s}}^{\mathrm{SF}} + B_{\mathsf{s}}^{\mathrm{RF}}) - \mathcal{W}(S)}{(1-\theta_c)(Y^{\mathrm{SF}} + \Upsilon Y^{\mathrm{RF}})} \tag{48}$$

As with proportional monetary transfers, the household's bankruptcy constraint (equation (34)) cannot be binding in this case, because the household will lend positive amounts to both the safe and the risky firm. That is, households do not borrow 'safe' to lend 'risky'. Distortions would be absent.

6.3 Monetized Subsidies for Bonds: Theoretically Sound, Achievable in Practice

A far less disruptive way to implement the optimal monetary policy would be by subsidizing bonds. In this paper's economy, that amounts to paying a percentage of the bond's market price to the issuer at the time of sale.

This mechanism would avoid any zero-lower-bound concerns since the lender would always receive a positive nominal interest rate (in equilibrium).[26]

Implementing this mechanism is no easy task, but many well-organized markets for risky bonds are already in place and the Central Bank has a natural connection with financial markets.

To illustrate this mechanism with this paper's model, one would have households receive a proportional subsidy θ_b for their bond purchases. That is, households would pay $(1-\theta_b)\mathcal{I}_i(S)$ for bond i (where $i = \mathsf{s}, \mathsf{u}$). In other words, households receive $\theta_b \mathcal{I}_i(S)$ monetary units for every $\mathcal{I}_i(S)$ monetary units they lend to firms. This size of the subsidy would be state-dependent.

The household's constraints (equations (2) and (3)) would become

$$\mathcal{P}(S)c + (1-\theta_b)\mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}\prime} + (1-\theta_b)\mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}\prime} \leq m + w \tag{49}$$

$$m' + \mathcal{P}(S)c + (1-\theta_b)\mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}\prime} + (1-\theta_b)\mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}\prime} \leq m + w + \mathcal{W}(S) + D(S) \tag{50}$$

Meanwhile, the Central Bank would have $b_{\mathsf{s}}^{\mathrm{CB}\prime}(S) = 0$ for all S, and instead use θ_b as its instrument:

$$V^{\mathrm{CB}}(S) = \max_{\theta_b}\{I(\mathcal{P}(\theta_b, S) = (1+\pi^{\mathrm{T}})\mathfrak{P}\} \tag{51}$$

where

$$\mathcal{P}(\theta_b, S) = \frac{M - \Upsilon(B_{\mathsf{u}}^{\mathrm{SF}} + B_{\mathsf{u}}^{\mathrm{RF}}) - (B_{\mathsf{s}}^{\mathrm{SF}} + B_{\mathsf{s}}^{\mathrm{RF}}) - (1-\theta_b)\mathcal{W}(S)}{Y^{\mathrm{SF}} + \Upsilon Y^{\mathrm{RF}}} \tag{52}$$

As with proportional monetary transfers, the household's bankruptcy constraint (equation (34)) cannot be binding in this case, because the household will lend positive amounts

[26] This is, of course, in an economy were *nominal* money balances never shrink over time, which happens to be the case for every single economy in the world throughout history.

to both the safe and the risky firm. Again, households will not borrow 'safe' to lend 'risky'. Distortions would be absent.

7 Implications for the Zero Lower Bound and the Credit-Spread

The Zero Lower Bound problem vanishes with this paper's proposal of Monetary Policy implementation: by giving a proportional monetary subsidy to bonds (primary market), the interest that lenders receive is always enough to guarantee their supply of loanable funds/money, and avoid the hoarding of it. The subsidy would allow for borrowers to pay negative nominal interest rates and lenders to receive positive nominal interest rates, if necessary.

For the Credit Spread, one is able to obtain an interesting implication under the strong assumption that Υ is an i.i.d. random variable, an assumption made throughout the paper for the sake of tractability.

Recall that, given a constant industry-failure probability, the optimal policy is such that the relative price of risky to risk-free bonds remains constant, just as in the underlying real-economy. This implies that the credit-spread increases with expansive optimal monetary policy, and decreases with contractive optimal monetary policy.[27]

8 Conclusion

Today's monetary systems are the result of historical accidents and disaster-responses. Hopefully, the present work will provide a platform to show that it is possible to implement monetary policies in ways that would speed-up recoveries and decrease the procyclicality of risk-taking behavior, all of this while reducing the misallocation of resources / eliminating distortions. The model of this paper is, naturally, a very crude representation of the world, particularly when it comes to the deterministic nature of Total Factor Productivity, the independence of industry failures, and the relationship between industry failures and Total Factor Productivity. Future improvements along these lines may render a model that is not only theoretically insightful, but that is also helpful as a practical tool.

[27]Where an expansive/contractive monetary policy is defined as an increase/decrease in the monetary subsidy rate.

References

[1] Altman, E. I., and E. Hotchkiss (2006): *Corporate Financial Distress and Bankruptcy*. Third edition. Hoboken, New Jersey: John Wiley & Sons Inc.

[2] Board of Governors of the Federal Reserve System, "Open Market Operations," http://www.federalreserve.gov/monetarypolicy/openmarket.htm (accessed April 3rd, 2013).

[3] Cecchetti, S. G. (2009): "Crisis and Responses: The Federal Reserve in the Early Stages of the Financial Crisis," *Journal of Economic Perspectives*, 23, 51–75.

[4] Chatterjee, S., and G. Gordon (2012): "Dealing with Consumer Default: Bankruptcy vs Garnishment," *Journal of Monetary Economics*, 59, S1–S16.

[5] Cochrane, J. H. (2005): *Asset Pricing*. Revised edition. Princeton, New Jersey: Princeton University Press.

[6] Cúrdia, V., and M. Woodford (2010): "Credit-Spreads and Monetary Policy," *Journal of Money, Credit and Banking*, 42, 3–35.

[7] European Central Bank (2015), "ECB announces expanded asset purchase programme," http://www.ecb.europa.eu/press/pr/date/2015/html/pr150122_1.en.html (accessed February 12th, 2015).

[8] European Central Bank, "Open Market Operations," http://www.ecb.europa.eu/mopo/implement/omo/html/index.en.html (accessed March 25th, 2014).

[9] Fama, E. F., and M. H. Miller (1972): *The Theory of Finance*. New York: Hold, Rhinehart and Winston.

[10] Federal Reserve Bank of New York (2013): *2012 Annual Report*. http://www.newyorkfed.org/aboutthefed/annual/annual12/annual.pdf

[11] Gilchrist, S., and E. Zakrajsek (2012): "Credit Spreads and Business Cycle Fluctuations," *American Economic Review*, 102, 1692–1720.

[12] Goodhart, C. (2010): "Is a Less Pro-Cyclical Financial System an Achievable Goal?," *National Institute Economic Review*, 211, 81–90.

[13] Goodhart, C., and D. Tsomocos (2009): "Default and DSGE Models," *VOX*, http://www.voxeu.org/article/why-dsge-models-should-include-defaults.

[14] Goodhart, C., Sunirand, P., and D. Tsomocos (2006): "A model to analse financial fragility," *Economic Theory*, 27, 107–142.

[15] Goodhart, C., and D. Tsomocos (2011): "The Role of Default in Macroeconomics," IMES Discussion Paper No. 2011-E-23.

[16] Gust, C., and D. Lopez-Salido (2010): "Monetary Policy and the Cyclicality of Risk," International Finance Discussion Papers No. 999, Board of Governors of the Federal Reserve System.

[17] Gertler, M., and C. S. Lown (1999): "The Information in the High-Yield Bond Spread for the Business Cycle: Evidence and some Implications," *Oxford Review of Economic Policy*, 15, 132–150.

[18] Olivella, V., and J. Roldán-Peña (2013): "Re-examining the Role of Financial Constraints in Business Cycles: Is Something Wrong with the Credit Multiplier?," Banco de México Working Paper No. 2013-05.

[19] Kotlikoff, L. J., and P. Mehrling (2008): "Finally, System-Risk Insurance," *The Financial Times*, November 24.
http://blogs.ft.com/economistsforum/2008/11/finally-system-risk-insurance/

[20] Lucas, R. E. (1982): "Interest Rates and Currency Prices in a Two-Country World," *Journal of Monetary Economics*, 10, 335–359.

[21] Rai, A. M. (2013): "The Impact of Policy Initiatives on Credit Spreads during the 2007-2009 Financial Crisis," *International Journal of Central Banking*, 9, 45–104.

[22] Reuters (2014): "Banks to Return 19 Billion Euros of Crisis Loans to ECB Next Week," *Reuters.com*, March 21. http://uk.reuters.com/article/2014/03/21/uk-ecb-banks-repayment-idUKBREA2K0Q020140321 (accessed March 25th, 2014).

[23] Svensson, L. (1985): "Money and Asset Prices in a Cash-in-Advance Economy," *Journal of Political Economy*, 93, 919–944.

[24] Trichet, J. C. (2009): "The ECB's Enhanced Credit Support", keynote address at the University of Munich, July 13.
http://www.ecb.europa.eu/press/key/date/2009/html/sp090713.en.html

[25] U.S. Bankruptcy Court and U.S. Securities and Exchange Commission (2004): *Bankruptcy Basics*. Revised second edition. La Vergne, Tennessee: BN Publishing.

[26] White, M. J. (1989): "The Corporate Bankruptcy Decision," *Journal of Economic Perspectives*, 3, 129–151.

Appendix

A The underlying real economy

In this section, we shall present and characterize the underlying real economy of the paper. This will be a useful point of departure that will help us identify the optimum of the paper's monetary economy.

A.1 Preferences

The representative household of the economy has preferences given by:

$$U_t = U(\{c_s\}_{s=t}^{\infty}) = \mathrm{E}_t \left[\sum_{h=t}^{\infty} \beta^{t-h} u(c_h) \right]$$

The utility function $u(\cdot)$ satisfies the conditions stated in the main body of this paper. The available amount of the unique, perishable, consumption good—denoted Y—is predetermined, and there is no technology to transfer it across time-periods. So the household will consume all of it:

$$c_t = Y_t$$

The household is endowed every period with one unit of labor, which it supplies inelastically. The true choice of the household lies in which amounts of labor it will employ on each of the available production technologies.

A.2 Production technologies

There are two available production technologies: a 'safe' technology and a 'risky' technology. Both production technologies take today's labor as input in order to deliver a possibly-random amount of output next period. The safe production technology delivers an amount

$$Y_{t+1}^{\mathbb{SF}} = A_t f(l_t^{\mathbb{SF}})$$

of output. A is the total factor productivity, which follows a deterministic, commonly-known, path; l is labor; $f(\cdot)$ is a function satisfying the conditions stated in the main body of this paper; and $\mathbb{SF}$ stands for 'safe firm'.

Let Υ_t be an i.i.d. Bernoulli random variable with probability of success υ strictly between zero and one. The risky production technology delivers an amount

$$\Upsilon_{t+1} Y^{\text{RF}}_{t+1} = \Upsilon_{t+1} A_t \gamma f(l^{\text{RF}}_t)$$

of output. RF stands for 'risky firm'.

Naturally, next period's amount of available consumption good will be the sum of the two outputs. And hence, next period's consumption is given by

$$c_{t+1} = Y_{t+1} \equiv Y^{\text{SF}}_{t+1} + \Upsilon_{t+1} Y^{\text{RF}}_{t+1}$$

Thus next period's consumption is a—random—function of today's labor allocation and technology:

$$c_{t+1} = A_t f(l^{\text{SF}}_t) + \Upsilon_{t+1} A_t \gamma f(l^{\text{RF}}_t)$$

A.3 Optimum

Notice that there is no terminal condition (hence no need for a transversality condition), and that the allocation of labor between the safe and risky technologies is always interior and unique by our assumptions. Furthermore, let us assume that the growth-factor of A is bounded from above by β^{-1}; this guarantees that the Total Utility is always finite and hence that the mathematical problem is well-defined. Furthermore, because there is no accumulation or any other persistent effect of choices, the optimal choice of $(l^{\text{SF}}_t, l^{\text{RF}}_t)$ can be found without paying attention to other values of the sequence (i.e., values for $t+k$).

The optimal allocation of labor will satisfy the first order conditions:

$$\begin{aligned} l^{\text{SF}}_t + l^{\text{RF}}_t &= 1 \\ \beta \, \mathrm{E}_t \left[u_c \left(A_t f(l^{\text{SF}}_t) + \Upsilon_{t+1} A_t \gamma f(l^{\text{RF}}_t) \right) A_t f_l(l^{\text{SF}}_t) \right] &= \lambda \\ \beta \, \mathrm{E}_t \left[u_c \left(A_t f(l^{\text{SF}}_t) + \Upsilon_{t+1} A_t \gamma f(l^{\text{RF}}_t) \right) \Upsilon_{t+1} A_t \gamma f_l(l^{\text{RF}}_t) \right] &= \lambda \end{aligned}$$

We can easily calculate the expectations since there are only two possible states. The last two conditions become:

$$\begin{aligned} \upsilon \left[u_c \left(A_t f(l^{\text{SF}}_t) + A_t \gamma f(l^{\text{RF}}_t) \right) A_t f_l(l^{\text{SF}}_t) \right] + (1-\upsilon) \left[u_c \left(A_t f(l^{\text{SF}}_t) \right) A_t f_l(l^{\text{SF}}_t) \right] &= \lambda / \beta \\ \upsilon \left[u_c \left(A_t f(l^{\text{SF}}_t) + A_t \gamma f(l^{\text{RF}}_t) \right) A_t \gamma f_l(l^{\text{RF}}_t) \right] &= \lambda / \beta \end{aligned}$$

Together they imply

$$1 + \frac{(1-\upsilon) \;\; u_c\big(A_t f(l^{\text{SF}}_t)\big) \;\; A_t f_l(l^{\text{SF}}_t)}{\upsilon \;\; u_c(A_t f(l^{\text{SF}}_t) + A_t \gamma f(l^{\text{RF}}_t)) \;\; A_t f_l(l^{\text{SF}}_t)} = \frac{u_c \left(A_t f(l^{\text{SF}}_t) + A_t \gamma f(l^{\text{RF}}_t) \right) A_t \gamma f_l(l^{\text{RF}}_t)}{u_c \left(A_t f(l^{\text{SF}}_t) + A_t \gamma f(l^{\text{RF}}_t) \right) A_t f_l(l^{\text{SF}}_t)}$$

or

$$1+\frac{(1-\upsilon)\ \ u_c\big(A_t f(l_t^{\mathrm{SF}})\big)}{\upsilon\ \ u_c(A_t f(l_t^{\mathrm{SF}})+A_t\gamma f(l_t^{\mathrm{RF}}))}=\frac{\gamma f_l(l_t^{\mathrm{RF}})}{f_l(l_t^{\mathrm{SF}})}$$

which together with the restriction that $l_t^{\mathrm{SF}}+l_t^{\mathrm{RF}}=1$ gives

$$1+\frac{(1-\upsilon)\ \ u_c\big(A_t f(l_t^{\mathrm{SF}})\big)}{\upsilon\ \ u_c(A_t f(l_t^{\mathrm{SF}})+A_t\gamma f(1-l_t^{\mathrm{SF}}))}=\frac{\gamma f_l(1-l_t^{\mathrm{SF}})}{f_l(l_t^{\mathrm{SF}})}$$

At this point, we invoke CRRA. It can be shown (and we omit the proof) that if $u(\cdot)$ is a utility function with CRRA, then $u_c(kx)/u_c(ky)=u_c(x)/u_c(y)$ for all $k,x,y>0$.

Therefore, the condition of the underlying real economy's optimum can be further simplified into

$$\boxed{1+\frac{(1-\upsilon)\ \ u_c\big(f(l_t^{\mathrm{SF}})\big)}{\upsilon\ \ u_c(f(l_t^{\mathrm{SF}})+\gamma f(1-l_t^{\mathrm{SF}}))}=\frac{\gamma f_l(1-l_t^{\mathrm{SF}})}{f_l(l_t^{\mathrm{SF}})}}$$

which proves that the optimal allocation of labor, if it exists, is constant over time (since it is unique and independent of the state). It can be shown that, for a positive coefficient of relative risk aversion (as was assumed), the left-hand-side strictly decreases in l^{SF} from infinity to some positive finite number, and the right-hand-side strictly increases in l^{SF} from zero to infinity. Thus showing existence.

Given the primitives $u(\cdot)$ and $f(\cdot)$, one is able to compute the optimal labor allocation for every state. And, given the sequence $\{A_t\}_{t=0}^{\infty}$, one can also compute the value function arbitrarily well.

For our purposes, this last equation matters because it tells us when the corresponding monetary equilibrium is optimal (i.e., has the same allocation of real resources that would arise in a barter economy).

B Interest Rates and Credit-Risk

B.1 Interest rates and arbitrage

By construction, the price of a safe bond equals the inverse of its nominal return:

$$\mathcal{I}_s=\frac{1}{1+i_s}\ , \tag{53}$$

where i_s is the nominal interest rate of the safe bond.

Notice that every state S has only two possible succeeding states. Since a contingent bond pays one monetary unit in one of the possible succeeding states and zero in the rest,[28] we shall call that state the contingent bond's pay-state.

For the state-contingent bond $j \in \{\mathsf{u}, \mathsf{d}\}$, the price equals the inverse of its pay-state nominal return:

$$\mathcal{I}_j = \frac{1}{1 + i_j} \,, \tag{54}$$

where i_j is the pay-state nominal interest rate of bond $j = \mathsf{u}, \mathsf{d}$.[29]

In general, we may call $\mathcal{I}_j$ the nominal discount factor (or price, for short) of bond $j = \mathsf{s}, \mathsf{u}, \mathsf{d}$.

