Exogenous uncertainty and the identification of Structural Vector Autoregressions with external instruments

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Abstract

We provide necessary and sufficient conditions for the identification of Structural Vector Autoregressions (SVARs) with external instruments considering the case in which r instruments are used to identify g structural shocks of interest, $r \geq g \geq 1$. Novel frequentist estimation methods are discussed by considering both a 'partial shocks' identification strategy, where only g structural shocks are of interest and are instrumented, and in a 'full shocks' identification strategy, where despite g structural shocks are instrumented, all n structural shocks of the system can be identified under certain conditions. The suggested approach is applied to empirically investigate whether financial and macroeconomic uncertainty can be approximated as exogenous drivers of U.S. real economic activity, or rather as endogenous responses to first moment shocks, or both. We analyze whether the dynamic causal effects of non-uncertainty shocks on macroeconomic and financial uncertainty are significant in the period after the Global Financial Crisis.

Keywords: Exogenous Uncertainty, External Instruments, Identification, proxy-SVAR, SVAR.

J.E.L.: C32, C51, E44, G10.

1 Introduction

Structural Vector Autoregressions (SVARs) provide stylized and parsimonious characterizations of shock transmission mechanisms and allow to track dynamic causal effects in empirical macro-

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1 Introduction

Structural Vector Autoregressions (SVARs) provide stylized and parsimonious characterizations of shock transmission mechanisms and allow to track dynamic causal effects in empirical macroeconomics. The identification of SVARs requires parameter restrictions on the matrix which maps the VAR disturbances to structural shocks, henceforth denoted with B, that are often implausible. The parameters in the matrix B capture the instantaneous impacts of the structural shocks on the variables and are crucial ingredients of the Impulse Response Functions (IRFs). One of the most interesting approaches developed in the recent literature to identify structural shocks by possibly avoiding recursive structures, or implausible assumptions on the elements of B is the so-called 'external instruments' or 'proxy-SVAR' (or 'SVAR-IV') approach, see Stock and Watson (2012, 2018) and Mertens and Ravn (2013, 2014). This method takes advantage of information developed from 'outside' the VAR in the form of variables which are correlated with the latent structural shocks of interest (relevance condition) and are uncorrelated with the other structural shocks of the system (exogeneity, or orthogonality condition).

The emerging literature on proxy-SVARs (throughout the paper we use the terms 'SVARs with external instruments' and 'proxy-SVARs' interchangeably) is mainly devoted to the use of one external instrument to identify a single structural shock of interest in isolation from all the other shocks of the system. For example, Stock and Watson (2012) identify six shocks (the oil shock, the monetary policy shock, the productivity shock, the uncertainty shock, the liquidity/financial risk shock and the fiscal policy shock) by exploiting many external instruments, but use them one at a time; see also Ramey (2016). Remarkable exceptions are Mertens and Ravn (2013) and Mertens and Montiel Olea (2018) who deal with the case of two instruments and two structural shocks ($r = g = 2$); see also Arias *et al.* (2018b).[1] Mertens and Ravn (2013) show that when $g > 1$, the restrictions provided by the external instruments do not suffice to identify the shocks and must be complemented with additional constraints. They obtain these constraints from a Choleski decomposition of a covariance matrix.

In general, there exists no result in the literature which provides a guidance for practitioners to address the following question: given $g \geq 1$ structural shocks of interest in a system of n variables and $r \geq g$ external instruments available for these shocks, how many restrictions do we need for the model to be identified and where do these restrictions need to be placed? One main contribution of this article is to provide such a general framework, i.e. we extend the identification analysis of proxy-SVARs to the case in which multiple instruments (r) are used to identify multiple shocks ($g \geq 1$). We show that when $g > 1$, the additional restrictions necessary

[1] Actually, Caldara and Kamps (2017) consider the case $r = g = 3$ and an identification strategy which can be classified as a 'full shocks' identification approach according to the characterization we provide in the paper.

to identify the shocks of interest (up to sign normalization) other the external instruments need not be Choleski-type constraints. We discuss novel frequentist estimation methods alternative to instrumental variables (IV) techniques: a classical minimum distance (CMD) approach and maximum likelihood (ML) approach, respectively. Further, we argue that one of the advantages of covering the case $r > g$ (i.e. the proxy-SVAR features more instruments than shocks of interest) is that a practitioner can potentially include up to $r - g$ weakly relevant (or not relevant at all) external instruments in the proxy-SVAR without affecting the inference if at least g instruments are strongly correlated with the structural shocks of interest.

Our approach is based on the idea of augmenting the SVAR with an auxiliary model for the r external instruments. We obtain a 'larger' m-dimensional SVAR, with $m = n + r$, which conveys conveniently all the information relevant to identify the g structural shocks of interest. This large system is called the AC-SVAR model, where 'AC' stands for 'augmented-constrained'. The AC-SVAR model is 'augmented' because it is obtained by appending the external instruments to the original SVAR equations. The AC-SVAR model is 'constrained' because it features a triangular structure in the autoregressive coefficients and a particularly constrained structure in the matrix that contains the structural parameters (the on-impact coefficients). We discuss two types of identification strategies which can be accommodated within the AC-SVAR framework depending on the information available to the practitioner: a 'partial shocks' identification approach, which is the typical case in the proxy-SVARs literature, and a 'full shocks' identification approach which occurs, under certain conditions, when all structural shocks of the system can be identified by the r external instruments.

In the 'partial shocks' identification approach, the objective is to identify the dynamic causal effects of $g \geq 1$ structural shocks of interest using $r \geq g$ valid external instruments, regardless of the other $n - g$ shocks of the system. When $r = g = 1$, the parameter which captures the correlation between the external instrument and the shock of interest is a scalar, say ϕ, and the parameters which are necessary to estimate the IRFs correspond to a column of the matrix B. When $g > 1$ and $r \geq g$ external instruments are used, $\phi = \Phi$ becomes a matrix with r rows and g columns and contains therefore more than one 'relevance' parameter. We propose a novel CMD estimation method for proxy-SVARs based on a set of moment conditions implied by the AC-SVAR model. In particular, we minimize the distance between a set of reduced form parameters, which can be easily and consistently estimated from the AC-SVAR model, and the parameters which capture the instantaneous impact of the instrumented shocks. The identification of the proxy-SVAR (up to sign) depends on a rank condition associated with the Jacobian matrix behind CMD estimation. We derive this Jacobian matrix analytically and discuss the necessary and sufficient condition and the necessary order condition for identification.

In the 'full shocks' identification approach, r valid external instruments are still used to identify (up to sign) $g \geq 1$ instrumented structural shocks, $r \geq g$, but it is now possible to infer, under some additional constraints, the dynamic causal effects of all n structural shocks of the system, including those associated with the $n - g$ non-instrumented shocks. The identification of the AC-SVAR model in the 'full shocks' approach amounts to the practice of identifying an enlarged 'B-model' using the terminology in Lütkepohl (2005) ('C-model' using the terminology in Amisano and Giannini, 1997). Estimation can be carried out by ML and requires minor adaptations to the algorithms discussed in e.g. Amisano and Giannini (1997) and Lütkepohl (2005) implemented in econometric packages.

Since we treat the proxy-SVAR as a 'large' SVAR, in our framework the issue of making bootstrap inference on the IRFs, discussed in Jentsch and Lunsford (2016) and Montiel Olea *et al.* (2018) and recently debated in Jentsch and Lunsford (2019) and Mertens and Ravn (2019), boils down to the problem of making bootstrap inference on the IRFs generated by SVARs, see e.g. Kilian and Lütkepohl (2017) for a review. For instance, in the empirical application of Section 8 we first check that the disturbances of the estimated AC-SVAR model are not characterized by conditional heteroskedasticity (because of the results in Brüggemann *et al.* (2016) on inference in SVARs with conditional volatility of unknown form), and then compute simultaneous confidence bands for the IRFs of interest by combining a standard residual-based recursive-design bootstrap algorithm with the 'sup-t' method discussed in Montiel Olea and Plagborg-Møller (2019).

Moreover, when the AC-SVAR system is overidentified, a convenient way to test the empirical validity of the proxy-SVAR model is to compute overidentification restrictions tests. Our analysis shows that these tests tend to reject the proxy-SVAR when the external instruments are erroneously assumed orthogonal to the non-instrumented shocks. Thus, we have analogs of the 'Sargan's specification test' in the instrumental variables framework, or the 'Hansen's J-test' in the generalized method of moments framework, and this appears a novelty in the literature on proxy-SVARs. Notably, in the full shocks identification approach, the quality of the identification can be evaluated not only by checking the relevance condition, but also the orthogonality between the external instruments and the non-instrumented shocks.

The second contribution of this article is empirical. We apply our methodology to address a recently debated issue of the uncertainty literature, i.e. whether uncertainty is a driver of the U.S. business cycle or rather a response to first moment shocks, or both. A well recognized fact in the literature is that uncertainty is recessionary in presence of real options effects (e.g. Bloom, 2009) or financial frictions (e.g. Christiano *et al.*, 2014). Instead, a less documented and controversial fact is whether uncertainty, a second moment variable, is also a response to first moment shocks, especially during periods of economic and financial turmoil. Indeed, uncertainty appears also to endogenously increase during recessions, as lower economic growth induces greater dispersion at the micro level and higher aggregate volatility.

Reverse causality between uncertainty and real economic activity using monthly or quarterly data can not be analyzed by recursive (triangular) SVARs which presume that some variables respond only with a lag to others. This issue has been analyzed in the recent literature by Ludvigson *et al.* (2018), Carriero *et al.* (2018) and Angelini *et al.* (2019). These authors use non-recursive SVARs and different identification methods and report mixed evidence. We focus on the U.S. economy after the Global Financial Crisis, in particular the 'Great Recession+Slow Recovery' period 2008-2015, and consider a small-scale monthly SVAR which includes measures of macroeconomic and financial uncertainty taken from Jurado *et al* (2015) and Ludvigson *et al.* (2018), respectively, and a measure of real economic activity, say the industrial production growth ($n = 3$). The scope of our analysis is to investigate whether the selected measures of macroeconomic and financial uncertainty respond on-impact (instantaneously) and/or with lags to a 'non-uncertainty' shock ($g = 1$). This requires a non-recursive (non-triangular) specification for the matrix B which makes our approach potentially attractive. The direction of causality we are primarily interested in runs from real economic activity to uncertainty, not the other way around, and this requires the use of valid external instruments for the variable of the system related to real economic activity. Thus, in our baseline specification we use two external instruments jointly ($r = 2$) to identify the 'non-uncertainty' shock of the system and track its dynamic impact on financial and macroeconomic uncertainty. This strategy differs from e.g. Stock and Watson (2012) who use valid external instruments (one at a time) to identify the effects of uncertainty shocks on macroeconomic variables. It differs also from Ludvigson *et al.* (2018)'s strategy, where two external instruments for financial and macroeconomic uncertainty shocks are employed to narrow the identification set obtained by directly restricting the structural shocks in correspondence of particular events (event constraints).

The external instruments we employ for the non-uncertainty shock are: (a) the time series innovations obtained from an auxiliary regression models for the changes in the log of new privately owned housing units started; (b) an oil supply shock identified along the lines of Kilian (2009); (c) the time series innovations obtained from an auxiliary regression model for the changes in the log of hours worked. The couples of external instruments (a,b) and (a,c) are used to identify the non-uncertainty shock of the system in a partial shocks identification strategy, but also in a full shocks identification strategy under an auxiliary hypothesis on the pass-through from financial to macroeconomic uncertainty. Empirical results show that macroeconomic and financial uncertainty do not respond significantly to the identified non-uncertainty shocks on-impact. Macroeconomic uncertainty does not respond at any lag to the identified real economic activity shocks while financial uncertainty displays a short-lived response after one period, so that the overall empirical evidence in favor of the hypothesis of 'endogenous uncertainty' appears scant. Notably, our analyses provide empirical support to the validity of the selected external instruments. Admittedly, however, our results can not be considered 'final' as they depend on the specific set of external instruments used to identify the non-uncertainty shock.

Our paper is naturally connected with the increasing strand of the macroeconometric literature which develops and applies estimation and inferential methods for SVARs with external instruments. We compare thoroughly our methodology with other approaches in Sections 7 and our empirical results with other works on exogenous/endogenous uncertainty in Section 8.3. Our approach is based on the maintained assumption that the external instruments are strongly correlated with the instrumented structural shocks, which might not be the case in applied work. Lunsford (2015) and Montiel Olea *et al.* (2018) discuss identification-robust inferential methods for weak external instruments. The extension of our approach to proxy-SVARs to weak instruments is left to future research.

The paper is organized as follows. Section 2 introduces the reference SVAR with external instruments and presents the main assumptions. Section 3 discusses the AC-SVAR representation of proxy-SVARs and Section 4 motivates two identification strategies featured by AC-SVAR models by considering an example centered of the concept of exogenous/endogenous uncertainty in SVAR models. Section 5 deals with the 'partial shocks' identification strategy and proposes a CMD estimation approach alternative to IV methods. Section 6 deals with the 'full shocks' identification strategy and discusses estimation through ML. Section 7 connects our approach to proxy-SVARs to the literature. Section 8 applies the suggested methodology to investigate the exogeneity/endogeneity of uncertainty in the U.S. in the period after the Global Financial Crisis. Section 9 contains some concluding remarks. Additional technical details, formal proofs, Monte Carlo experiments and further empirical results are confined in a Supplementary Appendix.

2 SVAR and the auxiliary model for the external instruments

We start from the SVAR system:

$$Y_t = \Pi X_t + \Upsilon_y D_{y,t} + u_t \ , \ \ u_t = B\varepsilon_t \ \ , \ \ t = 1, ..., T \tag{1}$$

where Y_t is the $n \times 1$ vector of endogenous variables, X_t:=$(Y'_{t-1}, ..., Y'_{t-k},)'$ is $nk \times 1$, $\Pi :=$ $(\Pi_1 : ... : \Pi_k)$ is the $n \times nk$ matrix containing the autoregressive (slope) parameters, $D_{y,t}$ is an d_y-dimensional vector containing deterministic components (constant, dummies, etc.) with parameters in the $n \times d_y$ matrix Υ_y; finally, u_t is the $n \times 1$ vector of *iid* reduced-form disturbances with positive definite covariance matrix $\Sigma_u := E(u_t u_t')$. The initial conditions Y_0, ..., Y_{1-k} are treated as fixed. The system of equations $u_t = B\varepsilon_t$ in eq. (1) maps the $n \times 1$ vector of *iid* structural shocks ε_t, which are assumed to have normalized unit covariance matrix $E(\varepsilon_t \varepsilon_t') := \Sigma_\varepsilon = I_n$, to the reduced form disturbances through the $n \times n$ matrix B.[2]

We call the elements in $(\Pi, \Upsilon_y, \Sigma_u)$ reduced-form parameters and the elements in B structural parameters or on-impact coefficients. Moreover, we use the terms 'identification of B', 'identification of the SVAR' and 'identification of the shocks' interchangeably. Let

$$A_y := \begin{pmatrix} \Pi_1 & \cdots & \Pi_k \\ I_{n(k-1)} & & 0_{n(k-1)\times n} \end{pmatrix} \tag{2}$$

be the VAR companion matrix. The responses of the variables in Y_{t+h} to one standard deviation structural shock ε_{jt} is captured by the IRFs:

$$IRF_j(h) := (J_n (A_y)^h J_n') b_j \ , \quad h = 0, 1, 2, ... \tag{3}$$

where b_j is the j-th column of B, $j = 1, ..., n$ and $J_n := (I_n : 0_{n\times n(k-1)})$ is a selection matrix such that $J_n J_n' = I_n$. Standard local and global identification results for the SVAR in eq. (1) are reviewed in the Supplementary Appendix A.2.

Our first assumption postulates the correct specification of the SVAR and the nonsingularity of the matrix of structural parameters B, the only formal requirement we place on this matrix, except where indicated.

Assumption 1 (DGP) *The data generating process belongs to the class of models in eq. (1) which satisfy the following conditions:*
(i) the companion matrix A_y in eq. (2) is stable, i.e. all of its eigenvalues lie inside the unit circle;
(ii) the matrix B is nonsingular.

Given Assumption 1 we consider, without loss of generality, the following partition of the vector of structural shocks:

$$\varepsilon_t := \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix} \begin{matrix} g \times 1 \\ (n-g) \times 1 \end{matrix} \tag{4}$$

[2] The structural shocks ε_t may also have diagonal covariance matrix $\Sigma_\varepsilon := diag(\sigma_1^2, ..., \sigma_n^2)$. In this case, the link between reduced form disturbances and structural shocks can be expressed in the form $u_t = B^* \Sigma_\varepsilon^{1/2} \varepsilon_t^*$, where $\varepsilon_t^* := \Sigma_\varepsilon^{-1/2} \varepsilon_t$ and B^* has exactly the same structure as B in eq. (1) except that the elements on the main diagonal are normalized to 1. Throughout the paper we follow the parameterization based on $u_t = B\varepsilon_t$ with $\Sigma_\varepsilon = I_n$, except where indicated.

where $\varepsilon_{1,t}$ is the $g \times 1$ subvector of structural shocks henceforth denoted 'instrumented structural shocks', and $\varepsilon_{2,t}$ is the $(n-g) \times 1$ subvector of other structural shocks, denoted 'non-instrumented shocks'. The instrumented structural shocks are ordered first for notational convenience only: the ordering of variables is irrelevant in our setup. Given the corresponding partition of reduced form VAR disturbances, $u_t := (u_{1,t}', u_{2,t}')$, where $u_{1,t}$ and $u_{2,t}$ have the same dimensions as $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$, we partition the matrix of structural parameters B conformably with eq. (4):

$$B := \left(\begin{array}{c:c} \underset{n \times g}{B_1} & \underset{n \times (n-g)}{B_2} \end{array} \right) = \left(\begin{array}{c:c} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array} \right) \quad \begin{array}{c:c} g \times g & g \times (n-g) \\ (n-g) \times g & (n-g) \times (n-g) \end{array} . \tag{5}$$

In eq. (5), the dimensions of submatrices have been reported below and alongside blocks. B_1 is the submatrix containing the on-impact coefficients associated with the instrumented structural shocks $\varepsilon_{1,t}$, and B_2 is the submatrix containing the on-impact coefficients associated with the non-instrumented shocks $\varepsilon_{2,t}$; $rank(B_1) = g$ and $rank(B_2) = n - g$ because of Assumption 1(ii).

The external instruments approach postulates that given the partitions in eq.s (4)-(5), there are available $r \geq g$ observable 'external' (to the SVAR) variables called instruments, that we collect in the $r \times 1$ vector $v_{Z,t}$, which can be used to identify the dynamic causal effect of $\varepsilon_{1,t}$ on Y_{t+h}, $h = 0, 1, ...$, without the need to impose implausible assumptions on the elements of B. Thus, we can consider the instrumented structural shocks in $\varepsilon_{1,t}$ as the shocks of primary interest in the analysis and for which r valid external instruments are employed. However, as it will be shown below, there are cases in which despite $\varepsilon_{1,t}$ is instrumented, also the 'other' structural shocks in $\varepsilon_{2,t}$ might be of interest and identified under certain conditions. The key properties of $v_{Z,t}$ are formalized in the next assumption.

Assumption 2 (Relevance and orthogonality) *The $r \times 1$ vector $v_{Z,t}$ is generated by the system of equations:*

$$v_{Z,t} = R_{\Phi} \varepsilon_t + \omega_t = \Phi \varepsilon_{1,t} + \omega_t \tag{6}$$

where $R_{\Phi} := \left(\Phi : 0_{r \times (n-g)} \right)$, Φ is an $r \times g$ matrix of full column rank, and ω_t is a $r \times 1$ measurement error term uncorrelated with ε_t ($E(\varepsilon_t \omega_t') = 0_{n \times g}$) with positive definite covariance matrix $E(\omega_t \omega_t') := \Sigma_{\omega} < \infty$.

Assumption 2 ensures that the external instruments in $v_{Z,t}$ satisfy the conditions $E(v_{Z,t} \varepsilon_{1,t}') = \Phi \neq 0_{r \times n}$ ('relevance') and $E(v_{Z,t} \varepsilon_{2,t}') = 0_{r \times (n-g)}$ ('exogeneity', or 'orthogonality').[3] These conditions are typically presented in the proxy-SVAR literature under the setup $r = g$, see e.g.

[3] Assumption 2 is consistent with a scenario in which the external instruments in $v_{Z,t}$ can potentially be correlated with past structural shocks, i.e. it may hold the condition $Cov(v_{Z,t}, \varepsilon_{t-i}) = E(v_{Z,t} \varepsilon_{t-i}') \neq 0_{r \times n}$, $i = 1, 2, ...$ which, because of eq. (6), requires $E(\omega_t \varepsilon_{t-i}') \neq 0_{r \times n}$, $i = 1, 2, ...$

Stock and Watson (2012, 2018), Mertens and Ravn (2013, 2014), and require that the elements in $v_{Z,t}$ are correlated with the instrumented structural shocks $\varepsilon_{1,t}$ and are orthogonal to the other shocks in $\varepsilon_{2,t}$.[4] The matrix $R_{\Phi} := \left(\Phi \;\vdots\; 0_{r\times(n-g)} \right)$ in eq. (6) characterizes the instruments validity as it collects the relevance and orthogonality conditions. We call Φ the 'relevance matrix' (or 'matrix of relevance parameters') thoughout the paper. The condition $rank(\Phi) = g$ in Assumption 2 ensures that each column of Φ carries independent - not redundant - information on the instrumented structural shocks. It will be shown in the next sections that $rank(\Phi) = g$ is a necessary condition for identification which becomes also sufficient when $g = 1$. The additive measurement error ω_t in eq. (6) captures the idea that the external instruments are imperfectly correlated with the instrumented structural shocks; the covariance matrix Σ_ω can be possibly diagonal.

The 'one shock-one instrument' case mainly treated in the proxy-SVAR literature obtains for $r = g = 1$ and implies that $R_\Phi := \phi \left(1 : 0_{1\times(n-1)}\right)$ in eq. (6) is a row and $\Phi = \phi$ is a scalar. In this paper we focus on the general case $r \geq g \geq 1$ and explore the consequences of such generalization on the identification and (frequentist) estimation of proxy-SVARs. By considering the general case $r \geq g \geq 1$, we mimick the situation that occurs in the instrumental variable regressions when the number of instruments can be larger than the number of estimated parameters. As it will be shown next, when $r > g$, $(r - g)$ external instruments might be weakly correlated (or not correlated at all) with the g instrumented shocks without consequences on asymptotic inference if at least g external instruments are strongly correlated with the instrumented shocks.

To present our method it is convenient to generalize the auxiliary model for the external instruments postulated in eq. (6). Indeed, the data generating process specified for $v_{Z,t}$ in Assumption 2 maintains that the dynamics of the external instruments is expressed in 'innovation form, as $v_{Z,t}$ depends on the instrumented structural shocks $\varepsilon_{1,t}$ and the measurement error ω_t. In some situations, however, the practitioner might observe an $r\times 1$ vector of 'raw' (stationary) time dependent time series whose innovation part might potentially serve as external instruments. To account for these situations, we interpret $v_{Z,t}$ as the innovation part of Z_t, i.e. the quantity $v_{Z,t} := Z_t - E(Z_t \mid \mathcal{F}_{t-1})$, where $\mathcal{F}_{t-1}$ is the econometrician's information set available at time $t-1$. A specification consistent with the decomposition $Z_t = E(Z_t \mid \mathcal{F}_{t-1}) + v_{Z,t}$ is given by the dynamic system:

$$Z_t = \Theta(L)Z_{t-1} + \Gamma(L)Y_{t-1} + \Upsilon_z D_{z,t} + \Upsilon_{z,y} D_{y,t} + v_{Z,t} \tag{7}$$

where $\Theta(L) := \Theta_1 + ... + \Theta_p L^{p-1}$ is a matrix polynomial in the lag operator L, whose coefficients

[4] Henceforth, the exogeneity of the external instruments with respect to the non-instrumented shocks will be denoted with the term 'ortogonality' in order to avoid ambiguities with the distinct concept of 'exogenous uncertainty' we discuss in the empirical section of the paper.

are in the $r \times r$ matrices Θ_i, $i = 1, ..., p$ (and can be possibly zero); $\Gamma(L) := \Gamma_1 + \Gamma_2 L + ... + \Gamma_q L^{q-1}$ is a matrix polynomial in the lag operator L whose coefficients are in the $r \times n$ matrices Γ_j, $j = 1, ..., q$ (and can be possibly zero); $D_{z,t}$ is an d_z-dimensional vector containing deterministic components (constant, dummies, etc.) specific to Z_t and not included in $D_{y,t}$; Υ_z and $\Upsilon_{z,y}$ are the $r \times d_z$ and $r \times d_y$ matrices of coefficients associated with $D_{z,t}$ and $D_{y,t}$, respectively (an can be possibly zero).

Equation (7) defines our auxiliary statistical model for the external instruments. It reads as a reduced form system where Z_t may depend on its own lags $Z_{t-1}, ..., Z_{t-p}$, the predetermined 'control' variables $Y_{t-1}, ..., Y_{t-k}$, and a set of deterministic terms. Obviously, $Z_t \equiv v_{Z,t}$ when all coefficient of the system in eq. (7) are zero, meaning that the elements in Z_t are already in innovations or *iid* shocks taken from other studies (see Section 8). When the Θ_i are non-zero, the stability requirement assumed for the SVAR (Assumption 1) is extended to Z_t and requires that the roots of $\det(I_r - \Theta_1 s - ... - \Theta_p s^p) = 0$ satisfy the condition $\mid s \mid > 1$. With this in mind, in the following we call 'external instruments' Z_t and $v_{Z,t}$ interchangeably.[5]

In the next section, the SVAR in eq. (1) will be combined with the auxiliary model in eq. (7) to form a 'larger' system which incorporates the dynamics of the external instruments in a coherent and efficient way.

