IFRS9 Expected Credit Loss Estimation: Advanced Models for Estimating Portfolio Loss and Weighting Scenario Losses

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Abstract

Estimation of portfolio expected credit loss is required for IFRS9 regulatory purposes. It starts with the estimation of scenario loss at loan level, and then aggregated and summed up by scenario probability weights to obtain portfolio expected loss. This estimated loss can vary significantly, depending on the levels of loss severity generated by the IFRS9 models, and the probability weights chosen. There is a need for a quantitative approach for determining the weights for scenario losses. In this paper, we propose a model to estimate the expected portfolio losses brought by recession risk, and a quantitative approach for determining the scenario weights. The model and approach are validated by an empirical example, where we stress portfolio expected loss by recession risk, and calculate the scenario weights accordingly.

Keywords: Scenario weight, stressed expected credit loss, loss severity, recession probability, Vasicek distribution, probit mixed model

1. Introduction

Estimation of portfolio expected credit loss is required for IFRS9 regulatory purposes ([2], [3]). It starts with the estimation of scenario loss at loan level, and then aggregated and summed over by scenario weights to obtain portfolio expected loss. In general, there are three scenarios under consideration: pessimistic, base, and optimistic.

Let \( p_1, p_2, \) and \( p_3 \) denote respectively the occurring probabilities for these three scenarios, and \( y_1, y_2, \) and \( y_3 \) the corresponding portfolio losses. In practice, the expected credit loss is estimated as:

\[
EL = w_1^0 y_1 + w_2^0 y_2 + w_3^0 y_3, \tag{1.1}
\]

where \( w_1^0, w_2^0, \) and \( w_3^0 \) are relative weights derived from \( p_1, p_2, \) and \( p_3 \) as:

\[
w_i^0 = \frac{p_i}{p_1 + p_2 + p_3}, \quad 1 \leq i \leq 3. \tag{1.2}
\]

Result for \( EL \) by (1.1) can vary significantly, depending on the values of \( p_1, p_2, \) and \( p_3 \) chosen, and the portfolio scenario losses \( y_1, y_2, \) and \( y_3 \) generated by IFRS9 models. In practice, this number is compared with historical portfolio losses to position its severity (in relative to the historical portfolio loss distribution). An issue arises, when the number calculated by (1.1) is nonintuitive, in which case, practitioners will manually search for weights \( w_1^0, w_2^0, \) and \( w_3^0 \) to come up with an acceptable number for \( EL \). Weights obtained in this way lack quantitative justification, and are somehow arbitrary.

We focus on portfolio loss rate and assume that the exposure for the portfolio is one dollar. We will use interchangeably the words “loss” and “loss rate”.

When assuming that the portfolio loss rate \( y \) is driven by a latent dynamic \( s \), the expected portfolio loss is equal to the integral \( \int_A y(s)p(s)ds \) for the function \( y(s)p(s) \) over the range \( A \) where \( s \) varies, where \( p(s) \) denotes the probability density of \( s \). Therefore, the portfolio expected loss is not necessarily equal to weighted sum \( [p(s_1)y(s_1) + p(s_2)y(s_2) + p(s_3)y(s_3)]/[p(s_1) + p(s_2) + p(s_3)] \), even when \( p(s_1), p(s_2), \) and \( p(s_3) \) correspond to the given occurring probabilities.

In this paper, we propose a quantitative approach (see Algorithm 3.4) to finding the weights \( w_1, w_2, \) and \( w_3 \) (not necessarily the same as \( w_1^0, w_2^0, \) and \( w_3^0 \) as described above) satisfying equation (1.3) below:

\[
EL = w_1 y_1 + w_2 y_2 + w_3 y_3, \tag{1.3}
\]
Proposition 2. Forward looking consideration is generally required for expected portfolio loss. Under this setting, portfolio expected loss needs to be assessed under a stressed condition. Scenario weights for the corresponding stressed portfolio loss are required. For this purpose, we propose a model (see model (3.11)) to stress the portfolio expected loss conditional on recession risk. Scenario weights are determined, given the recession probability, occurring probabilities \( p_1, p_2, p_3 \), and scenario losses \( y_1, y_2, y_3 \).

The paper is organized as follows. In section 2, we demonstrate an equivalent parameterization for Vasicek distribution. An one-factor probit mixed model is defined for each Vasicek distribution. In section 3, we propose a probit-type mixed model for portfolio expected loss conditional on recession risk. A generic algorithm to find scenario weights is proposed in this section. The proposed model and approach are validated in section 4 by an empirical example, where we stress portfolio expected loss by recession risk, and calculate scenario weights accordingly.

2. Preliminaries: Vasicek Distribution and One-Factor Probit Mixed Models

Vasicek distribution ([12], [13]) is skewed and leptokurtic ([13]). It is widely used for modeling of portfolio loss distributions ([4], [5], [6], [7], [11]).