Remark B.1. *In a free market, the no-nominal-arbitrage condition $\mathcal{I}_\mathsf{s} = \mathcal{I}_\mathsf{u} + \mathcal{I}_\mathsf{d}$ is satisfied whenever future financial wealth has a positive shadow value at both possible states of the world.*

Intuitively, it must be that in equilibrium the nominal cost of a future dollar is the same whether through the purchase of a safe bond or the purchase of both contingent bonds. This *no-nominal-arbitrage condition* must be true whenever financial wealth has a positive shadow value in every state for some agent in the economy.

Recall the identity that relates nominal interest rate i, real interest rate r, and inflation π:

$$1 + i \equiv (1 + r)(1 + \pi) \,, \tag{55}$$

where the three variables are, in general, random / state-contingent and we have lower-cased them only for the purpose of exposition.

We introduce the pay-state real interest rate of contingent bond j, denoted r_j, by considering both the price-level and the nominal return at their pay-state as well:

$$(1 + r_j) \equiv \frac{1 + i_j}{1 + \pi_j} = (1 + i_j)\frac{p}{p_j} \qquad \text{for } j = \mathsf{u}, \mathsf{d}; \tag{56}$$

where the constant p is this period's price-level, and the constant p_j is next period's price-level at the pay-state.

[28] We treat a payment of zero as no payment. These type of bonds are often known as "Arrow securities".

[29] Notice that, in every-day parlance, people do not specify 'pay-state' even though they implicitly mean it. That is, when people say "that junk bond gives a 60% nominal interest rate" the statement is typically contingent on the "junk bond" paying. Nevertheless, we shall keep the "pay-state" qualifier since—formally—the nominal interest rate (without any further qualification) of a contingent bond j is a random variable with both, -1 and i_j, as possible outcomes.

We define the real discount factor of bond j as the inverse of the pay-state real return:

$$\mathcal{R}_j \equiv \frac{1}{1+r_j} = \mathcal{I}_j \frac{p_j}{p} \ . \tag{57}$$

One could easily consider a security which delivered the same real return in both states, and had therefore a well-defined risk-free real discount factor $\mathcal{R}_s$. In fact, any agent can construct such a security in a synthetic fashion with the help of both contingent bonds.

Whether the security is synthetic or not, we have the following result.

Remark B.2. *In a free market, the no-real-arbitrage condition* $\mathcal{R}_{\mathsf{s}} = \mathcal{R}_{\mathsf{u}} + \mathcal{R}_{\mathsf{d}}$ *is satisfied whenever future consumption has a positive value at both possible states of the world.*

One therefore has that the equilibrium safe real interest rate can be computed from this expression, since the *no-real-arbitrage condition* must be true as long as consumption is valued in both states by some agent in the economy.

Notice that these two conditions (no-nominal-arbitrage and no-real-arbitrage) are independent if $p_{\mathsf{u}} \neq p_{\mathsf{d}}$. However, if the central bank is able to somehow guarantee $p_{\mathsf{u}} = p_{\mathsf{d}}$, then both conditions imply each other. Without a central bank, $p_{\mathsf{u}} = p_{\mathsf{d}}$ cannot be guaranteed and inefficiencies may arise because of this.

B.2 Market-based measures of credit-risk

Let us now define three different measures of credit-risk for the model. This, in turn, will help us characterize the behavior of credit-risk in our model.[30]

Definition. *The 'nominal risk-spread', denoted by ξ, is the difference between the pay-state nominal interest rate of the risky bond and the nominal interest rate of the safe bond:*

$$\xi \equiv i_{\mathsf{u}} - i_{\mathsf{s}} \ .$$

The following measure will be frequently employed in the main text, for the sake of exposition.

Definition. *The 'nominal risk-ratio', denoted by $\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}}$, is the ratio between the safe bond's price and the risky bond's price.*

When talking about distortions, real relative prices–like the one in the following definition–are crucial.

[30] For simplicity and consistency with the rest of the paper we use the bond $j = \mathsf{u}$ as "the" risky bond. But the terminology applies to any Arrow security.

Definition. *The 'real risk-ratio', denoted by ρ, is the ratio of the risky bond's real return to the safe bond's real return:*

$$\rho \equiv \frac{1+r_{\mathsf{u}}}{1+r_{\mathsf{s}}}$$

By definition (eq. 57), the right-hand-side is equal to the relative price of real safe-debt to real risky-debt. So the real risk-ratio can also be seen as the relative price of safe to risky debt in the underlying real economy: $\rho = \frac{\mathcal{R}_{\mathsf{s}}}{\mathcal{R}_{\mathsf{u}}}$. Naturally, we would expect this relative-price to be weakly greater than one.

Let the constant p_{u} be the price level for the succeeding state with $\Upsilon = 1$, and the constant p_{d} be the price level for the succeeding state with $\Upsilon = 0$. Then we get the following useful expression.

Claim 3. *Under the no-real-arbitrage and no-nominal-arbitrage conditions, the real risk-ratio can be expressed in nominal terms as*

$$\rho = \frac{\mathcal{I}_{\mathsf{s}} p_{\mathsf{d}}}{\mathcal{I}_{\mathsf{u}} p_{\mathsf{u}}} - \frac{p_{\mathsf{d}}}{p_{\mathsf{u}}} + 1 \tag{58}$$

Proof. From the no-real-arbitrage condition we have

$$\mathcal{R}_{\mathsf{s}} = \mathcal{R}_{\mathsf{u}} + \mathcal{R}_{\mathsf{d}} \tag{59}$$

$$\Rightarrow \frac{1}{1+r_{\mathsf{s}}} = \frac{1}{1+r_{\mathsf{u}}} + \frac{1}{1+r_{\mathsf{d}}} \tag{60}$$

using the definition of real return we have

$$= \frac{1+\pi_{\mathsf{u}}}{1+i_{\mathsf{u}}} + \frac{1+\pi_{\mathsf{d}}}{1+i_{\mathsf{d}}} \tag{61}$$

$$= \mathcal{I}_{\mathsf{u}} \frac{p_{\mathsf{u}}}{p} + \mathcal{I}_{\mathsf{d}} \frac{p_{\mathsf{d}}}{p} \tag{62}$$

using the no-nominal-arbitrage condition we have

$$= \mathcal{I}_{\mathsf{u}} \frac{p_{\mathsf{u}}}{p} + (\mathcal{I}_{\mathsf{s}} - \mathcal{I}_{\mathsf{u}}) \frac{p_{\mathsf{d}}}{p} \tag{63}$$

$$= \mathcal{I}_{\mathsf{u}} \frac{p_{\mathsf{u}}}{p} + \mathcal{I}_{\mathsf{s}} \frac{p_{\mathsf{d}}}{p} - \mathcal{I}_{\mathsf{u}} \frac{p_{\mathsf{d}}}{p} \tag{64}$$

so we get the intermediate result

$$\mathcal{R}_{\mathsf{s}} = \frac{\mathcal{I}_{\mathsf{u}}}{p}(p_{\mathsf{u}} - p_{\mathsf{d}}) + \mathcal{I}_{\mathsf{s}} \frac{p_{\mathsf{d}}}{p} \ . \tag{65}$$

Now, from the definition of real risk-ratio we have

$$\rho \equiv \frac{1+r_{\mathsf{u}}}{1+r_{\mathsf{s}}} = (1+r_{\mathsf{u}})\mathcal{R}_{\mathsf{s}} = \frac{1+i_{\mathsf{u}}}{1+\pi_{\mathsf{u}}}\mathcal{R}_{\mathsf{s}} = \frac{p}{p_{\mathsf{u}}}\frac{\mathcal{R}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}} \tag{66}$$

$$\Rightarrow \rho = \frac{(p_{\mathsf{u}} - p_{\mathsf{d}})}{p_{\mathsf{u}}} + \frac{\mathcal{I}_{\mathsf{s}} p_{\mathsf{d}}}{\mathcal{I}_{\mathsf{u}} p_{\mathsf{u}}} \tag{67}$$

$$\Rightarrow \rho = \frac{\mathcal{I}_{\mathsf{s}} p_{\mathsf{d}}}{\mathcal{I}_{\mathsf{u}} p_{\mathsf{u}}} - \frac{p_{\mathsf{d}}}{p_{\mathsf{u}}} + 1 \; . \tag{68}$$

□

Corollary 1. *If $p_{\mathsf{u}} = p_{\mathsf{d}}$ (say, if there is a monetary policy that somehow ensures this) then*

$$\rho \equiv \frac{\mathcal{R}_{\mathsf{s}}}{\mathcal{R}_{\mathsf{u}}} = \frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}} \tag{69}$$

That is, the real risk-ratio equals the nominal risk-ratio. Moreover, in this case we also have that the nominal risk-spread and the nominal risk-ratio are related according to the equivalent equalities

$$\rho = \xi \mathcal{I}_{\mathsf{s}} + 1 \quad \Leftrightarrow \quad \xi = (\rho - 1)(1 + i_{\mathsf{s}}) \tag{70}$$

C Equilibrium for the model without bankruptcy

C.1 Full Derivation of Kuhn-Tucker Conditions

For the household, recall that

$$V^{\mathrm{H}}(m, w, S) = \max_{c, m', b_{\mathsf{s}}^{\mathrm{H}\prime}, b_{\mathsf{u}}^{\mathrm{H}\prime}} \left\{ u(c) + \beta \, \mathrm{E}\left[V^{\mathrm{H}}(m', w', S') \right] \right\} \tag{71}$$

subject to

$$\mathcal{P}(S)c + \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\mathrm{H}\prime} + \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{H}\prime} \leq m + w \tag{72}$$

$$m' + \mathcal{P}(S)c + \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\mathrm{H}\prime} + \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{H}\prime} \leq m + w + \mathcal{W}(S) + D(S) \tag{73}$$

$$0 \leq m', c \tag{74}$$

where

$$w' \equiv b_{\mathsf{s}}^{\mathrm{H}\prime} + \Upsilon' b_{\mathsf{u}}^{\mathrm{H}\prime} \tag{75}$$

$$D(S) \equiv \left(\mathcal{P}(S) Y^{\mathrm{SF}} + B_{\mathsf{s}}^{\mathrm{SF}} + \Upsilon B_{\mathsf{u}}^{\mathrm{SF}} \right) + \left(\mathcal{P}(S) \Upsilon Y^{\mathrm{RF}} + B_{\mathsf{s}}^{\mathrm{RF}} + \Upsilon B_{\mathsf{u}}^{\mathrm{RF}} \right) \tag{76}$$

The corresponding Lagrangian is

$$
\begin{aligned}
&\mathcal{L}(c, m', b_{\mathsf{s}}^{\mathrm{H}\prime}, b_{\mathsf{u}}^{\mathrm{H}\prime}, \lambda_1, \lambda_2) = \\
&u(c) + \beta\,\mathrm{E}\left[V^{\mathrm{H}}(m', w', S')\right] \\
&+ \lambda_1[w + m - \mathcal{P}(S)c - \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}\prime} - \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}\prime}] \\
&+ \lambda_2[w + m + \mathcal{W}(S) + D(S) - m' - \mathcal{P}(S)c - \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}\prime} - \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}\prime}]\,.
\end{aligned}
\tag{77}
$$

The two last inequality restrictions are omitted: consumption is positive because $u(\cdot)$ has an infinite derivative at zero; new money balances are positive by construction (firms cannot keep money balances between periods).

The Kuhn-Tucker conditions are

$$c: \quad u_c - \lambda_1\mathcal{P}(S) - \lambda_2\mathcal{P}(S) = 0 \tag{78}$$

$$m': \quad \beta\,\mathrm{E}[V_{m'}^{\mathrm{H}}] - \lambda_2 = 0 \tag{79}$$

$$b_{\mathsf{s}}^{\mathrm{H}\prime}: \quad \beta\,\mathrm{E}[V_{w'}^{\mathrm{H}}] - \lambda_1\mathcal{I}_{\mathsf{s}}(S) - \lambda_2\mathcal{I}_{\mathsf{s}}(S) = 0 \tag{80}$$

$$b_{\mathsf{u}}^{\mathrm{H}\prime}: \quad \beta\,\mathrm{E}[V_{w'}^{\mathrm{H}}\Upsilon'] - \lambda_1\mathcal{I}_{\mathsf{u}}(S) - \lambda_2\mathcal{I}_{\mathsf{u}}(S) = 0 \tag{81}$$

$$\lambda_1: \quad \lambda_1[w + m - \mathcal{P}(S)c - \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}\prime} - \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}\prime}] \geq 0 \tag{82}$$

$$\lambda_2: \quad \lambda_2[w + m + \mathcal{W}(S) + D(S) - m' - \mathcal{P}(S)c - \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}\prime} - \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}\prime}] \geq 0 \tag{83}$$

There is an abuse of notation, by taking $ab \geq 0$ instead of $a, b \geq 0$ and $ab = 0$.

From updating the envelope conditions we get

$$V_{m'}^{\mathrm{H}} = \lambda_1' + \lambda_2' \tag{84}$$

$$V_{w'}^{\mathrm{H}} = \lambda_1' + \lambda_2' \tag{85}$$

Rewriting the KT conditions we get

$$c: \quad \frac{u_c}{\mathcal{P}(S)} = \lambda_1 + \lambda_2 \tag{86}$$

$$m': \quad \beta\,\mathrm{E}[\lambda_1' + \lambda_2'] = \lambda_2 \tag{87}$$

$$b_{\mathsf{s}}^{\mathrm{H}\prime}: \quad \frac{\beta\,\mathrm{E}[\lambda_1' + \lambda_2']}{\mathcal{I}_{\mathsf{s}}(S)} = \lambda_1 + \lambda_2 \tag{88}$$

$$b_{\mathsf{u}}^{\mathrm{H}\prime}: \quad \frac{\beta\,\mathrm{E}[(\lambda_1' + \lambda_2')\Upsilon']}{\mathcal{I}_{\mathsf{u}}(S)} = \lambda_1 + \lambda_2 \tag{89}$$

$$\lambda_1: \quad \lambda_1[w + m - \mathcal{P}(S)c - \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}\prime} - \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}\prime}] \geq 0 \tag{90}$$

$$\lambda_2: \quad \lambda_2[w + m + \mathcal{W}(S) + D(S) - m' - \mathcal{P}(S)c - \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}\prime} - \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}\prime}] \geq 0 \tag{91}$$

For the safe firm, recall that we have

$$V^{\mathrm{SF}}(S) = \max_{l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime}} \left\{ \mathrm{E}\left[\mathcal{M}(S', S)\Psi^{\mathrm{SF}}(l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime}, S', S) \right] \right\} \tag{92}$$

subject to

$$\mathcal{W}(S)l^{\mathrm{SF}} + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{SF}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{SF}\prime} = 0 \tag{93}$$

where

$$\Psi^{\mathrm{SF}}(l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime}, S', S) \equiv \mathcal{P}(S')Af(l^{\mathrm{SF}}) + b_{\mathsf{s}}^{\mathrm{SF}\prime} + \Upsilon' b_{\mathsf{u}}^{\mathrm{SF}\prime} \tag{94}$$

$$\mathcal{M}(S', S) \equiv \Upsilon' \frac{\mathcal{I}_{\mathsf{u}}(S)}{\upsilon} + (1 - \Upsilon') \frac{(\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))}{1 - \upsilon} \tag{95}$$

Using the definitions of the stochastic discount factor and the state-dependent profits we have that

$$\mathrm{E}\left[\mathcal{M}(S', S)\Psi^{\mathrm{SF}}(l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime}, S', S) \right] \tag{96}$$

$$= \mathcal{I}_{\mathsf{u}}(S)[p_{\mathsf{u}}Af(l^{\mathrm{SF}}) + b_{\mathsf{s}}^{\mathrm{SF}\prime} + b_{\mathsf{u}}^{\mathrm{SF}\prime}] + (\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))\, [p_{\mathsf{d}}Af(l^{\mathrm{SF}}) + b_{\mathsf{s}}^{\mathrm{SF}\prime}] \tag{97}$$

$$= [\mathcal{I}_{\mathsf{u}}(S)p_{\mathsf{u}} + (\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))\, p_{\mathsf{d}}]Af(l^{\mathrm{SF}}) \tag{98}$$

$$+ [\mathcal{I}_{\mathsf{u}}(S) + (\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))]b_{\mathsf{s}}^{\mathrm{SF}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{SF}\prime} \tag{99}$$

$$= [\mathcal{I}_{\mathsf{u}}(S)p_{\mathsf{u}} + (\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))\, p_{\mathsf{d}}]Af(l^{\mathrm{SF}}) + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{SF}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{SF}\prime} \tag{100}$$

$$= [\mathcal{I}_{\mathsf{u}}(S)p_{\mathsf{u}} + (\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))\, p_{\mathsf{d}}]Af(l^{\mathrm{SF}}) - \mathcal{W}(S)l^{\mathrm{SF}} \tag{101}$$

The last line substitutes the wage-bill in using the budget constraint. The first order condition (with respect to l^{SF}) is

$$[\mathcal{I}_{\mathsf{u}}(S)p_{\mathsf{u}} + (\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))\, p_{\mathsf{d}}]Af_l(l^{\mathrm{SF}*}) = \mathcal{W}(S) \tag{102}$$

which uniquely determines the choice of l^{SF}. Any non-positive duple $(b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime})$ satisfying $\mathcal{W}(S)l^{\mathrm{SF}*} + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{SF}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{SF}\prime} = 0$ maximizes the firm' objective.[31]

For the risky firm, we have

$$V^{\mathrm{RF}}(S) = \max_{l^{\mathrm{RF}}, b_{\mathsf{s}}^{\mathrm{RF}\prime}, b_{\mathsf{u}}^{\mathrm{RF}\prime}} \left\{ \mathrm{E}\left[\mathcal{M}(S', S)\Psi^{\mathrm{RF}}(l^{\mathrm{RF}}, b_{\mathsf{s}}^{\mathrm{RF}\prime}, b_{\mathsf{u}}^{\mathrm{RF}\prime}, S', S) \right] \right\} \tag{103}$$

subject to

$$\mathcal{W}(S)l^{\mathrm{RF}} + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{RF}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{RF}\prime} = 0 \tag{104}$$