3 The AC-SVAR representation

By coupling the SVAR in eq. (1) with the auxiliary model for the external instruments in eq. (7) we obtain the system:

$$\begin{pmatrix} Y_t \\ Z_t \end{pmatrix} = \sum_{j=1}^{\ell} \begin{pmatrix} \Pi_j & 0_{n\times r} \\ \Gamma_j & \Theta_j \end{pmatrix} \begin{pmatrix} Y_{t-j} \\ Z_{t-j} \end{pmatrix} + \begin{pmatrix} \Upsilon_y & 0_{n\times d_z} \\ \Upsilon_{z,y} & \Upsilon_z \end{pmatrix} \begin{pmatrix} D_{y,t} \\ D_{z,t} \end{pmatrix} + \begin{pmatrix} u_t \\ v_{Z,t} \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} u_t \\ v_{Z,t} \end{pmatrix} = \left(\begin{array}{c|c} B & 0_{n\times r} \\ \hline R_\Phi & \Sigma_\omega^{1/2} \end{array}\right) \begin{pmatrix} \varepsilon_t \\ \omega_t^\circ \end{pmatrix} \tag{9}$$

[5] Some of the external instruments in Z_t ($v_{Z,t}$) might be censored as in the case of narrative time series, see e.g. Mertens and Ravn (2013). The Supplementary Appendix A.11 summarizes how the approach presented in this paper can be amended to account for external instruments which are generated by censored autoregressive processes. A full treatment of this issue deserves a detailed analysis which goes beyond the scopes of the present article and is therefore postponed to future research. To our knowledge, Mertens and Raven (2013) and Jentsch and Lunsford (2019) are examples in which censoring is excplicitly accounted for in the current proxy-SVARs literature.

where $\ell := \max\{k,p,q\}$, $\Sigma_{\omega}^{1/2}$ denotes the symmetric square root of the matrix Σ_{ω} and the term $\omega_t^{\circ} := \Sigma_{\omega}^{-1/2}\omega_t$ can be here interpreted as a normalized measurement error.[6] System (8) reads an m-dimensional VAR, $m := n + r$, of lag order ℓ which incorporates a constrained (triangular) autoregressive structure: the lags of Z_t and the deterministic variables in $D_{z,t}$ are not allowed to enter the Y_t-equations of the original SVAR. The matrices Γ_j and Θ_j, $j = 1, ..., \ell$ and $\Upsilon_{z,y}$ and Υ_z are restricted to zero in eq. (8) when $Z_t \equiv v_{Z,t}$. System (9) maps the term $\xi_t := (\varepsilon_t', \omega_t^{\circ\prime})'$ (which includes the structural shocks) onto $\eta_t := (u_t', v_{Z,t}')'$.

It is seen that the joint system (8)-(9) forms a large 'B-model' (Lütkepohl, 2005) which we call the 'augmented-constrained' SVAR (AC-SVAR) model. For future reference, we compact the AC-SVAR model in the expression:

$$W_t = \widetilde{\Psi}F_t + \widetilde{\Upsilon}D_t + \eta_t \quad , \quad \eta_t = \widetilde{G}\xi_t \tag{10}$$

where $W_t := (Y_t', Z_t')'$ and $\eta_t := (u_t', v_{Z,t}')'$ are $m \times 1$, the reduced form disturbance η_t has covariance matrix $\Sigma_\eta := E(\eta_t\eta_t')$, $F_t := (W_{t-1}', ..., W_{t-\ell}')'$ is $f \times 1$ ($f = m\ell$), $\widetilde{\Psi} := (\widetilde{\Psi}_1, ..., \widetilde{\Psi}_\ell)$ is $m \times f$, $D_t := (D_{y,t}', D_{z,t}')'$ is $d \times 1$ ($d := d_y + d_z$), $\widetilde{\Upsilon}$ is $m \times d$ and, finally, $\xi_t := (\varepsilon_t', \omega_t^{\circ\prime})'$ is $m \times 1$. We use the symbol '$\sim$' over the matrices Ψ and Υ and G to remark that these are restricted. The structure of the matrix $\widetilde{G}$ deserves special attention:

$$\widetilde{G} := \left(\begin{array}{c|c} B & 0_{n\times r} \\ \hline R_\Phi & \Sigma_\omega^{1/2} \end{array}\right) = \left(\begin{array}{c|c|c} B_1 & B_2 & 0_{n\times r} \\ \hline \Phi & 0_{r\times(n-g)} & \Sigma_\omega^{1/2} \end{array}\right). \tag{11}$$

It is seen that $\widetilde{G}$ contains the structural parameters in B, the relevance and (the zero) orthogonality conditions embedded in R_Φ and the parameters of the matrix $\Sigma_\omega^{1/2}$. The covariance restrictions implied by the AC-SVAR model (the ones stemming from $\Sigma_\eta = \widetilde{G}\widetilde{G}'$) boild down to

$$\Sigma_u = BB' \qquad \text{SVAR symmetry} \tag{12}$$

$$\Sigma_{v_Z,u} = \Phi B_1' \qquad \text{External instruments} \tag{13}$$

$$\Sigma_{v_Z} = \Phi\Phi' + \Sigma_\omega \qquad \text{External instruments.} \tag{14}$$

In the next sections we use the AC-SVAR model and the mapping in eq.s (12)-(14) to derive general necessary and sufficient conditions for identification and to put forth a novel estimation approach for proxy-SVARs.

[6] Alternatively we might replace the square root matrix $\Sigma_\omega^{1/2}$ with e.g. the Choleski factor of Σ_ω, P_ω, and normalize the measurement error ω_t as $\omega_t^{\circ} := P_\omega^{-1}\omega_t$, without affecting results.

4 Motivating example: exogenous/endogenous uncertainty in a small-scale SVAR

In this section we motivate empirically two types of identification strategies that the AC-SVAR model may feature. We discuss the exogeneity/endogeneity of measures of uncertainty with respect to the business cycle in a small-scale SVAR model, a topic which will be addressed empirically in Section 8 on U.S. monthly data.[7]

Consider a SVAR model for $Y_t{:=}(a_t, U_{F,t}, U_{M,t})'$ $(n = 3)$, where a_t is a measure of real economic activity, $U_{F,t}$ a measure of financial uncertainty and $U_{M,t}$ a measure of macroeconomic uncertainty. The relationship between the reduced form disturbances and the structural shocks is given by:

$$\underbrace{\begin{pmatrix} u_{a,t} \\ u_{F,t} \\ u_{M,t} \end{pmatrix}}_{u_t} = \underbrace{\begin{pmatrix} b_{a,a} & b_{a,F} & b_{a,M} \\ b_{F,a} & b_{F,F} & b_{F,M} \\ b_{M,a} & b_{M,F} & b_{M,M} \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} \varepsilon_{a,t} \\ \varepsilon_{F,t} \\ \varepsilon_{M,t} \end{pmatrix}}_{\varepsilon_t} \tag{15}$$

where $u_t{:=}(u_{a,t}, u_{F,t}, u_{M,t})'$ is the vector of VAR reduced form disturbances and $\varepsilon_t := (\varepsilon_{a,t}, \varepsilon_{F,t}, \varepsilon_{M,t})'$ is the vector of structural shocks. As is know, this SVAR is not identified in the absence of (at least three) restrictions on the matrix B. To inform the discussion, we temporarily label $\varepsilon_{a,t}$ as the 'non-uncertainty shock', $\varepsilon_{F,t}$ as the 'financial uncertainty shock' and $\varepsilon_{M,t}$ as the 'macroeconomic uncertainty shock'.

Since the seminal paper of Bloom (2009), attention in the empirical literature on uncertainty has been focused on measuring the impact of uncertainty shocks on real economic activity, which requires the identification of the parameters $b_{a,F}$ and $b_{a,M}$ in eq. (15). For instance, Stock and Watson (2012) use either stock market volatility or the economic policy uncertainty index of Baker *et al.* (2016) as external instruments to identify the effects of uncertainty shocks. In their framework, the parameters of interest are in the column $b_M{:=}(b_{a,M}, b_{F,M}, b_{M,M})'$ (or $b_F{:=}(b_{a,F}, b_{F,F}, b_{M,F})'$) of the matrix B. In this paper, we are interested in the other direction of causality, namely the impact of $\varepsilon_{a,t}$ on the variables $U_{F,t+h}$ and $U_{M,t+h}$ for $h = 0, 1, ...$ The on-impact responses $(h = 0)$ are captured by the parameters $b_{F,a}$ and $b_{M,a}$ contained in the first column $b_a{:=}(b_{a,a}, b_{F,a}, b_{M,a})'$ of B; the lagged responses $(h = 1, 2, ...)$ can be inferred from the IRFs in eq. (3) by setting $b_j = b_a$. As in Angelini *et al.* (2019), we consider financial and macroeconomic uncertainty 'exogenous' if $U_{F,t}$ and $U_{M,t}$ do not respond to $\varepsilon_{a,t}$ on-impact, which corresponds to the hypothesis $b_{F,a} = 0$ and $b_{M,a} = 0$ in eq. (15). Conversely, we consider financial and macroeconomic uncertainty 'endogenous' if $U_{F,t}$ and $U_{M,t}$ respond significantly

[7] As an exercise, the Supplementary Appendix A.4 applies the two identification strategies to a monetary SVAR model largely analyzed in the literature.

on-impact to the non-uncertainty shock; see Carriero *et al.* (2018) for a similar characterization. Obviously, it may happen that $U_{F,t+h}$ and $U_{M,t+h}$ respond to $\varepsilon_{a,t}$ only after some periods the shock occurs, hence we distinguish between 'contemporaneous exogeneity' and lagged effects throughout the paper.

The reverse causality problem can not be addressed by placing 'conventional' restrictions on the matrix B in eq. (15). Ludvigson *et al.* (2018), Angelini *et al.* (2018) and Carriero *et al.* (2018) face this issue by estimating non-recursive SVARs identified by different methods briefly reviewed in Section 8.3. In this paper, we argue that our proxy-SVAR approach can be a valuable alternative to the existing methods.

Partition the structural relationships in eq. (15) as follows:

$$\underbrace{\begin{pmatrix} u_{a,t} \\ u_{F,t} \\ u_{M,t} \end{pmatrix}}_{u_t} = \underbrace{\begin{pmatrix} b_{a,a} \\ b_{F,a} \\ b_{M,a} \end{pmatrix}}_{B_1 \equiv b_a} \underbrace{(\varepsilon_{a,t})}_{\varepsilon_{1,t}} + \underbrace{\begin{pmatrix} b_{a,F} & b_{a,M} \\ b_{F,F} & b_{F,M} \\ b_{M,F} & b_{M,M} \end{pmatrix}}_{B_2} \underbrace{\begin{pmatrix} \varepsilon_{F,t} \\ \varepsilon_{M,t} \end{pmatrix}}_{\varepsilon_{2,t}} \tag{16}$$

which means that the structural shock of interest, and for which valid external instruments must be found is the non-uncertainty shock, i.e. $\varepsilon_{1,t} := (\varepsilon_{a,t})$ $(g = 1)$. Here $B_1 \equiv b_a$ coincides with the first column of B and contains the parameters of primary interest. Consider, as an example, the case in which $r = 2$ valid external instruments, collected in the vector $Z_t(v_{Z,t})$, are used for $\varepsilon_{a,t}$. The matrix $\widetilde{G}$ of the AC-SVAR model in eq. (11) reads:

$$\widetilde{G} := \left(\widetilde{G}_1 \,\vdots\, \widetilde{G}_2 \right) = \left(\begin{array}{c|c|c} B_1 & B_2 & 0_{3\times 2} \\ \hline \Phi & 0_{2\times 2} & \Sigma_{\varpi}^{1/2} \end{array} \right) = \left(\begin{array}{c|cc|cc} b_{a,a} & b_{a,F} & b_{a,M} & 0 & 0 \\ b_{F,a} & b_{F,F} & b_{F,M} & 0 & 0 \\ b_{M,a} & b_{M,F} & b_{M,M} & 0 & 0 \\ \hline \varphi_{1,a} & 0 & 0 & \varpi_{1,1} & \varpi_{2,1} \\ \varphi_{2,a} & 0 & 0 & \varpi_{2,1} & \varpi_{2,2} \end{array} \right) \tag{17}$$

where $\varphi_{1,a} := E(v_{Z_1,t}\varepsilon_{a,t})$ and $\varphi_{2,a} := E(v_{Z_2,t}\varepsilon_{a,t})$ are the relevance parameters contained in the 2×1 matrix Φ, and $\varpi_{1,1}$, $\varpi_{2,1}$, $\varpi_{2,2}$ are the 3 free elements of $\Sigma_{\varpi}^{1/2}$ (assumed here non-diagonal); recall from the previous section that Σ_{ϖ} is the covariance matrix of the measurement errors ω_t in eq. (6) and that with $\Sigma_{\varpi}^{1/2}$ we denote the 'square root' of Σ_{ϖ}. Observe that in eq. (17), $\widetilde{G}_1 := (b_a', \varphi')'$, where $b_a := (b_{a,a}, b_{F,a}, b_{M,a})'$ and $\varphi := (\varphi_{1,a}, \varphi_{2,a})'$.

The 'partial shocks' identification strategy identifies the parameters in the column $\widetilde{G}_1 := (b_a', \varphi')'$, hence the shock $\varepsilon_{a,t}$, regardless of the other shocks of the system. The idea is that $\widetilde{G}_1$ is the only ingredient (other than the reduced form parameters) necessary to track the dynamic causal effect of $\varepsilon_{a,t}$ on $U_{F,t+h}$ and $U_{M,t+h}$, $h = 0, 1, ...$ The other columns of $\widetilde{G}$, collected in $\widetilde{G}_2$, are not of interest. Since $Corr(v_{Z_1,t}, \varepsilon_{a,t}) = \varphi_{1,a}/\sigma_{v_{Z_1}}$ and $Corr(v_{Z_2,t}, \varepsilon_{a,t}) = \varphi_{2,a}/\sigma_{v_{Z2}}$, where $\sigma_{v_{Z_1}}^2$ and

$\sigma^2_{v_{Z_2}}$ are the main diagonal elements of Σ_{v_Z}, the quality of the identification can be evaluated in this case by computing the measures $\hat{\varphi}_{1,a}/\hat{\sigma}_{v_{Z_1}}$ and $\hat{\varphi}_{2,a}/\hat{\sigma}_{v_{Z_2}}$, where $\hat{\varphi}_{1,a}$, $\hat{\varphi}_{2,a}$, $\hat{\sigma}_{v_{Z_1}}$ and $\hat{\sigma}_{v_{Z_2}}$ are consistent estimates of the parameters $\varphi_{1,a}$, $\varphi_{2,a}$, $\sigma_{v_{Z_1}}$ and $\sigma_{v_{Z_2}}$, respectively. In Section 5 we study the identification and estimation of the proxy-SVAR considering the case $r \geq g \geq 1$.

Suppose now that we have a some (limited) information on the non-instrumented structural shocks whose instantaneous effects are captured by the columns of the matrix B_2 in eq. (17). In particular, based on the results in Angelini *et al.* (2019), we claim that in the period after the Global Financial Crisis, the contemporaneous pass-trough between financial and macroeconomic uncertainty is one-way and runs from financial uncertainty to macroeconomic uncertainty, which implies $b_{F,M} = 0$ in eq. (17). While the original SVAR for $Y_t{:=}(a_t, U_{F,t}, U_{M,t})'$ is not identified with $b_{F,M} = 0$ in eq. (15), the AC-SVAR model based on $\widetilde{G}$ in eq. (17) and the additional restriction $b_{F,M} = 0$ is identified (see Section 8.2). The consequence of this result is that although the structural shock of primary interest is the non-uncertainty shock $\varepsilon_{1,t} := (\varepsilon_{a,t})$, we can also identify the financial and macroeconomic uncertainty shocks in $\varepsilon_{2,t} := (\varepsilon_{F,t}, \varepsilon_{M,t})'$. We call this scenario the 'full shocks' identification strategy, which is studies in Section 6. Since both $\varepsilon_{1,t} := (\varepsilon_{a,t})$ and $\varepsilon_{2,t} := (\varepsilon_{F,t}, \varepsilon_{M,t})'$ can be identified, one can obtain $\hat{\varepsilon}_t := \widehat{B}^{-1}\hat{u}_t$, $t = 1, ..., T$, where $\hat{u}_t$ are the reduced form VAR residuals and $\widehat{B}$ is a consistent estimate of B recovered from $\widehat{\widetilde{G}}$. Accordingly, one way to evaluate the quality of the identification is computing the measures of strength $Corr(\hat{v}_{Z_1,t}, \hat{\varepsilon}_{a,t})$ and $Corr(\hat{v}_{Z_2,t}, \hat{\varepsilon}_{a,t})$ which should be significant with valid instruments, but also the correlations $Corr(\hat{v}_{Z_1,t}, \hat{\varepsilon}_{F,t})$, $Corr(\hat{v}_{Z_1,t}, \hat{\varepsilon}_{M,t})$, $Corr(\hat{v}_{Z_2,t}, \hat{\varepsilon}_{F,t})$ and $Corr(\hat{v}_{Z_2,t}, \hat{\varepsilon}_{M,t})$ which should not be statistically significant.[8] In this example, the 'price to pay' to move from a partial to a full shocks identification approach is given by the auxiliary restriction $b_{F,M} = 0$, which appears a modest cost relative to the benefits. In general, the investigator is required to take a (minimal) stand also on the structure of B_2 to identify all shocks.

5 Partial shocks identification and estimation strategy

In the partial shocks identification strategy, the objective of the analysis is to exploit the instruments in Z_t ($v_{Z,t}$) to solely identify the dynamic causal effects of the g instrumented shocks in $\varepsilon_{1,t}$, ignoring the other shocks collected in $\varepsilon_{2,t}$. This amounts to identify the submatrix B_1 in eq. (5), which in turn provides the IRFs in eq. (3) for $j = 1, ..., g$, $g < n$.

Our analysis starts from the AC-SVAR representation of the proxy-SVAR summarized in

[8] Alternatively, one can compute F-type tests from regressions of the form $\hat{v}_{Z,t} = C_1\hat{\varepsilon}_{1,t} + C_2\hat{\varepsilon}_{2,t} + \epsilon_t$, $t = 1, ..., T$, where the rejection of $H_0^R : C_1 = 0_{r\times g}$ indicates that the relevance condition is supported by the data, and the non-rejection of $H_0^O : C_2 = 0_{r\times(n-g)}$ indicates that the orthogonality condition is supported by the data.

eq.s (10)-(11). We are interested in the first g columns of the matrix $\widetilde{G}$:

$$\widetilde{G}_1 := \begin{pmatrix} B_1 \\ \text{- - -} \\ \Phi \end{pmatrix}, \tag{18}$$

while the remaining $m-g$ columns in $\widetilde{G}_2$ are not of interest. The identification of the g columns of $\widetilde{G}_1$ in eq. (18) reads as a partial identification exercise which requires imposing at least $1/2g(g-1)$ restrictions on $\widetilde{G}_1$ (i.e. on B_1 and Φ); obviously, no restriction is needed when $g=1$. This necessary order condition for identification clearly shows that when $g>1$, the r external instruments alone do not suffice to identify the shocks of interest, see e.g. Mertens and Ravn (2013), Mertens and Montiel Olea (2018) and Arias *et al.* (2018b). In principle, conditional on the validity of a rank condition we discuss below, the additional restrictions can be placed on the columns of B_1 leaving Φ free, or can be imposed on the columns of Φ (preserving the full column rank condition) leaving B_1 free, or possibly can be distributed on both B_1 and Φ.

It turns out that when the restrictions on $\widetilde{G}_1$ are homogenous (i.e. there are zero restrictions only) and separable across columns, one can check the identification of the proxy-SVAR by referring to the sufficient conditions for global identification established by Theorem 2 in Rubio-Ramirez *et al.* (2010). However, Theorem 2 in Rubio-Ramirez *et al.* (2010) provides only sufficient conditions for identification which are valid when the restrictions are homogeneous and separable across columns. To derive necessary and sufficient conditions for identification which are valid in more general situations, including the case of non-homogeneus, cross-columns restrictions, we find it convenient to exploit a set of moment conditions implied by the AC-SVAR model which pave the way for a CMD estimation approach.

The Supplementary Appendix A.5 shows that by using simple algebra the moment conditions in eq.s (12)-(13) can be transformed into:

$$\Xi = \Phi\Phi' \quad , \quad \Sigma_{v_Z,u} = \Phi B_1' \tag{19}$$

where $\Xi := \Sigma_{v_Z,u}\Sigma_u^{-1}\Sigma_{u,v_Z}$ is an $r \times r$ symmetric matrix (of rank g) which is positive definite when $r=g$ and is positive semidefinite when $r>g$. The advantage of the representation in eq. (19), relative to that in eq.s (12)-(13) is that the nuisance parameters in B_2 have been marginalized out. The moment conditions in eq. (19) can be formally compacted in the expression:

$$\zeta = f(\vartheta) \tag{20}$$

where $\zeta := (vech(\Xi)', vec(\Sigma_{v_Z,u})')'$ is a vector whose elements depend on the reduced form parameters $\sigma_\eta^+ := vech(\Sigma_\eta)$ of the AC-SVAR model, $f(\vartheta) := (vech(\Phi\Phi')', vec(\Phi B_1')')'$ is a nonlinear differentiable vector function and ϑ is the vector which collects the unrestricted (free) elements in the matrix $\widetilde{G}_1$ in eq. (18). The restrictions necessary to identify $\widetilde{G}_1$ are parameterized in explicit form by:

$$\beta_1 := vec(B_1) = S_{B_1}\alpha_1 + s_{B_1} \quad , \quad \phi := vec(\Phi) = S_{\Phi}\varphi \tag{21}$$

where α_1 is the $e_1 \times 1$ vector which collects the unrestricted (free) elements of β_1, $e_1 \leq ng$, $S_{B_1}$is an $ng \times e_1$ full column rank selection matrix and and s_{B_1} is an $ng \times 1$ vector containing zeros or known non-zero elements; φ is the $c \times 1$ vector which collects the unrestricted (free) elements of Φ, $c \leq rg$, and S_{Φ} is an $rg \times c$ full column-rank selection matrix. Obviously, when β_1 is unrestricted, $\beta_1 \equiv \alpha_1$, $e_1 \equiv ng$, $S_{B_1} \equiv I_{ng}$ and $s_{B_1} \equiv 0_{ng\times 1}$; when ϕ is unrestricted, $\phi \equiv \varphi$, $c = rg$ and $S_{\Phi} \equiv I_{rg}$. Because of the presence of the possibly non-zero term s_{B_1}, eq. (21) accommodates also non-homogeneous restrictions, which means that some elements of B_1 can be e.g. fixed to known non-zero constants. It is seen that $\vartheta := (\alpha_1', \varphi')'$ is $(e_1 + c) \times 1$.

Equation (20) defines a 'distance' between the $a \times 1$ ($a := 1/2r(r+1)+nr$) vector of reduced form parameters ζ and the $(e_1 + c) \times 1$ vector of parameters ϑ. From Rothenberg (1971) it follows that necessary and sufficient condition for ϑ being uniquely recovered from ζ is that the $a \times (e_1 + c)$ Jacobian matrix $\frac{\partial f}{\partial \vartheta'} := \digamma_{\vartheta}$ is regular and of full column rank in a neighborhood of the true value of ϑ.[9] We derive the Jacobian matrix $\digamma_{\vartheta}$ below.

Under Assumption 1, the estimator of the reduced form parameter σ_{η}^{+} of the AC-SVAR model is consistent and asymptotically Gaussian, hence we have the result:

$$T^{1/2}(\hat{\zeta}_T - \zeta_0) \to_d N(0_{a\times 1}, \Omega_{\zeta}) \tag{22}$$

where ζ_0 denotes the true value of ζ, Ω_{ζ} is an $a \times a$ covariance matrix which can be estimated consistently (see Supplementary Appendix A.5 for details) and '$\to_d$' denotes convergence in distribution. The convergence in eq. (22) involves the estimator of the reduced form parameters of the AC-SVAR model and is valid also if the matrix of relevance parameters Φ is zero, i.e. irrespective of whether the external instruments are strongly, weakly or not correlated at all with the structural shocks of interest. This result motivates a robust indirect test for the null hypothesis of 'no relevance' based on the idea that if in eq. (19) it is assumed that $B_1 \neq 0_{n\times g}$, the null hypothesis $H_0 : vec(\Phi) = 0_{rg\times 1}$ (no relevance) is equivalent to $H_0' : vec(\Sigma_{v_Z,u}) = 0_{rn\times 1}$ (no correlation between the external instruments and the VAR disturbances). We discuss a simple Wald-type test for H_0' in the Supplementary Appendix A.9.

Given eq.s (20) and the consistency of $\hat{\zeta}_T$, ϑ can be estimated by solving the CMD problem:

$$\min_{\vartheta}(\hat{\zeta}_T - f(\vartheta))'\widehat{\Omega}_{\zeta}^{-1}(\hat{\zeta}_T - f(\vartheta)) \tag{23}$$

[9] Let $M = M(\theta)$ be a matrix of rank m, whose elements depend on θ. M is 'regular' if $rank(M(\theta)) = m$ in a neighborhood of θ_0, where θ_0 is the true value of θ.

where $\widehat{\Omega}_\zeta$ is a consistent estimate of Ω_ζ. The properties of the estimator $\hat{\vartheta}_T$ obtained from eq. (23) depend on the identification of the proxy-SVAR. The next proposition formalizes the necessary and sufficient rank conditions and the necessary order condition for the identification of the proxy-SVAR.

Proposition 1 [Partial shocks identification] Given the SVAR in eq. (1), a vector of r external instruments $v_{Z,t}$ for the $1 \leq g < n$ strucutral shocks in $\varepsilon_{1,t}$ and Assumptions 1-2, consider the identification of the g columns of $\widetilde{G}_1$ in eq. (18). Let $\vartheta_0 := (\alpha_{1,0}', \varphi_0')'$ be the true value of $\vartheta := (\alpha_1', \varphi')'$. Then:

(a) necessary and sufficient rank condition for identification is

$$rank\,\{\mathcal{F}_{\vartheta_0}\} = e_1 + c$$

where $\mathcal{F}_{\vartheta_0}$ is the Jacobian matrix:

$$\mathcal{F}_\vartheta := \left(\begin{array}{c|c} 0_{1/2r(r+1)\times ng} & 2D_r^+(\Phi \otimes I_r) \\ \hline (I_n \otimes \Phi)K_{ng} & (B_1 \otimes I_r) \end{array}\right)\left(\begin{array}{c|c} S_{B_1} & 0_{ng\times c} \\ \hline 0_{rg\times e_1} & S_\Phi \end{array}\right) \tag{24}$$

evaluated in a neighborhood of ϑ_0 and is 'regular';[10]

(b) necessary order condition is that at least $1/2g(g-1)$ restrictions are placed on the g columns of $\widetilde{G}_1$, which is equivalent to the condition $e_1 + c \leq g(n+r) - 1/2g(g-1)$.