In this section, we demonstrate an alternative formulation for Vasicek distribution by using the mean and standard deviation of the probit form of the variable. An one-factor, probit-type, mixed model is defined by this parameterization. The parameter estimation by maximum likelihood approach for a probit-type mixed model becomes simpler when likelihood is formulated by this parameterization (see Proposition 3.1).

For a variable \( y \) (\( 0 < y < 1 \)) that follows a Vasicek distribution, its density is given by ([13]):

\[
g(y, p, \rho) = \sqrt{\frac{1-p}{\rho}} \exp\left\{-\frac{1}{2p} \left[\sqrt{1-\rho} \phi^{-1}(y) - \phi^{-1}(p)\right]^2 + \frac{1}{2} \phi^{-1}(y)]^2 \right\}, \tag{2.1}
\]

where \( \phi \) denotes the mean value of \( y \), and \( \rho \) is the asset correlation, which is related to the Asymptotic Single Risk factor model ([1], [5], [8], [13]).

Let \( \Phi \) denote the cumulative distribution function (CDF) for a standard normal variable and \( \phi \) its density. A random variable \( y \), \( 0 < y < 1 \), is said to follow a Vasicek distribution if its probit form \( \Phi^{-1}(y) \) is normally distributed ([9]). Let \( a \) and \( b \) denote respectively the mean and standard deviation of \( \Phi^{-1}(y) \). Then \( y \) can be formulated as an one-factor probit-type model:

\[
y = \Phi(a + bS), \quad S \sim N(0, 1). \tag{2.2}
\]

The probability density function is given by proposition below.

**Proposition 2.1** ([15]). The density of the variable \( y \) in (2.2) is given by

\[
f(y, a, b) = U_1/(bU_2), \tag{2.3}
\]

where

\[
U_1 = \Phi([\Phi^{-1}(y) - a]/b), \quad U_2 = \Phi[\Phi^{-1}(y)]. \tag{2.4}
\]

\[\square\]

**Proposition 2.2.** Densities (2.3) and (2.1) are equivalent under the relationships:
\[ a = \frac{\Phi^{-1}(p)}{\sqrt{1-p}} \quad \text{and} \quad b = \sqrt{\frac{p}{1-p}}. \] (2.5)

**Proof.** By (2.1), we have:

\[
g(y, p, \rho) = \frac{1}{\sqrt{2\pi\rho}} \exp\left(-\frac{1}{2\rho} \left[ \Phi^{-1}(y) - \frac{\Phi^{-1}(p)}{\sqrt{1-p}} \right]^2 + \frac{1}{2} \left[ \Phi^{-1}(y) \right]^2 \right) = \frac{1}{b} \exp\left(-\frac{1}{2} \left[ \frac{\Phi^{-1}(y) - a}{b} \right]^2 \right) \exp\left(\frac{1}{2} \left[ \Phi^{-1}(y) \right]^2 \right) = U_1 / (bU_2) = f(y, a, b). \quad \Box
\]

By (2.5), we have the following relationships:

\[
\rho = \frac{b^2}{1+b^2}, \quad \frac{1}{\sqrt{1-p}} = \sqrt{1+b^2}, \quad a = \Phi^{-1}(p)\sqrt{1+b^2}.
\] (2.6) (2.7) (2.8)

Denote by \( E_e[\Phi(a + be)] \) the expectation of \( \Phi(a + be) \) with respect to a random variable \( e \). The following lemma is useful.

**Lemma 2.3.** ([10]) \( E_e[\Phi(a + be)] = \Phi \left( \frac{a}{\sqrt{1+b^2}} \right) \), where \( e \sim N(0,1) \). \( \Box \)

The next proposition summarizes the properties for Vasicek distribution under (2.2).

**Proposition 2.4.** The following statements hold:

(a) \( E_x[\Phi(a + bs)] = \Phi \left( a/\sqrt{1+b^2} \right) = p. \)

(b) Density \( f(y, a, b) \) is unimodal if \( 0 < b < 1 \) with mode given by \( \Phi \left( \frac{a}{1-b^2} \right) \), and is U-shaped if \( b > 1 \).

(c) Assume \( b = 1 \). If \( a = 0 \) then \( f(y, a, b) \) is uniformly distributed over the interval \((0, 1)\); it is increasing if \( a > 0 \) and decreasing if \( a < 0 \).