[31] Remember firms are non-financial (i.e., b's must be non-positive).

where

$$\Psi^{\text{RF}}(l^{\text{RF}}, b_{\mathsf{s}}^{\text{RF}\prime}, b_{\mathsf{u}}^{\text{RF}\prime}, S', S) \equiv \mathcal{P}(S')\Upsilon'\gamma A f(l^{\text{RF}}) + b_{\mathsf{s}}^{\text{RF}\prime} + \Upsilon' b_{\mathsf{u}}^{\text{RF}\prime} \tag{105}$$

$$\mathcal{M}(S', S) \equiv \Upsilon' \frac{\mathcal{I}_{\mathsf{u}}(S)}{\upsilon} + (1 - \Upsilon') \frac{(\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))}{1 - \upsilon} \tag{106}$$

Using the definitions of the stochastic discount factor and the state-dependent profits we have that

$$\mathrm{E}\left[\mathcal{M}(S', S)\Psi^{\text{SF}}(l^{\text{SF}}, b_{\mathsf{s}}^{\text{SF}\prime}, b_{\mathsf{u}}^{\text{SF}\prime}, S', S)\right] \tag{107}$$

$$= \mathcal{I}_{\mathsf{u}}(S)[p_{\mathsf{u}}\gamma A f(l^{\text{RF}}) + b_{\mathsf{s}}^{\text{RF}\prime} + b_{\mathsf{u}}^{\text{RF}\prime}] + (\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))\, b_{\mathsf{s}}^{\text{RF}\prime} \tag{108}$$

$$= \mathcal{I}_{\mathsf{u}}(S)[p_{\mathsf{u}}\gamma A f(l^{\text{RF}}) + b_{\mathsf{u}}^{\text{RF}\prime}] + \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\text{RF}\prime} \tag{109}$$

$$= \mathcal{I}_{\mathsf{u}}(S) p_{\mathsf{u}}\gamma A f(l^{\text{RF}}) + \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\text{RF}\prime} + \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\text{RF}\prime} \tag{110}$$

$$= \mathcal{I}_{\mathsf{u}}(S) p_{\mathsf{u}}\gamma A f(l^{\text{RF}}) - \mathcal{W}(S) l^{\text{RF}} \tag{111}$$

$$\tag{112}$$

Again, the last line substitutes the wage-bill in by using the budget constraint. The first order condition (with respect to l^{RF}) is

$$\mathcal{I}_{\mathsf{u}}(S) p_{\mathsf{u}}\gamma A f_l(l^{\text{RF}*}) = \mathcal{W}(S) \tag{113}$$

So any non-positive duple $(b_{\mathsf{s}}^{\text{RF}\prime}, b_{\mathsf{u}}^{\text{RF}\prime})$ satisfying $\mathcal{W}(S) l^{\text{RF}*} + \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\text{RF}\prime} + \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\text{RF}\prime} = 0$ is optimal.

C.2 Proof of Proposition 1

We restate the proposition for convenience:

Proposition 1 In the model without bankruptcy, every equilibrium with positive rates:

- is optimal (i.e., the allocation of real resources is the same as that of the underlying real economy);
- has a perfectly enforced inflation target (i.e., $\mathcal{P}(S') = (1 + \pi^{\mathrm{T}})P(S)$ for all S' given S, for all S);
- has the same price-vector $\langle \mathcal{P}(S), \mathcal{I}_{\mathsf{s}}(S), \mathcal{I}_{\mathsf{u}}(S), \mathcal{W}(S) \rangle$ (i.e., there is price-vector determinacy);
- has a constant nominal risk-ratio $\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}}$.

The proof consists of the following steps:

1. Show that in an equilibrium with positive rates, the liquidity constraint of the representative household is binding.
2. Show that, in equilibrium, if the liquidity constraint is binding, then the price-level $\mathcal{P}(S)$ is uniquely determined and always satisfies the inflation target.
3. Show that, in equilibrium, if $\mathcal{P}(S)$ is uniquely determined and always satisfies the inflation target, then (1) the allocation of labor is the same as that of the underlying real economy and (2) the nominal risk-ratio $\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}}$ is constant.
4. Show that, in equilibrium, if $\mathcal{P}(S)$ is uniquely determined and always satisfies the inflation target and the allocation of labor is the same as that of the underlying real economy, then $\mathcal{I}_{\mathsf{s}}(S), \mathcal{I}_{\mathsf{u}}(S), \mathcal{W}(S)$ are uniquely determined as well.

Let us proceed.

In an equilibrium with positive rates, the liquidity constraint of the representative household is binding. From the FOCs of the household we have that

$$m' : \qquad \beta\,\mathrm{E}\left[\frac{u_{c'}}{\mathcal{P}(S')}\right] = \beta\,\mathrm{E}[\lambda_1' + \lambda_2'] = \lambda_2 \tag{114}$$

$$b_{\mathsf{s}}^{\mathrm{H}\prime} : \qquad \frac{\beta\,\mathrm{E}[\lambda_1' + \lambda_2']}{\mathcal{I}_{\mathsf{s}}(S)} = \lambda_1 + \lambda_2 \tag{115}$$

Clearly, the shadow value of end-of-period money, λ_2, is positive since the marginal utility of consumption is always positive. Together the two conditions imply

$$\lambda_2 = \mathcal{I}_{\mathsf{s}}(S)(\lambda_1 + \lambda_2) \tag{116}$$

or

$$\lambda_1 = \lambda_2 \left[\frac{1}{\mathcal{I}_{\mathsf{s}}(S)} - 1\right] > 0 \tag{117}$$

since $\mathcal{I}_{\mathsf{s}}(S) < 1$ by the assumption of positive rates.

In equilibrium, if the liquidity constraint is binding, then the price-level $\mathcal{P}(S)$ is uniquely determined and always satisfies the inflation target. Since we just proved that $\lambda_1 > 0$ then it must be that the household's liquidity constraint is satisfied with equality:

$$\mathcal{P}(S)c + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}\prime} = m + w \tag{118}$$

aggregating the identical households over the unit continuum we get

$$\mathcal{P}(S)\int c + \mathcal{I}_{\mathsf{s}}(S)\int b_{\mathsf{s}}^{\mathrm{H}\prime} + \mathcal{I}_{\mathsf{u}}(S)\int b_{\mathsf{u}}^{\mathrm{H}\prime} = \int m + \int w \tag{119}$$

using the clearing of financial markets and the firms' budget restrictions one can substitute the households' bond-purchases for wage payments minus the central bank's bond-purchases, and writing M instead of $\int m$ one gets

$$\mathcal{P}(S)\int c + \left(\mathcal{W}(S) - \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{CB}\prime}\right) = M + \int w \tag{120}$$

decomposing the households' financial wealth $\int w$ into the holdings of the corresponding maturing bonds, one gets

$$\mathcal{P}(S)Y + \left(\mathcal{W}(S) - \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{CB}\prime}\right) = M - \sum_j B_{\mathsf{s}}^j - \Upsilon \sum_k B_{\mathsf{u}}^k \tag{121}$$

where $k = \mathrm{SF}, \mathrm{RF}$ and $j = \mathrm{SF}, \mathrm{RF}, \mathrm{CB}$. Isolating nominal output on the left side we have

$$\mathcal{P}(S)Y = M - \sum_j B_{\mathsf{s}}^j - \Upsilon \sum_k B_{\mathsf{u}}^k - \mathcal{W}(S) + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{CB}\prime} \tag{122}$$

and finally, isolating the price level and disaggregating output by industry we have

$$\mathcal{P}(S) = \frac{M - \sum_j B_{\mathsf{s}}^j - \Upsilon \sum_k B_{\mathsf{u}}^k - \mathcal{W}(S) + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{CB}\prime}}{Y^{\mathrm{SF}} + \Upsilon Y^{\mathrm{RF}}} \tag{123}$$

Notice that all but $b_{\mathsf{s}}^{\mathrm{CB}\prime}$ are state variables or functions of the state variables. The central bank has therefore complete control over the price level through the purchase/sale of safe bonds; and he is indeed employing the right restriction to achieve its objective (namely this last equation). The central bank will choose one and only one price-level, namely the one that satisfies the inflation target (given the past price-level, which is a state variable as well) and therefore maximizes its objective function. In conclusion, the price level $\mathcal{P}(S)$ is uniquely determined and satisfies $\mathcal{P}(S') = (1 + \pi^{\mathrm{T}})\mathcal{P}(S)$ for all states S' successors of S, for all states S.

In equilibrium, if $\mathcal{P}(S)$ is uniquely determined and always satisfies the inflation target, then (1) the allocation of labor is the same as that of the underlying real economy and (2) the nominal risk-ratio $\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}}$ is constant. Consider the FOC for the safe firm:

$$[\mathcal{I}_{\mathsf{u}}(S)p_{\mathsf{u}} + (\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))\, p_{\mathsf{d}}]Af_l(l^{\mathrm{SF}*}) = \mathcal{W}(S) \tag{124}$$

And consider the FOC of the risky firm:

$$\mathcal{I}_{\mathsf{u}}(S)p_{\mathsf{u}}\gamma Af_l(l^{\mathrm{RF}*}) = \mathcal{W}(S) \tag{125}$$

Since the inflation target is always satisfied, we can write $p' \equiv p_u = p_d$. Both FOC's can be simplified to render

$$\mathcal{I}_{\mathsf{s}}(S)p' Af_l(l^{\mathrm{SF}*}) = \mathcal{W}(S) \tag{126}$$

$$\mathcal{I}_{\mathsf{u}}(S)p'\gamma Af_l(l^{\mathrm{RF}*}) = \mathcal{W}(S) \tag{127}$$

Together implying

$$\mathcal{I}_{\mathsf{s}}(S)p'Af_l(l^{\mathrm{SF}*}) = \mathcal{I}_{\mathsf{u}}(S)p'\gamma Af_l(l^{\mathrm{RF}*}) \tag{128}$$

or

$$\boxed{\rho = \frac{\mathcal{I}_{\mathsf{s}}(S)}{\mathcal{I}_{\mathsf{u}}(S)} = \frac{\gamma f_l(1-l^{\mathrm{SF}*})}{f_l(l^{\mathrm{SF}*})}} \tag{129}$$

On the other hand, from the household's FOC's we have

$$b_{\mathsf{s}}^{\mathrm{H}\prime}: \qquad \frac{\beta\,\mathrm{E}[\lambda_1' + \lambda_2']}{\mathcal{I}_{\mathsf{s}}(S)} = \lambda_1 + \lambda_2 \tag{130}$$

$$b_{\mathsf{u}}^{\mathrm{H}\prime}: \qquad \frac{\beta\,\mathrm{E}[(\lambda_1' + \lambda_2')\Upsilon']}{\mathcal{I}_{\mathsf{u}}(S)} = \lambda_1 + \lambda_2 \tag{131}$$

Together they imply

$$\frac{\mathcal{I}_{\mathsf{s}}(S)}{\mathcal{I}_{\mathsf{u}}(S)} = \frac{\mathrm{E}[\lambda_1' + \lambda_2']}{\mathrm{E}[(\lambda_1' + \lambda_2')\Upsilon']} \tag{132}$$

$$= \frac{\mathrm{E}[u_c(Y')/p']}{\mathrm{E}[\Upsilon'(u_c(Y')/p']} \tag{133}$$

$$= \frac{\upsilon\frac{u_c(Y^{\mathrm{SF}\prime}+Y^{\mathrm{RF}\prime})}{p'} + (1-\upsilon)\frac{u_c(Y^{\mathrm{SF}\prime})}{p'}}{\upsilon\frac{u_c(Y^{\mathrm{SF}\prime})}{p'}} \tag{134}$$

Or

$$\boxed{\rho = \frac{\mathcal{I}_{\mathsf{s}}(S)}{\mathcal{I}_{\mathsf{u}}(S)} = 1 + \frac{(1-\upsilon)\;\; u_c\big(Af(l^{\mathrm{SF}*})\big)}{\upsilon\;\; u_c(Af(l^{\mathrm{SF}*}) + A\gamma f(1-l^{\mathrm{SF}*}))}} \tag{135}$$

Together equations (129) and (135) and the CRRA properties of the utility function imply the condition:

$$\boxed{\frac{\gamma f_l(1-l^{\mathrm{SF}**})}{f_l(l^{\mathrm{SF}**})} = 1 + \frac{(1-\upsilon)\;\; u_c\big(f(l^{\mathrm{SF}**})\big)}{\upsilon\;\; u_c(f(l^{\mathrm{SF}**}) + \gamma f(1-l^{\mathrm{SF}**}))}} \tag{136}$$

which is exactly the same as that of the underlying-real-economy (section A.3) and therefore determines the unique and optimal labor allocation, which is constant.

Since labor is constant, both equation (129) and equation (135) show that the nominal risk-ratio is constant as well.

In equilibrium, if $\mathcal{P}(S)$ is uniquely determined and always satisfies the inflation target and the allocation of labor is the same as that of the underlying real economy, then $\mathcal{I}_{\mathsf{s}}(S), \mathcal{I}_{\mathsf{u}}(S), \mathcal{W}(S)$ are uniquely determined as well. Since labor allocation $l^{\mathrm{SF}*}$ is unique, so are output in the 'lucky state' u and output in the 'unlucky state' d, in equilibrium:

$$Y'_{\mathsf{u}} \equiv Y^{\mathrm{SF}'} + Y^{\mathrm{RF}'} = Af(l^{\mathrm{SF}**}) + A\gamma f(1 - l^{\mathrm{SF}**}) \tag{137}$$

$$Y'_{\mathsf{d}} \equiv Y^{\mathrm{SF}'} = Af(l^{\mathrm{SF}**}) \tag{138}$$

But then, from the household's FOC's, we have that

$$\mathcal{I}_{\mathsf{s}}(S) = \frac{\beta\, \mathrm{E}[\lambda'_1 + \lambda'_2]}{\lambda_1 + \lambda_2} \tag{139}$$

$$= \frac{\beta \left[\upsilon \frac{u_c(Y'_{\mathsf{u}})}{p'} + (1-\upsilon) \frac{u_c(Y'_{\mathsf{d}})}{p'} \right]}{\frac{u_c(Y)}{p}} \tag{140}$$

$$= (1 + \pi^{\mathrm{T}})\, \frac{\beta \left[\upsilon\, u_c(Y'_{\mathsf{u}}) + (1-\upsilon)\, u_c(Y'_{\mathsf{d}}) \right]}{u_c(Y)} \tag{141}$$

Thus $\mathcal{I}_{\mathsf{s}}$ is uniquely determined (since so is the right-hand-side). And similarly

$$\mathcal{I}_{\mathsf{u}}(S) = \frac{\beta\, \mathrm{E}[(\lambda'_1 + \lambda'_2)\Upsilon']}{\lambda_1 + \lambda_2} \tag{142}$$

$$= \frac{\beta \left[\upsilon \frac{u_c(Y'_{\mathsf{u}})}{p'} \right]}{\frac{u_c(Y)}{p}} \tag{143}$$

$$= (1 + \pi^{\mathrm{T}})\, \upsilon\, \frac{\beta u_c(Y'_{\mathsf{u}})}{u_c(Y)} \tag{144}$$

Thus $\mathcal{I}_{\mathsf{u}}$ is uniquely determined (since so is the right-hand-side).

Finally, given the uniqueness of $\mathcal{I}_{\mathsf{u}}, l^{\mathrm{SF}**}$, and p', the risky firm's FOC uniquely determines the equilibrium wage as well:

$$\mathcal{W}(S) = \mathcal{I}_{\mathsf{u}}(S) p' \gamma A f_l (1 - l^{\mathrm{SF}**}) \tag{145}$$

Q.E.D.

C.3 Proof of Claim 1

We state the definition of unconditional correlation now:

Definition. *The correlation between two random variables X and Z, given some distribution $F(X,Z)$, is denoted by corr(X,Z) and given by*

$$corr(X,Z) = \mathrm{E}\left[\left(\frac{X-\mathrm{E}[X]}{\sigma_X}\right)\left(\frac{Z-\mathrm{E}[Z]}{\sigma_Z}\right)\right]$$

where σ_X is the standard deviation of X, and σ_Z is the standard deviation of Z.

Thus, establishing the distribution $F(X,Z)$ is a basic requisite for calculating the correlation.

We state the definition of conditional correlation now:

Definition. *The correlation between two random variables X and Z conditional on a third random variable W, given some distribution $F(X,Z,W)$, is denoted by corr$(X,Z\,|W)$ and given by*

$$corr(X,Z\,|W) = \mathrm{E}\left[\left(\frac{X-\mathrm{E}[X]}{\sigma_X}\right)\left(\frac{Z-\mathrm{E}[Z]}{\sigma_Z}\right)\middle| W\right]$$

where σ_X is the standard deviation of X, and σ_Z is the standard deviation of Z.

Thus, establishing the distribution $F(X,Z,W)$ is a basic requisite for calculating the correlation.

Notice that, for any constant c, we have that $corr(X,cZ) = corr(X,Z)$ and $corr(X,cZ\,|W) = corr(X,Z,|W)$. This fact will be used extensively.

We state a more detailed definition of pro-cyclicality, and the claim to be proved, for convenience:

Definition. *A variable $X(S)$ is said to be pro-cyclical if it is positively correlated with the growth factor of output $\frac{Y(S')}{Y(S)}$ given a preceding state $S^{\backprime}$, for all $S^{\backprime}$:*

$$corr\left(X(S), \frac{Y(S')}{Y(S)}\,\middle|\, S^{\backprime}\right) > 0 \qquad \forall S^{\backprime}$$

where $F(S',S \mid S^{\backprime}) = F(S' \mid S,S^{\backprime})F(S \mid S^{\backprime}) = F(S' \mid S)F(S \mid S^{\backprime})$. The last equality follows from the marcovian nature of the model.