Proof: Supplementary Appendix A.3.

Some remarks are in order.

First, Proposition 1 provides an alternative to Mertens and Ravn's (2013) identification approach for proxy-SVARs with $g > 1$ multiple shocks. Mertens and Ravn (2013) show that when $g > 1$, the restrictions implied by the external instruments do not suffice alone to identify the g shocks of interest and must be complemented with additional constraints. In their setup, the $1/2g(g-1)$ additional constraints necessary to identify the proxy-SVAR stem from the mechanics of the IV approach and are obtained from a Choleski decomposition of a symmetric matrix. In our framework, the fact that it is necessary to impose at least $1/2g(g-1)$ restrictions to identify the g shocks of interest is a necessary order condition, but these restrictions need not be Choleski-type constraints (the Supplementary Appendix A.7 compares in detailed Mertens

[10] Given the $n \times g$ matrix M, K_{ng} denotes the $ng \times ng$ *commutation matrix* which satisfies $K_{ng}vec(M) = vec(M')$. $D_n^+ := (D_n' D_n)^{-1} D_n'$ denotes the Moore-Penrose inverse of D_n, where D_n is the $n^2 \times \frac{1}{2}n(n+1)$ *duplication matrix* such that $D_n vech(M) = vec(M)$, where $vech(M)$ is the column obtained from $vec(M)$ by eliminating all supra-diagonal elements. See Magnus and Neudecker (1999).

and Ravn's (2013) identification approach with ours). Proposition 1 establishes necessary and sufficient conditions for the identification of the parameters in Φ and B_1 (i.e. of $\widetilde{G}_1$) which hold up to sign normalization, which means that if e.g. a given $\tilde{\Phi}$ satisfies the restriction $\Xi = \tilde{\Phi}\tilde{\Phi}'$ in eq. (19), also the matrix $\tilde{\Phi}^* \neq \tilde{\Phi}$, obtained from $\tilde{\Phi}$ by changing the sign of one or more than one of its columns, will satisfy eq. (19).

Second, according to Proposition 1(*b*), the proxy-SVAR is exactly identified when $e_1 + c = g(n+r) - 1/2g(g-1)$, i.e. when there are exactly $1/2g(g-1)$ restrictions on the elements of $\widetilde{G}_1$ in eq. (18), and is overidentified (and therefore testable) when $e_1 + c < g(n+r) - 1/2g(g-1)$.[11]

Third, Proposition 1 clarifies that in general, the full column rank condition of the matrix Φ (Assumption 2) is only necessary for identification. Indeed, the (2,1) block $(I_n \otimes \Phi)K_{ng}$ of the Jacobian matrix in eq. (24) suggests that $rank\{\Phi\} = g$ is necessary for $rank\{(I_n \otimes \Phi)K_{ng}\} = ng$, which in turn is a necessary condition for $rank\{\mathcal{F}_{\vartheta_0}\} = e_1 + c$. The structure of $\mathcal{F}_{\vartheta}$ also shows that when $g = 1$, the full column rank condition of Φ is also sufficient for the identification of the proxy-SVAR (whatever r). This is easily seen in the 'one shock-one instrument' case $r = g = 1$, where $\Phi = \phi = \varphi$ is a scalar ($c = 1$) and if β_1 is unrestricted ($\beta_1 \equiv \alpha_1$) the Jacobian $\mathcal{F}_{\vartheta}$ collapses to the $(n+1) \times (n+1)$ matrix:

$$\mathcal{F}_{\vartheta} := \left(\begin{array}{cccc:c} 0 & 0 & \cdots & 0 & 2\varphi \\ \hdashline & & \varphi I_n & & \alpha_1 \end{array}\right);$$

it is seen that $\varphi = \phi \neq 0$ is necessary and sufficient for identification.[12] The form of the Jacobian in eq. (24) also shows that one of the advantages of using more than one instrument to identify a single shock of interest, $r > g = 1$, is that $rank(\Phi) = 1$ if at least one component in $\Phi := (\varphi_1, ..., \varphi_r)$ is different from zero, which means that provided at least one external instrument is strongly correlated with the shock of interest, $r - 1$ external instruments might potentially violate the relevance condition. This argument can be easily generalized to the case $r > g > 1$.

Coming back to the estimation problem (23), under Assumptions 1-2 and the conditions of Proposition 1 we have (Newey and McFadden, 1991):

$$T^{1/2}(\hat{\vartheta}_T - \vartheta_0) \to_d N(0_{(e_1+c)\times 1}, \Omega_{\vartheta}) \ , \ \ \Omega_{\vartheta} := \left(\mathcal{F}_{\vartheta}' \, \Omega_{\zeta}^{-1} \mathcal{F}_{\vartheta}\right)^{-1} \tag{25}$$

[11] When $r = g$ one has $a = 1/2g(g+1) + ng$ and $e_1 + c = g(n+g) - 1/2g(g-1) = 1/2g(g+1) + gn$ under exact identification, hence the Jacobian matrix in eq. (24) is square. Instead, when $r > g$, the Jacobian matrix in eq. (24) is 'tall' (meaning that it has more rows than columns) even in the case of exact identification, i.e. $a = 1/2r(r+1) + nr = r^2 - 1/2r(r-1) + nr > g(n+r) - 1/2g(g-1) = e_1 + c$.

[12] The structure of this Jacobian matrix shows that if the true value of φ satisfies the local-to-zero embedding: $\varphi_0 := T^{1/2}\varrho$, $\varrho \neq 0$, the proxy-SVAR is not identified asymptotically. We refer to Lunsford (2015) and Montiel Olea *et al.* (2018) for inference in proxy-SVARs in the presence of 'weak' instruments.

where the asymptotic covariance matrix Ω_{ϑ} can be estimated consistently by $\widehat{\Omega}_{\vartheta,T} := \left(\hat{F}'_{\vartheta}\ \widehat{\Omega}_{\zeta}^{-1}\hat{F}_{\vartheta}\right)^{-1}$ and $\hat{F}_{\vartheta}$ is taken from eq. (24) by replacing the unconstrained (free) elements in B_1 and Φ with the corresponding elements in $\hat{\vartheta}_T := (\hat{\alpha}'_{1,T}, \hat{\varphi}'_T)'$.

When according to Proposition 1 the proxy-SVAR model is overidentified, the CMD problem delivers a test of overidentifying restrictions because, under the null hypothesis $\zeta_0 = f(\vartheta_0)$ and Assumptions 1-2 the quantity $TQ(\hat{\vartheta}_T)$ converges asymptotically to a $\chi^2(l)$ random variable with $l := g(n+r) - 1/2g(g-1) - (e_1 + c)$ degree of freedoms. In the IV (GMM) framework, when the number of instruments (moment conditions) is larger than the number of estimated parameters, it is possible to compute Sargan's specification test (Hansen's J-test), which is typically interpreted as a specification test for the estimated model. The $TQ(\hat{\vartheta}_T)$ test can be used similarly. Our Monte Carlo experiments (summarized in the Supplementary Appendix A.8 to save space) show that the test $TQ(\hat{\vartheta}_T)$ rejects the overidentification restrictions when the external instruments are incorrectly assumed orthogonal to the non-instrumented structural shocks.

In case of exact identification, or when the overidentification restrictions are not rejected by the $TQ(\hat{\vartheta}_T)$ test, the IRFs of interest in eq. (3) can be estimated by replacing the companion matrix A_y with its consistent estimate $\widehat{A}_y$ derived from the AC-SVAR model and b_j with the j-th column of $\widehat{B}_1$, for $j = 1, .., g$, where $\widehat{B}_1$ is reconstructed from $\hat{\beta}_{1,T}{:=}S_{B_1}\hat{\alpha}_{1,T} + s_{B_1}$. We refer to Jentsch and Lunsford (2019) and Mertens and Ravn (2019) for a recent debate on bootstrap inference for IRFs in proxy-SVARs; see also Montiel Olea *et al.* (2018). Since in our framework the proxy-SVAR is specified as a large (constrained) SVAR 'B-model', bootstrap confidence bands for the IRFs can be obtained by applying the methods currently available for SVARs reviewed e.g. in Kilian and Lütkepohl (2017, Ch. 12). In particular, when the disturbances $\eta_t := (u'_t, v'_{Z,t})'$ in eq. (10) are conditionally homoskedastic, it is possible to combine residual-based bootstrap methods with the CMD approach by the algorithm summarized in the Supplementary Appendix A.10. Accordingly, the (innovations associated with the) external instruments are resampled jointly with the VAR residuals regardless of the number of instruments r and instrumented structural shocks g. For instance, in Section 8 we compute 90%-bootstrap simultaneous confidence bands for the IRFs by using the 'sup-t' bands discussed in Montiel Olea and Plagborg-Møller (2019). If instead the disturbances η_t in eq. (10) display conditional heteroskedasticity of unknown form, the results in Bruggemann *et al.* (2016) and Jentsch and Lunsford (2016, 2019) suggest that reliable inference must be based on the residual-based moving block bootstrap.

6 Full shocks identification and estimation strategy

In this case we investigate the conditions under which the instruments Z_t ($v_{Z,t}$) used to identify the structural shocks of interest $\varepsilon_{1,t}$ permit to identify the dynamic causal effects of all structural shocks in ε_t, including the non-instrumented ones in $\varepsilon_{2,t}$.[13] We formalize the identification analysis and estimation of proxy-SVARs in these situations and show that (frequentist) estimation of these models can be conveniently carried out by ML.

As in the partial shocks approach, our starting point is the AC-SVAR representation of the proxy-SVAR summarized in eq.s (10)-(11). This is a large 'B-model' whose identification depends on the (number and structure of) restrictions which characterize the matrix $\widetilde{G}$. Equation (11) shows that $\widetilde{G}$ incorporates by construnction $r(n-g)+nr$ zero restrictions plus the symmetry constraints stemming from the matrix $\Sigma_\omega^{1/2}$; however, these restrictions do not generally suffice to achieve the at least $1/2m(m-1)$ restrictions necessary to identify the m columns of $\widetilde{G}$. It turns out that aside from special cases discussed below we also need a few restrictions on B_2, other than on B_1 and Φ. In the uncertainty example discussed in Section 4, if one adds the constraint $b_{F,M}=0$ in the sub-matrix B_2 of $\widetilde{G}$ in eq. (17), and if e.g. the covariance matrix of measurement errors Σ_ω is diagonal (which implies $\omega_{12}=\omega_{21}=0$), there are 13 homogeneous restrictions in $\widetilde{G}$ which are separable across columns (there are therefore 3 overidentification restrictions); it is possible to prove that in this case the model is identified (globally) according to Theorem 1 in Rubio-Ramirez *et al.* (2010).

In general, when the restrictions on $\widetilde{G}$ are homogeneous and separable across columns, it is convenient to study the identification of the AC-SVAR model by checking whether the sufficient conditions for (global) identification in Theorem 1 of Rubio-Ramirez et al. (2010) are satisfied. We derive necessary and sufficient conditions for (local) identification which are valid in more general situations, including the case of non-homogeneus, cross-columns linear restrictions. To do so, we formalize the restrictions on B_1 and Φ as in eq. (21) but, in addition, we include also restrictions on B_2 and $\Sigma_\omega^{1/2}$ as follows:

$$\beta_2 := vec(B_2) = S_{B_2}\alpha_2 \ + s_{B_2}, \quad \omega := vech(\Sigma_\omega^{1/2}) = S_{\Sigma_\omega}\varpi. \tag{26}$$

In eq. (26), α_2 is the vector collecting the e_2 unrestricted (free) elements of B_2 (if any), S_{B_2} is an $n(n-g)\times e_2$ full column rank selection matrix and $s_{B_2}$is an $n(n-g)\times 1$ vector containing

[13] In the current proxy-SVAR literature, a concrete example where an identification strategy based on external instruments identifies all shocks of the systen is Caldara and Kamps's (2017) fiscal framework. In a system of $n=5$ variables, they use $r=3$ non-fiscal instruments to identify $g=3$ non-fiscal shocks (output, inflation and monetary policy) but simultaneously they jointly identify tax and spending shocks ($n-g=2$) under the additional constraint that government spending does not respond contemporaneously to taxes.

zeros and known elements; obviously $S_{B_2} \equiv I_{n(n-g)}$, $\beta_2 \equiv \alpha_2$ and $s_{B_2} = 0_{n(n-g)\times 1}$ when β_2 is unrestricted; ϖ is the vector containing the s_ω unrestricted (free) non-zero elements of $\Sigma_\omega^{1/2}$ and S_{Σ_ω} is an $1/2r(r+1) \times s_\omega$ full column rank selection matrix, where s_ω:=$1/2r(r+1)$ when Σ_ω is full and s_ω:=r when Σ_ω is diagonal. Summing up, the identification restrictions featured by the matrix $\widetilde{G}$ in eq. (11) can be represented (in explicit form) as:

$$vec(\widetilde{G}) = S_{\widetilde{G}}\theta + s_{\widetilde{G}} \tag{27}$$

where $\theta := (\alpha_1', \alpha_2', \varphi', \varpi')'$ has dimension $a_{\widetilde{G}} \times 1$, $a_{\widetilde{G}} := e_1 + e_2 + c + s_\omega$, S_G is an $m^2 \times a_{\widetilde{G}}$ full column rank selection matrix which depends on S_{B_1}, S_{B_2}, S_Φ and S_{Σ_ω}, respectively, and $s_{\widetilde{G}}$ is a known $m^2 \times 1$ vector. The next proposition provides the necessary and sufficient rank conditions for the (local) identification of the AC-SVAR model and the necessary order conditions.

Proposition 2 [Full shocks identification] Given the SVAR in eq. (1), a vector of r external instruments $v_{Z,t}$ for the $1 \leq g < n$ strucutral shocks in $\varepsilon_{1,t}$ and Assumptions 1-2, consider the identification of all shocks in $\varepsilon_t := (\varepsilon_{1,t}',\varepsilon_{2,t}')'$, i.e. the identification of the first n columns of $\widetilde{G}$ in eq. (11). Let $\theta_0 := (\alpha_{1,0}', \alpha_{2,0}', \varphi_0', \varpi_0')'$ be the true value of $\theta := (\alpha_1', \alpha_2', \varphi', \varpi')'$. Then:

(a) necessary and sufficient rank condition for identification is:

$$rank\left\{D_m^+(\widetilde{G}_0 \otimes I_m)S_{\widetilde{G}}\right\} = a_{\widetilde{G}} \tag{28}$$

where $\widetilde{G}_0$ is the matrix $\widetilde{G}$ evaluated in a neighborhood of θ_0 and is 'regular';

(b) necessary order condition for identification is: $a_{\widetilde{G}} \leq \frac{1}{2}m(m+1)$.

Proof: Supplementary Appendix A.3.

Some remarks are in order.

First, according to Proposition 2, the identification of the shocks in $\varepsilon_t := (\varepsilon_{1,t}',\varepsilon_{2,t}')'$ based on r instruments for $\varepsilon_{1,t}$ may occur in two situations: (i) when $g < (n-1)$ provided a few restrictions are also placed on B_2 (see the example discussed in Section 4); (ii) when $g = (n-1)$ (all structural shocks of the system are instrumented but one) provided the rank condition in eq. (28) holds.[14]

Second, when the AC-SVAR model is overidentified, the system features $l := \frac{1}{2}m(m+1) - a_{\widetilde{G}}$ testable restrictions which can be used to assess the empirical validity of the estimated proxy-SVAR, see the next section.

[14] When $n = g-1$ and $B_2 := b_2$ is left unrestricted, the total number of restrictions featured by the matrix $\widetilde{G}$ in eq. (11), denoted ϱ_1, is such that $\varrho_1 \geq 1/2m(m-1)$ for $r \geq g$, which means that the necessary order condition is always satisfied.

Third, since the analysis is based on the factorization $\Sigma_\eta = \widetilde{G}\widetilde{G}'$, also in this case identification holds up to sign normalization.

If the conditions of Proposition 2 are valid, the estimation of the AC-SVAR model in eq.s (10)-(11) amounts to the estimation of a particular 'B-model' and can be carried out by ML by simply adapting the algorithms reported in Amisano and Giannini (1997) and Lütkepohl (2005).[15] The Supplementary Appendix A.6 reviews the specification steps necessary to obtain the (concentrated) log-likelihood associated with the reduced form model, denoted $L_T(\sigma_\eta^+)$, and the (concentrated) log-likelihood associated with the structural form, denoted $L_T^s(\theta)$. Under Assumptions 1-2 and Proposition 2 the ML estimator $\hat{\theta}_T := \max_\theta L_T^s(\theta)$ is consistent and asymptotically Gaussian. Moreover, when the AC-SVAR model is overidentified, it is possible to compute the LR test $LR_T := -2(L_T^s(\hat{\theta}_T) - L_T(\hat{\sigma}_{\eta,T}^+))$ which is distributed asymptotically, under the null of correct specification, as a $\chi^2(l)$ variable with $l := \frac{1}{2}m(m+1) - a_{\widetilde{G}}$ degrees of freedom. This overidentification restrictions test can be interpreted similarly to the $TQ(\hat{\vartheta}_T)$ test discussed in Section 5, hence LR_T tends to reject the proxy-SVAR model when e.g. the external instruments are wrongly assumed orthogonal to the non-insrumented shocks, see the Monte Carlo resuls in the Supplementary Appendix A.8.

In case of exact identification, or if the overidentification restrictions are not rejected by the LR test, the IRFs are estimated from eq. (3) by replacing A_y with the consistent estimate $\hat{A}_y$ derived from the reduced form of the AC-SVAR model, and by replacing b_j with the j-th column of $\widehat{\widetilde{G}} = \widetilde{G}(\hat{\theta}_T)$, $j = 1, ..., n$. In this case the computation of bootstrap confidence bands for the IRFs requires bootstrapping a SVAR model: see Section 5 and the Supplementary Appendix A.10.

7 Connections with the literature

The approach presented in the previous sections has several connections with the proxy-SVAR literature. Stock and Watson (2012, 2018), Mertens and Ravn (2013, 2014) and Montiel Olea *et al.* (2018) are seminal contributions in proxy-SVARs are estimated by IV methods; see also Jentsch and Lunsford (2016).[16] Plagborg-Møller and Wolf (2018) is the only contribution in the frequentist framework (other than ours) where an auxiliary model for the external instruments plays an active role in the analysis. They consider a proxy-SVAR model similar to system (7) but

[15] Any econometric package which features the estimation of SVARs can be used or adapted to this scope. In practice, it is necessary to estimate a SVAR model for $W_t := (Y_t', Z_t')'$ by incorporating zero restrictions in the autoregressive coefficients.

[16] Important applied developments based on IV methods include, among others, Gertler and Karadi (2015), Carriero *et al.* (2015) and Caldara and Kamps (2017).

with infinite lags for the variables. Plagborg-Møller and Wolf (2018) cover the case $r \geq g = 1$ and discuss inference on variance amd historical decompositions in a general semiparametric moving average model. Extending our approach to the infinite order case as in Plagborg-Møller and Wolf (2018) requires moving to a frequency domain approach.

In the Bayesian framework, the idea of appending the external instruments to the original SVAR model is not new. Caldara and Herbst (2019) consider the 'one shock-one instrument' case $r = g = 1$ and add an external instrument to the original SVAR system to identify a monetary policy shock (in their framework $Z_t \equiv v_{Z,t}$, hence the parameters Γ_js, Θ_js and Υ_z and $\Upsilon_{z,y}$ are zero in eq. (7)). Arias *et al.* (2018b) consider the case $r = g \geq 1$ and a dynamic representation of the proxy-SVAR similar to ours. However, while our AC-SVAR specification corresponds to a large (and constrained) 'B-model', Arias *et al.* (2018b)'s parameterization reads as an large (and constrained) 'A-model' (Lütkepohl, 2005). Their identification strategy features both zero and sign restrictions and modifies Arias *et al.* (2018a)'s algorithm to account for the highly constrained parametric structure of the proxy-SVAR model. In line with (and independently from) our analysis, Arias *et al.* (2018b) recognize that when $g > 1$, the additional (zero or sign) restrictions necessary to identify the structural shocks need not be Choleski-type constraints. Interestingly, these authors also observe that the additional identification restrictions that complement the restrictions implied by the external instruments can possibly be extended to the part of the system which pertains to the non-instrumented shocks, which is exactly the logic of the full shocks identification strategy developed in Section 6.

Compared to the above mentioned contributions, we show that proxy-SVARs with $r \geq g \geq 1$ can be conveniently represented as 'B-models' with advantages in the identification and estimation. In our framework, the analysis of proxy-SVARs is not necessarily confined to partial identification strategies but depends on the information available to the practitioner. The suggested CMD and ML estimation approaches are novel in the proxy-SVAR literature and straighforward to implement.

8 Empirical application

In this section we apply our methodology to investigate a recently debated issue of the empirical uncertainty literature, i.e. whether uncertainty is an exogenous source of business cycle fluctuations, or an endogenous response to first moment shocks, or both. In Section 4 we have already anticipated some of the technical challenges that the empirical investigation of this problem rises in the context of small-scale proxy-SVARs. Well documented facts suggest that heightened uncertainty triggers a contraction in real economic activity, and that uncertainty tends to be higher during economic recessions, see e.g. Bloom (2009), Stock and Watson (2012), Christiano *et al.* (2014), Jurado *et al* (2015), Carriero *et al.* (2015), Caggiano *et al.* (2017) and Angelini *et al.* (2019), just to mention a few. It is less clear, however, whether the higher uncertainty observed in correspondence of periods of high economic and financial turmoil is rather a consequence of first moment shocks hitting the business cycle, not the cause. The empirical assessment of the exogeneity/endogeneity of uncertainty is not only important to discriminate among two broad classes of theories about the origins of uncertainty, excellently reviewed in Ludvigson *et al.* (2018), but also for its policy implications. Indeed, as suggested by Bloom (2009, pages 626-7), uncertainty shocks may induce a trade-off between policy 'correctness' and 'decisiveness' - it may be better to act decisively (but occasionally incorrectly) than to deliberate on policy, generating policy-induced uncertainty.

We address the exogeneity/endogeneity problem by using a small-scale SVAR model for $Y_t{:=}(a_t, U_{F,t}, U_{M,t})'$ $(n = 3)$, including a measure of real economic activity (a_t), a measure of financial uncertainty $(U_{F,t})$ and a measure of macroeconomic uncertainty $(U_{M,t})$. As explained in Ludvigson *et al.* (2018), the joint use of macroeconomic and financial uncertainty is crucial to disentangle the contributions of two distinct sources of uncertainty and study their pass through to the business cycle. The direction of causality we are concerned with runs from non-uncertainty shocks to financial and macroeconomic uncertainty, not the other way around. As argued in Section 4, one way to achieve this objective is to use a set of external instruments for $\varepsilon_{a,t}$, see the matrices B and $\widetilde{G}$ in eq.s (15)-(17).

In Section 8.1 we summarize the data, in Section 8.2 we present the empirical results obtained with our baseline AC-SVAR model and in Section 8.3 we compare our findings with those obtained by other authors.

8.1 Data

Real economic activity a_t is proxied by the growth rate of the log of the U.S. industrial production index, $a_t := \Delta ip_t$ (source Fred); financial uncertainty $U_{F,t}$ is proxied by a measure of 1-month ahead financial uncertainty taken from Ludvigson *et al.* (2018); macroeconomic uncertainty $U_{M,t}$ is proxied by a measure of 1-month ahead macroeconomic uncertainty taken from Jurado *et al.* (2015).[17] We consider $T = 88$ monthly observations and the same variables as in Ludvigson *et al.* (2018), but differently from these authors, we do not estimate the model on the entire period 1960-2015, but on the subsample 2008M1-2015M4 that we term the 'Great Recession + Slow Recovery' period. Our choice of considering only the period after the Global Financial Crisis is

[17] We consider a version of the index $U_{M,t}$ 'purged' from possible effects of financial variables, see Angelini *et al.* (2019) for details

motivated by the empirical results in Angelini *et al.* (2019), who show that the unconditional error covariance matrix of the VAR for $Y_t{:=}(a_t, U_{F,t}, U_{M,t})'$ is affected by at least two major structural breaks in the period 1960-2015. The 'Great Recession + Slow Recovery' period 2008M1-2015M4 is particularly informative to infer whether uncertainty measures respond on-impact to non-uncertainty shocks as it broadly coincides with the zero lower bound constraint on the short-term nominal interest rate. According to Plante *et al.* (2018)'s argument, uncertainty should be triggered by first moment shocks in this period because of the Fed's inability to offset adverse shocks by conventional policies; see also Basu and Bundick (2015).[18]

8.2 Non-uncertainty shock, empirical results

We are primarily interested in the parameters in the column $b_a{:=}(b_{a,a}, b_{F,a}, b_{M,a})'$ which enters the matrix $\widetilde{G}$ in eq. (17) and capture the instantaneous impact of the shock $\varepsilon_{a,t}$. The specification in eq. (17) pertains to an AC-SVAR model for $W_t := (Y_t', Z_t')'$, where Z_t contains $r = 2$ external instruments for the non-uncertainty shock of the system. In principle, we might include variables in Z_t selected from a set of external instruments correlated with real economic activity, including proxies for the technology shock, the oil shock, investors confidence shocks, loan demand and supply shocks, to give a few examples relating to both aggregate supply and aggregate demand shocks.[19] Unfortunately, given the monthly frequency of our variables and the estimation sample we consider, it is not immediate to find monthly analogs of the series of shocks largely available in the literature at the quarterly frequency, see e.g. Ramsey (2016). However, the flexibility of the AC-SVAR approach allows us to use 'raw' time series in Z_t and employ, under Assumption 2, the reduced form innovations $v_{Z,t} := Z_t - E(Z_t \mid \mathcal{F}_{t-1})$ as external instruments.