**Proof.** By applying Lemma 2.3 to (2.2) and using (2.8), we have:

\[
E_x[\Phi(a + bs)] = \Phi \left( a/\sqrt{1+b^2} \right) = p.
\]

This proves (a). Let \( z = \Phi^{-1}(y) \). By (2.3), we have:

\[
\log \left( \frac{U_1}{U_2} \right) = \frac{z^2 + 2az - a^2 + b^2z^2}{2b^2} = \frac{-(1-b^2)(z-\frac{a}{1-b^2})^2 + b^2a^2}{2b^2}.
\] (2.9) (2.10)

For statement (c), it follows from the fact that the coefficient of term \( z^2 \) in (2.9) is zero. For (b), we have \( -(1-b^2) > 0 \) when \( b > 1 \), therefore \( U_1/U_2 \) is U-shaped by (2.10). When \( 0 < b < 1 \), we have \( -(1-b^2) < 0 \), therefore \( \log \left( \frac{U_1}{U_2} \right) \) reaches its unique maximum at \( z = \frac{a}{1-b^2} \) by (2.10), resulting in a value for the mode at \( \Phi \left( \frac{a}{1-b^2} \right). \) \( \Box \)

**Remark 2.5.** Results in Proposition 2.4 are consistent with results given in [13] where the density distribution is parameterized as (2.1). For most cases, this follows from the fact that \( \rho < \frac{1}{2} \) if and only if \( b < 1 \). Here we show only the consistency for distribution mode. The mode in [13] is given as \( \Phi \left( \frac{1-p}{1-2p} \Phi^{-1}(p) \right) \), while it is
given as \( \Phi \left( \frac{a}{1-b^2} \right) \) in Proposition 2.4. First, by (2.6), we have \( 1 - 2\rho = \frac{1-b^2}{1+b^2} \). Therefore, by (2.7) and (2.8), we have

\[
\sqrt{\frac{1-\rho}{1-2\rho}} \Phi^{-1}(p) = \sqrt{\frac{1+b^2}{1-b^2}} \Phi^{-1}(p) = \frac{a}{1-b^2}
\]

\[
\Rightarrow \Phi \left( \sqrt{\frac{1-\rho}{1-2\rho}} \Phi^{-1}(p) \right) = \Phi \left( \frac{a}{1-b^2} \right).
\]

This proves the consistency for distribution mode between the two formulations.

3. Estimating Portfolio Loss Rate and Weighting Portfolio Scenario Losses

3.1. The Proposed Loss Rate Model

We assume that there exists a latent risk factor \( S \) that drives up the portfolio loss, and the loss rate \( y \) follows a Vasicek distribution. Thus, by (2.2), portfolio loss rate is given by a probit-type mixed model of the form below:

\[
y = \Phi(a + bS), \quad b > 0, S \sim N(0,1).
\]

(3.1)

Parameters \( a \) and \( b \) can be estimated as described in the next proposition below. This is the loss rate model for the portfolio.

Proposition 3.1 ([15]). Given a time series sample \( \{y_t\} \), where \( y_t \) denotes the loss rate for a portfolio observed at time \( t \), the maximum likelihood estimates for \( a \) and \( b \) under model (3.1) are the mean and standard deviation for the sample \( \{z_t\} \) given by the probit form values of \( \{y_t\} \), i.e. \( z_t = \Phi^{-1}(y_t) \). \( \blacksquare \)

3.2. Levels of Loss Severity

Given \( 0 < \alpha < 1 \), the \( \alpha \)-quantile value for a loss rate variable \( y \) is the loss rate \( y_\alpha \) satisfying \( P(y \leq y_\alpha) = \alpha \). In this case, we say that \( \alpha \) is the severity level for the loss rate \( y_\alpha \). The level of severity measures the relative position for the loss rate in its overall loss distribution.

Proposition 3.2. Given a loss rate \( y_0 \) under model (3.1), the severity level of \( y_0 \) is \( \Phi([\Phi^{-1}(y_0) - a]/b) \).

Given \( 0 < \alpha < 1 \), the \( \alpha \)-quantile value is given by \( y_\alpha = \Phi(a + bS_\alpha) \), where \( S_\alpha = \Phi^{-1}(\alpha) \) is the \( \alpha \)-quantile value for \( S \sim N(0,1) \).

Proof. By model (3.1), \( y = \Phi(a + bS), S \sim N(0,1) \). The first statement follows as:

\[
P(y \leq y_\alpha) = P \left( S \leq \frac{\Phi^{-1}(y_\alpha)-a}{b} \right) = \Phi \left( \frac{\Phi^{-1}(y_\alpha)-a}{b} \right).
\]

For the second statement, \( \alpha \)-quantile value \( y_\alpha \) satisfies the equation below:

\[
\alpha = P(y \leq y_\alpha) = P[a + bS \leq \Phi^{-1}(y_\alpha)]
\]
\[
\Rightarrow P(S \leq [\Phi^{-1}(y_\alpha) - a]/b] = \alpha
\]
\[
\Rightarrow S_\alpha = [\Phi^{-1}(y_\alpha) - a]/b
\]
\[
\Rightarrow y_\alpha = \Phi(a + bS_\alpha). \quad \blacksquare
\]