For the sequential case, a stochastic process $\{X_t\}_{t=0}^{\infty}$ is said to be pro-cyclical if corr $\left(X_t, \frac{Y_{t+1}}{Y_t}\right) > 0$ for all t, given the stochastic process for the output's growth factor $\{\frac{Y_{t+1}}{Y_t}\}_{t=0}^{\infty}$. A counter-cyclical variable is defined by using the opposite strict-inequality signs.

Claim 1 At the optimum, the monetary economy with an inflation-targeting central bank has a pro-cyclical nominal risk-spread.

The proof consists of showing the perfect correlation of the nominal risk-spread to several expressions and noting that the last of these expressions is positively correlated with $Y(S')/Y(S)$. Let us proceed.

Recall from subsection B.2 that the nominal credit-spread ξ is given by

$$\xi = (\rho - 1)(1 + i_{\mathsf{s}}) \tag{146}$$

where ρ is the real risk-ratio, which–under perfect enforcement of the inflation target–equals the nominal risk-ratio $\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}}$. We just showed that the equilibria for the model without bankruptcy have a constant nominal risk-ratio, so all the variation in the nominal risk-spread comes from changes in $1 + i_{\mathsf{s}}$. Hence

$$corr(\xi, 1 + i_{\mathsf{s}}) = corr(\xi, {\mathcal{I}_{\mathsf{s}}}^{-1}) = 1 \tag{147}$$

Next, notice from the household's KT conditions that

$$\mathcal{I}_{\mathsf{s}} = \frac{\beta}{1+\pi} \frac{\mathrm{E}[u_c(Y(S')) \mid S]}{u_c(Y(S))} \tag{148}$$

$$= \frac{\beta}{1+\pi} \mathrm{E}[u_c(Y(S'))/u_c(Y(S)) \mid S] \tag{149}$$

due to the properties of CRRA we can rewrite this as

$$= \frac{\beta}{1+\pi} \mathrm{E}[u_c(Y(S')/Y(S)) \mid S] \tag{150}$$

hence

$${\mathcal{I}_{\mathsf{s}}}^{-1} = \frac{1+\pi}{\beta} \left(\mathrm{E}[u_c(Y(S')/Y(S)) \mid S]\right)^{-1} \tag{151}$$

It follows that

$$corr(\xi, (\mathrm{E}[u_c(Y(S')/Y(S)) \mid S])^{-1}) = 1 \tag{152}$$

And therefore the correlation that we are looking for is

$$corr((\mathrm{E}[u_c(Y(S')/Y(S)) \mid S])^{-1},\ Y(S')/Y(S)) \tag{153}$$

conditional on a given preceding state S`.

We see that for *any* preceding state, there will be a positive correlation. The proof for the sequential case follows from this last equation as well. **Q.E.D.**

D Equilibrium in the presence of bankruptcy rights

D.1 Full Derivation of Kuhn-Tucker Conditions

For the household, recall that

$$V^{\mathrm{H}}(m,w,S) = \max_{c,m',b_{\mathsf{s}}^{\mathrm{H}'},b_{\mathsf{u}}^{\mathrm{H}'}} \left\{ u(c) + \beta\,\mathrm{E}\left[V^{\mathrm{H}}(m',w',S')\right]\right\} \tag{154}$$

subject to[32]

$$\mathcal{P}(S)c + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}'} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}'} \le m + w \tag{155}$$

$$m' + \mathcal{P}(S)c + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}'} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}'} \le m + w + \mathcal{W}(S) + D(S) \tag{156}$$

$$0 \le m', c \tag{157}$$

$$0 \le m' + b_{\mathsf{s}}^{\mathrm{H}'} + b_{\mathsf{u}}^{\mathrm{H}'} - e(S) \tag{158}$$

$$0 \le m' + b_{\mathsf{s}}^{\mathrm{H}'} - e(S) \tag{159}$$

where

$$w' \equiv b_{\mathsf{s}}^{\mathrm{H}'} + \Upsilon' b_{\mathsf{u}}^{\mathrm{H}'} \tag{160}$$

$$D(S) \equiv \left(\mathcal{P}(S)Y^{\mathrm{SF}} + B_{\mathsf{s}}^{\mathrm{SF}} + \Upsilon B_{\mathsf{u}}^{\mathrm{SF}}\right) + \left(\mathcal{P}(S)\Upsilon Y^{\mathrm{RF}} + B_{\mathsf{s}}^{\mathrm{RF}} + \Upsilon B_{\mathsf{u}}^{\mathrm{RF}}\right) \tag{161}$$

The corresponding Lagrangian is

$$\begin{aligned}
&\mathcal{L}(c,m',b_{\mathsf{s}}^{\mathrm{H}'},b_{\mathsf{u}}^{\mathrm{H}'},\lambda_1,\lambda_2,\lambda_3) = \\
&u(c) + \beta\,\mathrm{E}\left[V^{\mathrm{H}}(m',w',S')\right] \\
&+ \lambda_1[w + m - \mathcal{P}(S)c - \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}'} - \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}'}] \\
&+ \lambda_2[w + m + \mathcal{W}(S) + D(S) - m' - \mathcal{P}(S)c - \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}'} - \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}'}] \\
&+ \lambda_3[m' + b_{\mathsf{s}}^{\mathrm{H}'} - e(S)]
\end{aligned} \tag{162}$$

The non-negativity restrictions on m' and c are omitted (for the same reasons as in the model without bankruptcy): consumption is positive because $u(\cdot)$ has an infinite derivative at zero; new money balances are positive by construction (firms cannot keep money balances between periods). The absence of the next-to-last restriction is due to subtler reasons:

[32]Notice that the last two inequalities together are equivalent to the stochastic inequality in the main text.

Firms cannot lend, $b_{\mathsf{u}}^{\mathrm{SF}\prime}, b_{\mathsf{s}}^{\mathrm{RF}\prime} \leq 0$; and central banks do not deal with risky assets, by assumption. Hence the market clearing condition $b_{\mathsf{u}}^{\mathrm{H}\prime} + b_{\mathsf{u}}^{\mathrm{SF}\prime} + b_{\mathsf{s}}^{\mathrm{RF}\prime} = 0$ implies that in equilibrium $b_{\mathsf{u}}^{\mathrm{H}\prime} = -(b_{\mathsf{u}}^{\mathrm{SF}\prime} + b_{\mathsf{s}}^{\mathrm{RF}\prime}) \geq 0$. So we can safely omit the inequality restriction where $b_{\mathsf{u}}^{\mathrm{H}\prime}$ is added on one side of another existing restriction.

The Kuhn-Tucker conditions are

$$\begin{aligned}
c &: & u_c - \lambda_1 \mathcal{P}(S) - \lambda_2 \mathcal{P}(S) &= 0 & (163)\\
m' &: & \beta\, \mathrm{E}[V_{m'}^{\mathrm{H}}] - \lambda_2 + \lambda_3 &= 0 & (164)\\
b_{\mathsf{s}}^{\mathrm{H}\prime} &: & \beta\, \mathrm{E}[V_{w'}^{\mathrm{H}}] - \lambda_1 \mathcal{I}_{\mathsf{s}}(S) - \lambda_2 \mathcal{I}_{\mathsf{s}}(S) + \lambda_3 &= 0 & (165)\\
b_{\mathsf{u}}^{\mathrm{H}\prime} &: & \beta\, \mathrm{E}[V_{w'}^{\mathrm{H}} \Upsilon'] - \lambda_1 \mathcal{I}_{\mathsf{u}}(S) - \lambda_2 \mathcal{I}_{\mathsf{u}}(S) &= 0 & (166)\\
\lambda_1 &: & \lambda_1[w + m - \mathcal{P}(S)c - \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\mathrm{H}\prime} - \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{H}\prime}] &\geq 0 & (167)\\
\lambda_2 &: & \lambda_2[w + m + \mathcal{W}(S) + D(S) - m' - \mathcal{P}(S)c - \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\mathrm{H}\prime} - \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{H}\prime}] &\geq 0 & (168)\\
\lambda_3 &: & \lambda_3[m' + b_{\mathsf{s}}^{\mathrm{H}\prime} - e(S)] &\geq 0 & (169)
\end{aligned}$$

There is an abuse of notation, by stating $ab \geq 0$ instead of $a, b \geq 0$ and $ab = 0$.

From updating the envelope conditions we get

$$V_{m'}^{\mathrm{H}} = \lambda_1' + \lambda_2' \quad (170)$$

$$V_{w'}^{\mathrm{H}} = \lambda_1' + \lambda_2' \quad (171)$$

Rewriting the KT conditions we get

$$\begin{aligned}
c &: & \frac{u_c}{\mathcal{P}(S)} &= \lambda_1 + \lambda_2 & (172)\\
m' &: & \beta\, \mathrm{E}[\lambda_1' + \lambda_2'] &= \lambda_2 - \lambda_3 & (173)\\
b_{\mathsf{s}}^{\mathrm{H}\prime} &: & \frac{\beta\, \mathrm{E}[\lambda_1' + \lambda_2'] + \lambda_3}{\mathcal{I}_{\mathsf{s}}(S)} &= \lambda_1 + \lambda_2 & (174)\\
b_{\mathsf{u}}^{\mathrm{H}\prime} &: & \frac{\beta\, \mathrm{E}[(\lambda_1' + \lambda_2')\Upsilon']}{\mathcal{I}_{\mathsf{u}}(S)} &= \lambda_1 + \lambda_2 & (175)\\
\lambda_1 &: & \lambda_1[w + m - \mathcal{P}(S)c - \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\mathrm{H}\prime} - \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{H}\prime}] &\geq 0 & (176)\\
\lambda_2 &: & \lambda_2[w + m + \mathcal{W}(S) + D(S) - m' - \mathcal{P}(S)c - \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\mathrm{H}\prime} - \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{H}\prime}] &\geq 0 & (177)\\
\lambda_3 &: & \lambda_3[m' + b_{\mathsf{s}}^{\mathrm{H}\prime} - e(S)] &\geq 0 & (178)
\end{aligned}$$

It will prove timely to state the following result:

Claim 4. *In the presence of bankruptcy rights, every equilibrium has* $\mathcal{I}_{\mathsf{s}}(S) > \mathcal{I}_{\mathsf{u}}(S) \quad \forall S$.

Proof. We prove this using two exhaustive and mutually-exclusive cases.

Case 1: $\lambda_3 = 0$ In this case, the fact alone that the time-invariant probability υ is strictly between zero and one, together with equations (267) and (268), give the result.

Case 2: $\lambda_3 > 0$ In this case, the difference between $\mathcal{I}_{\mathsf{s}}(S)$ and $\mathcal{I}_{\mathsf{u}}(S)$ is only strengthen by the fact that $\lambda_3 > 0$ and, in order to satisfy equations (267) and (268), $\mathcal{I}_{\mathsf{s}}(S)$ will have to be even greater than for $\lambda_3 = 0$. □

For the safe firm, recall that we have

$$V^{\mathrm{SF}}(S) = \max_{l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime}} \left\{ \mathrm{E}\left[\mathcal{M}(S', S) \Psi^{\mathrm{SF}}(l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime}, S', S) \right] \right\} \tag{179}$$

subject to

$$\mathcal{W}(S) l^{\mathrm{SF}} + \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\mathrm{SF}\prime} + \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{SF}\prime} = 0 \tag{180}$$

and now also subject to[33]

$$0 \leq p_{\mathsf{u}} A f(l^{\mathrm{SF}}) + b_{\mathsf{s}}^{\mathrm{SF}\prime} + b_{\mathsf{u}}^{\mathrm{SF}\prime} \tag{181}$$

$$0 \leq p_{\mathsf{d}} A f(l^{\mathrm{SF}}) + b_{\mathsf{s}}^{\mathrm{SF}\prime} \tag{182}$$

$$b_{\mathsf{s}}^{\mathrm{SF}\prime} \leq 0 \tag{183}$$

$$b_{\mathsf{u}}^{\mathrm{SF}\prime} \leq 0 \tag{184}$$

where

$$\Psi^{\mathrm{SF}}(l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime}, S', S) \equiv \mathcal{P}(S') A f(l^{\mathrm{SF}}) + b_{\mathsf{s}}^{\mathrm{SF}\prime} + \Upsilon' b_{\mathsf{u}}^{\mathrm{SF}\prime} \tag{185}$$

$$\mathcal{M}(S', S) \equiv \Upsilon' \frac{\mathcal{I}_{\mathsf{u}}(S)}{\upsilon} + (1 - \Upsilon') \frac{(\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))}{1 - \upsilon} \tag{186}$$

Using the definitions of the stochastic discount factor and the state-dependent profits we have that

$$\mathrm{E}\left[\mathcal{M}(S', S) \Psi^{\mathrm{SF}}(l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime}, S', S) \right] \tag{187}$$

$$= \mathcal{I}_{\mathsf{u}}(S)[p_{\mathsf{u}} A f(l^{\mathrm{SF}}) + b_{\mathsf{s}}^{\mathrm{SF}\prime} + b_{\mathsf{u}}^{\mathrm{SF}\prime}] + (\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S)) [p_{\mathsf{d}} A f(l^{\mathrm{SF}}) + b_{\mathsf{s}}^{\mathrm{SF}\prime}] \tag{188}$$

$$= [\mathcal{I}_{\mathsf{u}}(S) p_{\mathsf{u}} + (\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S)) p_{\mathsf{d}}] A f(l^{\mathrm{SF}}) \tag{189}$$

$$+ [\mathcal{I}_{\mathsf{u}}(S) + (\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))] b_{\mathsf{s}}^{\mathrm{SF}\prime} + \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{SF}\prime} \tag{190}$$

$$= [\mathcal{I}_{\mathsf{u}}(S) p_{\mathsf{u}} + (\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S)) p_{\mathsf{d}}] A f(l^{\mathrm{SF}}) + \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\mathrm{SF}\prime} + \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{SF}\prime} \tag{191}$$

[33] Notice that the two first inequalities together are equivalent to the stochastic inequality in the main text; and this time we are writing down the non-positivity constraints for safe and risky debt because they might be actually needed.

This time we will have to write down a (long) Lagrangian:

$$
\begin{aligned}
&\mathcal{L}(l^{\text{SF}}, b_{\mathsf{s}}^{\text{SF}\prime}, b_{\mathsf{u}}^{\text{SF}\prime}, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \\
&\mathcal{I}_{\mathsf{s}}(S)p'Af(l^{\text{SF}}) + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\text{SF}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\text{SF}\prime} \\
&+ \lambda_1[-\mathcal{W}(S)l^{\text{SF}} - \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\text{SF}\prime} - \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\text{SF}\prime}] \\
&+ \lambda_2[p'Af(l^{\text{SF}}) + b_{\mathsf{s}}^{\text{SF}\prime} + b_{\mathsf{u}}^{\text{SF}\prime}] \\
&+ \lambda_3[-b_{\mathsf{s}}^{\text{SF}\prime}] \\
&+ \lambda_4[-b_{\mathsf{u}}^{\text{SF}\prime}]
\end{aligned}
\tag{192}
$$

Notice that the second inequality restriction was omitted. This is because it can be–and will be–shown that (regardless of the firms' choice problems) in an equilibrium with positive rates the inflation target is perfectly enforced. We therefore know $p_{\mathsf{u}} = p_{\mathsf{d}}$ and use $p' \equiv p_{\mathsf{u}} = p_{\mathsf{d}}$ instead. And, since $b_{\mathsf{s}}^{\text{SF}\prime} \leq 0$, the second inequality restriction is implied by the first one and can be dropped.

The Kuhn-Tucker conditions are

$$l^{\text{SF}}: \quad \mathcal{I}_{\mathsf{s}}(S)p'Af_l(l^{\text{SF}}) - \lambda_1\mathcal{W}(S) + \lambda_2 p'Af_l(l^{\text{SF}}) = 0 \tag{193}$$

$$b_{\mathsf{s}}^{\text{SF}\prime}: \quad \mathcal{I}_{\mathsf{s}}(S) - \lambda_1\mathcal{I}_{\mathsf{s}}(S) + \lambda_2 - \lambda_3 = 0 \tag{194}$$

$$b_{\mathsf{u}}^{\text{SF}\prime}: \quad \mathcal{I}_{\mathsf{u}}(S) - \lambda_1\mathcal{I}_{\mathsf{u}}(S) + \lambda_2 - \lambda_4 = 0 \tag{195}$$

$$\lambda_1: \quad \mathcal{W}(S)l^{\text{SF}} + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\text{SF}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\text{SF}\prime} = 0 \tag{196}$$

$$\lambda_2: \quad \lambda_2[p'Af(l^{\text{SF}}) + b_{\mathsf{s}}^{\text{SF}\prime} + b_{\mathsf{u}}^{\text{SF}\prime}] \geq 0 \tag{197}$$

$$\lambda_3: \quad \lambda_3[-b_{\mathsf{s}}^{\text{SF}\prime}] \geq 0 \tag{198}$$

$$\lambda_4: \quad \lambda_4[-b_{\mathsf{u}}^{\text{SF}\prime}] \geq 0 \tag{199}$$

There is an abuse of notation, by stating $ab \geq 0$ instead of $a, b \geq 0$ and $ab = 0$.