We consider the following external instruments for the real economic activity shock $\varepsilon_{a,t}$: (a) innovations obtained from an auxiliary model for $\Delta house_t$, where $house_t$ is the log of new privately owned housing units started (source: Fred); (b) an oil supply shock constructed by following Kilian's (2009) identification strategy, denoted oil_t (see Supplementary Appendix A.12.1 for details); (c) innovations obtained from an auxiliary model for $\Delta hours_t$, where $hours_t$ is the log of hours worked (source: Fred). The baseline AC-SVAR model estimated in this section

[18] Interestingly, Pellegrino (2017) compares the real effects of a monetary shock in tranquil and turbulent periods by distinguishing the cases of endogenous and exogenous uncertainty. He reports that the responses of real variables to a monetary policy shock gets halved when uncertainty is treated as an endogenous variable.

[19] We prefer not to consider explicitly a monetary policy shock among the list of candidate external instruments for the real economic activity shock. As is known, assessing the impact of unconventional policy (given the sample 2008M1-2015M4) is more challenging than it is for conventional policy, see, among others, Gertler and Karadi (2015) and Roger *et al.* (2018). In the Supplementary Appendix A.12.5 we check to what extent the empirical results obtained in this section are affected by the inclusion of Wu and Xia (2016)'s 'shadow policy rate' (other than the inflation rate) in the system.

employes (a,b) as external instruments for $\varepsilon_{a,t}$. Oil shocks might be weak instruments for real economic activity (Stock and Watson, 2012), but this should not in principle affect standard asymptotic inference if the other instrument is strong, see Section 5. However, in the Supplementary Appendix A.12.2 we also estimate an analog of the baseline AC-SVAR model based on the external instruments (a, c), i.e. not including oil shocks.

While oil_t is in shock form, $\Delta house_t$ is a 'raw' variable from which we derive the innovations $v_{Z_1,t} := \Delta house_t - E(\Delta house_t \mid \mathcal{F}_{t-1})$ by estimating an auxiliary dynamic model which is appendend to the original SVAR model. Since $\Delta house_t$ can be considered a predictor of real economic activity, it is reasonable to conjecture that the innovations $v_{Z_1,t} := \Delta house_t - E(\Delta house_t \mid \mathcal{F}_{t-1})$ are not contemporaneosuly correlated with financial and macroeconomic uncertainty shocks.[20].

The baseline AC-SVAR model is given by the VAR for $Y_t{:=}(a_t, U_{F,t}, U_{M,t})'$ along with the auxiliary model for $Z_t := (Z_{1,t}, Z_{2,t})' = (\Delta house_t,\ oil_t)'$ which jointly form the system:

$$Y_t \;=\; \Pi_1 Y_{t-1} + \Pi_2 Y_{t-2} + \Pi_3 Y_{t-3} + \Pi_4 Y_{t-4} + \Upsilon_y + u_t \tag{29}$$

$$Z_t \;=\; \Theta_1 Z_{t-1} + \Gamma_1 Y_{t-1} + \Gamma_2 Y_{t-2} + \Upsilon_z + v_{Z,t} \tag{30}$$

where $m = n + r = 5$ and $\ell^{op} = k = 4$. Here the second rows of the matrices Θ_1, Γ_1, Γ_2 and the vector Υ_z (the ones associated with the $Z_{2,t}$-equation) are fixed to zero because $Z_{2,t} \equiv v_{Z_2,t} = oil_t$. The reduced form disturbances $\eta_t := (u_t', v_{Z,t}')'$ of system (29)-(30) are linked to $\xi_t := (\varepsilon_t', \omega_t^{\circ\prime})'$ by the relationship $\eta_t = \widetilde{G}\xi_t$, where the form of the matrix $\widetilde{G}$ is given in eq. (17).

System (29)-(30) is estimated by setting to zero the autoregressive coefficients in Θ_1, Γ_1 and Γ_2 which are not statistically significant at the 5% significance level. A battery of diagnostic tests on the disturbances of the AC-SVAR model are summarized in Panel A of Table 1, where we report equation-wise: a test of normality; a test for the absence of serial autocorrelation; and a test for the absence of conditional heteroskedasticity. To save space, we have not reported the estimated reduced form parameters, including the covariance matrix Σ_η. Panel A of Table 1 shows that aside from the non-normality of the residuals associated with the a_t-equation, the specified model is neither affected by serial autocorrelation, nor by conditional heteroskedasticity in the residuals. As observed in the previous sections, the absence of conditional heteroskedasticity in the residuals is particularly important for bootstrap inference on the IRFs, see below. The last column in Panel A of Table 1 also summarizes F-type tests for the null hypothesis that (four) lags of Z_t are not statistically significant in the Y_t-equations. These Granger causality tests largely support the triangular structure that characterizes the

[20]It is reasonable to assume that $E(v_{Z_1,t}\varepsilon_{2,t}') = 0_{1\times 2}$, while it is not possible to rule out the condition $E(v_{Z_1,t}\varepsilon_{2,t-i}') \neq 0_{1\times 2}$, $i = 1, 2, ...$ (which however does not violate Assumption 2). The orthogonality condition $E(v_{Z_1,t}\varepsilon_{2,t}') = 0_{1\times 2}$ will be tested empirically.

autoregressive parameters of the AC-SVAR model, see eq.s (8)-(10). This result remarks that the only possible connection between the two external instruments in $v_{Z,t} := (v_{Z_1,t}, v_{Z_2,t})'$ and the instrumented structural shock $\varepsilon_{a,t}$ is by the covariance matrix $\Sigma_{v_Z,u} := E(v_{Z,t}u_t')$ which is a key element of system (29)-(30) and of the proxy-SVAR approach. A Wald test for the null hypothesis $H_0' : vec(\Sigma_{v_Z,u}) = 0_{6\times 1}$ (no correlation between the external instruments and the VAR disturbances) against $H_1' : vec(\Sigma_{v_Z,u}) \neq 0_{6\times 1}$, see Section 5 and the Supplementary Appendix A.9, is equal to W_T^{rel}:=8.55 with a p-value of 0.01, hence the reduced form evidence tends to indirectly reject the hypothesis that the relevance parameters are zero at the 5% level of significance.

Partial shocks identification strategy

In the partial shocks identification strategy, we are solely interested the identification and estimation of the parameters $\widetilde{G}_1$:=$(b_a', \varphi')' = \vartheta$, where b_a:=$(b_{a,a}, b_{F,a}, b_{M,a})'$ and $\phi = vec(\Phi) := (\varphi_{1,a}, \varphi_{2,a})'$. Proposition 1 ensures that the model is identified if at least one among $\varphi_{1,a}$ and $\varphi_{2,a}$ is non-zero in the population. Panel B of Table 1, left-side, summarizes the CMD estimates (with analytic standard errors) obtained with the procedure described in Section 5. The estimated on-impact coefficients $b_{F,a}$ and $b_{M,a}$ have negative sign, as expected, but are not statistically significant. The overidentification restrictions test $TQ(\hat{\vartheta}_T)_{exog}$ has p-value 0.54 and strongly supports the hypothesis of 'contemporaneous exogeneity' of financial and macroeconomic uncertainty, $b_{F,a} = 0$ and $b_{M,a} = 0$. The immediate interpretation of this result is that financial and macroeconomic uncertainty do not respond contemporaneously to the identified non-uncertainty shock. We inspect the implied IRFs next.

We evaluate the quality of the identification by checking directly whether the (estimated) innovations in new privately owned housing units started ($\hat{v}_{Z_1,t}$) and the oil supply shock ($v_{Z_2,t}$) are relevant for the real economic activity shock. Panel C of Table 1, left-side, reports the estimated correlations between $\hat{v}_{Z_1,t}$ and $\varepsilon_{a,t}$ (given by given by the ratio $\hat{\varphi}_{1,a}/\hat{\sigma}_{v_{Z_1}}$) and between $v_{Z_2,t}$ and $\varepsilon_{a,t}$ (given by the ratio $\hat{\varphi}_{2,a}/\hat{\sigma}_{v_{Z_2}}$). These are 0.29 and 0.27, respectively, and are both statistically significant at the 5% level of significance.

Full shocks identification strategy

If we add the restriction $b_{F,M} = 0$ in the matrix $\widetilde{G}$ in eq. (17) also the financial and macroeconomic uncertainty shocks can be identified from the AC-SVAR model (despite these shocks are not instrumented). The restriction $b_{F,M} = 0$ involves the submatrix B_2 of B which collects the instantaneous impacts of the non-instrumented structural shocks. We borrow the restriction $b_{F,M} = 0$ from Angelini *et al.* (2019) who investigate the endogeneity/exogeneity of uncertainty by exploiting the breaks in unconditional volatility of a VAR for Y_t:=$(a_t, U_{F,t}, U_{M,t})'$ across three main macroeconomic regimes of U.S. business cycle. Given the two external instruments

$Z_t := (Z_{1,t}, Z_{2,t})' = (\Delta house_t,\ oil_t)'$ and driven by some preliminary evidence, we impose a diagonal structure to the covariance matrix of measurement errors Σ_ω, i.e. we set $\varpi_{2,1} = 0$ in eq. (17). According to Proposition 2(b), with $b_{F,M} = 0$ and $\varpi_{2,1} = 0$ the necessary order condition for identification is satisfied because there are $a_{\widetilde{G}} = 12$ unrestricted (free) elements in $\widetilde{G}$ and $1/2m(m+1) = 15$ covariance restrictions, hence the system features 3 testable overidentification restrictions if also the rank condition in Proposition 2(a) holds. It is possible to show that also the identification rank condition is satisfied. Actually, it can be easily checked that the AC-SVAR model with the matrix $\widetilde{G}$ in eq. (17) subject to $b_{F,M} = 0$ and $\varpi_{2,1} = 0$ satisfies the sufficient conditions for global identification of Theorem 1 in Rubio-Ramirez *et al.* (2010).

Panel B of Table 1, right-side, summarizes the ML estimates (with analytic standard errors) of the parameters in the matrix $\widetilde{G}$, see Section 6. The table also reports the results of two LR tests: one (LR_T) is a test for the 3 overidentification restrictions featured by the estimated AC-SVAR model and the other (LR_{exog}) is a test for the hypothesis of 'contemporaneous exogeneity' of financial and macroeconomic uncertainty, $b_{F,a} = 0$ and $b_{M,a} = 0$. Both tests provide ample empirical support to the estimated proxy-SVAR model (p-value 0.92) and to the hypothesis of 'contemporaneous exogeneity' (p-value 0.53), confirming the finding already obtained with the partial shocks identification strategy. In this framework we can also evaluate the instantaneous impacts of financial and macroeconomic uncertainty shocks on industrial production growth. The estimated coefficient $b_{a,F}$ is positive (0.049) but is not significant at the 5% level of significance, while the estimated coefficient $b_{a,M}$ is negative (-0.313) and significant, meaning that a one standard deviation macroeconomic uncertainty shock leads to an instantaneous decline in industrial production growth.

Panel C of Table 1, right-side, reports the estimated correlations between $\hat{v}_{Z,t} := (\hat{v}_{Z_1,t}, v_{Z_2,t})'$ and $\varepsilon_t := (\varepsilon_{a,t}, \varepsilon_{F,t}, \varepsilon_{M,t})'$. It is seen that the estimated correlations between $\hat{v}_{Z,t}$ and $\hat{\varepsilon}_{a,t}$ are 0.29 and 0.25, respectively, and are both statistically significant at the 5% level of significance. Instead, the estimated correlations between $\hat{v}_{Z,t}$ and $\hat{\varepsilon}_{2,t} := (\varepsilon_{F,t}, \varepsilon_{M,t})'$ are close to zero and not statistically significant at the 5% level.

Dynamic causal effects

The IRFs generated by the estimated AC-SVAR model in the full shocks identification strategy are plotted in Figure 1 with associated 90%-bootstrap simultaneous confidence bands. The IRFs are estimated by imposing the 'contemporaneous exogeneity' restrictions $b_{F,a} = 0$ and $b_{M,a} = 0$, not rejected by formal testing. Since the tests in Panel A of Table 1 rule out the occurrence of conditional heteroskedasticity in the disturbances, the simultaneous 'sup-t' bootstrap confidence bands for the IRFs are computed by combining a nonparametric *iid* resampling scheme for $\hat{\eta}_t := (\hat{u}_t', \hat{v}_{Z,t}')'$ with Algorithm 3 in Montiel Olea and Plagborg-Møller (2019); see the Supplementary Appendix A.10 for details.

The first column of Figure 1 plots the responses of the variables in $Y_{t+h}:=(a_{t+h}, U_{F,t+h}, U_{M,t+h})'$ to one standard deviation non-uncertainty shock $\varepsilon_{a,t}$, $h = 0, 1, ...$ Given the estimates in Panel B of Table 1, the IRFs in the first column of Figure 1 are expected to be numerically similar to the ones computed from a partial shocks identification strategy. It is seen that while macroeconomic uncertainty does not respond significantly at any lag to the identified non-uncertainty shock, financial uncertainty responds after one month, but such response is short-lived and lasts one month. The no response of macroeconomic uncertainty and the very short-lived response of financial uncertainty to the identified non-uncertainty shock in Figure 1 are at odds with Plante *et al.* (2018)'s hypothesis of 'endogenous uncertainty'.

In the full shocks identification framework, we can also track the dynamic responses of $Y_{t+h}:=(a_{t+h}, U_{F,t+h}, U_{M,t+h})'$ to the identified uncertainty shocks $\varepsilon_{2,t} := (\varepsilon_{F,t}, \varepsilon_{M,t})'$, $h = 0, 1, ...$ Given our scopes, these responses are not of strict interest but for completeness we plot them in the second and third columns of Figure 1. The estimated dynamic causal effects confirm a well established fact of the uncertainty literature: financial and macroeconomic uncertainty shocks exert non-negligible contractionary effects on industrial production growth after the Global Financial Crisis, with financial uncertainty fostering greater macroeconomic uncertainty.

The Supplementary Appendix A.12.2 shows that the main results obtained in the paper are confirmed by changing the oil supply shock with innovations taken from an auxiliary model for changes in hours worked, i.e. the instrument (c) in place of (b). Overall, our empirical analyses support the common practice of ordering (financial and macroeconomic) uncertainty first in SVARs, i.e. as 'the most exogenous' variables of the system.

8.3 Comparison with existing works

The empirical results discussed in the previous section allow us to make direct contact with Angelini *et al.* (2019), Carriero *et al.* (2018) and Ludvigson *et al.* (2018).

Angelini *et al.* (2019) identify a small-scale SVAR for $Y_t:=(a_t, U_{F,t}, U_{M,t})'$ on the period 1960M8-2015M4 by applying a novel 'identification-through-heteroskedasticity' method which exploits the changes in the unconditional volatility of macroeconomic variables across the main U.S. macroeconomic regimes. In line with our results, they find that macroeconomic uncertainty can be approximated as an exogenous driver of real economic activity and that financial uncertainty displays a delayed and short-lived response to industrial production shocks.

Carriero *et al.* (2018) use a novel stochastic volatility approach in SVAR models which include measures of macroeconomic and financial uncertainty (one at a time), along with measures of real economic activity. Their empirical evidence is partly consistent with ours: they document that macroeconomic uncertainty is broadly exogenous to business cycle fluctuations but find that financial uncertainty might, at least in part, arise as an endogenous response to some macroeconomic developments. Carriero *et al.* (2018) do not model financial and macroecononomic uncertainty jointly, and this might explain why their findings are not fully consistent with ours.

Ludvigson *et al.* (2018) employ a SVAR for $Y_t{:=}(a_t, U_{F,t}, U_{M,t})'$ on the period 1960M8-2015M4 and apply a novel set-identification strategy which combines sign-restrictions, imposed directly on the structural shocks in correspondence of particular events (event constraints), with the use of external instruments (correlation constraints). They instrument the uncertainty shocks by exploiting a measure of the aggregate stock market return and the log difference in the real price of gold, respectively, and this is one key difference with respect to our identification strategy. The specific events constraints they impose to identify the uncertainty shocks pertain mostly to financial uncertainty: the 1987 stock market crash and the 2007-09 financial crisis. They report that while financial uncertainty can be approximated as an exogenous driver of real economic activity, macroeconomic uncertainty is often an endogenous response to output shocks, and this is another major difference with respect to our empirical findings. Ludvigson *et al.* (2018) also find that, while financial uncertainty shocks are contractionary shocks, macro uncertainty shocks have positive effects on real activity, in line with 'growth options' theories. In Ludvigson *et al.* (2018), the main role of the external instruments is to narrow the identification set constructed with the event constraints: the external instruments need not be orthogonal to the non-instrumented structural shocks but the relevance condition is relaxed to a set of inequality restrictions.

9 Concluding remarks

We have presented a general framework to analyze the identification of proxy-SVARs when $r \geq g$ external instruments are used to identify $1 \leq g < n$ structural shocks of interest. We have discussed 'partial' and 'full' shocks identification strategies and developed novel frequentist estimation methods based on CMD and ML, respectively.

We have applied the suggested proxy-SVAR methodology to analyze whether commonly employed measures of macroeconomic and financial uncertainty respond to non-uncertainty shocks in the U.S., after the Global Financial Crisis. The empirical evidence supports the view that financial and macroeconomic uncertainty can be approximated as exogenous drivers of real economic activity. Our results, however, can not be considered 'final' as they depend on the specific set of external instruments used to identify the non-uncertainty shock.

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Table 1. Estimated baseline AC-SVAR model

Panel A

	N_{JB}	AR_4	$ARCH_4$	$F_T : Z_t \nrightarrow Y_t$
a_t-eq.	318.21[0.00]	2.31[0.68]	6.06[0.19]	1.39[0.22]
U_{Ft}-eq.	0.85[0.50]	1.35[0.85]	2.45[0.65]	0.24[0.98]
U_{Mt}-eq.	2.22[0.23]	0.20[0.99]	3.98[0.41]	0.39[0.92]
Z_{1t}-eq.	0.89[0.50]	0.47[0.98]	8.47[0.06]	-
Z_{2t}-eq.	1.11[0.50]	3.14[0.54]	0.61[0.96]	-

Panel B

Partial Shocks Identification

$$\widehat{\widetilde{G}}_1 = \left(\begin{array}{c} \underset{0.1142}{0.5614} \\ \underset{0.0026}{-0.0058} \\ \underset{0.0019}{0.0008} \\ \hline \underset{0.0111}{0.0245} \\ \underset{0.0700}{0.1470} \end{array}\right)$$

$TQ(\vartheta)_{exog} = 1.23[0.54]$

Full Shocks Identification

$$\widehat{\widetilde{G}} = \left(\begin{array}{ccc|cc} \underset{0.0941}{0.5614} & \underset{0.1502}{0.0493} & \underset{0.1444}{-0.3135} & 0 & 0 \\ \underset{0.0060}{-0.0058} & \underset{0.0022}{0.0215} & 0 & 0 & 0 \\ \underset{0.0027}{0.0008} & \underset{0.0012}{0.0037} & \underset{0.0008}{0.0092} & 0 & 0 \\ \hline \underset{0.0089}{0.0245} & 0 & 0 & \underset{0.0061}{0.0811} & 0 \\ \underset{0.0571}{0.1469} & 0 & 0 & 0 & \underset{0.0394}{0.5224} \end{array}\right)$$

$LR_T = 0.47[0.93]$ $LR_{exog} = 1.27[0.53]$

Panel C

Correlations (relevance)

	$\hat{\varepsilon}_a$
$\hat{\upsilon}_{Z_1}$	0.29[0.01]
$\hat{\upsilon}_{Z_2}$	0.27[0.01]

Correlations (relevance, orthogonality)

	$\hat{\varepsilon}_a$	$\hat{\varepsilon}_F$	$\hat{\varepsilon}_M$
$\hat{\upsilon}_{Z_1}$	0.29[0.01]	−0.03[0.75]	0.05[0.61]
$\hat{\upsilon}_{Z_2}$	0.25[0.02]	−0.12[0.28]	0.04[0.68]

NOTES: Estimated AC-SVAR model for $Y_t := (a_t, U_{Ft}, U_{Mt})'$ and external instrument $Z_t := (\Delta house_t, oil_t)'$, period 2008:M1-2015:M4 (T=88). Panel A: diagnostic tests. 'N_{JB}' is the Jarque-Bera test for the null of Gaussian disturbances.'AR_4' is the LM-type test for the null of absence of residual autocorrelation against the alternative of autocorrelated disturbances up to 4 lags. '$ARCH_4$' is a test for the null of absence of the ARCH-type conditional heteroskedasticity up to 4 lags. '$F_T : Z_t \nrightarrow Y_t$' is a Granger causality F-type test for the null hypothesis that Z_t do not Granger cause the corresponding equation in Y_t. Numbers in brackets are p-values. Panel B: estimates. Left side, CMD estimates of $\widetilde{G}_1$ with associated standard errors, '$TQ(\vartheta)_{exog}$' is the overidentification restriction test for the null $b_{F,a} = 0$ and $b_{M,a} = 0$. Right side, ML estimates of $\widetilde{G}$ with associated standard errors, 'LR_T' is a test for the 3 overidenfication restrictions featured by the estimated model, 'LR_{exog}' is the overidentification test for the null $b_{F,a} = 0$ and $b_{M,a} = 0$. Panel C: ex-post correlations. Left side, ex-post correlations between the structural shock $\hat{\varepsilon}_a$ and the reduced form shocks $\hat{\upsilon}_{Z_1}$ and $\hat{\upsilon}_{Z_2}$ (relevance). Right side, ex-post correlations between the structural shocks $\hat{\varepsilon}_t := (\hat{\varepsilon}_{at}, \hat{\varepsilon}_{Ft}, \hat{\varepsilon}_{Mt})'$ and the reduced form shocks $\hat{\upsilon}_{Z_1}$ and $\hat{\upsilon}_{Z_2}$ (relevance and orthogonality).

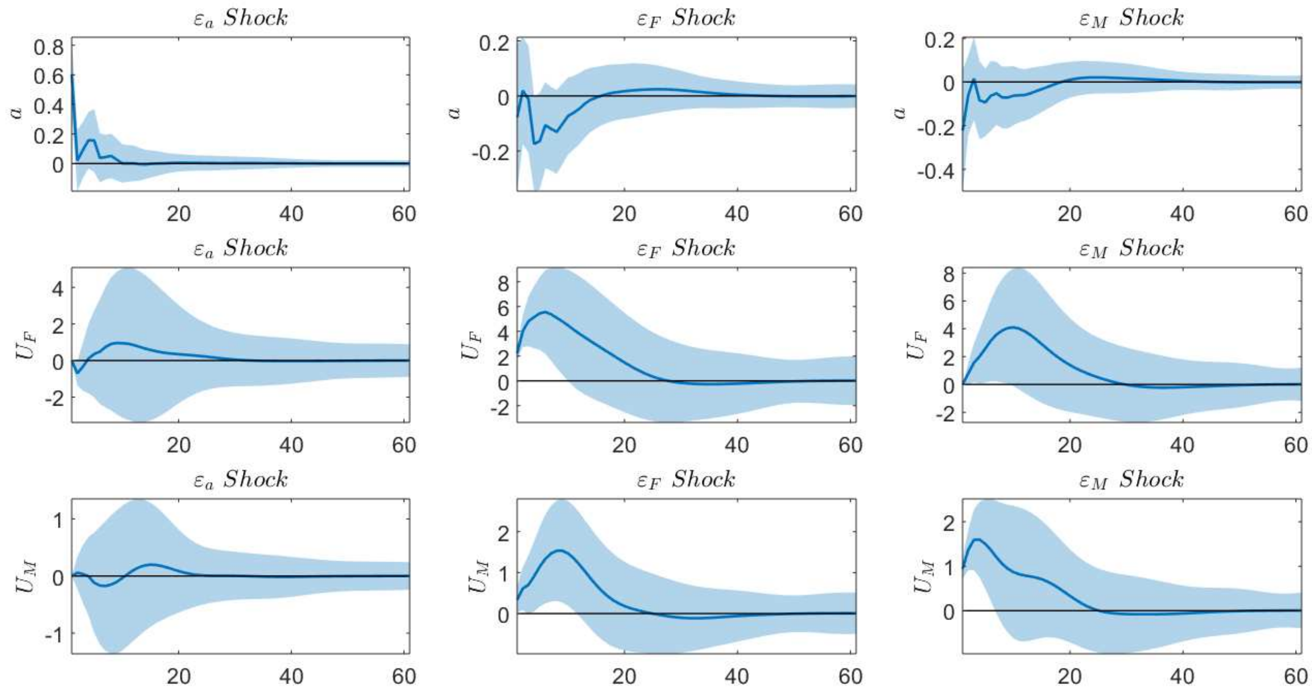

FIGURE 1: *IRFs obtained from the baseline AC-SVAR model for* $Y_t := (a_t, U_{Ft}, U_{Mt})'$ *and external instrument* $Z_t := (\Delta house_t, oil_t)'$, *period 2008:M1-2015:M4 (T=88). Blue shaded areas denote 90%-bootstrap simultaneous 'sup-t' confidence bands (Algorithm 3 in Montiel Olea and Plagborg-Møller, 2019). Responses are measured with respect to one standard deviation changes in the structural shocks. The on-impact coefficients are estimated by imposing the null hypothesis* $b_{F,a} = 0$ *and* $b_{M,a} = 0$ *of exogenous financial and macro uncertainty.*

SUPPLEMENTARY APPENDIX for

Exogenous uncertainty and the identification of Structural Vector Autoregressions with external instruments

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A.1 Introduction

This Supplementary Appendix complements the results of the paper. Section A.2 reviews some standard results on the local and global identification of SVARs which are mentioned and used in the paper. Section A.3 sketches the proofs of Proposition 1 and Proposition 2 in the paper.

Section A.4 revisits the identification of a monetary SVAR model similar to one analyzed in Rubio-Ramirez *et al* (2010) by using external instruments and considering both a partial shocks identification strategy (Section A.4.1) and a full shocks identification strategy (Section A.4.2).