Remark 3.3. Given an occurring probability \( \alpha \) for a scenario for the base or the optimistic scenario, we assume that this occurring probability is given as \( P(S < S_\alpha) = \alpha \), under model (3.1). Therefore, it corresponds to a loss at severity level \( \alpha \) by Proposition 3.2. While for the pessimistic scenario, the occurring probability \( \alpha \) is given as \( P(S > S_\alpha) = \alpha \), thus it corresponds to a loss at severity level \( (1 - \alpha) \). For example, if \( p_1, p_2, \) and \( p_3 \) are
respectively the scenario occurring probabilities for the pessimistic, the base, and the optimistic scenarios, and their values are respectively 10%, 60%, and 30%, then their corresponding loss severity levels are assumed to be 90%, 60%, and 30%.

3.3. Weighting Scenario Losses Given the Noncyclic Expected Loss

By Proposition 2.4 (a), the noncyclic expected loss under model (3.1) is:

\[ EL = \Phi\left(\frac{a}{\sqrt{1+b^2}}\right). \]  

(3.2)

Given the noncyclic expected loss \( EL \), and scenario losses \( y_1, y_2, y_3 \), as well as the corresponding occurring probabilities \( p_1, p_2, \) and \( p_3 \), respectively for pessimistic, base, and optimistic scenarios, we need an algorithm to find weights \( w_1, w_2, \) and \( w_3 \) satisfying the equation below:

\[ EL = w_1 y_1 + w_2 y_2 + w_3 y_3. \]  

(3.3)

3.4. The Proposed Generic Algorithm for Finding Scenario Weights Given Portfolio Expected Loss

We introduce a parameter \( \lambda \) to denote the ratio \( \frac{w_2}{1-w_1} \), i.e., \( \lambda = \frac{w_2}{1-w_1} \). Then \( 0 \leq \lambda \leq 1 \), this is because \( 1 - w_1 = w_2 + w_3 \). Thus \( \frac{w_2}{1-w_1} = 1 - \lambda \), and we have:

\[ w_2 = (1 - w_1)\lambda, \quad w_3 = (1 - w_1)(1 - \lambda). \]  

(3.4)

**Algorithm 3.4** (generic scenario weight algorithm). Suppose the following are given:

(a) The value of noncyclic expected loss \( EL \).

(b) Scenario losses \( y_1, y_2, y_3 \), respectively for pessimistic, base, and optimistic scenarios.

(c) The corresponding occurring probabilities \( p_1, p_2, \) and \( p_3 \).

Follow the steps below to find weights \( w_1, w_2, \) and \( w_3 \) satisfying (3.3):

1a. Formulate \( w_1, w_2, \) and \( w_3 \) as a function of \( \lambda \). First, solve equation (3.3) for \( w_1 \):

\[ EL = w_1 y_1 + w_2 y_2 + w_3 y_3 \]
\[ \Rightarrow EL = w_1 y_1 + (\lambda - \lambda w_1) y_2 + (1 - \lambda)(1 - \lambda) y_3 \]
\[ \Rightarrow w_1 = \frac{[EL - \lambda y_2 - (1 - \lambda) y_3]}{[y_1 - \lambda y_2 - (1 - \lambda) y_3]}. \]  

(3.5)

Next calculate \( w_2 \) and \( w_3 \) as:

\[ w_2 = (1 - w_1)\lambda, \quad w_3 = (1 - w_1)(1 - \lambda). \]  

(3.6)

Note that weights \( w_1, w_2, \) and \( w_3 \) satisfy (3.3) for each given \( \lambda \), where \( 0 \leq \lambda \leq 1 \), as long as:

\[ EL \leq y_1 \] and \( y_1 - \lambda y_2 - (1 - \lambda) y_3 > 0. \]  

(3.7)

1b. By (3.7), there exist many values of \( \lambda \) that satisfy (3.3). Calculate \( w_1^0, w_2^0, \) and \( w_3^0 \) by (1.2), and search by an optimization for a value \( \lambda^* \) such that the corresponding weights \( w_1, w_2, \) and \( w_3 \) given by (3.5) and (3.6) are as close as possible in Euclidian distance to the preferable weights \( w_1^0, w_2^0, \) and \( w_3^0 \).

**Remark 3.5.** Given components (a), (b) and (c), scenario weights \( w_1, w_2, \) and \( w_3 \) can be found directly. No model is involved in steps 1a and 1b. Hence, one can obtain the scenario weights for a given stressed portfolio expected loss, by replacing portfolio expected loss \( EL \) in (a) with the stressed portfolio expected loss.