Substituting out λ_1, we get the following conditions

$$\mathcal{I}_{\mathsf{s}}(S)p'Af_l(l^{\text{SF}}) - \left(1 + \frac{\lambda_2 - \lambda_3}{\mathcal{I}_{\mathsf{s}}}\right)\mathcal{W}(S) + \lambda_2 p'Af_l(l^{\text{SF}}) = 0 \tag{200}$$

$$\frac{\lambda_2 - \lambda_3}{\mathcal{I}_{\mathsf{s}}(S)} = \frac{\lambda_2 - \lambda_4}{\mathcal{I}_{\mathsf{u}}(S)} \tag{201}$$

$$\mathcal{W}(S)l^{\text{SF}} + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\text{SF}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\text{SF}\prime} = 0 \tag{202}$$

$$\lambda_2[p'Af(l^{\text{SF}}) + b_{\mathsf{s}}^{\text{SF}\prime} + b_{\mathsf{u}}^{\text{SF}\prime}] \geq 0 \tag{203}$$

$$\lambda_3[-b_{\mathsf{s}}^{\text{SF}\prime}] \geq 0 \tag{204}$$

$$\lambda_4[-b_{\mathsf{u}}^{\text{SF}\prime}] \geq 0 \tag{205}$$

We will now show that $\lambda_2 = \lambda_3 = \lambda_4 = 0$.

First, notice that multipliers λ_3 and λ_4 cannot be simultaneously positive because that would entail zero production and would contradict the infinite marginal productivity of labor at zero. Therefore, $\min\{\lambda_3, \lambda_4\} = 0$. Next, if $\max\{\lambda_3, \lambda_4\} > \min\{\lambda_3, \lambda_4\} = 0$, then it must be that $\lambda_2 > \max\{\lambda_3, \lambda_4\}$ as well. Otherwise, one would get a contradiction from the safe firm's KT condition (201) and the positivity of bond prices. Next, if $\lambda_2 > \max\{\lambda_3, \lambda_4\} > \min\{\lambda_3, \lambda_4\} = 0$, then it must be that $\lambda_2 > \lambda_4 > \lambda_3 = 0$. This follows from the fact that $\mathcal{I}_{\mathsf{s}}(S) > \mathcal{I}_{\mathsf{u}}(S)$ (Claim 4) and from the safe firm's condition (201). But $\lambda_2 > \lambda_4 > \lambda_3 = 0$ cannot be true because then (200) and (203) cannot both be satisfied. Finally, $\lambda_2 > \lambda_4 = \lambda_3 = 0$ cannot be true either because again (200) and (203) cannot both be satisfied. It must therefore be true that $\lambda_2 = \lambda_3 = \lambda_4 = 0$.

It follows from the previous paragraph that we can further simplify and reduce the conditions to get

$$\mathcal{I}_{\mathsf{s}}(S)p'Af_l(l^{\text{SF}}) - \mathcal{W}(S) = 0 \tag{206}$$

$$\mathcal{W}(S)l^{\text{SF}} + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\text{SF}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\text{SF}\prime} = 0 \tag{207}$$

$$[p'Af(l^{\text{SF}}) + b_{\mathsf{s}}^{\text{SF}\prime} + b_{\mathsf{u}}^{\text{SF}\prime}] \geq 0 \tag{208}$$

$$-b_{\mathsf{s}}^{\text{SF}\prime} \geq 0 \tag{209}$$

$$-b_{\mathsf{u}}^{\text{SF}\prime} \geq 0 \tag{210}$$

Now, notice

$$p'Af(l^{\text{SF}}) + b_{\mathsf{s}}^{\text{SF}\prime} + b_{\mathsf{u}}^{\text{SF}\prime} \tag{211}$$

$$> \mathcal{I}_{\mathsf{s}}p'Af(l^{\text{SF}}) + \mathcal{I}_{\mathsf{s}}b_{\mathsf{s}}^{\text{SF}\prime} + \mathcal{I}_{\mathsf{s}}b_{\mathsf{u}}^{\text{SF}\prime} \tag{212}$$

$$\geq \mathcal{I}_{\mathsf{s}}p'Af(l^{\text{SF}}) + \mathcal{I}_{\mathsf{s}}b_{\mathsf{s}}^{\text{SF}\prime} + \mathcal{I}_{\mathsf{u}}b_{\mathsf{u}}^{\text{SF}\prime} \tag{213}$$

$$= \mathcal{I}_{\mathsf{s}}p'Af(l^{\text{SF}}) - \mathcal{W}l^{\text{SF}} \tag{214}$$

but by the shape of $f(\cdot)$ and condition (187) we know that $\mathcal{I}_{\mathsf{s}}p'Af(l^{\text{SF}*}) - \mathcal{W}l^{\text{SF}*} > 0$ so $p'Af(l^{\text{SF}}) + b_{\mathsf{s}}^{\text{SF}\prime} + b_{\mathsf{u}}^{\text{SF}\prime} > 0$ and we can ignore the restriction to get the smaller set of conditions:

$$\mathcal{I}_{\mathsf{s}}(S)p'Af_l(l^{\text{SF}}) - \mathcal{W}(S) = 0 \tag{215}$$

$$\mathcal{W}(S)l^{\text{SF}} + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\text{SF}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\text{SF}\prime} = 0 \tag{216}$$

$$-b_{\mathsf{s}}^{\text{SF}\prime} \geq 0 \tag{217}$$

$$-b_{\mathsf{u}}^{\text{SF}\prime} \geq 0 \tag{218}$$

Thus, we see that the safe firm is indifferent between issuing safe or risky debt. But we assumed that, if indifferent, firms would finance themselves exclusively with safe debt. Therefore the final conditions are

$$\mathcal{I}_{\mathsf{s}}(S)p'Af_l(l^{\mathrm{SF}}) - \mathcal{W}(S) = 0 \tag{219}$$

$$\mathcal{W}(S)l^{\mathrm{SF}} + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{SF}\prime} = 0 \tag{220}$$

For the risky firm, we have

$$V^{\mathrm{RF}}(S) = \max_{l^{\mathrm{RF}}, b_{\mathsf{s}}^{\mathrm{RF}\prime}, b_{\mathsf{u}}^{\mathrm{RF}\prime}} \left\{ \mathrm{E}\left[\mathcal{M}(S', S)\Psi^{\mathrm{RF}}(l^{\mathrm{RF}}, b_{\mathsf{s}}^{\mathrm{RF}\prime}, b_{\mathsf{u}}^{\mathrm{RF}\prime}, S', S)\right]\right\} \tag{221}$$

subject to

$$\mathcal{W}(S)l^{\mathrm{RF}} + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{RF}\prime} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{RF}\prime} = 0 \tag{222}$$

and now also subject to[34]

$$0 \leq p_{\mathsf{u}}\gamma Af(l^{\mathrm{RF}}) + b_{\mathsf{s}}^{\mathrm{RF}\prime} + b_{\mathsf{u}}^{\mathrm{RF}\prime} \tag{223}$$

$$0 \leq b_{\mathsf{s}}^{\mathrm{RF}\prime} \tag{224}$$

$$b_{\mathsf{s}}^{\mathrm{RF}\prime} \leq 0 \tag{225}$$

$$b_{\mathsf{u}}^{\mathrm{RF}\prime} \leq 0 \tag{226}$$

where

$$\Psi^{\mathrm{RF}}(l^{\mathrm{RF}}, b_{\mathsf{s}}^{\mathrm{RF}\prime}, b_{\mathsf{u}}^{\mathrm{RF}\prime}, S', S) \equiv \mathcal{P}(S')\Upsilon'\gamma Af(l^{\mathrm{RF}}) + b_{\mathsf{s}}^{\mathrm{RF}\prime} + \Upsilon' b_{\mathsf{u}}^{\mathrm{RF}\prime} \tag{227}$$

$$\mathcal{M}(S', S) \equiv \Upsilon'\frac{\mathcal{I}_{\mathsf{u}}(S)}{\upsilon} + (1-\Upsilon')\frac{(\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))}{1-\upsilon} \tag{228}$$

Using the definitions of the stochastic discount factor and the state-dependent profits we have that

$$\mathrm{E}\left[\mathcal{M}(S', S)\Psi^{\mathrm{SF}}(l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, b_{\mathsf{u}}^{\mathrm{SF}\prime}, S', S)\right] \tag{229}$$

$$= \mathcal{I}_{\mathsf{u}}(S)[p_{\mathsf{u}}\gamma Af(l^{\mathrm{RF}}) + b_{\mathsf{s}}^{\mathrm{RF}\prime} + b_{\mathsf{u}}^{\mathrm{RF}\prime}] + (\mathcal{I}_{\mathsf{s}}(S) - \mathcal{I}_{\mathsf{u}}(S))\, b_{\mathsf{s}}^{\mathrm{RF}\prime} \tag{230}$$

$$= \mathcal{I}_{\mathsf{u}}(S)[p_{\mathsf{u}}\gamma Af(l^{\mathrm{RF}}) + b_{\mathsf{u}}^{\mathrm{RF}\prime}] + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{RF}\prime} \tag{231}$$

$$= \mathcal{I}_{\mathsf{u}}(S)p_{\mathsf{u}}\gamma Af(l^{\mathrm{RF}}) + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{RF}\prime} + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{RF}\prime} \tag{232}$$

[34]Notice that the two first inequalities together are equivalent to the stochastic inequality in the main text; and this time we are writing down the non-positivity constraints for safe and risky debt.

This time we will have to write down a (long) Lagrangian:

$$
\begin{aligned}
&\mathcal{L}(l^{\mathrm{RF}}, b_{\mathsf{u}}^{\mathrm{RF}\prime}, \lambda_1, \lambda_2, \lambda_3) = \\
&\mathcal{I}_{\mathsf{u}}(S)p'\gamma A f(l^{\mathrm{RF}}) + \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{RF}\prime} \\
&+ \lambda_1[-\mathcal{W}(S)l^{\mathrm{RF}} - \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{RF}\prime}] \\
&+ \lambda_2[p'\gamma A f(l^{\mathrm{RF}}) + b_{\mathsf{u}}^{\mathrm{RF}\prime}] \\
&+ \lambda_3[-b_{\mathsf{u}}^{\mathrm{RF}\prime}]
\end{aligned} \tag{233}
$$

Notice that it is way simpler than the original problem seemed. This is because from the second and third inequality restrictions we get that $b_{\mathsf{s}}^{\mathrm{RF}\prime} = 0$. Thus $b_{\mathsf{s}}^{\mathrm{RF}\prime}$ can be dropped from the whole problem. We also set $p' \equiv p_{\mathsf{u}}$ for consistency with the safe firm's Lagrangian.

The Kuhn-Tucker conditions are

$$
\begin{aligned}
l^{\mathrm{RF}}: &\quad \mathcal{I}_{\mathsf{u}}(S)p'\gamma A f_l(l^{\mathrm{RF}}) - \lambda_1 \mathcal{W}(S) + \lambda_2 p'\gamma A f_l(l^{\mathrm{RF}}) = 0 && (234)\\
b_{\mathsf{u}}^{\mathrm{RF}\prime}: &\quad \mathcal{I}_{\mathsf{u}}(S) - \lambda_1 \mathcal{I}_{\mathsf{u}}(S) + \lambda_2 - \lambda_3 = 0 && (235)\\
\lambda_1: &\quad \mathcal{W}(S)l^{\mathrm{RF}} + \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{RF}\prime} = 0 && (236)\\
\lambda_2: &\quad \lambda_2[p'\gamma A f(l^{\mathrm{RF}}) + b_{\mathsf{u}}^{\mathrm{RF}\prime}] \geq 0 && (237)\\
\lambda_3: &\quad \lambda_3[-b_{\mathsf{u}}^{\mathrm{RF}\prime}] \geq 0 && (238)
\end{aligned}
$$

There is an abuse of notation, by stating $ab \geq 0$ instead of $a, b \geq 0$ and $ab = 0$.

Next, substituting out λ_1, we get the following conditions

$$
\begin{aligned}
\mathcal{I}_{\mathsf{u}}(S)p'\gamma A f_l(l^{\mathrm{RF}}) - \left(1 + \frac{\lambda_2 - \lambda_3}{\mathcal{I}_{\mathsf{u}}(S)}\right)\mathcal{W}(S) + \lambda_2 p'\gamma A f_l(l^{\mathrm{RF}}) &= 0 && (239)\\
\mathcal{W}(S)l^{\mathrm{RF}} + \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{RF}\prime} &= 0 && (240)\\
\lambda_2[p'\gamma A f(l^{\mathrm{RF}}) + b_{\mathsf{u}}^{\mathrm{RF}\prime}] &\geq 0 && (241)\\
\lambda_3[-b_{\mathsf{u}}^{\mathrm{RF}\prime}] &\geq 0 && (242)
\end{aligned}
$$

It can be shown that the two last constraints are always satisfied with strict inequality.[35] The choice problem reduces to:

$$
\begin{aligned}
\mathcal{I}_{\mathsf{u}}(S)p'\gamma A f_l(l^{\mathrm{RF}}) - \mathcal{W}(S) &= 0 && (243)\\
\mathcal{W}(S)l^{\mathrm{RF}} + \mathcal{I}_{\mathsf{u}}(S) b_{\mathsf{u}}^{\mathrm{RF}\prime} &= 0 && (244)
\end{aligned}
$$

[35] Because of the production function's shape, maximizing $\mathcal{I}_{\mathsf{u}}(S)p_{\mathsf{u}}\gamma A f(l^{\mathrm{RF}}) - \mathcal{W}(S)l^{\mathrm{RF}}$ always leads to a strictly negative $b_{\mathsf{u}}^{\mathrm{RF}\prime}$ and strictly positive profits (no bankruptcy).

D.2 Proof of the Lemma

We restate the Lemma for convenience:

Lemma In the presence of bankruptcy rights, equilibria with positive rates

1. are–in general–suboptimal, and
2. have a perfectly enforced inflation target (i.e., $\mathcal{P}(S') = (1+\pi^{\mathrm{T}})P(S)$ for all S' given S, for all S).

The proof consists of the following steps:

1. Show that in an equilibrium with positive rates, the liquidity constraint of the representative household is binding.
2. Show that, in equilibrium, if the liquidity constraint is binding, then the price-level $\mathcal{P}(S)$ is uniquely determined and always satisfies the inflation target.
3. Show that, in equilibrium, if $\mathcal{P}(S)$ is uniquely determined and always satisfies the inflation target, then the allocation of labor is–in general–not the same as that of the underlying real economy.

Let us proceed.

In an equilibrium with positive rates, the liquidity constraint of the representative household is binding. From the FOCs of the household we have that

$$m' : \qquad \beta\,\mathrm{E}\left[\frac{u_{c'}}{\mathcal{P}(S')}\right] = \beta\,\mathrm{E}[\lambda_1' + \lambda_2'] = \lambda_2 - \lambda_3 \tag{245}$$

$$b_{\mathsf{s}}^{\mathrm{H}\prime} : \qquad \beta\,\mathrm{E}[\lambda_1' + \lambda_2'] = (\lambda_1 + \lambda_2)\mathcal{I}_{\mathsf{s}}(S) - \lambda_3 \tag{246}$$

Clearly, $\lambda_2 - \lambda_3$ is positive since the marginal utility of consumption is always positive. Together the two conditions imply

$$\lambda_2 = \mathcal{I}_{\mathsf{s}}(S)(\lambda_1 + \lambda_2) \tag{247}$$

From the KT condition on consumption (and the fact that marginal utility of consumption is always positive) we have that

$$\frac{u_c}{P(S)} = \lambda_1 + \lambda_2 > 0 \tag{248}$$

and substituting λ_1 out using our previous result (equation 247) we have

$$= \lambda_2 \left[\frac{1}{\mathcal{I}_{\mathsf{s}}(S)} - 1 \right] + \lambda_2 > 0 \tag{249}$$

$$= \frac{\lambda_2}{\mathcal{I}_{\mathsf{s}}(S)} > 0 \tag{250}$$

Therefore $\lambda_2 > 0$, and $\lambda_1 = \lambda_2 \left[\frac{1}{\mathcal{I}_{\mathsf{s}}(S)} - 1 \right] > 0$ since $0 < \mathcal{I}_{\mathsf{s}}(S) < 1$ by the assumption of positive rates and the fact that prices are positive.

In equilibrium, if the liquidity constraint is binding, then the price-level $\mathcal{P}(S)$ is uniquely determined and always satisfies the inflation target. Since we just proved that $\lambda_1 > 0$ then it must be that the household's liquidity constraint is satisfied with equality:

$$\mathcal{P}(S)c + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{H}'} + \mathcal{I}_{\mathsf{u}}(S)b_{\mathsf{u}}^{\mathrm{H}'} = m + w \tag{251}$$

aggregating the identical households over the unit continuum we get

$$\mathcal{P}(S)\int c + \mathcal{I}_{\mathsf{s}}(S)\int b_{\mathsf{s}}^{\mathrm{H}'} + \mathcal{I}_{\mathsf{u}}(S)\int b_{\mathsf{u}}^{\mathrm{H}'} = \int m + \int w \tag{252}$$

using the clearing of financial markets and the firms' budget restrictions one can substitute the households' bond-purchases for wage payments minus the central bank's bond-purchases, and writing M instead of $\int m$ one gets

$$\mathcal{P}(S)\int c + \left(\mathcal{W}(S) - \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{CB}'}\right) = M + \int w \tag{253}$$

decomposing the households' financial wealth $\int w$ into the holdings of the corresponding maturing bonds, one gets

$$\mathcal{P}(S)Y + \left(\mathcal{W}(S) - \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{CB}'}\right) = M - \sum_j B_{\mathsf{s}}^j - \Upsilon \sum_k B_{\mathsf{u}}^k \tag{254}$$

where $k = \mathrm{SF}, \mathrm{RF}$ and $j = \mathrm{SF}, \mathrm{RF}, \mathrm{CB}$. Isolating nominal output on the left side we have

$$\mathcal{P}(S)Y = M - \sum_j B_{\mathsf{s}}^j - \Upsilon \sum_k B_{\mathsf{u}}^k - \mathcal{W}(S) + \mathcal{I}_{\mathsf{s}}(S)b_{\mathsf{s}}^{\mathrm{CB}'} \tag{255}$$

and finally, isolating the price level and disaggregating output by industry we have

$$\mathcal{P}(S) = \frac{M - \sum_j B_{\mathsf{s}}^j - \Upsilon \sum_k B_{\mathsf{u}}^k - \mathcal{W}(S) + \mathcal{I}_{\mathsf{s}}(S) b_{\mathsf{s}}^{\mathrm{CB}\prime}}{Y^{\mathrm{SF}} + \Upsilon Y^{\mathrm{RF}}} \tag{256}$$

Notice that all but $b_{\mathsf{s}}^{\mathrm{CB}\prime}$ are state variables or functions of the state variables. The central bank has therefore complete control over the price level through the purchase/sale of safe bonds; and he is indeed employing the right restriction to achieve its objective (namely this last equation). The central bank will choose one and only one price-level, namely the one that satisfies the inflation target (given the past price-level, which is a state variable as well) and therefore maximizes its objective function. In conclusion, the price level $\mathcal{P}(S)$ is uniquely determined and satisfies $\mathcal{P}(S') = (1 + \pi^{\mathrm{T}})\mathcal{P}(S)$ for all states S' successors of S, for all states S.