Section A.5 derives the mapping between the reduced form coefficients and the structural parameters at the basis of the partial shocks identification approach formalized in Section 5 of the paper, and then derives the asymptotic distribution of the estimator of the reduced form parameters $\hat{\zeta}_T$ necessary to estimate the proxy-SVAR model by the CMD approach. Section A.6 summarizes the main steps behind the specification of the AC-SVAR model and derives the structural log-likelihood function maximized in Section 6 of the paper in the ML estimation approach.

Section A.7 remarks the formal differences between our approach and Mertens and Raven's (2013) approach when $g > 1$.

Section A.8 presents the results of some Monte Carlo simulations where we investigate the performance of the overidentification restrictions tests when the external instruments are erroneously assumed orthogonal to the non-instrumented shocks; the design of the experiment is

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outlined in Section A.8.1 while the results obtained with the partial and full shocks identification strategies are in Sections A.8.2 and A.8.3, respectively.

Section A.9 discusses a simple Wald test for the null hypothesis that the relevance parameters of the proxy-SVAR model are zero. Section A.10 reviews the bootstrap algorithms used in the paper to compute confidence bands for the IRFs of interest.

Section A.11 analyzes how the estimation procedures presented in Sections 5 and 6 of the paper can be amended to account for an external instrument which is generated by a censored autoregressive process.

Section A.12 complements the empirical results reported in Section 8 of the paper along several directions. Section A.12.1 documents the construction of the supply oil shock used as external instrument in the baseline AC-SVAR model estimated in the paper; Section A.12.2 replicates the empirical analyses in Section 8 of the paper by using as external instruments for the real economic activity shock innovations built on the changes in the log of new privately owned housing units started and the changes in the log of hours worked, respectively; Section A.12.3 compares the results one would obtain with small-scale Choleski-SVARs with those reported in the paper; Section A.12.4 considers the 'one shock-one instrument' scenario ($r = g = 1$) and the case of an invalid (non orthogonal) external instrument with the idea of checking the empirical reliability of our approach; Section A.12.5 checks whether the empirical results obtained with the baseline proxy-SVAR in the paper are robust to the inclusion of Wu and Xia (2016)'s shadow federal funds rate and the inflation rate as control variables in the system.

A.2 Preliminaries: 'standard' identification results

In this section we review some standard results on the local and global identification of SVARs which are mentioned in the paper.

The reference SVAR is in eq. (1) of the paper, which is a B-model' in the terminology of Lütkepohl (2005), or a 'C-model' in the terminology of Amisano and Giannini (1997). As is known, the moment conditions

$$E(u_t u_t') =: \Sigma_u = BB' \tag{A.1}$$

impose $\frac{1}{2}n(n+1)$ restrictions on the n^2 elements of B which leave $\frac{1}{2}n(n-1)$ elements unidentified. Identification can be achieved either by specifying a recursive (triangular) structure for B which places $\frac{1}{2}n(n-1)$ zero restrictions, or by imposing at least $\frac{1}{2}n(n-1)$ linear restrictions of general form which can be written in explicit form as:

$$vec(B){=}S_B\ \theta + s_B \tag{A.2}$$

where S_B is a $n^2 \times a_B$ selection matrix, θ is an $a_B \times 1$ vector containing the 'free' structural parameters in B ($a_B \leq \frac{1}{2}n(n+1)$) and s_B is a $n^2 \times 1$ vector containing known elements. (Note that when $s_B \neq 0_{n^2 \times 1}$there are non-homogeneous restrictions). By complementing the moment conditions in eq. (A.1) with the restrictions in eq. (A.2), the identification can be framed within a 'classical' Rothenberg's (1971) paradigm. In particular, the *necessary and sufficient* rank condition for identification is given by:

$$rank\left\{2D_n^+(\breve{B}_0 \otimes I_n)S_B\right\} = a_B \tag{A.3}$$

where $\breve{B}_0 := \breve{B}(\theta_0)$ is a 'regular' matrix, i.e. whose rank does not change in a neighborhood of θ_0, which depends on θ and satisfies the identification restrictions $vec(\breve{B}_0)$=$S_B\ \theta_0 + s_B$; the *necessary order* condition is $a_B \leq \frac{1}{2}n(n+1)$; see also Hamilton (1994, Ch. 11).

The necessary and sufficient rank condition in eq. (A.3) is a *local* identification condition. Rubio-Ramírez *et al.* (2010) have established *sufficient* rank conditions for *global* identification for cases where the restrictions on B are homogeneous ($s_B = 0_{n^2 \times 1}$) and separable across equations (meaning that cross-restrictions that involve elements in different columns of B are ruled out). Furthermore, Rubio-Ramírez *et al.* (2010) have shown that if their sufficient condition for global identification is satisfied at an arbitrary point in the parameter space, it will be satisfied almost everywhere. More specifically, for a specified matrix $B := \breve{B}$ which incorporates a total of κ zero restrictions, we define the set of admissible restrictions: $\mathcal{R} := \left\{\breve{B},\ Q_j\breve{b}_j = 0_{n\times 1},\ j = 1, ..., n\right\}$ where $\breve{b}_j$ is the j-th column of $\breve{B}$, and Q_j are, for $j = 1, ..., n$, selection matrices (i.e. containing zeros and ones) of dimensions $n \times n$ and $rank(Q_j)$=κ_j, such that $\kappa = \sum_{j=1}^{n} \kappa_j$. Thus, the matrix Q_j selects the zero elements that characterize the column $\breve{b}_j$ for $j = 1, ..., n$. Assume without loss of generality that the n selection matrices in $\mathcal{R}$ are ordered such that $\kappa_1 \geq \kappa_2 \geq ... \geq \kappa_n$, i.e. the columns of $\breve{B}$ which feature a larger number of zero restrictions are ordered first. Then define the n matrices

$$M_j(\breve{B}) := \begin{pmatrix} Q_j\breve{B} \\ (I_j \ \vdots\ 0_{j\times(n-j)}) \end{pmatrix},\ \ j = 1, ..., n \tag{A.4}$$

which have dimensions $(n+j) \times n$, respectively. Theorem 1 in Rubio-Ramírez *et al.* (2010) establishes that:

$$rank\left\{M_j(\breve{B})\right\} = n \ \text{ for } \ j = 1, ..., n \tag{A.5}$$

is a *sufficient rank* condition for the global identification of the SVAR based on $B := \breve{B}$. Furthermore, Theorem 3 in Rubio-Ramírez *et al.* (2010) shows that when the condition in eq. (A.5) is valid for $B := \breve{B}$, it will be satisfied almost everywhere in the set $\mathcal{R}$.

A.3 Proofs

Proof of Proposition 1 (a) The Jacobian matrix in eq. (24) of the paper follows from standard matrix derivative rules, see Magnus and Neudecker (1999). The result follows trivially from the structure of the derived Jacobian matrix. (b) From the definitions in eq. (21) of the paper it follows that $ng - e$ is the number of restrictions which characterize B_1 and $rg - c$ is the number of restrictions which characterize Φ. Since there must be at least $1/2g(g-1)$ restrictions on the columns of $\widetilde{G}_1$ in eq. (18) of the paper, it follow that $(ng - e) + (rg - c) \geq 1/2g(g-1)$.■

Proof of Proposition 2 (a) The proof follows from a straightforward application of the results in eq.s (A.2)-(A.3) to the matrix

$$\widetilde{G} := \left(\begin{array}{c:c} B & 0_{n\times g} \\ \hdashline R_\Phi & \Sigma_\omega^{1/2} \end{array}\right) = \left(\begin{array}{c:c:c} B_1 & B_2 & 0_{n\times r} \\ \hdashline \Phi & 0_{r\times(n-g)} & \Sigma_\omega^{1/2} \end{array}\right).$$

In particular, define the $m^2 \times 1$ vector $\gamma := (vec(B_1)', vec(B_2)', vec(\Phi)', vec(0_{r\times(n-g)})', vec(0_{n\times r})', vec(\Sigma_\omega^{1/2})')'$ which contains the same information as $vec(\widetilde{G})$ but in a different order. The linear relationship between γ and $\theta := (\alpha_1', \alpha_2', \varphi', \varpi')'$ is given by:

$$\gamma = \begin{pmatrix} S_{B_1} & 0_{ng\times e_2} & 0_{ng\times c} & 0_{ng\times s_\omega} \\ 0_{n(n-g)\times e_1} & S_{B_2} & 0_{n(n-g)\times c} & 0_{n(n-g)\times s_\omega} \\ 0_{rg\times e_1} & 0_{rg\times e_2} & S_\Phi & 0_{rg\times s_\omega} \\ 0_{r(n-g)\times e_1} & 0_{r(n-g)\times e_2} & 0_{r(n-g)\times c} & 0_{r(n-g)\times s_\omega} \\ 0_{nr\times e_1} & 0_{nr\times e_2} & 0_{nr\times c} & 0_{nr\times s_\omega} \\ 0_{r^2\times e_1} & 0_{r^2\times e_2} & 0_{r^2\times c} & S_{\Sigma_\omega} \end{pmatrix} \theta + \mu \qquad \text{(A.6)}$$

which we simplify by the expression $\gamma = \Omega\theta + \mu$, with Ω given as above and μ contains known elements by which we can impose non-homogeneous restrictions. Then, introduce the $m^2 \times m^2$ permutation matrix $\mathcal{P}$ such that:

$$vec(\widetilde{G}) \ = \mathcal{P}\gamma,$$

i.e. the matrix $\mathcal{P}$ applied to γ returns $vec(\widetilde{G})$. By pre-multiplying both sides of eq. (A.6) by $\mathcal{P}$, yields the linear restrictions

$$vec(\widetilde{G}) = \mathcal{P}\gamma = \mathcal{P}\Omega\theta + \mathcal{P}\mu = S_{\widetilde{G}}\theta + s_{\widetilde{G}}$$

which show that the selection matrix $S_{\widetilde{G}}$ is a permutation of Ω, $S_{\widetilde{G}} := \mathcal{P}\Omega$, and $s_{\widetilde{G}}$ is a permutation of μ, $s_{\widetilde{G}} := \mathcal{P}\mu$. The result is thus obtained.

(b). The necessary order condition follows from the fact that the matrix $D_m^+(\widetilde{G}_0 \otimes I_m)S_{\widetilde{G}}$ is $\frac{1}{2}m(m+1)\times\ a_{\widetilde{G}}$. ■

A.4 The identification of a monetary SVAR by external instruments

In this section we implement the identification approach discussed in the paper to the case of monetary SVAR analyzed in other studies, including Rubio-Ramirez *et al.* (2010, Section 5.2). The SVAR model is based on the following specification of the matrix B:[1]

$$
B := \begin{array}{cc} & \begin{array}{ccc:cc} \text{MP} & \text{MD} & \text{Inf} & \text{PS}_1 & \text{PS}_2 \end{array} \\ \begin{array}{c} R \\ \log M \\ \log Y \\ \log P \\ \log P_c \end{array} & \left(\begin{array}{ccc:cc} b_{11} & b_{12} & b_{13} & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & 0 & b_{45} \\ b_{51} & 0 & b_{53} & 0 & 0 \end{array}\right) \end{array} \tag{A.7}
$$

where $b_{31} = b_{41} = b_{51} = 0$ in the original specification. The five variables ($n = 5$) are the nominal short-term interest rate (R), log M3 ($\log M$), log gross domestic product ($\log Y$), log GDP deflator ($\log P$), and log of commodity prices ($\log P_c$). The first column of B (MP) captures a central bank's contemporaneous behaviour, the second column (MD) describes a money demand equation and the third column (Inf) the commodity (information) market; the last two columns (PS_1, PS_2) pertain to the production sector.

Rubio-Ramirez *et al.* (2010) show that the so-defined SVAR (with $b_{31} = b_{41} = b_{51} = 0$) is globally identified. The zero constraints $b_{31} = b_{41} = b_{51} = 0$ maintain that $\log Y$, $\log P$ and $\log P_c$ does not respond on-impact (i.e. within the month) to the monetary policy shock, and have been disputed in the recent empirical monetary policy literature, see e.g. Ramey (2016, Section 3.3.1) and references therein for a review. If $b_{31} \neq 0, b_{41} \neq 0$, $b_{51} \neq 0$, the SVAR is no longer identified (indeed there are 8 restrictions while at least $10{=}1/2n(n-1)$ restrictions are needed).

Suppose that we analyze the more general setup based on $b_{31} \neq 0, b_{41} \neq 0$, $b_{51} \neq 0$, and that the scope of the analysis is to identify three structural shocks of interest ($g = 3$): the monetary policy shock ($\varepsilon_{MP,t}$), the money demand shock ($\varepsilon_{MD,t}$) and the commodity market shock ($\varepsilon_{Inf,t}$), hence $\varepsilon_{1,t} := (\varepsilon_{MP,t}, \varepsilon_{MD,t}, \varepsilon_{Inf,t})'$. This amounts to identify the three columns in $B_1 :=$ (MP, MD, Inf), while the columns of $B_2 := (\text{PS}_1, \text{PS}_2)$, i.e. the shocks to the production sector, are not of interest. The SVAR with B in eq. (A.7) is not identified but the proxy-SVAR approach may help to solve the problem. More specifically, it is in principle possible to identify B_1 (with $b_{31} \neq 0, b_{41} \neq 0$, $b_{51} \neq 0$) conditionally on the existence of at least three valid external

[1] Compared to Rubio-Ramirez *et al.* (2010) we order the variables differently; moreover, while their parameterization is based on the 'A-model' $Au_t = \varepsilon_t$, we use a 'B-model' formuation.

instruments which are correlated with the shocks of interest $\varepsilon_{1,t} := (\varepsilon_{MP,t}, \varepsilon_{MD,t}, \varepsilon_{Inf,t})'$ but are orthogonal to $\varepsilon_{2,t} := (\varepsilon_{PS_1,t}, \varepsilon_{PS_2,t})'$.

To simplify the exposition (and to save space), we consider the case $r = g = 3$, and assume that there exist three hypothetical external instruments $Z_t := (Z_{1,t}, Z_{2,t}, Z_{3,t})'$ whose innovation components $v_{Z,t} := (v_{Z1,t}, v_{Z2,t}, v_{Z3,t})'$ (see eq.s (6)-(7) in the paper) satisfy the conditions:

$$E\left\{\begin{pmatrix} v_{Z1,t} \\ v_{Z2,t} \\ v_{Z3,t} \end{pmatrix}\left(\varepsilon_{1,t}', \varepsilon_{2,t}'\right)\right\} := \left(\begin{pmatrix} \varphi_{11} & \varphi_{12} & 0 \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ 0 & \varphi_{32} & \varphi_{33} \end{pmatrix} \begin{array}{c:c} & \end{array} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = \left(\Phi \;\vdots\; 0_{3\times 2} \right) = R_{\Phi}. \tag{A.8}$$

In eq. (A.8) $v_{Z1,t}$ is correlated with $\varepsilon_{MP,t}$ and $\varepsilon_{MD,t}$ but not with $\varepsilon_{Inf,t}$ and the other (non-instrumented) shocks; $v_{Z2,t}$ is correlated with all three structural shocks of interest, $\varepsilon_{MP,t}$, $\varepsilon_{MD,t}$ and $\varepsilon_{Inf,t}$ and is orthogonal to the non-instrumented shocks; finally, $v_{Z3,t}$ is assumed to be correlated with the money demand shock $\varepsilon_{MD,t}$ and the commodity market shock $\varepsilon_{Inf,t}$, but is orthogonal to $\varepsilon_{MP,t}$ and the other (non-instrumented) shocks.

In this example, we have $n = 5$ variables, $g = 3$ structural shocks of interest and $r = g$ external instruments, hence the AC-SVAR model is 8-dimensional, i.e. $m = n + r = 8$. The matrix $\widetilde{G}$ in eq. (11) of the paper boils down to:

$$\widetilde{G} := \left(\begin{array}{c:c:c} B_1 & B_2 & 0_{5\times 3} \\ \hdashline \Phi & 0_{3\times 2} & \Sigma_{\omega}^{1/2} \end{array}\right) = \left(\begin{array}{ccc:cc:ccc} \text{MP} & \text{MD} & \text{Inf} & \text{PS}_1 & \text{PS}_2 & - & - & - \\ b_{11} & b_{12} & b_{13} & 0 & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & 0 & 0 & 0 \\ b_{41} & b_{42} & b_{43} & 0 & b_{45} & 0 & 0 & 0 \\ b_{51} & 0 & b_{53} & 0 & 0 & 0 & 0 & 0 \\ \hdashline \varphi_{11} & \varphi_{12} & 0 & 0 & 0 & \varpi_{11} & \varpi_{21} & \varpi_{31} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & 0 & 0 & \varpi_{21} & \varpi_{22} & \varpi_{32} \\ 0 & \varphi_{32} & \varphi_{33} & 0 & 0 & \varpi_{31} & \varpi_{32} & \varpi_{33} \end{array}\right) \tag{A.9}$$

where $\varpi_{i,l}$ are the elements of $\Sigma_{\omega}^{1/2}$, hence pertain to the covariance matrix of the measurement errors. The matrix $\widetilde{G}$ is at the basis of the two types of identification strategy one can implement within the AC-SVAR model. In Section A.4.1 we use the external instruments $v_{Z,t} := (v_{Z1,t}, v_{Z2,t}, v_{Z3,t})'$ to identify $\varepsilon_{1,t} := (\varepsilon_{MP,t}, \varepsilon_{MD,t}, \varepsilon_{Inf,t})'$ in a partial shocks identification strategy, while in Section A.4.2 we study the circumstances under which it is possible to identify $\varepsilon_{1,t} := (\varepsilon_{MP,t}, \varepsilon_{MD,t}, \varepsilon_{Inf,t})'$ jointly with $\varepsilon_{2,t} := (\varepsilon_{PS_1,t}, \varepsilon_{PS_2,t})'$ by the external instruments $v_{Z,t} := (v_{Z1,t}, v_{Z2,t}, v_{Z3,t})'$.

A.4.1 Partial shocks identification strategy

The focus is on eq.s (A.7)-(A.8). The objective is the identification of the three shocks of interest in $\varepsilon_{1,t} := (\varepsilon_{MP,t}, \varepsilon_{MD,t}, \varepsilon_{Inf,t})'$ for which three valid instruments are used. The on-impact effects of the instrumented shocks are captured by the first three columns of $\widetilde{G}$ in eq. (A.9), collected in $\widetilde{G}_1$:

$$\widetilde{G}_1 := \begin{pmatrix} B_1 \\ \hdashline \Phi \end{pmatrix} := \begin{array}{c} \begin{array}{ccc} \text{MP} & \text{MD} & \text{Inf} \end{array} \\ \left(\begin{array}{ccc} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \\ b_{51} & 0 & b_{53} \\ \hdashline \varphi_{11} & \varphi_{12} & 0 \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ 0 & \varphi_{32} & \varphi_{33} \end{array}\right) \end{array}.$$

Recall that $\widetilde{G}_1$ is part of the matrix $\widetilde{G} := \left(\widetilde{G}_1 \;\vdots\; \widetilde{G}_2 \right)$ (see eq.(11) of the paper), where $\widetilde{G}_2 :=$ (PS_1,PS_2) collects the production section of the system which is not of interest.

The order condition of Proposition 1(b) of the paper is respected as $\widetilde{G}_1$ incorporates $1/2g(g-1) = 3$ restrictions; alternatively, $e_1 = 13$ is number of unrestricted elements in $vec(B_1)$ and $c = 7$ is number of unrestricted relevance parameters in $vec(\Phi)$, hence $e_1{+}c = 21{=}g(n{+}r){-}1/2g(g{-}1)$. Since the restrictions that characterize the columns of $\widetilde{G}_1$ are homogeneous and separable across columns, we skip for the moment the check for the necessary and sufficient rank condition of Proposition 1(a) of the paper, which require computing the Jacobian matrix in eq. (24), and rather check wether the sufficient conditions for global identification in partially identified models reported in Theorem 2 of Rubio-Ramirez *et al.* (2010) are satisfied.

The zero restrictions on the columns of $\widetilde{G}_1$ are:

$$\kappa_1 = 1 \geq \kappa_2 = 1 \geq \kappa_3 = 1$$

for a total of $\kappa = \kappa_1 + \kappa_2 + \kappa_3 = 3$ zero restrictions, equal in this case to $\frac{1}{2}g(g-1)$. Following Rubio-Ramirez *et al.* (2010), we derive the matrices Q_1, Q_2 and Q_3 that satisfy the conditions

$rank(Q_i) = \kappa_i$, $Q_i \tilde{g}_i = 0_{m\times 1}$, $i = 1, 2, 3$, where $\tilde{g}_{1,i}$, $i = 1, 2, 3$, are the columns of $\widetilde{G}_1$. One has:

$$Q_1 := \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \quad Q_1\widetilde{G}_1 = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&\varphi_{32}&\varphi_{33} \end{pmatrix};$$

$$Q_2 := \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \quad Q_2\widetilde{G}_1 = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ b_{51}&0&b_{53}\\ 0&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix};$$

$$Q_3 := \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \quad Q_3\widetilde{G}_1 = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ \varphi_{11}&\varphi_{12}&0\\ 0&0&0\\ 0&0&0 \end{pmatrix};$$

To evaluate the sufficient condition of Theorem 2 of Rubio-Ramirez *et al.* (2010) we fill the rank matrices $M_i(\widetilde{G}_1)$, $i = 1, ..., 3$ as:

$$M_1(\widetilde{G}_1) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \varphi_{32} & \varphi_{33} \\ \hdashline 1 & 0 & 0 \end{pmatrix} \text{ which has rank 3 for } \varphi_{32} \neq \varphi_{33};$$

$$M_2(\widetilde{G}_1) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{51} & 0 & b_{53} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ which has rank 3 if } b_{53} \neq 0$$

$$M_3(\widetilde{G}_1) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varphi_{11} & \varphi_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ which has rank 3.}$$

According to Theorems 2 and 3 in Rubio-Ramírez *et al.* (2010), the three columns in $\widetilde{G}_1$ are globally identified for almost all structural parameters.

A.4.2 Full shocks identification strategy

The scope of the analysis is still to identify the three structural shocks in $\varepsilon_{1,t} := (\varepsilon_{MP,t}, \varepsilon_{MD,t}, \varepsilon_{Inf,t})'$ by using the $r = g = 3$ external instruments in $v_{Z,t} := (v_{Z1,t}, v_{Z2,t}, v_{Z3,t})'$. The novelty is that we now analyze the conditions which permit the identification also of the shocks in $\varepsilon_{2,t} := (\varepsilon_{PS_1,t}, \varepsilon_{PS_2,t})'$. To fully appreciate the advantages of the full shocks approach, we relax all zero restrictions which characterize the production sector $B_2 :=$ (PS$_1$, PS$_2$) in eq. (A.7), except the constraint $b_{15} = 0$. Moreoer, we assume that the covariance matrix of measurement errors, $\Sigma_\omega := E(\omega_t \omega_t')$, is diagonal, so that also the matrix $\Sigma_\omega^{1/2}$ has a diagonal form. With these changes, the matrix $\widetilde{G}$ in eq. (A.9) becomes:

$$\widetilde{G} := \left(\begin{array}{c|c|c} B_1 & B_2 & 0_{5\times 3} \\ \hline \Phi & 0_{3\times 2} & \Sigma_\omega^{1/2} \end{array}\right) = \begin{array}{c} \begin{array}{ccc|cc|ccc} \text{MP} & \text{MD} & \text{Inf} & \text{PS}_1 & \text{PS}_2 & \text{-} & \text{-} & \text{-} \end{array} \\ \left(\begin{array}{ccc|cc|ccc} b_{11} & b_{12} & b_{13} & b_{14} & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & 0 & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & 0 & 0 & 0 \\ b_{51} & 0 & b_{53} & b_{54} & b_{55} & 0 & 0 & 0 \\ \hline \varphi_{11} & \varphi_{12} & 0 & 0 & 0 & \omega_{11} & 0 & 0 \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & 0 & 0 & 0 & \omega_{22} & 0 \\ 0 & \varphi_{32} & \varphi_{33} & 0 & 0 & 0 & 0 & \omega_{33} \end{array}\right) \end{array}. \qquad \text{(A.10)}$$

As we know, in this example $B := \left(B_1 \,\vdots\, B_2 \right)$ is not identified from the n-dimensional SVAR for Y_t. However, if we consider the m-dimensional AC-SVAR model based on $W_t := (Y_t', Z_t')'$, $m = n + r = 8$, and the matrix $\widetilde{G}$ in eq. (A.10), we observe that the model features 31 zero restrictions, three more than the $1/2m(m-1) = 28$ restrictions necessary to achieve identification. Thus, provided a rank condition holds, the matrix $\widetilde{G}$ matrix in eq. (A.10) is identified.

Again, in this model the restrictions that characterize the columns of $\widetilde{G}$ in eq. (A.10) are separable across columns and homogeneous which suggests that it might be convenient to check whether the sufficient conditions for global identification in Rubio-Ramírez *et al.* (2010) are satisfied.[2]

[2] It is possible to prove that regardless of whether Σ_ω is diagonal or 'full', the AC-SVAR model based on the matrix $\widetilde{G}$ in eq. (A.10) satisfies the necessary and sufficient conditions rank condition of Proposition 2(a) of the paper.

Let

$$\widetilde{G}^{*} := \begin{pmatrix} 0 & 0 & 0 & 0 & b_{14} & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{25} & b_{24} & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{35} & b_{34} & b_{31} & b_{32} & b_{33} \\ 0 & 0 & 0 & b_{45} & b_{44} & b_{41} & b_{42} & b_{43} \\ 0 & 0 & 0 & b_{55} & b_{54} & b_{51} & 0 & b_{53} \\ \omega_{11} & 0 & 0 & 0 & 0 & \varphi_{11} & \varphi_{12} & 0 \\ 0 & \omega_{22} & 0 & 0 & 0 & \varphi_{21} & \varphi_{22} & \varphi_{23} \\ 0 & 0 & \omega_{33} & 0 & 0 & 0 & \varphi_{32} & \varphi_{33} \end{pmatrix} \tag{A.11}$$

be the counterpart of $\widetilde{G}$ obtained by ordering the columns such that the ones with a larger number of zero restrictions come first (formally, $\widetilde{G}^{*} := \mathcal{P}\widetilde{G}$, $\mathcal{P}$ being a permutation matrix). It is seen that the zero restrictions on the $m = 8$ columns of $\widetilde{G}^{*}$ are:

$$\kappa_1 = 7 \geq \kappa_2 = 7 \geq \kappa_3 = 7 > \kappa_4 = 4 > \kappa_5 = 3 > \kappa_6 = 1 \geq \kappa_7 = 1 \geq \kappa_8 = 1$$

for a total of $\kappa = \kappa_1 + ... + \kappa_8 = 31$ zero restrictions, more than the $\frac{1}{2}m(m-1)$=28 restrictions required to exactly identify the system. It turns out that provided a rank condition is also satisfied, the model is overidentified and features 3 testable over-identification restrictions (which arise from the postulated diagonal form of Σ_ω). We now establish that the model based on the $\widetilde{G}^{*}$ satisfies the conditions in Theorem 1 in Rubio-Ramírez *et al.* (2010).