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1 Step 1a is proposed by Carlos Lopez
3.5. The Proposed Model for Recession Risk

Assume that there exists a latent risk factor $S_R$ that drives up the recession risk for an economy. Let $p(S_R)$ denote the recession probability conditional on $S_R$. We assume $p(S_R)$ is given by a probit-type mixed model of the form below for parameters $c$ and $d$:

$$p(S_R) = \Phi(c + dS_R), \quad S_R \sim N(0,1).$$  \hspace{1cm} (3.8)

Given a time series sample $\{R_t\}$, where $R_t$ is an indicator variable with value 1 if a recession occurs at time $t$ or 0 otherwise, the likelihood for observing $R_t$ at time $t$ is:

$$(p(S_R))^R_t \left(1 - p(S_R)\right)^{1-R_t} = [\Phi(c + dS_R)]^R_t [1 - \Phi(c + dS_R)]^{1-R_t}.$$  \hspace{1cm} (3.9)

Here we assume that $R_t$ follows a Bernoulli distribution with probability $p(S_R)$. Parameters $c$ and $d$ can be estimated by maximizing the total sample log-likelihood by using, for example, the SAS procedure PROC NL MIXED ([14]).

3.6. Correlation between Risk Factors $S_R$ and $S$

Under probit model (3.1), loss rate is driven by the latent risk factor $S \sim N(0,1)$, while for recession probability, it is driven by the latent risk factor $S_R \sim N(0,1)$ under model (3.8). We assume that the pair $(S_R, S)$ is bivariate normal. Let $\rho_S$ denote the correlation between factors $S_R$ and $S$. Then the latent factor $S$ for loss rate splits into two parts:

$$S = \rho_S S_R + \sqrt{1 - \rho_S^2} \varepsilon, \quad \varepsilon \sim N(0,1),$$

where $\varepsilon$ is independent of $S_R$. Thus by model (3.1), the loss rate $y$, stressed by recession risk, can be rewritten and transformed to:

$$y_{\text{stressed}} = \Phi\left(a + b(\rho_S S_R + \sqrt{1 - \rho_S^2} \varepsilon)\right), \quad \varepsilon \sim N(0,1).$$  \hspace{1cm} (3.10)

3.7. The Proposed Model for Stressed Expected Loss Rate Given Recession Risk $S_R$

By applying Lemma 2.3 to (3.10), we have the stressed expected loss rate, conditional on $S_R$, as:

$$EL(S_R)_{\text{stressed}} = E_{\varepsilon} [y(S \mid S_R)] = \Phi\left(\frac{a + b\rho_S S_R}{\sqrt{1 + b^2(1 - \rho_S^2)}}\right).$$  \hspace{1cm} (3.11)

Note that this $EL(S_R)_{\text{stressed}}$ differs from the noncyclic expected loss $EL$. It is the expectation of $y$ given $S_R$. Given models (3.8) and (3.11), one can stress the expected loss based on a given level of forward looking recession probability as follows: Find the value for $S_R$ corresponding to the given probability $\beta$ by model (3.8), then use (3.11) to obtain the stressed expected loss.

3.8. The Proposed Methods for Estimating Correlation $\rho_S$

Given model (3.1) and (3.8) the remaining parameter in (3.11) to be estimated is the correlation parameter $\rho_S$. By (3.10), the mean and standard deviation of $\Phi^{-1}(y)$, conditional on $S_R$, are respectively $a + b\rho_S S_R$ and $b\sqrt{1 - \rho_S^2}$. Thus, by Proposition 2.1, the likelihood conditional on $S_R$ for observing loss rate $y_t$ at a time $t$ is

$$\frac{u_1}{b \sqrt{1 - \rho_S^2} u_2},$$

where:

6
\[ U_1 = \phi\left[\frac{\Phi^{-1}(y_t) - (a + b \rho_s S_R)}{b \sqrt{1 - \rho_s^2}}\right], \]  
(3.12)\[ U_2 = \phi[\Phi^{-1}(y_t)]. \]  
(3.13)

Assume that, for the pair of outcome \((R_t, y_t)\), \(R_t\) is independent of \(y_t\), conditional on \(S_R\). Then the joint likelihood for observing the pair of outcome \((R_t, y_t)\) at time \(t\) is:

\[ LK_t = [p(S_R)]^{R_t}[1 - p(S_R)]^{1 - R_t}\left[\frac{U_1}{b \sqrt{1 - \rho_s^2 u_2}}\right]. \]  
(3.14)

Use (3.12) and (3.13) for \(U_1\) and \(U_2\), and use model (3.8) for \(p(S_R)\). Given parameters \(a, b, c\), and \(d\), one can estimate \(\rho_s\) by maximizing the total sample likelihood by using, for example, the SAS procedure PROC NL MIXED.