In equilibrium, if $\mathcal{P}(S)$ is uniquely determined and always satisfies the inflation target, then the allocation of labor is–in general–not the same as that of the underlying real economy. In a similar fashion as the proof of proposition 1, we obtain the implied real risk-ratio for the representative household, then for the firms, and finally we equate them. At that point, we notice that the resulting condition for labor allocation is–in general–not the same as that in the model without bankruptcy.

Let us proceed.

From the household's KT conditions (i.e. equations (172) through (178)) we have:

$$\rho = \frac{\mathcal{I}_{\mathsf{s}}(S)}{\mathcal{I}_{\mathsf{u}}(S)} = \frac{\beta\,\mathrm{E}\left[u_c(Y')/p'\right] + \lambda_3}{\beta\,\mathrm{E}\left[\Upsilon' u_c(Y')/p'\right]} \tag{257}$$

$$= \frac{\mathrm{E}\left[u_c(Y')\right] + \lambda_3 p'/\beta}{\mathrm{E}\left[\Upsilon' u_c(Y')\right]} \tag{258}$$

$$= \frac{\upsilon \;\; u_c\big(Af(l^{\mathrm{SF}*}) + A\gamma f(1 - l^{\mathrm{SF}*})\big) + (1-\upsilon) \;\; u_c\big(Af(l^{\mathrm{SF}*})\big) + \lambda_3 p'/\beta}{\upsilon \;\; u_c(Af(l^{\mathrm{SF}*}) + A\gamma f(1 - l^{\mathrm{SF}*}))} \tag{259}$$

$$= 1 + \frac{(1-\upsilon)\, u_c\big(Af(l^{\mathrm{SF}*})\big) + \lambda_3 p'/\beta}{\upsilon \;\; u_c\left(Af(l^{\mathrm{SF}*}) + A\gamma f(1 - l^{\mathrm{SF}*})\right)} \tag{260}$$

On the other hand, from the safe firm condition (272) we have:

$$\mathcal{I}_{\mathsf{s}}(S) p' A f_l(l^{\mathrm{SF}*}) - \mathcal{W}(S) = 0 \tag{261}$$

Together with the risky firm's condition

$$\mathcal{I}_{\mathsf{u}}(S) p' \gamma A f_l(l^{\mathrm{RF}*}) - \mathcal{W}(S) = 0 \tag{262}$$

we have that

$$\rho = \frac{\mathcal{I}_{\mathsf{s}}(S)}{\mathcal{I}_{\mathsf{u}}(S)} = \frac{\gamma f_l(1 - l^{\text{SF}*})}{f_l(l^{\text{SF}*})} \tag{263}$$

Equating the household's ρ with the firms' ρ and using the properties of CRRA we get

$$1 + \frac{(1-\upsilon)\ u_c\big(f(l^{\text{SF}**})\big) + K^H}{\upsilon\ \ u_c(f(l^{\text{SF}**}) + \gamma f(1 - l^{\text{SF}**}))} = \frac{\gamma f_l(1 - l^{\text{SF}**})}{f_l(l^{\text{SF}**})} \tag{264}$$

where $K^H \equiv \lambda_3 u_c(A)p'/\beta$. The last equation differs from that of the underlying real economy only due to the term $K^H \geq 0$.

It can be shown that, for some primitives of the model, there are states where the household's bankruptcy constraint binds ($K^H > 0$).[36] The resulting condition for labor allocation in those states is different to that of the model without bankruptcy and leads to a higher safe-firm labor l^{SF} and a higher real risk-ratio ρ than at the unique optimum. More labor at the safe firm and less at the risky firm implies, on average, a lower output for the next period. This can be interpreted as a slower recovery. Notice that there are purely walrasian mechanisms at work here; there are no rigidities or unexpected changes in the price-level. **Q.E.D.**

D.3 Proof of the Theorem

We restate the Theorem for convenience:

Theorem In the presence of bankruptcy rights, equilibria with positive rates and significantly indebted households

1. have a counter-cyclical nominal risk-ratio $\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}}$, and
2. have a counter-cyclical amount of real resources allocated to the safe industry; indeed implying that the economy takes more risk in expansions, and has slower recession recoveries, on average.

The proof consists of showing that, in an equilibrium with significantly indebted households, if $\mathcal{P}(S)$ is uniquely determined and always satisfies the inflation target, then both the nominal risk-ratio $\frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}}$ and the labor allocated to the safe industry are counter-cyclical. Let us proceed.

[36]See the Theorem's proof.

D.3.1 The Equation System for the Theorem

We will now write down the equations needed for the Theorem: the household's conditions, the safe firm's conditions, the risky firm's conditions, and the market clearing conditions. We do this taking into account that the first and second statements of the Lemma are true (hence $\mathcal{P}(S) = (1 + \pi^{\mathrm{T}})\mathfrak{P}$), and simplifying wherever possible (we also drop the (S) of state-dependent variables to shorten the expressions).

The household's conditions are

$$\frac{u_c}{(1+\pi^{\mathrm{T}})\mathfrak{P}} = \lambda_1 + \lambda_2 \tag{265}$$

$$\beta\,\mathrm{E}[\lambda_1' + \lambda_2'] = \lambda_2 - \lambda_3 \tag{266}$$

$$\frac{\beta\,\mathrm{E}[\lambda_1' + \lambda_2'] + \lambda_3}{\mathcal{I}_{\mathsf{s}}} = \lambda_1 + \lambda_2 \tag{267}$$

$$\frac{\beta\,\mathrm{E}[(\lambda_1' + \lambda_2')\Upsilon']}{\mathcal{I}_{\mathsf{u}}} = \lambda_1 + \lambda_2 \tag{268}$$

$$w + m - (1+\pi^{\mathrm{T}})\mathfrak{P}c - \mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{H}\prime} - \mathcal{I}_{\mathsf{u}} b_{\mathsf{u}}^{\mathrm{H}\prime} = 0 \tag{269}$$

$$w + m + \mathcal{W} + D - m' - (1+\pi^{\mathrm{T}})\mathfrak{P}c - \mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{H}\prime} - \mathcal{I}_{\mathsf{u}} b_{\mathsf{u}}^{\mathrm{H}\prime} = 0 \tag{270}$$

$$m' + b_{\mathsf{s}}^{\mathrm{H}\prime} - e = 0 \tag{271}$$

The safe firm's conditions are

$$\mathcal{I}_{\mathsf{s}}(1+\pi^{\mathrm{T}})^2\mathfrak{P}Af_l(l^{\mathrm{SF}}) - \mathcal{W} = 0 \tag{272}$$

$$\mathcal{W}l^{\mathrm{SF}} + \mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{SF}\prime} = 0 \tag{273}$$

The risky firm's conditions are

$$\mathcal{I}_{\mathsf{u}}(1+\pi^{\mathrm{T}})^2\mathfrak{P}\gamma Af_l(l^{\mathrm{RF}}) - \mathcal{W} = 0 \tag{274}$$

$$\mathcal{W}l^{\mathrm{RF}} + \mathcal{I}_{\mathsf{u}} b_{\mathsf{u}}^{\mathrm{RF}\prime} = 0 \tag{275}$$

The market-clearing conditions are (at this point we can ignore integration signs)

$$c = Y^{\mathrm{SF}} + \Upsilon Y^{\mathrm{RF}} \equiv Y \tag{276}$$

$$l^{\mathrm{SF}} + l^{\mathrm{RF}} = 1 \tag{277}$$

$$b_{\mathsf{s}}^{\mathrm{CB}\prime} + b_{\mathsf{s}}^{\mathrm{H}\prime} + b_{\mathsf{s}}^{\mathrm{SF}\prime} = 0 \tag{278}$$

$$b_{\mathsf{u}}^{\mathrm{H}\prime} + b_{\mathsf{u}}^{\mathrm{RF}\prime} = 0. \tag{279}$$

These are 15 equations with the following 15 unknowns:
$c, \lambda_1, \lambda_2, \lambda_3, \mathcal{I}_{\mathsf{s}}, \mathcal{I}_{\mathsf{u}}, b_{\mathsf{s}}^{\mathrm{H}\prime}, b_{\mathsf{u}}^{\mathrm{H}\prime}, \mathcal{W}, m', l^{\mathrm{SF}}, b_{\mathsf{s}}^{\mathrm{SF}\prime}, l^{\mathrm{RF}}, b_{\mathsf{u}}^{\mathrm{RF}\prime}, b_{\mathsf{s}}^{\mathrm{CB}\prime}$.

At the risk of making an unnecessary statement, the following 6 extra equations (two laws of motion, two updates of previous equations, and two identities) are also part of the system

$$Y^{\mathrm{SF}\prime} = Af(l^{\mathrm{SF}}) \tag{280}$$

$$Y^{\mathrm{RF}\prime} = A\gamma f(l^{\mathrm{RF}}) \tag{281}$$

$$\lambda_1' + \lambda_2' = \frac{u_{c'}}{(1+\pi^{\mathrm{T}})^2\mathfrak{P}} \tag{282}$$

$$c' = Y^{\mathrm{SF}\prime} + \Upsilon' Y^{\mathrm{RF}\prime} \equiv Y' \tag{283}$$

$$w = -[B_{\mathsf{s}}^{\mathrm{SF}} + B_{\mathsf{s}}^{\mathrm{CB}} + \Upsilon B_{\mathsf{u}}^{\mathrm{RF}}] \tag{284}$$

$$D = (1+\pi^{\mathrm{T}})\mathfrak{P}Y + B_{\mathsf{s}}^{\mathrm{SF}} + \Upsilon B_{\mathsf{u}}^{\mathrm{RF}} \tag{285}$$

The 6 extra 'unknowns' being $\lambda_1' + \lambda_2', Y^{\mathrm{SF}\prime}, Y^{\mathrm{RF}\prime}, c', w, D$.

A smaller system of 11 equations and 11 unknowns can be obtained through the following straightforward and uncontroversial substitutions:

c is substituted out using equation (276)

l^{RF} is substituted out using equation (277)

$b_{\mathsf{s}}^{\mathrm{SF}\prime}$ is substituted out using equation (278)

$b_{\mathsf{u}}^{\mathrm{RF}\prime}$ is substituted out using equation (279)

$Y^{\mathrm{SF}\prime}$ is substituted out using equation (280)

$Y^{\mathrm{RF}\prime}$ is substituted out using equation (281)

$\lambda_1' + \lambda_2'$ is substituted out using equation (282)

c' is substituted out using equations (283), (280), and (281)

w is substituted out using equation (284)

D is substituted out using equation (285)

It seems timely to substitute out m for M and m' for M'. The resulting equation system follows.

The household's conditions

$$\frac{u_c(Y)}{(1+\pi^{\mathrm{T}})\mathfrak{P}} = \lambda_1 + \lambda_2 \tag{286}$$

$$\beta\,\mathrm{E}\left[\frac{u_c(Af(l^{\mathrm{SF}}) + \Upsilon' A\gamma f(1-l^{\mathrm{SF}}))}{(1+\pi^{\mathrm{T}})^2\mathfrak{P}}\right] = \lambda_2 - \lambda_3 \tag{287}$$

$$\frac{\beta\,\mathrm{E}\left[\frac{u_c(Af(l^{\mathrm{SF}})+\Upsilon' A\gamma f(1-l^{\mathrm{SF}}))}{(1+\pi^{\mathrm{T}})^2\mathfrak{P}}\right] + \lambda_3}{\mathcal{I}_{\mathsf{s}}} = \lambda_1 + \lambda_2 \tag{288}$$

$$\frac{\beta\,\mathrm{E}\left[\left(\frac{u_c(Af(l^{\mathrm{SF}})+\Upsilon' A\gamma f(1-l^{\mathrm{SF}}))}{(1+\pi^{\mathrm{T}})^2\mathfrak{P}}\right)\Upsilon'\right]}{\mathcal{I}_{\mathsf{u}}} = \lambda_1 + \lambda_2 \tag{289}$$

$$M - (1+\pi^{\mathrm{T}})\mathfrak{P}Y - \mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{H}'} - \mathcal{I}_{\mathsf{u}} b_{\mathsf{u}}^{\mathrm{H}'} = B_{\mathsf{s}}^{\mathrm{SF}} + B_{\mathsf{s}}^{\mathrm{CB}} + \Upsilon B_{\mathsf{u}}^{\mathrm{RF}} \tag{290}$$

$$M + \mathcal{W} - M' - \mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{H}'} - \mathcal{I}_{\mathsf{u}} b_{\mathsf{u}}^{\mathrm{H}'} = B_{\mathsf{s}}^{\mathrm{CB}} \tag{291}$$

$$M' + b_{\mathsf{s}}^{\mathrm{H}'} - e = 0 \tag{292}$$

The safe firm's conditions

$$\mathcal{I}_{\mathsf{s}}(1+\pi^{\mathrm{T}})^2\mathfrak{P}Af_l(l^{\mathrm{SF}}) - \mathcal{W} = 0 \tag{293}$$

$$\mathcal{W}l^{\mathrm{SF}} = \mathcal{I}_{\mathsf{s}}(b_{\mathsf{s}}^{\mathrm{H}'} + b_{\mathsf{s}}^{\mathrm{CB}'}) \tag{294}$$

And the risky firm's conditions

$$\mathcal{I}_{\mathsf{u}}(1+\pi^{\mathrm{T}})^2\mathfrak{P}\gamma Af_l(1-l^{\mathrm{SF}}) - \mathcal{W} = 0 \tag{295}$$

$$\mathcal{W}(1-l^{\mathrm{SF}}) = \mathcal{I}_{\mathsf{u}} b_{\mathsf{u}}^{\mathrm{H}'} \tag{296}$$

The 11 unknowns are:
$\lambda_1, \lambda_2, \lambda_3, l^{\mathrm{SF}}, \mathcal{I}_{\mathsf{s}}, \mathcal{I}_{\mathsf{u}}, b_{\mathsf{s}}^{\mathrm{H}'}, b_{\mathsf{u}}^{\mathrm{H}'}, \mathcal{W}, M', b_{\mathsf{s}}^{\mathrm{CB}'}$

We focus on a smaller system of 8 equations and 8 unknowns by substituting out $\lambda_1 + \lambda_2$ using equation (286), and by dropping equations (287) and (288). These two last equations are a dead end because they don't provide any worthwhile information: even the fact that $0 \leq \lambda_3 \leq \lambda_2$ seems useless. The smaller system follows:

Household

$$\frac{\beta\,\mathrm{E}\left[\left(\frac{u_c(Af(l^{\mathrm{SF}})+\Upsilon' A\gamma f(1-l^{\mathrm{SF}}))}{(1+\pi^{\mathrm{T}})^2\mathfrak{P}}\right)\Upsilon'\right]}{\mathcal{I}_{\mathsf{u}}} = \frac{u_c(Y)}{(1+\pi^{\mathrm{T}})\mathfrak{P}} \tag{297}$$

$$M - (1+\pi^{\mathrm{T}})\mathfrak{P}Y - \mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{H}'} - \mathcal{I}_{\mathsf{u}} b_{\mathsf{u}}^{\mathrm{H}'} = B_{\mathsf{s}}^{\mathrm{SF}} + B_{\mathsf{s}}^{\mathrm{CB}} + \Upsilon B_{\mathsf{u}}^{\mathrm{RF}} \tag{298}$$

$$M + \mathcal{W} - M' - \mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{H}'} - \mathcal{I}_{\mathsf{u}} b_{\mathsf{u}}^{\mathrm{H}'} = B_{\mathsf{s}}^{\mathrm{CB}} \tag{299}$$

$$M' + b_{\mathsf{s}}^{\mathrm{H}'} - e = 0 \tag{300}$$

Safe firm

$$\mathcal{I}_{\mathsf{s}}(1+\pi^{\mathrm{T}})^2\mathfrak{P}Af_l(l^{\mathrm{SF}}) - \mathcal{W} = 0 \tag{301}$$

$$\mathcal{W}l^{\mathrm{SF}} = \mathcal{I}_{\mathsf{s}}(b_{\mathsf{s}}^{\mathrm{H}'} + b_{\mathsf{s}}^{\mathrm{CB}'}) \tag{302}$$

Risky firm

$$\mathcal{I}_{\mathsf{u}}(1+\pi^{\mathrm{T}})^2\mathfrak{P}\gamma Af_l(1-l^{\mathrm{SF}}) - \mathcal{W} = 0 \tag{303}$$

$$\mathcal{W}(1-l^{\mathrm{SF}}) = \mathcal{I}_{\mathsf{u}} b_{\mathsf{u}}^{\mathrm{H}'} \tag{304}$$

The 8 unknowns are:
$l^{\mathrm{SF}}, \mathcal{I}_{\mathsf{s}}, \mathcal{I}_{\mathsf{u}}, b_{\mathsf{s}}^{\mathrm{H}'}, b_{\mathsf{u}}^{\mathrm{H}'}, \mathcal{W}, M', b_{\mathsf{s}}^{\mathrm{CB}'}$