First, we derive the restrictions matrices Q_1,...,Q_8 that satisfy the conditions $rank(Q_i) = \kappa_i$, $Q_i\tilde{g}_i^{*} = 0_{8\times 1}$, $i = 1, ..., 8$, where $\tilde{g}_i^{*}$ are the columns of $\widetilde{G}^{*}$, see eq.s (A.4)-(A.5). One has:

$$Q_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad Q_1\widetilde{G}^{*} = \begin{pmatrix} 0 & 0 & 0 & 0 & b_{14} & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{25} & b_{24} & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{35} & b_{34} & b_{31} & b_{32} & b_{33} \\ 0 & 0 & 0 & b_{45} & b_{44} & b_{41} & b_{42} & b_{43} \\ 0 & 0 & 0 & b_{55} & b_{54} & b_{51} & 0 & b_{53} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_{22} & 0 & 0 & 0 & \varphi_{21} & \varphi_{22} & \varphi_{23} \\ 0 & 0 & \omega_{33} & 0 & 0 & 0 & \varphi_{32} & \varphi_{33} \end{pmatrix}$$

$$Q_2 := \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \quad Q_2\widetilde{G}^* = \begin{pmatrix} 0&0&0&0&b_{14}&b_{11}&b_{12}&b_{13}\\ 0&0&0&b_{25}&b_{24}&b_{21}&b_{22}&b_{23}\\ 0&0&0&b_{35}&b_{34}&b_{31}&b_{32}&b_{33}\\ 0&0&0&b_{45}&b_{44}&b_{41}&b_{42}&b_{43}\\ 0&0&0&b_{55}&b_{54}&b_{51}&0&b_{53}\\ \omega_{11}&0&0&0&0&\varphi_{11}&\varphi_{12}&0\\ 0&0&0&0&0&0&0&0\\ 0&0&\omega_{33}&0&0&0&\varphi_{32}&\varphi_{33} \end{pmatrix};$$

$$Q_3 := \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \quad Q_3\widetilde{G}^* = \begin{pmatrix} 0&0&0&0&b_{14}&b_{11}&b_{12}&b_{13}\\ 0&0&0&b_{25}&b_{24}&b_{21}&b_{22}&b_{23}\\ 0&0&0&b_{35}&b_{34}&b_{31}&b_{32}&b_{33}\\ 0&0&0&b_{45}&b_{44}&b_{41}&b_{42}&b_{43}\\ 0&0&0&b_{55}&b_{54}&b_{51}&0&b_{53}\\ \omega_{11}&0&0&0&0&\varphi_{11}&\varphi_{12}&0\\ 0&\omega_{22}&0&0&0&\varphi_{21}&\varphi_{22}&\varphi_{23}\\ 0&0&0&0&0&0&0&0 \end{pmatrix};$$

$$Q_4 := \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \quad Q_4\widetilde{G}^* = \begin{pmatrix} 0&0&0&0&b_{14}&b_{11}&b_{12}&b_{13}\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \omega_{11}&0&0&0&0&\varphi_{11}&\varphi_{12}&0\\ 0&\omega_{22}&0&0&0&\varphi_{21}&\varphi_{22}&\varphi_{23}\\ 0&0&\omega_{33}&0&0&0&\varphi_{32}&\varphi_{33} \end{pmatrix};$$

$$Q_5 \; : \; = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \quad Q_5\widetilde{G}^* = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \omega_{11}&0&0&0&0&\varphi_{11}&\varphi_{12}&0\\ 0&\omega_{22}&0&0&0&\varphi_{21}&\varphi_{22}&\varphi_{23}\\ 0&0&\omega_{33}&0&0&0&\varphi_{32}&\varphi_{33} \end{pmatrix};$$

$$Q_6 \; : \; = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \; Q_6\widetilde{G}^* := \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&\omega_{33}&0&0&0&\varphi_{32}&\varphi_{33} \end{pmatrix}$$

$$Q_7 \; : \; = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \quad Q_7\widetilde{G}^* = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&b_{55}&b_{54}&b_{51}&0&b_{53}\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$Q_8 \; : \; = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \quad Q_8\widetilde{G}^* = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \omega_{11}&0&0&0&0&\varphi_{11}&\varphi_{12}&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

Secondly, we evaluate the sufficient condition in eq. (A.5) by filling the rank matrices $M_i(\widetilde{G}^*)$,

$i = 1, ..., 8$ as:

$$M_1(\widetilde{G}^*) := \left(\begin{array}{c:ccccccc} 0 & 0 & 0 & 0 & b_{14} & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{25} & b_{24} & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{35} & b_{34} & b_{31} & b_{32} & b_{33} \\ 0 & 0 & 0 & b_{45} & b_{44} & b_{41} & b_{42} & b_{43} \\ 0 & 0 & 0 & b_{55} & b_{54} & b_{51} & 0 & b_{53} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_{22} & 0 & 0 & 0 & \varphi_{21} & \varphi_{22} & \varphi_{23} \\ 0 & 0 & \omega_{33} & 0 & 0 & 0 & \varphi_{32} & \varphi_{33} \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \text{ which has rank 8 for } \omega_{22} \neq 0 \text{ and } \omega_{33} \neq 0;$$

$$M_2(\widetilde{G}^*) := \left(\begin{array}{cc:cccccc} 0 & 0 & 0 & 0 & b_{14} & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{25} & b_{24} & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{35} & b_{34} & b_{31} & b_{32} & b_{33} \\ 0 & 0 & 0 & b_{45} & b_{44} & b_{41} & b_{42} & b_{43} \\ 0 & 0 & 0 & b_{55} & b_{54} & b_{51} & 0 & b_{53} \\ \omega_{11} & 0 & 0 & 0 & 0 & \varphi_{11} & \varphi_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 0 & 0 & 0 & \varphi_{32} & \varphi_{33} \\ \hdashline \multicolumn{2}{c:}{I_2} & \multicolumn{6}{c}{0_{2\times 6}} \end{array}\right) \text{ which has rank 8 for } \omega_{33} \neq 0;$$

$$M_3(\widetilde{G}^*) := \left(\begin{array}{ccc:ccccc} 0 & 0 & 0 & 0 & b_{14} & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{25} & b_{24} & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{35} & b_{34} & b_{31} & b_{32} & b_{33} \\ 0 & 0 & 0 & b_{45} & b_{44} & b_{41} & b_{42} & b_{43} \\ 0 & 0 & 0 & b_{55} & b_{54} & b_{51} & 0 & b_{53} \\ \omega_{11} & 0 & 0 & 0 & 0 & \varphi_{11} & \varphi_{12} & 0 \\ 0 & \omega_{22} & 0 & 0 & 0 & \varphi_{21} & \varphi_{22} & \varphi_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline \multicolumn{3}{c:}{I_3} & \multicolumn{5}{c}{0_{3\times 5}} \end{array}\right) \text{ which has rank 8;}$$

$$M_4(\widetilde{G}^*) := \left(\begin{array}{cccc:cccc}
0 & 0 & 0 & 0 & b_{14} & b_{11} & b_{12} & b_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\omega_{11} & 0 & 0 & 0 & 0 & \varphi_{11} & \varphi_{12} & 0 \\
0 & \omega_{22} & 0 & 0 & 0 & \varphi_{21} & \varphi_{22} & \varphi_{23} \\
0 & 0 & \omega_{33} & 0 & 0 & 0 & \varphi_{32} & \varphi_{33} \\
\hdashline
\multicolumn{4}{c:}{I_4} & \multicolumn{4}{c}{0_{4\times 4}}
\end{array}\right) \text{ which has rank 8 for } b_{14} \neq 0;$$

$$M_5(\widetilde{G}^*) := \left(\begin{array}{ccccc:ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\omega_{11} & 0 & 0 & 0 & 0 & \varphi_{11} & \varphi_{12} & 0 \\
0 & \omega_{22} & 0 & 0 & 0 & \varphi_{21} & \varphi_{22} & \varphi_{23} \\
0 & 0 & \omega_{33} & 0 & 0 & 0 & \varphi_{32} & \varphi_{33} \\
\hdashline
\multicolumn{5}{c:}{I_5} & \multicolumn{3}{c}{0_{5\times 3}}
\end{array}\right) \text{ which has rank 8 for } \varphi_{11} \neq 0, \varphi_{21} \neq 0;$$

$$M_6(\widetilde{G}^*) := \left(\begin{array}{cccccc:cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \omega_{33} & 0 & 0 & 0 & \varphi_{32} & \varphi_{33} \\
\hdashline
\multicolumn{6}{c:}{I_6} & \multicolumn{2}{c}{0_{6\times 2}}
\end{array}\right) \text{ which has rank 8 for } \varphi_{32} \neq 0, \varphi_{33} \neq 0;$$

$$
M_7(\widetilde{G}^*) := \left(\begin{array}{ccccccc:c}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b_{55} & b_{54} & b_{51} & 0 & b_{53} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
\multicolumn{7}{c:}{I_7} & 0_{7\times 1}
\end{array}\right) \text{ which has rank 8 for } b_{53} \neq 0;
$$

$$
M_8(\widetilde{G}^*) := \left(\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\omega_{11} & 0 & 0 & 0 & 0 & \varphi_{11} & \varphi_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\multicolumn{8}{c}{I_8}
\end{array}\right) \text{ which has rank 8.}
$$

According to Theorems 1 and 3 in Rubio-Ramírez *et al.* (2010), the AC-SVAR model is globally identified for almost all structural parameters.

A.5 The mapping between the reduced form and structural parameters in the partial shocks identification approach and the asymptotic distribution of $\hat{\zeta}_T$

In this section we focus on the mapping in eq.s (12)-(13) of the paper which links the reduced form parameters of the AC-SVAR model to the parameters in the matrices B_1 and Φ.

One difficulty with the moment conditions in eq. (12) of the paper is that the covariance restrictions:

$$\Sigma_u = BB' = B_1B_1' + B_2B_2'$$

involve also the parameters in B_2 which are not of interest and must be marginalized out. To get rid of B_2 we use the nonsingularity of $\Sigma_u = BB'$ (Assumption 1 in the paper) and the transformation:

$$\Sigma_{v_Z,u}\Sigma_u^{-1}\Sigma_{u,v_Z} = \Phi B_1'(BB')^{-1}B_1\Phi'$$

$$= \Phi B_1'(B')^{-1}B^{-1}B_1\Phi' = \Phi B_1'(B')^{-1}B^{-1}B_1\Phi' = \Phi\Phi'. \tag{A.12}$$

Equation (A.12) has been derived by using the fact that the condition $B^{-1}B = I_n = (B^{-1}B_1 : B^{-1}B_2)$ implies the restrictions:

$$B^{-1}B_1 = \begin{pmatrix} I_g \\ 0_{(n-g)\times g} \end{pmatrix} \quad , \quad B^{-1}B_2 = \begin{pmatrix} 0_{g\times(n-g)} \\ I_{(n-g)} \end{pmatrix}.$$

Equation (A.12) is then re-written as:

$$\Xi = \Phi\Phi' \tag{A.13}$$

where $\Xi := \Sigma_{v_Z,u}\Sigma_u^{-1}\Sigma_{u,v_Z}$ is an $r \times r$ symmetric matrix (of rank g) which is positive definite when $r = g$, and is positive semidefinite when $r > g$. Reporting for convenience eq. (12) of the paper here:

$$\Sigma_{v_Z,u} = \Phi B_1', \tag{A.14}$$

we have a new set of moment conditions which involve the elements in B_1 and Φ alone. The moment conditions in eq.s (A.13)-(A.14) correspond to eq. (19) in the paper, and form the basis of the results established in Proposition 1.

Next, we prove the result in eq. (22) of the paper. Let $\sigma_{\eta,0}^+$ be the true value of $\sigma_\eta^+ := vech(\Sigma_\eta)$ and $\hat{\sigma}_{\eta,T}^+$ the corresponding ML estimator obtained from the AC-SVAR model (see Section A.6). Under Assumption 1 and for $T \to \infty$:

$$T^{1/2}(\hat{\sigma}_{\eta,T}^+ - \sigma_{\eta,0}^+) \to_d N(0_{1/2m(m+1)\times 1}, \Omega_\eta) \quad , \quad \Omega_\eta := 2D_m^+(\Sigma_\eta \otimes \Sigma_\eta)(D_m^+)' \tag{A.15}$$

where $D_m^+ := (D_m'D_m)^{-1}D_m'$ is the Moore-Penrose inverse of D_m, D_m is the $m^2 \times \frac{1}{2}m(m+1)$ *duplication matrix,* see Magnus and Neudecker (1999), and the symbol '$\to_d$' denotes convergence in distribution. The asymptotic covariance matrix Ω_η can be estimated consistently by $\widehat{\Omega}_{\eta,T} := 2D_m^+(\widehat{\Sigma}_{\eta,T} \otimes \widehat{\Sigma}_{\eta,T})(D_m^+)'$ and $\widehat{\Sigma}_{\eta,T} := (1/T)\sum_{t=1}^T \hat{\eta}_t\hat{\eta}_t'$. The convergence in eq. (A.15) is valid regardless of Assumption 2, i.e. the matrix of relevance parameters Φ might be also zero in eq. (6) of the paper.

Now, define now the $1/2m(m+1) \times 1$ vector $\lambda := (vech(\Sigma_u)', vec(\Sigma_{v_Z,u})', vech(\Sigma_{v_Z})')'$, which contains the same elements as the vector σ_η^+ but disposed in different order. Formally, $\lambda = \mathcal{P}_\sigma\sigma_\eta^+$, where $\mathcal{P}_\sigma$ is a permutation matrix. Given eq. (A.15) and Assumption 1 we have:

$$T^{1/2}(\hat{\lambda}_T - \lambda_0) \to_d N(0_{1/2m(m+1)\times 1}, \Omega_\lambda) \ , \ \ \Omega_\lambda := \mathcal{P}_\sigma\Omega_\eta\mathcal{P}_\sigma' \tag{A.16}$$

where $\hat{\lambda}_T := \mathcal{P}_\sigma\hat{\sigma}_{\eta,T}^+$, $\lambda_0 := \mathcal{P}_\sigma\sigma_{\eta,0}^+$ and Ω_λ can be estimated consistently by $\widehat{\Omega}_{\lambda,T} := \mathcal{P}_\sigma\widehat{\Omega}_{\eta,T}\mathcal{P}_\sigma'$. The $a \times 1$ vector $\zeta := (vech(\Xi)', vec(\Sigma_{v_Z,u}))'$ which plays a crucial role in the CMD estimation approach discussed in Section 5 of the paper depends on λ, i.e. $\zeta = w(\lambda)$, where $w(\cdot)$ is a differentiable vector function. Thus, $\zeta_0 = w(\lambda_0)$, $\hat{\zeta}_T = w(\hat{\lambda}_T)$ and from eq. (A.16) and the delta-method, $\hat{\zeta}_T$ is a maximum likelihood estimator with asymptotic covariance matrix $\Omega_\zeta := \digamma_\lambda \Omega_\lambda \digamma_\lambda'$, where $\digamma_\lambda := \partial\zeta/\partial\lambda'$ is the $a \times 1/2m(m+1)$ Jacobian matrix:

$$\digamma_\lambda := \left(\begin{array}{c|c|c} -D_r^+(\Sigma_{v_Z,u}\Sigma_u^{-1} \otimes \Sigma_{v_Z,u}\Sigma_u^{-1})D_n^{+\prime} & 2D_r^+(\Sigma_{v_Z,u}\Sigma_u^{-1} \otimes I_r) & 0_{1/2r(r+1)\times 1/2r(r+1)} \\ \hline 0_{nr\times 1/2n(n+1)} & I_{nr} & 0_{nr\times 1/2r(r+1)} \end{array} \right).$$

The matrix $\digamma_\lambda$ can be estimated consistently by replacing $\Sigma_{v_Z,u}$ and Σ_u with the corresponding elements taken from $\hat{\sigma}^+_{\eta,T} := vech(\widehat{\Sigma}_{\eta,T})$.

The result in eq. (22) of the paper holds also when the external instruments are available on a shorter sample relative to the sample length T used to estimate the covariance matrix Σ_u of the original VAR. Let T^v :=int$[frT]$ denote the number of (non zero) observations used to estimate $\Sigma_{v_Z,u}$ and Σ_{v_Z}, where $0 < fr < 1$ is a positive fraction of the full sample size. Then $\widehat{\Sigma}^{fr}_{v_Z,u} := \frac{1}{T^v}\sum_{t=1}^{T_v} v_{Z,t}\hat{u}_t'$ is consistent for $\Sigma_{v_Z,u}$ and $\widehat{\Sigma}^{fr}_{v_Z} := \frac{1}{T^v}\sum_{t=1}^{T_v} v_{Z,t}v_{Z,t}'$ is consistent for Σ_{v_Z} as $T \to \infty$. The convergences in eq.s (A.15) and (A.16) are still valid by using $\widehat{\Sigma}^{fr}_{v_Z,u}$ and $\widehat{\Sigma}^{fr}_{v_Z}$ in $\hat{\sigma}^+_{\eta,T}$ and $\widehat{\Omega}_{\eta,T}$. The CMD estimation procedure summarized in Section 5 of the paper can still be applied.

A.6 Specification steps for the AC-SVAR model and its likelihood function

In this section we summarize the specification steps behind the econometric analysis of the AC-SVAR model and discuss its log-likelihood function.

The reduced form covariance parameters of the AC-SVAR model in eq. (10) of the paper are in $\sigma^+_\eta := vech(\Sigma_\eta)$ and can be estimated by the following steps:

Step-1 Estimate the m-dimensional VAR system for $W_t := (Y_t', Z_t')'$ (by OLS), and use standard methods to determine the VAR lag order $\ell := \ell^{op}$, where $\ell^{op} \geq k$;

Step-2 Given $\ell := \ell^{op}$, re-estimate the VAR system for $W_t := (Y_t', Z_t')'$ with ℓ^{op} lags by imposing the set of zero restrictions that characterize the autoregressive parameters $\tilde{\Psi}$ and $\tilde{\Upsilon}$. From eq. (10) of the paper, it turns out that the AC-SVAR model represented in compact form is given by the system:

$$W = F\widetilde{\Psi}' + D\widetilde{\Upsilon}' + \eta$$

where the matrix W is $T \times m$ with rows given by $W_t' := (Y_t', Z_t')$, $t = 1, ..., T$, F is $T \times f$ with rows given by $F_t' := (W_{t-1}', ..., W_{t-\ell}')$, $t = 1, ..., T$, D is $T \times d$ with rows given by $D_t' := (D_{y,t}', D_{z,t}')$, $t = 1, ..., T$ and η is $T \times m$ with rows given by $\eta_t' := (u_t', v_{Z,t}')$, $t = 1, ..., T$.

Let δ be the vector collecting the non-zero elements contained in the matrices $\widetilde{\Psi}$ and $\widetilde{\Upsilon}$. Henceforth we use the notation $\widetilde{\Psi}(\delta)$ and $\widetilde{\Upsilon}(\delta)$ to remark that these matrices depend on δ. Constrained OLS estimation of δ and σ_{η}^{+} is not fully efficient in this setup hence estimation can be carried out either through an iterated version of (Feasible) GLS (Lütkepohl, 2005) or, assuming that $\eta_t := (u_t', v_{Z,t}')'$ is Gaussian, by maximizing the log-likelihood function:

$$L_T(\delta, \sigma_{\eta}^{+}) := \frac{-Tm}{2}\log(2\pi) - \frac{T}{2}\log(\det(\Sigma_{\eta})) - \frac{1}{2}tr\left\{\Sigma_{\eta}^{-1}(W - F\widetilde{\Psi}(\delta)' - D\widetilde{\Upsilon}(\delta)')'(W - F\widetilde{\Psi}(\delta)' - D\widetilde{\Upsilon}(\delta)')\right\} \tag{A.17}$$

which provides $\hat{\delta}_T$, $\widehat{\widetilde{\Psi}} := \widetilde{\Psi}(\hat{\delta}_T)$, $\widehat{\widetilde{\Upsilon}} := \widetilde{\Upsilon}(\hat{\delta}_T)$ and $\hat{\sigma}_{\eta,T}^{+} := vech(\widehat{\Sigma}_{\eta,T})$, $\widehat{\Sigma}_{\eta,T} := (1/T)\sum_{t=1}^{T}\hat{\eta}_t\hat{\eta}_t'$, $\hat{\eta}_t := (W_t - \widehat{\widetilde{\Psi}}F_t - \widehat{\widetilde{\Upsilon}}D_t)$. This estimator will be fully efficient if the disturbances of the AC-SVAR model are Gaussian in the data generating process.

The estimator $\hat{\sigma}_{\eta,T}^{+}$ is consistent and asymptotically Gaussian under Assumption 1 of the paper. When $Z_t \equiv v_{Z,t}$, the last r elements of the vector $\hat{\eta}_t$ coincide with Z_t and the two steps above are not needed as $\hat{\sigma}_{\eta,T}^{+}$ is directly obtained from $\widehat{\Sigma}_{\eta,T} := (1/T)\sum_{t=1}^{T}\hat{\eta}_t\hat{\eta}_t'$.

Under the conditions of Proposition 2, the (concentrated) log-likelihood of the AC-SVAR model can be obtained, given $\hat{\delta}_T$ and $\widehat{\widetilde{\Psi}} := \widetilde{\Psi}(\hat{\delta}_T)$, $\widehat{\widetilde{\Upsilon}} := \widetilde{\Upsilon}(\hat{\delta}_T)$, by expressing the log-likelihood in eq. (A.17) in the form:

$$L_T(\hat{\delta}_T, \sigma_{\eta}^{+}) := \frac{-Tm}{2}\log(2\pi) - \frac{T}{2}\log(\det(\Sigma_{\eta})) - \frac{T}{2}r\left\{\Sigma_{\eta}^{-1}\widehat{\Sigma}_{\eta,T}\right\}$$

and then by imposing the restriction $\Sigma_{\eta} := \Sigma_{\eta}(\theta) = \widetilde{G}(\theta)\widetilde{G}(\theta)'$:

$$L_T^{s}(\theta) := \frac{-Tm}{2}\log(2\pi) - T\log(\det(\widetilde{G}(\theta)) - \frac{T}{2}tr\left\{\left[\widetilde{G}(\theta)^{-1}\right]'\widetilde{G}(\theta)^{-1}\widehat{\Sigma}_{\eta,T}\right\} \tag{A.18}$$

where $\widetilde{G}(\theta)$ is restricted as in eq. (27) of the paper. The log-likelihood in eq. (A.18) can be maximized by the methods discussed e.g. in Amisano and Giannini (1997). Under Assumptions 1-2 of the paper, the estimator $\hat{\theta}_T := \max_{\theta} L_T^{s}(\theta)$ is consistent and asymptotically Gaussian.

A.7 Comparison with Mertens and Ravn (2013) when $g > 1$

In this section we consider the case $g > 1$ (multiple structural shocks of interest) and compare Mertens and Ravn's (2013) approach with ours.

To simplify the exposition we focus, as in Mertens and Ravn (2013), on the case $r = g = 2$, i.e. we assume that two valid external instruments are used to identify two structural shocks of interest collected in $\varepsilon_{1,t}$.

We start from the mapping between structural shocks and reduced form disturbances:

$$u_t = B\varepsilon_t.$$

By partitioning u_t and B conformably with $\varepsilon_t := (\varepsilon_{1,t}', \varepsilon_{2,t}')'$, the mapping reads:

$$\begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} = \begin{pmatrix} B_1 & \vdots & B_2 \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix} \equiv \begin{pmatrix} B_{11} & \vdots & B_{12} \\ B_{21} & \vdots & B_{22} \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix} \tag{A.19}$$

where B_{11} is 2×2 nonsingular, B_{21} is $(n-2) \times 2$, B_{12} is $2 \times (n-2)$ and B_{22} is $(n-2) \times (n-2)$ nonsingular. The objective is to identify the parameters in the $n \times 2$ sub-matrix B_1 whose columns capture the instantaneous impact of the shocks in $\varepsilon_{1,t}$. The relationships between the two external instruments in $v_{Z,t}$ and $\varepsilon_{1,t}$ is captured by the system:

$$v_{Z,t} = \Phi\varepsilon_{1,t} + \omega_t \tag{A.20}$$

where Φ is the 2×2 matrix of relevance parameters and ω_t is a 2×1 measurement error with 2×2 (symmetric) covariance matrix Σ_ω.

By simple algebra, eq. (A.19) can be rearranged in the form:

$$\begin{aligned} u_{1,t} &= \eta u_{2,t} + \ S_1\varepsilon_{1,t} \\ u_{2,t} &= \zeta u_{1,t} + S_2\varepsilon_{2,t} \end{aligned} \tag{A.21}$$

where

$$\begin{aligned} \eta &= B_{12}\left(B_{22}\right)^{-1} \\ S_1 &= B_{11} - B_{12}\left(B_{22}\right)^{-1} B_{21} \\ \zeta &= B_{21}\left(B_{11}\right)^{-1} \\ S_2 &= B_{22} - B_{21}\left(B_{11}\right)^{-1} B_{12}. \end{aligned} \tag{A.22}$$

Mertens and Ravn (2013) show that IV methods allow to identify η, ζ and S_1S_1', but not S_1. In other words, in the absence of further restrictions, S_1 is not separately identified from S_1S_1'. In particular, Mertens and Ravn (2013) show that B_1 can be represented as:

$$B_1 = \begin{pmatrix} I_2 + \eta(I_{n-2} - \zeta\eta)^{-1}\zeta \\ (I_{n-2} - \zeta\eta)^{-1}\zeta \end{pmatrix} S_1 \tag{A.23}$$

which shows that the identification of B_1 depends on the identification of S_1.[3] S_1 can be separately identified from S_1S_1' if at least $1/2g(g-1)(=1)$ restrictions are placed on its columns.