### 3.9. The Proposed Methods for Determining Scenario Loss Weights for Stressed Expected Loss

Given the stressed expected loss \(EL(S_R)_{\text{stressed}}\), and scenario losses \(y_1, y_2, y_3\), and the corresponding occurring probabilities \(p_1, p_2, p_3\), one can find weights \(w_1, w_2, w_3\) by Algorithm 3.4, by replacing \(EL\) with \(EL(S_R)_{\text{stressed}}\), as indicated in Remark 3.5.

### 4. Empirical Results and Discussions

IFRS9 expected loss evaluation consists of two stages: (a) Stage-one for portfolio expected loss in 12 months, (b) Stage-two for expected lifetime loss for impairment loans after 12 months. We focus only on scenario weights for stage-one loss. For impairment loan losses after 12 months, one could use the scenario weights for 12-month loss and assume that the relative scenario weights keep the same.

#### 4.1. The Loss Rate Model

The loss sample contains the historical loss for a portfolio at each quarter in period 2003Q4-2017Q1. For each quarter, the loss rate is cumulated over four quarters from the beginning of the quarter to get a time series of annual loss rate at quarterly basis. We use this annual loss rate to model 12-month loss rate for IFRS9 stage-one loss. By model (3.1), we assume that the loss rate is of the form:

\[ y = \Phi(\alpha + \beta S), S \sim N(0, 1). \]  
(4.1)

The average loss rate for this sample is 0.34%. This is the noncyclic (unstressed) \(EL\). The estimated values for \(\alpha\) and \(\beta\) are respectively -2.7243 and 0.1279, estimated respectively as the mean and standard deviation of the probit form of the observed annual loss rates, under Proposition 3.1.

#### 4.2. Results of Generic Algorithm for Noncyclic Expected Loss

Given occurring probabilities \(p_1, p_2, p_3\), and scenario loss rates \(y_1, y_2, y_3\), for pessimistic, the base, and the optimistic scenarios, follow the steps below to find \(w_1, w_2,\) and \(w_3\):

1. Calculate the noncyclic expected loss \(EL\) by (3.2), and set the corresponding relative weights \(w_1^0, w_2^0,\) and \(w_3^0\) by (1.2);  
2. Run steps 1a-1b in Algorithm 3.4 to find \(w_1, w_2,\) and \(w_3\) satisfying:

\[ EL = w_1 y_1 + w_2 y_2 + w_3 y_3. \]  
(4.2)
The table below shows the scenario weights, with values for \( p_1, p_2, \) and \( p_3 \) being set at 10\%, 60\%, and 30\%. The corresponding loss severity levels are set at \( 1 - p_1, p_2, \) and \( p_3 \) accordingly by Remark 3.3.

<table>
<thead>
<tr>
<th>EL</th>
<th>Scenario Loss Weight</th>
<th>Scenario Occuring Probability</th>
<th>Loss Severity Level</th>
<th>Scenario Loss Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pessimistic</td>
<td>Base</td>
<td>Optimistic</td>
<td>Pessimistic</td>
</tr>
<tr>
<td>0.34%</td>
<td>4%</td>
<td>61%</td>
<td>35%</td>
<td>5%</td>
</tr>
<tr>
<td>0.34%</td>
<td>10%</td>
<td>60%</td>
<td>30%</td>
<td>10%</td>
</tr>
<tr>
<td>0.34%</td>
<td>17%</td>
<td>59%</td>
<td>24%</td>
<td>15%</td>
</tr>
</tbody>
</table>

The column \( EL \) under “Loss Severity Level” in the table shows that the severity level for \( EL \) (0.34\%) is 57\%, higher than 50\%. We observe that, in these cases, the weights found differ but are close to the occurring probabilities \( p_1, p_2, \) and \( p_3 \).

4.3. The Model for Expected Loss Given Recession Risk

The historical recession sample contains a recession indicator at each quarter between 1984Q1-2018Q1. To align with the annual loss rate sample, an annual recession indicator is set for the year starting from the beginning of the quarter. For simplicity, the indicator is set to 1 if the number of quarters in recession in the year is above 2, otherwise it is set to 0. There are other ways to set up this recession indicator (see Remark 4.1 below). This gives rise to a time series sample with an annual recession indicator at quarterly basis, which can be joined by the time key to the annual loss rate time series sample, to form a training sample for model (3.11). The average recession rate for this sample is 9.735\%.

By model (3.8), we assume that the recession probability is:

\[
p(S_R) = \Phi(c + dS_R), \quad S_R \sim N(0,1),
\]

(4.3)
driven by a latent factor \( S_R \sim N(0,1) \). The values for \( c \) and \( d \) are respectively -4.1793 and 3.0636, estimated by maximizing the total sample log-likelihood summed up from the logarithm of (3.9), using SAS procedure PROC NLMIXED.