We can simplify an equation and transform two other equations into two more useful equations to get:

$$\mathcal{I}_{\mathsf{u}} = \frac{\beta\upsilon}{(1+\pi^{\mathrm{T}})} \frac{u_c\left(Af(l^{\mathrm{SF}}) + A\gamma f(1-l^{\mathrm{SF}})\right)}{u_c(Y)} \quad (305)$$

$$M - (1+\pi^{\mathrm{T}})\mathfrak{P}Y - \mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{H}\prime} - \mathcal{I}_{\mathsf{u}} b_{\mathsf{u}}^{\mathrm{H}\prime} = B_{\mathsf{s}}^{\mathrm{SF}} + B_{\mathsf{s}}^{\mathrm{CB}} + \Upsilon B_{\mathsf{u}}^{\mathrm{RF}} \quad (306)$$

$$M' = M - B_{\mathsf{s}}^{\mathrm{CB}} + \mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{CB}\prime} \quad (307)$$

$$M' + b_{\mathsf{s}}^{\mathrm{H}\prime} - e = 0 \quad (308)$$

$$\mathcal{I}_{\mathsf{s}}(1+\pi^{\mathrm{T}})^2\mathfrak{P}Af_l(l^{\mathrm{SF}}) = \mathcal{W} \quad (309)$$

$$\mathcal{I}_{\mathsf{u}}(1+\pi^{\mathrm{T}})^2\mathfrak{P}\gamma Af_l(1-l^{\mathrm{SF}}) = \mathcal{W} \quad (310)$$

$$\mathcal{W} = \mathcal{I}_{\mathsf{s}}(b_{\mathsf{s}}^{\mathrm{H}\prime} + b_{\mathsf{s}}^{\mathrm{CB}\prime}) + \mathcal{I}_{\mathsf{u}} b_{\mathsf{u}}^{\mathrm{H}\prime} \quad (311)$$

$$\frac{l^{\mathrm{SF}}}{(1-l^{\mathrm{SF}})} = \frac{\mathcal{I}_{\mathsf{s}}}{\mathcal{I}_{\mathsf{u}}} \frac{(b_{\mathsf{s}}^{\mathrm{H}\prime} + b_{\mathsf{s}}^{\mathrm{CB}\prime})}{b_{\mathsf{u}}^{\mathrm{H}\prime}} \quad (312)$$

A smaller system of 5 equations is obtained by substituting out $b_{\mathsf{s}}^{\mathrm{H}\prime}$ using equation (308), substituting out $\mathcal{W}$ using equation (309), and substituting out $\mathcal{I}_{\mathsf{u}} b_{\mathsf{u}}^{\mathrm{H}\prime}$ using equation (311). After some algebraic manipulations and a slight change in the equation order we get:

$$\mathcal{I}_{\mathsf{u}} = \frac{\beta\upsilon}{(1+\pi^{\mathrm{T}})} \frac{u_c\left(Af(l^{\mathrm{SF}}) + A\gamma f(1-l^{\mathrm{SF}})\right)}{u_c(Y)} \quad (313)$$

$$\mathcal{I}_{\mathsf{s}} = \frac{\gamma f_l(1-l^{\mathrm{SF}})}{f_l(l^{\mathrm{SF}})} \mathcal{I}_{\mathsf{u}} \quad (314)$$

$$M' = M - B_{\mathsf{s}}^{\mathrm{CB}} + \mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{CB}\prime} \quad (315)$$

$$M - B_{\mathsf{s}}^{\mathrm{CB}} + \mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{CB}\prime} = \mathcal{I}_{\mathsf{s}}(1+\pi^{\mathrm{T}})^2\mathfrak{P}Af_l(l^{\mathrm{SF}}) + (1+\pi^{\mathrm{T}})\mathfrak{P}Y + B_{\mathsf{s}}^{\mathrm{SF}} + \Upsilon B_{\mathsf{u}}^{\mathrm{RF}} \quad (316)$$

$$b_{\mathsf{s}}^{\mathrm{CB}\prime} = \frac{1}{1-\mathcal{I}_{\mathsf{s}}} \left[(M - B_{\mathsf{s}}^{\mathrm{CB}}) + l^{\mathrm{SF}} \frac{\mathcal{W}}{\mathcal{I}_{\mathsf{s}}} - e\right] \quad (317)$$

where $\frac{\mathcal{W}}{\mathcal{I}_{\mathsf{s}}} = (1+\pi^{\mathrm{T}})^2\mathfrak{P}Af_l(l^{\mathrm{SF}})$. The 5 unknowns are:
$l^{\mathrm{SF}}, \mathcal{I}_{\mathsf{s}}, \mathcal{I}_{\mathsf{u}}, M', b_{\mathsf{s}}^{\mathrm{CB}\prime}$

Next, we will directly reduce this system into one equation with one unknown. First, let us put $\mathcal{I}^{\mathsf{s}}$ in terms of l^{SF}:

$$\mathcal{I}_{\mathsf{s}} = \frac{\gamma f_l(1-l^{\text{SF}})}{f_l(l^{\text{SF}})} \frac{\beta \upsilon}{(1+\pi^{\text{T}})} \frac{u_c\left(Af(l^{\text{SF}}) + A\gamma f(1-l^{\text{SF}})\right)}{u_c(Y)} \tag{318}$$

This is obtained by substituting $\mathcal{I}_{\mathsf{u}}$ in equation (314), using equation (313). Let $\mathcal{I}_{\mathsf{s}}(l^{\text{SF}}, S)$ be the function implicitly defined by this last equation. That is

$$\boxed{\mathcal{I}_{\mathsf{s}}(l^{\text{SF}}, S) = \frac{\gamma f_l(1-l^{\text{SF}})}{f_l(l^{\text{SF}})} \frac{\beta \upsilon}{(1+\pi^{\text{T}})} \frac{u_c\left(Af(l^{\text{SF}}) + A\gamma f(1-l^{\text{SF}})\right)}{u_c(Y)}} \tag{319}$$

Consider an appropriately modified version of equation (316):

$$\underbrace{M - B^{\text{CB}}_{\mathsf{s}} + \mathcal{I}_{\mathsf{s}}(l^{\text{SF}}, S)\, b^{\text{CB}\prime}_{\mathsf{s}}}_{M'} = \underbrace{\mathcal{I}_{\mathsf{s}}(l^{\text{SF}}, S)\,(1+\pi^{\text{T}})^2 \mathfrak{P} A f_l(l^{\text{SF}})}_{\mathcal{W}} + \underbrace{(1+\pi^{\text{T}})\mathfrak{P}Y + B^{\text{SF}}_{\mathsf{s}} + \Upsilon B^{\text{RF}}_{\mathsf{u}}}_{D} \tag{320}$$

Notice that it stands for

$$M' = \mathcal{W} + D \tag{321}$$

since

$$M' = M - B^{\text{CB}}_{\mathsf{s}} + \mathcal{I}_{\mathsf{s}} b^{\text{CB}\prime}_{\mathsf{s}} \tag{322}$$

$$\mathcal{W} = \mathcal{I}_{\mathsf{s}}(1+\pi^{\text{T}})^2 \mathfrak{P} A f_l(l^{\text{SF}}) \tag{323}$$

$$D = (1+\pi^{\text{T}})\mathfrak{P}Y + B^{\text{SF}}_{\mathsf{s}} + \Upsilon B^{\text{RF}}_{\mathsf{u}} \tag{324}$$

We already knew that $M' = \mathcal{W} + D$ from the two household's restrictions (269) and (270). Now, however, we almost have this expression in terms of a single unknown: l^{SF}. We are just one step away. Let us consider equation (317). And notice

$$\mathcal{I}_{\mathsf{s}} b^{\text{CB}\prime}_{\mathsf{s}} = \frac{\mathcal{I}_{\mathsf{s}}}{1-\mathcal{I}_s}\left[(M - B^{\text{CB}}_{\mathsf{s}}) + l^{\text{SF}}\frac{\mathcal{W}}{\mathcal{I}_{\mathsf{s}}} - e\right] \tag{325}$$

Now we can substitute $\mathcal{I}_{\mathsf{s}} b^{\text{CB}\prime}_{\mathsf{s}}$ in equation (316) by using equation (325), and modify it appropriately to finally get

$$
\overbrace{\frac{1}{[1-\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}},S)\,]}\left[M-B^{\mathrm{CB}}_{\mathsf{s}}+l^{\mathrm{SF}}\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}},S)\,(1+\pi^{\mathrm{T}})^2\mathfrak{P}Af_l(l^{\mathrm{SF}})-\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}},S)\,e(S)\right]}^{M'(l^{\mathrm{SF}},S)}
$$

$$
=\underbrace{\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}},S)\,(1+\pi^{\mathrm{T}})^2\mathfrak{P}Af_l(l^{\mathrm{SF}})}_{\mathcal{W}(l^{\mathrm{SF}},S)}+\underbrace{(1+\pi^{\mathrm{T}})\mathfrak{P}Y+B^{\mathrm{SF}}_{\mathsf{s}}+\Upsilon B^{\mathrm{RF}}_{\mathsf{u}}}_{D(S)} \tag{326}
$$

One single equation with one single unknown: l^{SF}.

We can manipulate equation (326) to get all the l^{SF}-dependent terms on the left hand side (LHS):

$$
\frac{1}{[1-\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}},S)\,]}\left[M-B^{\mathrm{CB}}_{\mathsf{s}}+\left(\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}},S)+l^{\mathrm{SF}}-1\right)\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}},S)\,(1+\pi^{\mathrm{T}})^2\mathfrak{P}Af_l(l^{\mathrm{SF}})-\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}},S)\,e(S)\right]
$$

$$
=(1+\pi^{\mathrm{T}})\mathfrak{P}Y+B^{\mathrm{SF}}_{\mathsf{s}}+\Upsilon B^{\mathrm{RF}}_{\mathsf{u}} \tag{327}
$$

D.3.2 Uniqueness and Existence (preliminaries)

Since both existence and uniqueness depend critically on equation (327), it will be useful to establish the monotonicity of some of the terms in this equation with respect to l^{SF}.

$\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}},S)$ is increasing in l^{SF}. To see this, recall that (from our assumption that the household's liquidity constraint is binding) the equation system is only defined for $l^{\mathrm{SF}}\geq l^{\mathrm{SF}}_{\mathrm{Pareto}}$. This is very useful because $u_c(Af(l^{\mathrm{SF}})+A\gamma f(1-l^{\mathrm{SF}}))$ is strictly increasing in l^{SF} for all l^{SF} greater or equal to $l^{\mathrm{SF}}_{\mathrm{Pareto}}$.[37] The first term of equation (319) is always increasing in l^{SF} by the properties of $f(\cdot)$. Thus $\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}},S)$ is strictly increasing in l^{SF} for all $l^{\mathrm{SF}}\in[l^{\mathrm{SF}}_{\mathrm{Pareto}},1]$.

$\mathcal{W}(l^{\mathrm{SF}},S)$ is increasing in l^{SF}. To see this, recall that

$$
\mathcal{W}(l^{\mathrm{SF}},S)=\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}},S)\,(1+\pi^{\mathrm{T}})^2\mathfrak{P}Af_l(l^{\mathrm{SF}}) \tag{328}
$$

[37] The intuition is that $u_c(Af(l^{\mathrm{SF}})+A\gamma f(1-l^{\mathrm{SF}}))$ is the marginal utility in the lucky state where the risky technology does deliver. Thus, starting from the (ex-ante Pareto) l^{SF}, an increase in the safe-firm's labor reduces the output.

substituting in $\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}, S)$ we have

$$\mathcal{W}(l^{\mathrm{SF}}, S) = \frac{\gamma f_l(1-l^{\mathrm{SF}})}{f_l(l^{\mathrm{SF}})} \frac{\beta \upsilon}{(1+\pi^{\mathrm{T}})} \frac{u_c\left(Af(l^{\mathrm{SF}}) + A\gamma f(1-l^{\mathrm{SF}})\right)}{u_c(Y)} (1+\pi^{\mathrm{T}})^2 \mathfrak{P} A f_l(l^{\mathrm{SF}}) \tag{329}$$

or

$$\mathcal{W}(l^{\mathrm{SF}}, S) = \gamma f_l(1-l^{\mathrm{SF}}) \beta \upsilon \frac{u_c\left(Af(l^{\mathrm{SF}}) + A\gamma f(1-l^{\mathrm{SF}})\right)}{u_c(Y)} (1+\pi^{\mathrm{T}}) \mathfrak{P} A \tag{330}$$

Once again, we are faced with the fact that $u_c(Af(l^{\mathrm{SF}})+A\gamma f(1-l^{\mathrm{SF}}))$ is strictly increasing in l^{SF} for all l^{SF} greater or equal to $l^{\mathrm{SF}}_{\mathrm{Pareto}}$. And, on top of that, the first term is strictly increasing in l^{SF} for all l^{SF}. So $\mathcal{W}(l^{\mathrm{SF}}, S)$ is strictly increasing in l^{SF} for all $l^{\mathrm{SF}} \in [l^{\mathrm{SF}}_{\mathrm{Pareto}}, 1]$.

The LHS of equation (327) **is strictly increasing in** l^{SF} **depending on** $e(S)$**.** Thus, the role of $e(S)$ in determining uniqueness is very important. Generally speaking, $e(S) \equiv e(l^{\mathrm{SF}}(S), S)$ has to be consistent with a LHS that is strictly increasing in l^{SF} in order to get uniqueness. It turns out though, that for the most natural benchmark case (namely $e(S)$ equal to the nominal income) everything falls in place with ease. And, any value in the neighborhood of the nominal income will lead to the same results as well.

D.3.3 Uniqueness and Existence when $e(S) = \varepsilon \mathcal{N}(S)$ for some $\varepsilon \in (0, 1]$

We now use $e(S) = \varepsilon \mathcal{N}(S)$ with $\varepsilon \in (0, 1]$. Plugging $e(S) = \varepsilon \mathcal{N}(S) \equiv \varepsilon [\mathcal{W}(S) + D(S)]$, which also equals $\varepsilon M'$, into equation (327) we have

$$\begin{aligned} &\frac{1}{[1-\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}, S)]} \Big[M - B^{\mathrm{CB}}_{\mathsf{s}} + \left(\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}, S) + l^{\mathrm{SF}} - 1\right) \mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}, S)\, (1+\pi^{\mathrm{T}})^2 \mathfrak{P} A f_l(l^{\mathrm{SF}}) \\ &- \mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}, S)\, \varepsilon \left[\mathcal{W}(l^{\mathrm{SF}}, S) + D(S)\right] \Big] = (1+\pi^{\mathrm{T}}) \mathfrak{P} Y + B^{\mathrm{SF}}_{\mathsf{s}} + \Upsilon B^{\mathrm{RF}}_{\mathsf{u}} \end{aligned} \tag{331}$$

By construction, the l^{SF} that solves the system is the equilibrium one: $l^{\mathrm{SF}}(S)$. So really $\mathcal{W}(S) = \mathcal{W}(l^{\mathrm{SF}}(S), S)$. Also, recall that $\mathcal{W}(l^{\mathrm{SF}}, S) = \mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}, S)\, (1+\pi^{\mathrm{T}})^2 \mathfrak{P} A f_l(l^{\mathrm{SF}})$ and $D(S) \equiv (1+\pi^{\mathrm{T}}) \mathfrak{P} Y + B^{\mathrm{SF}}_{\mathsf{s}} + \Upsilon B^{\mathrm{RF}}_{\mathsf{u}}$. Hence, after some manipulations, we get

$$\boxed{M - B^{\mathrm{CB}}_{\mathsf{s}} + \left([1-\varepsilon]\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}, S) + l^{\mathrm{SF}} - 1\right) \mathcal{W}(l^{\mathrm{SF}}, S) = D(S)} \tag{332}$$

Uniqueness follows necessarily if the LHS is strictly increasing. Existence is obtained if there is a value of l^{SF} for which the equation is satisfied.[38]

[38] This sentence is somewhat sloppy. A rigorous analogue will appear shortly.

Uniqueness given condition $-\frac{f_{ll}(x)x}{f_l(x)} < 1 - \alpha_2 x \frac{\gamma f_l(x) - f_l(1-x)}{\gamma f(x) + f(1-x)}$ is established through a somewhat long argument.

First, have a look at equation (332) and notice that, for uniqueness, it is enough that $\big([1-\varepsilon]\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}, S) + l^{\mathrm{SF}} - 1\big)\,\mathcal{W}(l^{\mathrm{SF}}, S)$ be increasing in l^{SF} (other terms are constant given a fixed state S).