[3]Instead, when $g = 1$ no further restriction is needed. In this case, however, the shock is identified up to sign normalization.

It turns out that B_1 can be identified in eq. (A.23) if at least $1/2g(g-1)(=1)$ restrictions are placed on S_1. Mertens and Ravn (2013) manage these restrictions by taking S_1 lower (upper) triangular. Equation (A.22) shows that by taking S_1 lower (upper) triangular amounts to place $1/2g(g-1)(=1)$ indirect (nonlinear) restrictions on B_1, and this fact is not at odds with the necessary order condition stated in Proposition 1 of the paper.

In Section 5 of the paper, we discuss our 'partial shocks' identification strategy and observe that by modelling eq.s (A.19) and (A.20) jointly, one obtains the AC-SVAR representation of the proxy-SVAR:

$$\underbrace{\begin{pmatrix} u_t \\ v_{Z,t} \end{pmatrix}}_{\eta_t} = \underbrace{\left(\begin{array}{c|c|c} B_1 & B_2 & 0_{n\times 2} \\ \hline \Phi & 0_{2(n-2)} & \Sigma_{\omega}^{1/2} \end{array}\right)}_{\widetilde{G}} \underbrace{\begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \omega_t^0 \end{pmatrix}}_{\xi_t}$$

where $\eta_t := (u_t', v_{Z,t}')'$ is $m \times 1$, $m = n+2$, Σ_ω is the covariance matrix of the measurement error, $\Sigma_\omega^{1/2}$ is such that $\Sigma_\omega = \Sigma_\omega^{1/2}\Sigma_\omega^{1/2}$ and the term $\omega_t^\circ := \Sigma_\omega^{-1/2}\omega_t$ can be interpreted as a normalized measurement error. We are interested in the identification of the first $g(=2)$ columns of the matrix of instantaneous impact coefficients:

$$\widetilde{G} := \left(\begin{array}{c|c} \widetilde{G}_1 & \widetilde{G}_2 \end{array}\right) = \left(\begin{array}{c|c|c} B_1 & B_2 & 0_{n\times 2} \\ \hline \Phi & 0_{2\times(n-2)} & \Sigma_{\omega}^{1/2} \end{array}\right). \tag{A.24}$$

In eq. (A.24) $\widetilde{G}_1$ is the submatrix of $\widetilde{G}$ collecting the first $g(=2)$ columns and $\widetilde{G}_2$ collects the remaining $m-g$ columns. In particular, we have:

$$\widetilde{G}_1 := \begin{pmatrix} B_1 \\ \hline \Phi \end{pmatrix} \equiv \begin{pmatrix} B_{11} \\ B_{21} \\ \hline \Phi \end{pmatrix} \begin{array}{c} 2\times 2 \\ (n-2)\times 2 \\ \hline 2\times 2 \end{array}. \tag{A.25}$$

We treat the identification of $\widetilde{G}_1$ as *a partial identification problem where the restrictions are directly placed on the elements of* $\widetilde{G}_1$.

According to Proposition 1(b) of the paper, necessary condition for identification is that at least $1/2g(g-1)(=1)$ restrictions are placed on the two columns of $\widetilde{G}_1$. These restrictions may vary with the particular application and may involve B_1 but also Φ (provided nonsingularity is preserved). Proposition 1(a) establishes where these restrictions need to be placed for identification (up to sign). For example, the identification restrictions can be placed e.g. on B_1 by leaving Φ unrestricted: by setting e.g. $b_{11} = 0$ we have:

$$\widetilde{G}_1 := \begin{pmatrix} B_{11} \\ B_{21} \\ \hdashline \Phi \end{pmatrix} := \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ \hdashline \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}; \tag{A.26}$$

or, alternatively, by setting e.g. $b_{41} = 0$ we have:

$$\widetilde{G}_1 := \begin{pmatrix} B_{11} \\ B_{21} \\ \hdashline \Phi \end{pmatrix} := \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ 0 & b_{42} \\ \hdashline \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}.$$

Thus, while in Mertens and Ravn (2013), the $1/2g(g-1)(=1)$ 'additional' identification restrictions depend on a Choleski decomposition, in our approach there is no Choleski decomposition involved in the identification process. Our approach is more flexible as the identification restrictions can possibly be placed on Φ alone, leaving B_1 unrestricted: e.g. by setting $\varphi_{21} = 0$ we have:

$$\widetilde{G}_1 := \begin{pmatrix} B_1 \\ \hdashline \Phi \end{pmatrix} := \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ \hdashline \varphi_{11} & 0 \\ \varphi_{21} & \varphi_{22} \end{pmatrix}.$$

Summing up, in Mertens and Ravn (2013) the fact that in order to achieve identification when $g > 1$ it is necessary to complement the restrictions provided by the external instruments with at least $1/2g(g-1)$ additional restrictions stems from the representation in eq. (A.23) and the fact that S_1 is not separately identified from S_1S_1'. Mertens and Ravn (2013) solve the problem by taking S_1 as the Choleski factor of the matrix S_1S_1'. This identifies B_1 indirectly. Our 'partial shocks' identification approach is motivated by the observation that given the matrix $\widetilde{G} := \left(\widetilde{G}_1 \,\vdots\, \widetilde{G}_2 \right)$ and the interest in the parameters of $\widetilde{G}_1$ alone, it is necessary to impose at least $1/2g(g-1)$ *direct* restrictions on the g columns of $\widetilde{G}_1$. Proposition 1 in the paper answers the following questions: given $g \geq 1$ structural shocks of interest in a system of n variables and

$r \geq g$ external instruments available for these shocks, how many restrictions do we need for the model to be identified and where do these restrictions need to be placed? Proposition 1 provides necessary and sufficient conditions for the identification of $\widetilde{G}_1$ (hence of B_1 and Φ) *based on linear restrictions which are valid up to sign normalization but do not depend on Choleski-type restrictions.*

A.8 Monte Carlo results

One of the merits of the AC-SVAR approach is the possibility of easily testing overidentified proxy-SVARs. Under the maintained hypothesis of relevant external instruments, the overidentification restrictions should be rejected under two circumstances: (a) the overidentification restrictions are false and the external instruments are orthogonal to the non-instrumented shocks, (b) the external instruments are wrongly assumed orthogonal to the non-instrumented shocks.

In this section we focus on (b) and conduct a set of Monte Carlo experiments to analyze the performance of the overidentification restrictions tests discussed in the paper. Section A.8.1 presents the design of Monte Carlo experiments and the hypothesis of interest. Section A.8.2 summarize the results obtained with the CMD-based test for overidentification restrictions in the partial shocks identification approach and Section A.8.3 summarizes the results obtained with the LR for overidentification restrictions in the full shocks identification approach.

A.8.1 Design and hypothesis of interest

We consider a SVAR based on the vector $Y_t := (Y_{A,t}, Y_{B,t}, Y_{C,t})'$ ($n := 3$) and the system:

$$Y_t = \Upsilon_y + A_y Y_{t-1} + u_t, \quad u_t = B\varepsilon_t, \quad \varepsilon_t := \begin{pmatrix} \varepsilon_{A,t} \\ \varepsilon_{B,t} \\ \varepsilon_{C,t} \end{pmatrix} \sim \text{iid}N(0, I_3) \tag{A.27}$$

with initial value $Y_0 := 0_{3\times1}$. The population values of Υ_y (constant), A_y (companion matrix) and $B := \left(\begin{array}{c:c} B_1 & B_2 \end{array} \right)$ (matrix of structural parameters) are given by:

$$\Upsilon_y := \begin{pmatrix} 0.33 \\ 0.2 \\ -0.3 \end{pmatrix}, \quad A_y := \begin{pmatrix} -0.3 & -0.25 & 0 \\ 0.95 & 0.5 & 0.2 \\ 0.6 & 0 & 0.8 \end{pmatrix}, \quad \lambda_{\max}(A_y) = 0.738$$

$$B := \left(\begin{array}{c:c} B_1 & B_2 \end{array} \right) = \left(\begin{array}{c:cc} 0.6 & -0.85 & -0.8 \\ 0 & 0.55 & 0.45 \\ 0 & 0.32 & 0.23 \end{array} \right) \tag{A.28}$$

where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue in absolute value of the matrix in the argument.

The matrix B in eq. (A.28) depicts a situation in which $Y_{B,t}$ and $Y_{C,t}$ does not respond on-impact to the shock $\varepsilon_{1,t} := \varepsilon_{A,t}$ (as $b_{B,A} = 0$ and $b_{C,A} = 0$ in B) while the shocks $\varepsilon_{2,t} := (\varepsilon_{B,t}, \varepsilon_{C,t})'$ affect $Y_{A,t}$ on-impact: in particular, $b_{A,B} = -0.85 < 0$ and $b_{A,C} = -0.8 < 0$. In line with the empirical section of the paper, we denote this situation as 'contemporaneous exogeneity' of $Y_{B,t}$ and $Y_{C,t}$, meaning that $Y_{B,t}$ and $Y_{C,t}$ do not respond instantaneously to the shock $\varepsilon_{A,t}$ but may respond after some periods.

The hypothesis we are interested in our experiments is the joint restriction $b_{B,A} = 0$ and $b_{C,A} = 0$. It is worth noting that the SVAR defined by the B matrix in eq. (A.28) (with $b_{B,A} = 0$ and $b_{C,A} = 0$) is not identified in the absence of an additional restriction or the use of at least one external instrument.

To test the hypothesis of 'contemporaneous exogeneity' of $Y_{B,t}$ and $Y_{C,t}$ we need to identify the first column B, B_1, which captures the instantaneous impact of the shock $\varepsilon_{A,t}$. In the next two subsections we follow two distinct identification strategies to identify $\varepsilon_{A,t}$ and test $b_{B,A} = 0$ and $b_{C,A} = 0$. In one case, we instrument the shock of interest $\varepsilon_{A,t}$ directly (Section A.8.2) in a partial shocks identification strategy, using the results in Proposition 1 of the paper. In the other identification strategy we mimic a scenario in which it is difficult to find valid external instruments for $\varepsilon_{A,t}$ but it is relatively easier to find valid external instruments for the shocks $\varepsilon_{B,t}$ and $\varepsilon_{C,t}$ so that $\varepsilon_{A,t}$ is identified 'residually' through a full shocks identification strategy, using the results of Proposition 2 of the paper (Section A.8.3).

A.8.2 Partial shocks identification strategy

We consider one external instrument $Z_t := Z_{1,t}$ for $\varepsilon_{A,t}$ ($r = g = 1$) and the auxiliary model:

$$Z_{1,t} = \Upsilon_z + \Gamma Y_{t-1} + v_{Z_1,t}, \quad v_{Z,t} = R_\Phi \varepsilon_t + \omega_t, \quad \omega_t \sim \text{iid}N(0, \Sigma_\omega) \tag{A.29}$$

with population parameter values:

$$\Upsilon_z := 0, \quad \Gamma := \begin{pmatrix} 0.15 & 0.36 & 0 \end{pmatrix}$$

$$R_\Phi := \left(\begin{array}{c:c} \Phi & O \end{array} \right) = \left(\begin{array}{c:cc} 0.53 & o_1 & o_2 \end{array} \right), \quad \Sigma_\omega := \begin{pmatrix} 0.5 \end{pmatrix}.$$

The matrix R_Φ incorporates the relevance condition ($\Phi = \phi = \varphi = 0.53$) and the orthogonality condition if o_1 and o_2 are set to zero, i.e. if $O := E\left(v_{Z_1,t}\varepsilon_{2,t}'\right) := (o_1, o_2) = (0, 0)$.

The population value fixed of the relevance parameter $\Phi = \phi = \varphi = 0.53$ implies a correlation with the instrumented structural shock equal to $Corr(v_{Z_1,t}, \varepsilon_{1,t}) = Cov(v_{Z_1,t}, \varepsilon_{1,t}) / \left(Var(v_{Z_1,t})\right)^{1/2} = 0.749$. Instead, the parameters o_1 and o_2 in O control for the correlation between the external instrument and the non-instrumented shocks $\varepsilon_{2,t} := (\varepsilon_{B,t}, \varepsilon_{C,t})'$. The instrument $v_{Z_1,t}$ is orthogonal to $\varepsilon_{2,t}$ when $O := (o_1, o_2) = (0, 0)$; in this case the AC-SVAR model obtained by coupling systems (A.27) and (A.29) is based on the matrix $\widetilde{G}$ (see eq. (11) in the paper):

$$\widetilde{G} := \left(\begin{array}{c:c} \widetilde{G}_1 & \widetilde{G}_2 \end{array} \right) = \left(\begin{array}{c:c:c} B_1 & B_2 & 0_{n\times 1} \\ \hdashline \varphi & O & \sigma_{v_{Z_1}} \end{array} \right) = \left(\begin{array}{c:cc:c} 0.6 & -0.85 & -0.8 & 0 \\ 0 & 0.55 & 0.45 & 0 \\ 0 & 0.32 & 0.23 & 0 \\ \hdashline 0.53 & o_1 & o_2 & 0.78 \end{array} \right) \tag{A.30}$$

and we are interested in the first column $\widetilde{G}_1$ which is overidentified according to Proposition 1 in the paper (it incorporates two testable overidentification restrictions). On the other hand, if $O := (o_1, o_2) \neq (0, 0)$, the external instruments are not orthogonal to the non-instrumented shocks.

We generate samples of size $T = 100, 250$ and 500, M=1000 times from the AC-SVAR model. On each sample we estimate the parameters $\vartheta := (b_{A,A}, b_{B,A}, b_{C,A}, \varphi)'$ by the CMD approach in Section 5 of the paper and test the two contemporaneous exogeneity restrictions $b_{B,A} = 0$ and $b_{C,A} = 0$ by the test $TQ(\hat{\vartheta}_T)$ at the 5% nominal level of significance. We consider three different scenarios: (i) the case of orthogonal external instrument, $O := (o_1, o_2) = (0, 0)$; (ii) the case in which the external instrument is not orthogonal to $\varepsilon_{C,t}$, $O := (o_1, o_2) = (0, -0.1)$; (iii) the case in which the external instruments is not orthogonal to both $\varepsilon_{B,t}$ and $\varepsilon_{C,t}$, $O := (o_1, o_2)' = (-0.75, -0.25)'$. In (i), the rejection frequency of the CMD-based test $TQ(\hat{\vartheta}_T)$ coincides with the empirical size of the overidentification restrictions test.

The rejection frequencies of the CMD-based test are reported in Panel A of Table A.1. Results confirm that the test for the null hypothesis of 'contemporaneous exogeneity' of $Y_{B,t}$ and $Y_{A,t}$ delivers correct empirical size when the external instrument for $\varepsilon_{A,t}$ is orthogonal to the non-instrumented shocks $\varepsilon_{B,t}$ and $\varepsilon_{C,t}$ (case (i)). Instead, the true null hypothesis tends to be rejected when the selected external instrument is not orthogonal to the non-instrumented shocks (cases (ii) and (iii)).

A.8.3 Full shocks identification strategy

In this section we identify the shock $\varepsilon_{A,t}$ by changing the identification strategy radically. In particular, we consider a scenario in which it is easier to find valid external instruments for $\varepsilon_{B,t}$ and $\varepsilon_{C,t}$ rather than for $\varepsilon_{A,t}$. Thus, we assume that the instrumented shocks are $\varepsilon_{1,t} := (\varepsilon_{B,t}, \varepsilon_{C,t})'$ ($g = 2$) and that $\varepsilon_{2,t} := \varepsilon_{A,t}$ is treated as 'the other structural shock' despite it is the shock whose dynamic causal effects we are actually interested in. The shock $\varepsilon_{A,t}$ will be identified 'residually' by exploiting the advantages of the full shocks identification strategy summarized in Proposition 2 of the paper. In line with the 'new' definition of the shocks $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ and for convenience only, henceforth we change the order of the variables in the VAR by putting $Y_{B,t}$ and $Y_{C,t}$ first and $Y_{A,t}$ last.

It is assumed that the there exist two external instruments for $\varepsilon_{1,t} := (\varepsilon_{B,t}, \varepsilon_{C,t})'$ ($r = g = 2$) collected in the vector $Z_t := (Z_{1,t}, Z_{2,t})'$ generated by the auxiliary model:

$$Z_t = \Upsilon_z + \Gamma Y_{t-1} + v_{Z,t}, \quad v_{Z,t} = \Phi\varepsilon_{1,t} + \omega_t, \quad \omega_t \sim \text{iid}N(0, \Sigma_\omega) \tag{A.31}$$

with parameters population values:

$$\Upsilon_z := \begin{pmatrix} 0 \\ -0.05 \end{pmatrix}, \quad \Gamma := \begin{pmatrix} 0.15 & 0.36 & 0 \\ 0.12 & 0 & 0 \end{pmatrix}$$

$$R_\Phi := \left(\Phi \;\vdots\; O \right) = \left(\begin{array}{cc|c} 0.53 & 0.26 & o_1 \\ 0 & 0.74 & o_2 \end{array} \right), \quad \Sigma_\omega := \begin{pmatrix} 0.5 & 0.20 \\ & 0.85 \end{pmatrix}. \tag{A.32}$$

In eq. (A.32), the matrix $R_\Phi := (\Phi \vdots O)$ incorporates the relevance condition, captured by the 2×2 matrix Φ, and the orthogonality condition if $O := E\left(v_{Z,t}\varepsilon_{1,t}\right) := (o_1, o_2)' = (0, 0)'$. The parameters o_1 and o_2 in O control the correlation between $v_{Z,t}$ and the 'other' shock $\varepsilon_{2,t} := \varepsilon_{A,t}$. Instead, the specified matrix Φ implies that the correlations between $v_{Z,t}$ and the instrumented shocks are equal to

$$Corr(v_{Z,t}, \varepsilon'_{1,t}) = \begin{pmatrix} Corr(v_{Z_1,t}, \varepsilon_{B,t}) & Corr(v_{Z_1,t}, \varepsilon_{C,t}) \\ Corr(v_{Z_2,t}, \varepsilon_{B,t}) & Corr(v_{Z_2,t}, \varepsilon_{C,t}) \end{pmatrix} = \begin{pmatrix} 0.75 & 0.36 \\ 0 & 0.798 \end{pmatrix}.$$

The form of the matrix $\widetilde{G}$ in eq. (11) of the paper is in this case given by:

$$\widetilde{G} := \left(\widetilde{G}_1 \;\vdots\; \widetilde{G}_2 \right) = \left(\begin{array}{c|c|c} B_1 & B_2 & 0_{3\times 2} \\ \hline \Phi & O & \Sigma_\omega^{1/2} \end{array} \right) := \left(\begin{array}{cc|c|cc} b_{A,A} & b_{A,B} & b_{A,C} & 0 & 0 \\ b_{B,A} & b_{B,B} & b_{B,C} & 0 & 0 \\ b_{C,A} & b_{C,B} & b_{C,C} & 0 & 0 \\ \hline \varphi_{11} & \varphi_{12} & o_1 & \varpi_{11} & \varpi_{21} \\ 0 & \varphi_{22} & o_2 & \varpi_{21} & \varpi_{22} \end{array} \right) \tag{A.33}$$

$$= \left(\begin{array}{cc|c|cc} 0.55 & 0.45 & 0 & 0 & 0 \\ 0.32 & 0.23 & 0 & 0 & 0 \\ -0.85 & -0.8 & 0.6 & 0 & 0 \\ \hline 0.53 & 0.26 & o_1 & 0.85 & 0.38 \\ 0 & 0.74 & o_2 & 0.38 & 1.39 \end{array} \right).$$

It is possible to check that when the instruments in $v_{Z,t}$ are orthogonal to $\varepsilon_{A,t}$, $O := (o_1, o_2)' = (0, 0)'$ and $\widetilde{G}$ incorporates 12 restrictions, two more of the 10 restrictions necessary to respect the order condition for identification (similarly, using the notation of Proposition 2 of the paper: $a_{\widetilde{G}} = 13 < 1/2m(m+1) = 15$). According to Proposition 2(a) in the paper, the AC-SVAR is identified (overidentified). On the other hand, if one relaxes the hypothesis of 'contemporaneous exogeneity' of $Y_{B,t}$ and $Y_{C,t}$, i.e. replaces the two zeros corresponding to $b_{B,A} = 0$ and $b_{C,A} = 0$ with non-zero values, the AC-SVAR is exactly identified ($a_{\widetilde{G}} = 15 = 1/2m(m+1) = 15$ and the rank condition of Proposition 2(a) still holds). Thus, in this framework the shock $\varepsilon_{2,t} := \varepsilon_{A,t}$ is identified in the AC-SVAR model even if we are instrumenting $\varepsilon_{1,t} := (\varepsilon_{B,t}, \varepsilon_{C,t})'$.

We generate samples of size T = 100, 250 and 500, M =1000 times from system (A.27)-(A.31) and on each generated sample we test the (true) restrictions $b_{B,A} = 0$ and $b_{C,A} = 0$ by the LR test (see Section 6 of the paper) at the 5% nominal level of significance. Panel B of Table A.1 summarizes the rejection frequencies of the LR test in scenarios: (i) the case of orthogonal external instruments, $O := (o_1, o_2)' = (0, 0)'$; (ii) the case in which the second external instrument is not orthogonal to $\varepsilon_{2,t} := (\varepsilon_{A,t})$, $O := (o_1, o_2)' = (0, -0.1)'$; (iii) the case in which both external instruments are not orthogonal to $\varepsilon_{2,t} := (\varepsilon_{A,t})$, $O := (o_1, o_2)' = (-0.75, -0.25)'$. In (i) the rejection frequency coincides with the empirical size of the LR test and Table A.1 shows the test approaches the nominal size as the sample size increases. In (ii), the LR test tends to reject the (true) null hypothesis as the sample size increases. In (iii), the rejection frequency of the LR test approaches 1 also in small samples.

Overall, also in this case the LR test for the overidentification restrictions tend to reject the true null hypothesis when the external instruments are erroneously considered orthogonal to the non-instrumented shocks.

A.9 A Wald test for the 'no relevance' condition

In this section we focus on a simple indirect test for the null hypothesis that the relevance parameters of a proxy-SVAR model where multiple instruments (r) are used for multiple shocks (g) are zero.

Our starting point are the moment conditions in eq. (A.14) that play key role in the identification and estimation of proxy-SVARs. Equation (A.14) shows that if one assumes that $B_1 \neq 0_{n\times g}$, the null hypothesis $H_0 : \Phi = 0_{r\times g}$ (no relevance) is equivalent (implies and is implied by) to the null hypothesis $H_0' : \Sigma_{v_Z,u} = 0_{r\times n}$. It turns out that testing the problem:

$$H_0 : vec(\Phi) = 0_{rg\times 1} \text{ against } H_1 : vec(\Phi) \neq 0_{rg\times 1}$$

is equivalent to the testing the problem:

$$H_0' : vec(\Sigma_{v_Z,u}) = 0_{nr\times 1} \text{ against } H_1' : vec(\Sigma_{v_Z,u}) = 0_{nr\times 1}. \tag{A.34}$$

The advantage is that a test for the null H_0' against H_1' in eq. (A.34) requires standard asymptotics. Indeed, observe that $vec(\Sigma_{v_Z,u})$ is a component of the $a \times 1$ ($a := 1/2r(r+1) + nr$) vector $\zeta := (vech(\Xi)', vec(\Sigma_{v_Z,u})')'$ (see Section 5 in the paper) which in turn is a function of the reduced form parameters of the AC-SVAR model. Under Assumption 1 in the paper:

$$T^{1/2}(\hat{\zeta}_T - \zeta_0) \to_d N(0_{a\times 1}, \Omega_\zeta)$$

where an expression for the covariance matrix Ω_ζ has been derived in Section A.5. The asymptotic normality of $T^{1/2}(\hat{\zeta}_T - \zeta_0)$ holds irrespective of whether the external instruments are strongly, weakly or not correlated at all with the structural shocks of interest.

The testing problem in eq. (A.34) can be formulated as:

$$H_0^{''} : S\zeta = 0_{nr\times 1} \text{ against } H_1' : S\zeta \neq 0_{nr\times 1}$$

where $S := (0_{nr\times(a-nr)} \vdots I_{nr})$ is an $nr \times a$ selection matrix which sets the last nr components of ζ to zero. Let $\widehat{\Omega}_\zeta$ be a consistent estimator of Ω_ζ. Under H_0'' and Assumption 1 in the paper, the Wald test statistic:

$$W_T^{rel} := T\hat{\zeta}_T' S' \left(S\widehat{\Omega}_\zeta S'\right)^{-1} S\hat{\zeta}_T \tag{A.35}$$

is asymptotically distributed as a $\chi^2(nr)$. Conversely, the test statistic W_T^{rel} diverges under the alternative H_1'.

A.10 Baseline bootstrap algorithm

Since the AC-SVAR model reads as a large (constrained) 'B-model', bootstrap confidence bands for the IRFs of interest can be computed by applying the methods available for SVARs reviewed e.g. in Kilian and Lütkepohl (2017, Ch. 12). In particular, when the reduced form disturbances η_t are conditionally homoskedastic, we adapt a residual-based recursive-design bootstrap algorithm to the special features of the partial shocks approach discussed in Section 5 of the paper, using the algorithm sketched below. If instead the disturbances η_t display conditional heteroskedasticity, the residual-based bootstrap methods based on *iid* resampling or the wild-bootstrap tend to underestimate the uncertainty associated with the estimated dynamic causal effects, see Bruggemann *et al.* (2016) and Jentsch and Lunsford (2016, 2019). These authors show that when η_t features conditional heteroskedasticity of unknown form, reliable bootstrap asymptotic inference on the IRFs can be achieved by a residual-based moving block bootstrap procedure. We refer to Jentsch and Lunsford (2016, 2019) and Mertens and Ravn (2019) for a discussion on the inference on IRFs focused on proxy-SVARs.