The remaining parameter in (3.11) to be estimated is the correlation parameter \( \rho_s \) between the risk factor \( S_R \) for recession probability and factor \( S \) for loss rate. By using the overlapped period 2003Q4-2017Q1 between the loss rate sample and the recession indicator sample, we estimate \( \rho_s \) by maximizing the total sample log-likelihood summed up from the logarithm of (3.14). The estimate for \( \rho_s \) is 0.5797.

4.4. Results of Generic Algorithm for Stressed Periods

Given the occurring probabilities \( p_1, p_2, \) and \( p_3, \) and scenario loss rates \( y_1, y_2, \) and \( y_3, \) for pessimistic, the base, and the optimistic scenarios, we follow the steps below to find the scenario loss weights for a given recession probability \( \beta \):

3a. Set up the reference weights \( w^0_1, w^0_2, \) and \( w^0_3 \) by (1.2).
3b. For the recession probability \( \beta \), find the corresponding \( S_R \) by setting \( S_R = [\Phi^{-1}(\beta) - c]/d \).
3c. Calculate the stressed expected loss \( EL(S_R)_{stressed} \) by (3.11). Run 1a-1b in Algorithm 3.4 by replacing \( EL \) with \( EL(S_R)_{stressed} \) to get weights \( w_1, w_2, \) and \( w_3 \).

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2 Source: the authors (2018)
The table below shows the results for a series of given recession probabilities. The occurring probabilities, when recession probability is below 99%, are kept as 10%, 60%, and 30%, respectively for pessimistic, base, and optimistic scenarios, and are reset to 5%, 65%, and 30% when recession probability exceeds or equals to 99%. Loss severity levels for three scenarios under “Loss Severity Level” are set by Remark 3.3 at $1 - p_1, p_2,$ and $p_3$. The change of occurring probability for pessimistic scenario is shown in the column under “Loss Severity Level” for pessimistic scenario (highlighted cells).

As expected, the resulting scenario weights differ significantly from the weights given by occurring probabilities, and the weight for pessimistic scenario keeps on increasing when recession probability increases.

Table 2. Scenario loss weights for expected loss stressed by recession probability

<table>
<thead>
<tr>
<th>No.</th>
<th>Recession Probability</th>
<th>Stressed EL</th>
<th>Scenario Loss Weight</th>
<th>Scenario Loss Rate</th>
<th>Loss Severity Level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Base</td>
<td>Optimistic</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>10.00%</td>
<td>0.41%</td>
<td>38.88%</td>
<td>54.96%</td>
<td>6.16%</td>
</tr>
<tr>
<td>2</td>
<td>20.00%</td>
<td>0.43%</td>
<td>44.94%</td>
<td>52.41%</td>
<td>2.76%</td>
</tr>
<tr>
<td>3</td>
<td>30.00%</td>
<td>0.44%</td>
<td>50.47%</td>
<td>47.05%</td>
<td>2.48%</td>
</tr>
<tr>
<td>4</td>
<td>40.00%</td>
<td>0.45%</td>
<td>55.38%</td>
<td>42.39%</td>
<td>2.23%</td>
</tr>
<tr>
<td>5</td>
<td>50.00%</td>
<td>0.45%</td>
<td>60.04%</td>
<td>37.96%</td>
<td>2.00%</td>
</tr>
<tr>
<td>6</td>
<td>60.00%</td>
<td>0.46%</td>
<td>64.77%</td>
<td>33.47%</td>
<td>1.76%</td>
</tr>
<tr>
<td>7</td>
<td>70.00%</td>
<td>0.47%</td>
<td>69.92%</td>
<td>28.57%</td>
<td>1.50%</td>
</tr>
<tr>
<td>8</td>
<td>80.00%</td>
<td>0.48%</td>
<td>76.06%</td>
<td>22.74%</td>
<td>1.20%</td>
</tr>
<tr>
<td>9</td>
<td>90.00%</td>
<td>0.50%</td>
<td>84.79%</td>
<td>14.46%</td>
<td>0.76%</td>
</tr>
<tr>
<td>10</td>
<td>95.00%</td>
<td>0.51%</td>
<td>92.16%</td>
<td>7.45%</td>
<td>0.39%</td>
</tr>
<tr>
<td>11</td>
<td>99.00%</td>
<td>0.53%</td>
<td>72.45%</td>
<td>26.17%</td>
<td>1.38%</td>
</tr>
<tr>
<td>12</td>
<td>99.90%</td>
<td>0.56%</td>
<td>85.05%</td>
<td>14.20%</td>
<td>0.75%</td>
</tr>
<tr>
<td>13</td>
<td>99.99%</td>
<td>0.59%</td>
<td>95.88%</td>
<td>3.92%</td>
<td>0.21%</td>
</tr>
</tbody>
</table>

Note that, in both tables 1 and 2, scenario loss rates $y_1, y_2,$ and $y_3$ are set by the occurring probabilities $p_1, p_2,$ and $p_3$ at several levels at $1 - p_1, p_2,$ and $p_3$ using loss rate model (4.1) (i.e. model (3.1)). In practice, loss rates $y_1, y_2,$ and $y_3$ are generated by IFRS9 models for PD, LGD, and EAD, the corresponding loss severity levels may not match up to the severity levels at $1 - p_1, p_2,$ and $p_3$.