Next, let us differentiate this last expression with respect to l^{SF}. If the resulting expression is positive, uniqueness is guaranteed. Hence the following condition:

$$\left([1-\varepsilon]\frac{\partial \mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}, S)}{l^{\mathrm{SF}}} + 1\right)\mathcal{W}(l^{\mathrm{SF}}, S) + \big([1-\varepsilon]\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}, S) + l^{\mathrm{SF}} - 1\big)\frac{\partial \mathcal{W}(l^{\mathrm{SF}}, S)}{\partial l^{\mathrm{SF}}} > 0 \quad (333)$$

This last condition can be rewritten as

$$\mathcal{W} > [\varepsilon - 1]\frac{\partial \mathcal{I}_{\mathsf{s}}}{\partial l^{\mathrm{SF}}}\mathcal{W} + \big([\varepsilon - 1]\mathcal{I}_{\mathsf{s}} + [1 - l^{\mathrm{SF}}]\big)\frac{\partial \mathcal{W}}{\partial l^{\mathrm{SF}}} \quad (334)$$

At this point, we recall that the four expressions $\frac{\partial \mathcal{I}_{\mathsf{s}}}{l^{\mathrm{SF}}}$, $\mathcal{W}$, $\mathcal{I}_{\mathsf{s}}$, and $\frac{\partial \mathcal{W}}{\partial l^{\mathrm{SF}}}$, are all positive in equilibrium, and that $l^{\mathrm{SF}} \in (0,1)$ in equilibrium as well. So, assuming $\varepsilon \in (0,1]$, it would be sufficient for uniqueness to have the following simpler condition satisfied:

$$\boxed{\mathcal{W} > (1 - l^{\mathrm{SF}})\frac{\partial \mathcal{W}}{\partial l^{\mathrm{SF}}}} \quad (335)$$

Now, since

$$\mathcal{W} = \frac{\gamma\beta\upsilon\mathfrak{P}A(1+\pi^{\mathrm{T}})}{u_c(Y)} f_l(1 - l^{\mathrm{SF}})u_c\big(Af(l^{\mathrm{SF}}) + \gamma Af(1 - l^{\mathrm{SF}})\big) \quad (336)$$

and

$$u_c(x) = \alpha_1 x^{-\alpha_2}\,, \quad (337)$$

one has that

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial l^{\mathrm{SF}}} =& \frac{\gamma\beta\upsilon\mathfrak{P}A^{1-\alpha_2}(1+\pi^{\mathrm{T}})}{Y^{-\alpha_2}}\cdot \\ &\Big\{f_{ll}(1-l^{\mathrm{SF}})(-1)\alpha_1\left[f(l^{\mathrm{SF}}) + \gamma f(1-l^{\mathrm{SF}})\right]^{-\alpha_2} \\ &+ f_l(1-l^{\mathrm{SF}})\alpha_1(-\alpha_2)\left[f(l^{\mathrm{SF}}) + \gamma f(1-l^{\mathrm{SF}})\right]^{-\alpha_2-1}\left[f_l(l^{\mathrm{SF}}) + \gamma f_l(1-l^{\mathrm{SF}})(-1)\right]\Big\}\end{aligned} \quad (338)$$

Substituting $\mathcal{W}$ and $\frac{\partial \mathcal{W}}{\partial l^{\mathrm{SF}}}$ back into the inequality, and manipulating somewhat, one gets

$$-\frac{f_{ll}(1-l^{\mathrm{SF}})(1-l^{\mathrm{SF}})}{f_l(1-l^{\mathrm{SF}})} < 1 - \alpha_2(1-l^{\mathrm{SF}})\cdot\frac{\gamma f_l(1-l^{\mathrm{SF}}) - f_l(l^{\mathrm{SF}})}{\gamma f(1-l^{\mathrm{SF}}) + f(l^{\mathrm{SF}})} \quad (339)$$

which is true by the assumptions made for the production function $f(\cdot)$.

We are favoring parsimony over generality by considering the case $\varepsilon \leq 1$, and $e(S) = \varepsilon\mathcal{N}$. It is likely that a less stringent condition suffices for an even wider set of the parameter ε.

Existence requires the right-hand-side of equation (332) to be greater or equal to the left-hand-side evaluated at $l^{\mathbb{SF}}_{\text{Pareto}}$. Notice that this is a necessary condition for the household's bankruptcy constraint to be satisfied with equality for the given state, in equilibrium. *It is by no means* a necessary condition for an equilibrium with positive rates to exist. If this condition were not satisfied for a given state S_0, it would only mean that households are not significantly indebted in the corresponding equilibrium.

Since, by assumption, the equilibria we focus on have significantly indebted households, the existence condition is satisfied for every state. We can therefore proceed with our analysis. But before we do so, there is a subtler issue we would like to point out.

The assumption that has been made throughout the paper, namely that equilibria have positive rates, may constrain $l^{\mathbb{SF}}$ from above since $\mathcal{I}_{\mathsf{s}}(l^{\mathbb{SF}}, S)$ is increasing in $l^{\mathbb{SF}}$ and positive rates imply $\mathcal{I}_{\mathsf{s}}(l^{\mathbb{SF}}, S) < 1$. This is true for all cases: without bankruptcy restrictions or with bankruptcy restrictions (with or without indebted households). Let this upper bound for $l^{\mathbb{SF}}$ be written as $\overline{l^{\mathbb{SF}}}(S)$.[39] It may seem superfluous to bring this up now, but since we just showed that the LHS of equation (332) evaluated at $l^{\mathbb{SF}}_{\text{Pareto}}$ has to be smaller than or equal to the RHS for existence, transparency requires us to remember that the LHS evaluated at $\overline{l^{\mathbb{SF}}}(S)$ has to be weakly greater than the RHS for existence as well, which we assume. Together, these two conditions are sufficient for existence.

D.3.4 D, $\mathcal{I}_{\mathsf{s}}$, $\mathcal{W}$, and M' are greater in the lucky state, in equilibrium

Having established that the equilibrium exists and is unique, we wish to distinguish the "lucky state" from the "unlucky state", qualitatively. Consider changing Υ from zero to one, other state-variables equal (that is, fixing the subvector $S_{-\Upsilon}$).

First, by the properties of $f(\cdot)$ and the corresponding first order conditions, dividends $D(S)$ are greater in the lucky state. This is because the safe firm's ex-post profits are the same in both lucky and unlucky state; but the risky firm's ex-post profits are only positive in the lucky state. So dividends (which equal ex-post profits) are greater in the lucky state.

Next, we use a two-stage thought experiment:

[39] That is, $\overline{l^{\mathbb{SF}}}(S)$ is implicitly defined by $\mathcal{I}_{\mathsf{s}}(\overline{l^{\mathbb{SF}}}, S) = 1$.

Let S_{u} denote the state S with $\Upsilon = 1$ (i.e., the state is "up"); and similarly let S_{d} denote the state S with $\Upsilon = 0$ (i.e., the state is "down"). Assume both states are otherwise equal. Let $l^{\mathrm{SF}}_{\mathsf{d}}$ be the solution to equation (332) when $S = S_{\mathsf{d}}$. And define $\mathcal{I}_{\mathsf{s,d}} \equiv \mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}_{\mathsf{d}}, S_{\mathsf{d}})$. Now, consider the equation $\mathcal{I}_{\mathsf{s,d}} = \mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}, S)$, and define $l^{\mathrm{SF}}_{\bar{\mathsf{u}}}$ implicitly by $\mathcal{I}_{\mathsf{s,d}} = \mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}_{\bar{\mathsf{u}}}, S_{\mathsf{u}})$. In other words, start from an equilibrium situation in the unlucky state, change Υ from zero to one (change to the lucky twin-state), and set l^{SF} to the value such that the price of the safe asset $\mathcal{I}_{\mathsf{s}}$ stays as before; that value is denoted $l^{\mathrm{SF}}_{\bar{\mathsf{u}}}$.

A quick inspection of equation (319) reveals that $l^{\mathrm{SF}}_{\bar{\mathsf{u}}} < l^{\mathrm{SF}}_{\mathsf{d}}$. Furthermore, inspecting equation (328) reveals that $\mathcal{W}(l^{\mathrm{SF}}_{\bar{\mathsf{u}}}, S_{\mathsf{u}}) > \mathcal{W}(l^{\mathrm{SF}}_{\mathsf{d}} S_{\mathsf{d}})$ but, since our assumptions guarantee $xf_l(x)$ is increasing in x,[40] we have that $l^{\mathrm{SF}}_{\bar{\mathsf{u}}}\mathcal{W}(l^{\mathrm{SF}}_{\bar{\mathsf{u}}}, S_{\mathsf{u}}) < l^{\mathrm{SF}}_{\mathsf{d}}\mathcal{W}(l^{\mathrm{SF}}_{\mathsf{d}}, S_{\mathsf{d}})$.

Notice that equation (332) can be rewritten as

$$M - B^{\mathrm{CB}}_{\mathsf{s}} + \big([1-\varepsilon]\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}, S) - 1\big)\, \mathcal{W}(l^{\mathrm{SF}}, S) + l^{\mathrm{SF}}\mathcal{W}(l^{\mathrm{SF}}, S) = D(S) \tag{340}$$

Starting from the equilibrium situation under $S = S_{\mathsf{d}}$ we have

$$M - B^{\mathrm{CB}}_{\mathsf{s}} + \big([1-\varepsilon]\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}_{\mathsf{d}}, S_{\mathsf{d}}) - 1\big)\, \mathcal{W}(l^{\mathrm{SF}}_{\mathsf{d}}, S_{\mathsf{d}}) + l^{\mathrm{SF}}_{\mathsf{d}}\mathcal{W}(l^{\mathrm{SF}}_{\mathsf{d}}, S_{\mathsf{d}}) = D(S_{\mathsf{d}}) \tag{341}$$

Next, changing Υ from zero to one (S_{d} to S_{u}) and keeping $\mathcal{I}_{\mathsf{s}}$ constant (setting l^{SF} to $l^{\mathrm{SF}}_{\bar{\mathsf{u}}}$) we have

$$M - B^{\mathrm{CB}}_{\mathsf{s}} + \underbrace{\big([1-\varepsilon]\mathcal{I}_{\mathsf{s}}(l^{\mathrm{SF}}_{\bar{\mathsf{u}}}, S_{\mathsf{u}}) - 1\big)}_{\text{negative and stayed constant}} \cdot \underbrace{\mathcal{W}(l^{\mathrm{SF}}_{\bar{\mathsf{u}}}, S_{\mathsf{u}})}_{\text{positive and increased}} + \underbrace{l^{\mathrm{SF}}_{\bar{\mathsf{u}}}\mathcal{W}(l^{\mathrm{SF}}_{\bar{\mathsf{u}}}, S_{\mathsf{u}})}_{\text{positive and decreased}} < \underbrace{D(S_{\mathsf{u}})}_{\text{positive and increased}} \tag{342}$$

It follows that the equilibrium value of l^{SF} at state $S = S_{\mathsf{u}}$, denoted $l^{\mathrm{SF}}_{\mathsf{u}}$, must be strictly greater than $l^{\mathrm{SF}}_{\bar{\mathsf{u}}}$ to achieve equality (satisfy the equilibrium equation). As a result, we now know that $\mathcal{I}_{\mathsf{s}}$ and $\mathcal{W}$ are greater in the lucky state. And, since $M' = \mathcal{W} + D$, we know that M' is greater in the lucky state as well.

D.3.5 $l^{\mathrm{SF}*}$ is greater in the lucky state

Start by recalling the equations

$$\mathcal{W} = M' - D \tag{343}$$

$$\mathcal{W} = l^{\mathrm{SF}}\mathcal{W} + l^{\mathrm{RF}}\mathcal{W} \tag{344}$$

$$\mathcal{W} = \mathcal{I}_{\mathsf{s}}(b^{\mathrm{CB}\prime}_{\mathsf{s}} + b^{\mathrm{H}\prime}_{\mathsf{s}}) + \mathcal{I}_{\mathsf{u}} b^{\mathrm{H}\prime}_{\mathsf{u}} \tag{345}$$

$$\mathcal{W} = \mathcal{I}_{\mathsf{s}} b^{\mathrm{CB}\prime}_{\mathsf{s}} + \mathcal{I}_{\mathsf{s}}(e - M') + \mathcal{I}_{\mathsf{u}} b^{\mathrm{H}\prime}_{\mathsf{u}} \tag{346}$$

[40] Our assumptions guarantee that $-\frac{f_{ll}(x)x}{f_l(x)} < 1$. But then $-f_{ll}(x)x < f_l(x)$ or, equivalently, $0 < f_l(x) + f_{ll}(x)x$; this last expression is the derivative of $xf_l(x)$ with respect to x; so $xf_l(x)$ is indeed increasing in x.

Remember e is the exempted amount for households in case of bankruptcy.

From the first equation of the previous group, we immediately know that

$$\Delta\mathcal{W} = \Delta M' - \Delta D \tag{347}$$

where the operator Δ satisfies $\Delta h = h\,|_{\Upsilon=1} - h\,|_{\Upsilon=0}$. That is, the operator Δ stands for 'the change in this expression with respect to a change in Υ from zero to one'. And we use Δ as an operator that goes before over addition, and goes after multiplication.

Now, since $M' = (M - B_{\mathsf{s}}^{\mathrm{CB}}) + \mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{CB}\prime}$, we also know that

$$\Delta M' = \Delta\mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{CB}\prime} \tag{348}$$

From equation (344), it follows that

$$\Delta\mathcal{W} = \Delta l^{\mathrm{SF}}\mathcal{W} + \Delta l^{\mathrm{RF}}\mathcal{W} \tag{349}$$

$$= \mathcal{I}_{\mathsf{s}} b_{\mathsf{s}}^{\mathrm{CB}\prime} + \mathcal{I}_{\mathsf{s}}(e - M') + \Delta l^{\mathrm{RF}}\mathcal{W} \tag{350}$$

$$= \Delta M' + \Delta\mathcal{I}_{\mathsf{s}}(e - M') + \Delta l^{\mathrm{RF}}\mathcal{W} \tag{351}$$

Together with equation (347) we have

$$\Delta M' - \Delta D = \Delta M' + \Delta\mathcal{I}_{\mathsf{s}}(e - M') + \Delta l^{\mathrm{RF}}\mathcal{W} \tag{352}$$

or

$$-\Delta D = \Delta\mathcal{I}_{\mathsf{s}}(e - M') + \Delta l^{\mathrm{RF}}\mathcal{W} \tag{353}$$

$$\tag{354}$$

Hence

$$\Delta l^{\mathrm{RF}}\mathcal{W} = \Delta\mathcal{I}_{\mathsf{s}}(e - M') - \Delta D \tag{355}$$

$$= \Delta\mathcal{I}_{\mathsf{s}}(\varepsilon\mathcal{N} - M') - \Delta D \tag{356}$$

$$= \Delta\mathcal{I}_{\mathsf{s}}(\varepsilon M' - M') - \Delta D \tag{357}$$

$$= \Delta\mathcal{I}_{\mathsf{s}}(\varepsilon - 1)M' - \Delta D \tag{358}$$

$$= (\varepsilon - 1)\Delta\mathcal{I}_{\mathsf{s}} M' - \Delta D \tag{359}$$

Since $\Delta\mathcal{I}_{\mathsf{s}} M', \Delta D > 0$ and $(\varepsilon - 1) \leq 0$ we have that $\Delta l^{\mathrm{RF}}\mathcal{W} \leq 0$. And since we know from earlier that $\Delta\mathcal{W} > 0$, it must be that $\Delta l^{\mathrm{RF}} < 0$. **Q.E.D.**

D.4 Proof of the Claim

We restate the Claim for convenience:

Claim (Theorem extension on counter-cyclical Credit-Spreads) For each set of primitives, there is a real number $\overline{\alpha} > 0$ such that: if the Constant Relative Risk Aversion coefficient $\alpha_2 < \overline{\alpha}$, then the equilibrium nominal risk-spread (a.k.a. credit-spread) is counter-cyclical as well.

To see this, recall that–under perfect enforcement of inflation–the nominal credit-spread $\xi \equiv i_{\mathsf{u}} - i_{\mathsf{s}}$ is given by

$$\begin{aligned} \xi &= (\rho - 1)(1 + i_{\mathsf{s}}) && (360)\\ &= (1 - \rho^{-1})\rho\mathcal{I}_{\mathsf{s}}^{-1} && (361)\\ &= (1 - \rho^{-1})\rho(\rho\mathcal{I}_{\mathsf{u}})^{-1} && (362)\\ &= (1 - \rho^{-1})\mathcal{I}_{\mathsf{u}}^{-1} && (363)\end{aligned}$$

Substituting in the equilibrium terms for ρ and $\mathcal{I}_{\mathsf{u}}$ we have

$$\xi = \left(1 - \gamma^{-1}\frac{f_l(l^{\mathrm{SF}})}{f_l(1 - l^{\mathrm{SF}})}\right)\frac{(1 + \pi^{\mathrm{T}})}{\beta\upsilon}\frac{u_c(Y)}{u_c\left(Af(l^{\mathrm{SF}}) + A\gamma f(1 - l^{\mathrm{SF}})\right)} \tag{364}$$

Recalling that constants (like $\frac{(1+\pi^{\mathrm{T}})}{\beta\upsilon}$) are irrelevant for correlations, we can consider the following modified credit-spread:

$$\xi^* = \left(1 - \gamma^{-1}\frac{f_l(l^{\mathrm{SF}})}{f_l(1 - l^{\mathrm{SF}})}\right)\frac{u_c(Y)}{u_c\left(Af(l^{\mathrm{SF}}) + A\gamma f(1 - l^{\mathrm{SF}})\right)} \tag{365}$$

Finally, writing down the marginal utilities in explicit form, we have

$$\xi^* = \underbrace{\left(1 - \gamma^{-1}\frac{f_l(l^{\mathrm{SF}})}{f_l(1 - l^{\mathrm{SF}})}\right)}_{\text{counter-cyclical}}\underbrace{\left(\frac{Af(l^{\mathrm{SF}}) + A\gamma f(1 - l^{\mathrm{SF}})}{Y}\right)^{\alpha_2}}_{\text{pro-cyclical}} \tag{366}$$

The variance of the pro-cyclical factor can be arbitrarily dampened by choosing a smaller α_2. The equilibrium l^{SF} becomes smaller, and hence the radicand of the pro-cyclical factor has a greater variance; however this growth in variance is bounded (it cannot exceed the variance that would follow from a risk-neutral choice). On the other hand, the dampening that occurs by a bigger root degree (the smaller exponent α_2) is unbounded.

Finally, the variance of the pro-cyclical factor does go to zero (and the variance of the counter-cyclical factor does not go to zero) because $\Delta l^{\mathbb{SF}}$ does not go to zero and does not grow without bounds. The latter must be true since $l^{\mathbb{SF}}$ lies by construction between zero and one, so the variance of the radicand is bounded; the former though, is somewhat more subtle: If the equilibrium $l^{\mathbb{SF}}$ does shrink, then we know that the variance of output increases (since then the risky component of output is greater), and so does the variance of dividends too. It then follows, from equation (359), that the variance of $l^{\mathbb{SF}}$ cannot go to zero. **Q.E.D.**