In this case, we propose a lookup table for scenario weights, as described below:

Follow steps 3a-3c to generate a lookup table that contains scenario weights $w_1, w_2,$ and $w_3$ for each given $EL(S_R)$ and each loss rate triple $(y_1, y_2, y_3)$, where $y_i$ and $EL(S_R)$ vary within an appropriate range, for example, the severity level of $y_1$ (pessimistic) from 70% to 99%, $y_2$ (base) from 40% to 60%, and $y_3$ (optimistic) from 1% to 20%.

When scenario loss rates are generated by IFRS9 models, lookup to the table for a desired level of $EL(S_R)_\text{stressed}$, and the closest triple $(y_1, y_2, y_3)$ in the table (compared to losses generated by IFRS9 models). The corresponding values for $w_1, w_2,$ and $w_3$ are then the scenario weights required.

Remark 4.1. The annual recession indicator for a year can also be set in the following way: it has weight $\frac{i}{4}$ for value 1 and weight $(1 - \frac{i}{4})$ for value 0, where $i$ is the number of quarters in recession in the year. This is equivalent to four Bernoulli trials in a year, with $i$ being the number of times that a trial has value 1. Therefore, a sample for an annual recession indicator can be generated to have four observations at each year, where $i$ number of observations (among four) have the value 1 for the indicator.

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3 Source: the authors (2018)
Conclusions. For IFRS9 portfolio expected loss estimation, loan losses generated by IFRS9 models at scenario level are summed up by using the occurring probability weights for the scenarios. Portfolio expected loss estimated in this way can vary significantly, depending on the scenario losses the IFRS9 models generate and the occurring probability weights chosen. The models and approaches proposed in this paper provide a quantitative method for stressing portfolio loss based on recession risk, and a tool for finding the weights for noncyclic portfolio expected loss, and the expected loss under a stressed setting.

Future researches. Models proposed in this paper are limited to probit-type. For future researches, the following questions would be interesting:

(a) How do we perform a similar research, when logit-type mixed models are used (fatter tails than probit form)? In general, the latent recession factor $S_R$ can be non-normal. How do we perform a similar analysis when $S_R$ follows a distribution like log-logistic, Cauchy, or Burr is assumed?
(b) Could a single latent factor be enough to capture the low-probability, high-impact experienced at the recession periods? Would a mixed model with multiple latent risk factors work better?

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REFERENCES
Appendix

Proof of Proposition 2.1. The cumulative distribution for $y$ given by

$$F(y, a, b) = P[\Phi(a + bs) \leq y]$$

$$= P[s \leq [\Phi^{-1}(y) - a]/b]$$

$$= \Phi([\Phi^{-1}(y) - a]/b).$$

Since $\Phi[\Phi^{-1}(y)] = y$, the derivative for $\Phi^{-1}(y)$ with respect to $y$ is:

$$\frac{d\Phi^{-1}(y)}{dy} = \frac{1}{\phi[\Phi^{-1}(y)]}.$$  

By taking the derivative of $F(y, a, b)$ with respect to $y$, we have $f(y, a, b) = U_1/(bU_2)$, where $U_1$ and $U_2$ are given as (2.4).

Proof of Proposition 3.1. By (2.3), the log-likelihood at time $t$ is given by $\log(U_1) - \log(b) - \log(U_2)$. By (2.4), we can drop off the term $\log(U_2)$, since its partial derivatives are zero with respect to $a$ and $b$. The total sample log-likelihood at all times reduces to

$$LL = \sum_{1 \leq t \leq T} \log(U_1) - \log(b)$$

$$= \sum_{1 \leq t \leq T} [\frac{(z_t - a)^2}{2b^2} - \log(b)].$$

Setting the partial derivatives (with respectively to $a$ and $b$) of $LL$ to zero, we have

$$0 = \frac{\partial LL}{\partial a} = \sum_{1 \leq t \leq T} \frac{2(z_t - a)}{2b^2}$$

$$\Rightarrow a = \frac{1}{T} \sum_{1 \leq t \leq T} z_t.$$  

$$0 = \frac{\partial LL}{\partial b} = \sum_{1 \leq t \leq T} \frac{2(z_t - a)^2}{2b^3} - \frac{1}{b}$$

$$\Rightarrow b^2 = \frac{1}{T} \sum_{1 \leq t \leq T} (z_t - a)^2.$$