In the empirical application in Section 8 of the paper, we test for the occurrence of unconditional heteroskedasticity in the residuals associated with the estimated AC-SVAR model and do not reject the null hypothesis of conditional homoskedasticity (see Panel A of Table 1 in the paper). Based on this result, we build bootstrap-based 90% simultaneous confidence bands for the IRFs by combining a standard residual-based recursive-design bootstrap (which incorporates Kilian's (1998) 'bootstrap-after-bootstrap' correction) with the 'sup-t' discussed in Montiel Olea and Plagborg-Møller (2019). In the rest of this section we summarize the main steps of the algorithm used in the paper.

Let $\widehat{\widetilde{\Psi}} := \widetilde{\Psi}(\hat{\delta}_T)$, $\widehat{\widetilde{\Upsilon}} := \widetilde{\Upsilon}(\hat{\delta}_T)$ and $\hat{\sigma}^{+}_{\eta,T} := vech(\widehat{\Sigma}_{\eta,T})$, $\widehat{\Sigma}_{\eta,T} := (1/T)\sum_{t=1}^{T}\hat{\eta}_t\hat{\eta}_t'$, $\hat{\eta}_t := (W_t - \widehat{\widetilde{\Psi}}F_t - \widehat{\widetilde{\Upsilon}}D_t)$, $t = 1, ..., T$, be the estimates of the reduced form parameters associated with the AC-SVAR model, see Section A.6. To simplify exposition we temporarily assume that the η_t are conditionally homoskedastic.

Consider first the partial shocks identification approach (Section 5 of the paper). Let $\vartheta := (\alpha_1', \varphi')'$ be the unrestricted (free) structural parameters associated with the AC-SVAR model and assume that the conditions of Proposition 1 hold, i.e. the proxy-SVAR is identified. As discussed in Section 5 of the paper, under Assumptions 1-2 and the conditions of Proposition 1, the parameters in $\vartheta := (\alpha_1', \varphi')'$ can be estimated consistently by solving the problem:

$$\min_{\vartheta}(\hat{\zeta}_T - f(\vartheta))'\widehat{\Omega}_{\zeta}^{-1}(\hat{\zeta}_T - f(\vartheta)). \tag{A.36}$$

Obtained $\hat{\vartheta}_T := (\hat{\alpha}_{1,T}', \hat{\varphi}_T')'$, the g columns of the matrix B_1 are estimated consistently by $\widehat{B}_1$, where $\widehat{B}_1$ is reconstructed from $\hat{\beta}_{1,T}{:=}S_{B_1}\hat{\alpha}_{1,T} + s_{B_1}$ (recall that $\beta_1 := vec(B_1) = S_{B_1}\alpha_1 + s_{B_1}$). The estimated IRFs are given by:

$$\widehat{IRF}_j(h){:=}(J_n\left(\widehat{A}_y\right)^h J_n')\hat{b}_j\ ,\quad h = 0, 1, 2, ... \tag{A.37}$$

where the estimate of the companion matrix $\widehat{A}_y$ is obtained from the estimated autoregressive parameters $\widehat{\widetilde{\Psi}} := \widetilde{\Psi}(\hat{\delta}_T)$, and $\hat{b}_j$ is the j-th column of $\widehat{B}_1$, $j = 1, .., g$. In the full shocks identification approach, under Assumptions 1-2 and the conditions of Proposition 2 in the paper, the parameters are estimated by solving $\hat{\theta}_T := \max_{\theta} L_T^s(\theta)$, where the log-likelihood $L_T^s(\theta)$ is given in eq. (A.18). The IRFs are estimated as in eq. (A.37), the difference being now that $\hat{b}_j$, $j = 1, ..., n$ are the columns of $\widehat{B}$, where $\widehat{B}$ is part of $\widehat{\widetilde{G}} := \widetilde{G}(\hat{\theta}_T)$, see eq. (11) in the paper.

Let $\hat{\eta}_t$, $t = 1, ..., T$ be the residuals of the estimated AC-SVAR model. The algorithm for the residual-based recursive-design bootstrap is a follows (the lag length of the AC-SVAR model is fixed at the values determined on the sample):

BS-Step 1 Resample with replacement T residuals from the sequence $\hat{\eta}_t^c$, $t = 1, ..., T$, where $\hat{\eta}_t^c$ are the residuals $\hat{\eta}_t$ centered, obtaining the sequence η_t^*, $t = 1, ..., T$, and from this generate the bootstrap sample $W_1^*, ..., W_T^*$ conditional on the ℓ initial (sample) observations using the estimates $\widehat{\widetilde{\Psi}}$, $\widehat{\widetilde{\Upsilon}}$ as parameters values;

BS-Step 2 Use the sequence $W_1^*, ..., W_T^*$ generated in the previous step to estimate $\widetilde{\Psi}$ (A_y), $\widetilde{\Upsilon}$ and Σ_η by ML (FGLS) as detailed in Section A.6, obtaining $\hat{\delta}_T^b$, $\widehat{\widetilde{\Psi}}^b := \widetilde{\Psi}(\hat{\delta}_T^b)$ $(\widehat{A}_y^b)$, $\widehat{\widetilde{\Upsilon}}^b := \widetilde{\Upsilon}(\hat{\delta}_T^b)$ and $\widehat{\Sigma}_{\eta,T}^b$, respectively;[4] in the partial shocks identification approach, from $\widehat{\Sigma}_{\eta,T}^b$ derive the bootstrap estimators $\hat{\zeta}_T^b$ and $\widehat{\Omega}_\zeta^b$ as described in Section A.5 and then solve the CMD problem in eq. (A.36), obtaining $\hat{\vartheta}_T^b := (\hat{\alpha}_{1,T}^{b\prime}, \hat{\varphi}_T^{b\prime})'$. In the full shocks identification approach, $\hat{\theta}_T^b := \max_\theta L_T^{*s}(\theta)$, where the log-likelihood $L_T^{*s}(\theta)$ is the bootstrap analog of that in eq. (A.18), and then obtain $\widehat{\widetilde{G}}^b := \widetilde{G}(\hat{\theta}_T^b)$ and $\widehat{\Sigma}_{\eta,T}^b = \widetilde{G}(\hat{\theta}_T^b)\widetilde{G}(\hat{\theta}_T^b)'$;

BS-Steps 3 Compute the IRFs in eq. (A.37);

BS-Step 4 Repeat the steps BS-Step1-BS-Step2 for $b = 1, ..., N$, obtaining the sequence $\widehat{IRF}_j^1(h), ..., \widehat{IRF}_j^N(h)$, where $j = 1, .., g$ (partial shocks approach) or $j = 1, .., n$ (full shocks approach).

BS-Step 5 Compute bootstrap confidence intervals for the IRFs of interest using one of the methods listed in Section 12.2.6 of Kilian and Lütkepohl (2017, Ch. 12), or apply steps 6 to 13 of Algorithm 3 in Montiel Olea and Plagborg-Møller (2019) to compute simultaneous confidence 'sup-t' bands.

If the disturbances η_t of the AC-SVAR model display conditional homoskedasticity of unknown form it is necessary to replace the BS-Step 1 with a moving block bootstrap procedure along the lines described in Bruggemann *et al.* (2016).

A.11 The AC-SVAR model with a censored external instrument

The covariance matrix $\Sigma_{u,v_Z} := E(u_t v_{Z,t}')$ plays a key role in the proxy-SVAR analysis, see Sections 5 and 6 in the paper, and must be estimated consistently. In the presence of censored external instruments, the quantity $\widehat{\Sigma}_{u,v_Z,T} := \frac{1}{T}\sum_{t=1}^T \hat{u}_t \hat{v}_{Z,t}'$ does not estimate Σ_{u,v_Z} consistently. In general, any estimation procedure for proxy-SVARs produces inconsistent estimates of the parameters of interest (and accordingly of the IRFs) if the censoring mechanism is not properly accounted for. In the current proxy-SVAR literature, Mertens and Ravn (2013) address this issue.

In this section we discuss how the econometric analysis of the AC-SVAR model presented in the paper can be modified to account for a censored external instrument. This topic deserves a detailed treatment that goes well beyond the scopes of the paper and Supplementary Appendix.

To simplify the presentation and without any loss of generality, we consider the 'one shock-one instrument case', $r = g = 1$, so that Z_t is a scalar. In particular, we focus on the case

[4] Here we apply Kilian's (1998) bias correction.

in which the dynamics of Z_t is generated by a censored autoregressive process of order one (CAR(1)) with Gaussian disturbances. Our analysis is partly inspired by Park *et al.* (2007)'s estimation method of censored ARMA-type processes.

The data generating process is given by the SVAR in eq. (1) of the paper (we drop deterministic terms to simplify):

$$Y_t = \Pi X_t + u_t \quad , \quad u_t = B_1 \varepsilon_{1,t} + B_2 \varepsilon_{2,t} \quad , \quad t = 1, ..., T$$

that we couple with the following auxiliary model for the external instrument:

$$Z_t = \max \{\tau, Z_t^*\} \tag{A.38}$$

$$Z_t^* = \theta_1 Z_{t-1}^* + v_{Z,t} \quad , \quad v_{Z,t} = \phi \varepsilon_{1,t} + \omega_t \sim N(0, \sigma_{v_Z}^2) \quad , \quad t = 1, ..., T. \tag{A.39}$$

Equations (A.38)-(A.39) define a CAR(1) process where τ is a non-stochastic threshold (cutoff) value which could be 0, and the uncensored process Z_t^* is a 'standard' AR(1) model with autoregressive parameter $|\theta_1| < 1$, $E(Z_t^*) = 0$ and $\sigma_{v_Z}^2 := E(v_{Z,t}^2) = \phi^2 + \varpi^2$. Equation (A.38) states that the observed instruments coincide with values of Z_t^* only when these values are larger than τ, otherwise the process is censored to τ.

Obviously, the dynamics of the external instrument might not necessarily be autoregressive. For instance, eq.s (A.38)-(A.39) might be replaced by the censoring mechanism:

$$v_{Z,t} = \max \left\{\tau, v_{Z,t}^*\right\} \tag{A.40}$$

$$v_{Z,t}^* = \phi \varepsilon_{1,t} + \omega_t \sim N(0, \sigma_{v*_Z}^2). \tag{A.41}$$

Next we prefer to devote our attention to the CAR(1) specification by keeping in mind that the algorithm that follows can be easily extended to eq.s (A.40)-(A.41).

Before discussing the estimation of the AC-SVAR model under this data generating process, it is worth considering the main details of the estimation of the parameters θ_1 and $\sigma_{v_Z}^2$ (τ is treated as given) in the CAR(1) model in eq.s (A.38)-(A.39).

Let $Z := (Z_1, ..., Z_T)'$ be the $T \times 1$ vector containing the observations on Z_t. Let P be a $T \times T$ permutation matrix such that:

$$PZ = \begin{pmatrix} P_o \\ P_{cen} \end{pmatrix} Z = \begin{pmatrix} Z_o \\ Z_{cen} \end{pmatrix} \quad \begin{matrix} T_o \times 1 \\ T_{cen} \times 1 \end{matrix} \tag{A.42}$$

i.e. P applied to Z collects the T_o uncensored observations in Z_o first, and then the T_{cen} censored observations collected in Z_{cen}. Hence, T_{cen} represents the number of censored observations and T_o is the number of uncensored observations where, obviously, $T = T_o + T_{cen}$. Let $Z^* := (Z_1^*, ..., Z_T^*)'$

be the $T \times 1$ vector containing the (virtual) observations associated with the uncensored external instrument. It turns out that:

$$Z^* \sim N_T(0_{T\times 1}, V)$$

where $N_T(0_{T\times 1}, V)$ denotes a multivariate normal distribution with expected value $0_{T\times 1}$ and $T \times T$ covariance matrix $V := [V_{i,j}]$. The covariance matrix V summarizes the autocorrelation structure of the AR(1) process, in particular:

$$V_{i,j} := \frac{\sigma^2_{v_Z}}{1-(\theta_1)^2} (\theta_1)^{|i-j|} \ , \quad i, j = 1, ...T. \tag{A.43}$$

By applying the (above defined) permutation matrix P to Z^* yields:

$$PZ^* = \begin{pmatrix} Z^*_o \\ Z^*_{cen} \end{pmatrix} \sim N_T(0_{T\times 1}, PVP') \ ,$$

where

$$PVP' := \left(\begin{array}{c:c} P_o V P'_o & P_o V P'_{cen} \\ \hdashline P_{cen} V P'_o & P_{cen} V P'_{cen} \end{array}\right) =: \left(\begin{array}{c:c} V_{o,o} & V_{o,cen} \\ \hdashline V_{cen,o} & V_{cen,cen} \end{array}\right). \tag{A.44}$$

Note that the first T_o components of the vectors:

$$PZ^* = \begin{pmatrix} Z^*_o \\ Z^*_{cen} \end{pmatrix} \quad \text{and} \quad PZ = \begin{pmatrix} Z_o \\ Z_{cen} \end{pmatrix}$$

are equal. The two vector differ because $Z^*_{cen} \neq Z_{cen}$.

The idea behind Park *et al.* (2007)'s approach is to replace the censored observations in Z_{cen} with sampling values from the conditional distribution:

$$Z^*_{cen} \mid Z_o, Z_{cen} \sim N^0_{T_{cen}}(\rho, \Delta, R_{cen}) \tag{A.45}$$

where $N^0_{T_{cen}}(\rho, \Delta, R_{cen})$ denotes a truncated multivariate normal distribution of dimension T_{cen}, $R_{cen} := (\tau, \infty)^{T_{cen}} = (\tau, \infty) \times (\tau, \infty) \times ... \times (\tau, \infty)$ is the censoring region and ρ and Δ are the conditional mean and covariance matrix given by:

$$\rho := V_{cen,o} (V_{o,o})^{-1} Z_o \quad , \quad \Delta := V_{cen,cen} - V_{cen,o} (V_{o,o})^{-1} V_{o,cen} \tag{A.46}$$

where $V_{cen,o}$, $V_{o,o}$ and $V_{o,cen}$ are defined in eq. (A.44). Note that ρ depends on Z_o, the uncensored process. In the statistical language, the T_{cen} observations generated from the conditional multivariate truncated normal distribution in eq. (A.45) represent 'imputed' values which reproduce what would have been the sequence of observations from the Gaussian AR(1) process in the absence of censoring.

We now extend this idea to the estimation of the AC-SVAR model. Our procedure reads as a two-stage method. In the first stage, the auxiliary model for the external instrument is estimated through an iterative procedure which makes use of simulation methods[5] and generates a time series of 'imputed' (uncensored) values for the external instrument eventually. In the second stage, the AC-SVAR model is estimated by using the series of external instruments reconstructed in the first-stage. In this second-stage, one can apply the partial shocks identification approach discussed in Section 5 of the paper or the full shocks identification approach discussed in Section 6 of the paper.

We consider the following steps.

Step 1 Given the observations $Z := (Z_1, ..., Z_T)'$ on the external instrument, compute the permutation matrix $P := \left(\begin{array}{c:c} P_o' & P_{cen}' \end{array} \right)'$ as in eq. (A.42), so that one obtains Z_o and Z_{cen};

Step 2 Suppose we have initial estimates (which need not be consistent at this initial stage) of θ_1 and σ_Z^2, denoted $\hat{\theta}_1^{(0)}$ and $\left(\hat{\sigma}_Z^2\right)^{(0)}$, respectively. These can be obtained either by applying the distribution-free Least Absolute Deviations (LAD) estimation method of Powell (1984), which provides consistent estimates under fairly general conditions, or by applying OLS to the T_o uncensored observations of the AR(1) process;[6]

Step 3 Given $\hat{\theta}_1^{(0)}$ and $\left(\hat{\sigma}_Z^2\right)^{(0)}$, construct the quantity: $\widehat{V}^{(0)} := [\widehat{V}_{i,j}^{(0)}]$, where (see eq. (A.43))

$$\widehat{V}_{i,j}^{(0)} := \frac{\left(\hat{\sigma}_Z^2\right)^{(0)}}{1 - \left(\hat{\theta}_1^{(0)}\right)^2} \left(\hat{\theta}_1^{(0)}\right)^{|i-j|}, \; i, j = 1, ..., T.$$

Then from $\widehat{V}^{(0)}$ obtain the matrices $\widehat{V}_{cen,o}^{(0)}$, $\widehat{V}_{o,o}^{(0)}$, $\widehat{V}_{o,cen}^{(0)}$ and $\widehat{V}_{cen,cen}^{(0)}$ defined in eq. (A.44), and use the expressions in eq. (A.46) to calculate the conditional mean and covariance matrix:

$$\hat{\rho}^{(0)} := \widehat{V}_{cen,o}^{(0)} \left(\widehat{V}_{o,o}^{(0)}\right)^{-1} Z_o \quad , \quad \widehat{\Delta}^{(0)} := \widehat{V}_{cen,cen}^{(0)} - \widehat{V}_{cen,o}^{(0)} \left(\widehat{V}_{o,o}^{(0)}\right)^{-1} \widehat{V}_{o,cen}^{(0)};$$

Step 4 Given $\hat{\rho}^{(0)}$ and $\widehat{\Delta}^{(0)}$ generate T_{cen} observations randomly from the multivariate truncated normal distribution $N_{T_{cen}}^0(\hat{\nu}^{(0)}, \widehat{\Delta}^{(0)}, R_{cen})$[7] and collect these observations in $Z_{cen}^{(1)}$;

Step 5 Construct the augmented data set by using the observed part and the imputed sample from the censored part, i.e.:

$$Z^{(1)} = P^{-1} \begin{pmatrix} Z_o \\ Z_{cen}^{(1)} \end{pmatrix} \quad \begin{array}{c} T_o \times 1 \\ T_{cen} \times 1 \end{array}$$

[5]We refer to Hajivassiliou and Ruud (1994) for a general review of estimation methods of limited dependent variables models (which includes censoring processes) based on simulation methods.

[6]In this second case such naive estimator of θ_1 and σ_Z^2 is not consistent.

[7]This can be done e.g. by the Gibbs sampling described in Gelfand and Smith (1990).

where $P := (P_o' \vdots P_{cen}')'$ is given in Step 1;

Step 6 Re-estimate the parameters θ_1 and σ_Z^2 by using the T observations in $Z^{(1)}$ for the external instrument. These parameters can be now estimated by 'standard' methods, i.e. by using OLS or (Gaussian) ML based on the $t = 1, ..., T$ observations in $Z^{(1)}$;

Step 7 Repeat the steps 3-6 using the updated estimates of θ_1 and σ_Z^2 at every iteration until the parameter estimates converge according to some pre-fixed convergence rule. Each iteration generates sequences of imputed observations $Z^{(j)}$, $j = 1, ..., stop$. Let $Z^{(stop)} := (Z_1^{(stop)}, ..., Z_T^{(stop)})'$ be the vector of T observations on the external instruments obtained at the convergence of the process;

Step 8 Consider the estimation of the reduced form parameters $(\widetilde{\Psi}, \Sigma_\eta)$ associated with the AC-SVAR model:

$$\begin{pmatrix} Y_t \\ Z_t^{(stop)} \end{pmatrix} = \begin{pmatrix} \Pi & 0_{n\times 1} \\ 0_{1\times n} & \theta_1 \end{pmatrix} \begin{pmatrix} X_t \\ Z_{t-1}^{(stop)} \end{pmatrix} + \begin{pmatrix} u_t \\ v_{Z,t}^{(stop)} \end{pmatrix} \quad , \quad \Sigma_\eta := \begin{pmatrix} \Sigma_u & \Sigma_{u,v_Z} \\ \Sigma_{v_Z,u} & \sigma_{v_Z}^2 \end{pmatrix} \tag{A.47}$$

using the (original) observations $Y_1, ..., Y_T$ for the SVAR and the observations $Z_1^{(stop)}, ..., Z_T^{(stop)}$ for the external instrument obtained in Step 7. In eq. (A.47), $v_{Z,t}^{(stop)}$ is the innovation: $v_{Z,t}^{(stop)} := Z_t^{(stop)} - \theta_1 Z_{t-1}^{(stop)}$. Estimation is performed by Gaussian (constrained) ML along the lines discussed in Section A.6. Note in particular, that the covariance matrix Σ_{u,v_Z}, which plays a key role in the proxy-SVAR approach, is estimated by $\widehat{\Sigma}_{u,v_Z} := \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t \hat{v}_{Z,t}^{(stop),\prime}$;

Step 9 In the partial shocks identification strategy the AC-SVAR model is estimated as detailed in Section 5 of the paper. In the full shocks identification strategy the AC-SVAR model is estimated as outlined in Section 6 of the paper.

Some remarks are in order.

First, it is seen that in the estimation of the parameters of the auxiliary model for the censored external instrument, the algorithm is based on two principles: data augmentation (imputation) through simulation of the censored part of the data (Steps 1-7), and estimation which follows 'conventional' methods because it is based on reconstructed (imputed) data treated as uncensored (Steps 8-9).

Second, Park *et al.* (2007) show by simulation studies that the distribution of the parameter estimates obtained with the simulation/imputation method does not deviate substantially from the distribution of the parameter estimates obtained by considering the uncensored process Z_t^*. Based on these results, it is reasonable to conjecture that the asymptotic distributions of the estimators discussed in Sections 5 and 6 of the paper should be still Gaussian. This conjecture, however, should be analyzed in detail analytically and through additional simulations studies specifically designed to the features of the AC-SVAR model.

Third, conditional on the conjecture above, bootstrap confidence bands for the IRFs can be still computed by the algorithms summarized in Section A.10.

Fourth, the suggested method can be extended along several directions. In principle, it is possible to specify more involved dynamic processes for Z_t^*. Moreover, one could potentially consider the case of multiple external instruments for g structural shocks of interest, i.e. $r := r_{unc} + r_{cen} \geq g \geq 1$, where r_{cen} is the number of censored external instruments (with censoring mechanism analog to that in eq.s (A.38)-(A.39)) and r_{unc} is the number of uncensored instruments. The analysis in this case may become considerably more involved, especially if the r_{cen} external instruments are driven by completely different censoring processes.

These extensions deserve detailed investigations which are left to future research.

A.12 Further empirical results

This section complements the empirical results of the paper.

A.12.1 The oil supply shock

In this section we explain how the oil supply shock, oil_t, used as external instrument (b) in Section 8 of the paper has been constructed.

Our analysis follows Kilian's (2009) identification strategy for uncovering oil supply shocks, see also Wieland (2019). We start from a vector of three monthly variables $X_t := (\Delta oilprod_t, rea_t, rpo_t)'$, where $\Delta oilprod_t$ is the growth in global oil production, rea_t is a measure of global real economic activity and rpo_t is the log real oil prices,[8] and then consider the SVAR representation:

$$A_0 X_t = \Upsilon_X + \sum_{j=1}^{24} A_j X_{t-j} + \varepsilon_t^X \tag{A.48}$$

which is taken from Kilian (2009). In eq. (A.48), A_0 is the 3×3 matrix of structural parameters, Υ_X is a 3×1 constant, A_j, $j = 1, ..., 24$ are 3×3 matrices of parameters associated with the lags of X_t and, finally, ε_t^X is the 3×1 vector of structural shocks with diagonal covariance matrix. The first element in ε_t^X is the shock to global oil production, which Wieland (2019) refers to as the 'oil supply shock'.

Assuming as in Kilian (2009) that oil production responds to other structural shocks (e.g. demand shocks) with at least one-month delay, it is possible to argue that the first row of A_0 is

[8] We thank Johannes Wieland for kindly providing us with the monthly dataset.

given by (1,0,0) (this includes the normalization to '1' of the first element on the main diagonal of A_0). The first equation of system (A.48) is given by:

$$\Delta oilprod_t = \Upsilon_{X,1} + \sum_{j=1}^{24} \alpha_{j,1} X_{t-j} + \varepsilon_{1,t}^{X} \tag{A.49}$$

where $\Upsilon_{X,1}$ is the first element of Υ_X, $\alpha_{j,1}$ are the 1×3 first row vectors of the matrices A_j, $j = 1, ..., 24$, and $\varepsilon_{1,t}^{X}$ is the first element of ε_t^X. The regression in eq. (A.49) is estimated on the period 1973M2-2015M9 and the time series oil_t used in the paper corresponds to $oil_t := \hat{\varepsilon}_{1,t}^{X}$, $t =$2008M1, ...,2015M4, $\hat{\varepsilon}_{1,t}^{X}$, $t =$2008M1, ...,2015M4, being the OLS residuals.

A.12.2 Housing starts and hours worked

In this section we modify the baseline AC-SVAR model estimated in Section 8 of the paper (with using the couple of external instruments (a,b)) by replacing the external instrument oil_t (b) with (c): the innovations obtained from an auxiliary model for $\Delta hours_t$, where $hours_t$ is the log of hours worked (source: Fred).

The AC-SVAR model is estimated on the period 2008M1-2015M4, and is obtained by appending an auxiliary model for $Z_t := (Z_{1,t}, Z_{2,t})' = (\Delta house_t,\ \Delta hours_t)'$ to the system for Y_t:=$(a_t, U_{F,t}, U_{M,t})'$ discussed in the paper. Thee two models form an AC-SVAR specification like the one in eq.s (29)-(30) of the paper.

Admittedly, differently from $v_{Z_1,t} := \Delta house_t - E(\Delta house_t \mid \mathcal{F}_{t-1})$, we do not have a sound economic argument to motivate the orthogonality of the innovations $v_{Z_2,t} := \Delta hours_t - E(\Delta hours_t \mid \mathcal{F}_{t-1})$ to financial and macroeconomic uncertainty shocks. However, the relevance and orthogonality conditions are verified ex-post as in the paper.

In this case, a simple Wald test for the null hypothesis $H_0' : vec(\Sigma_{v_Z,u}) = 0_{6\times 1}$ (no correlation between the external instruments and the VAR disturbances) against $H_1' : vec(\Sigma_{v_Z,u}) \neq 0_{6\times 1}$, see Section 5 in the paper and Section A.9, is equal to W_T^{rel}:=23.05 with a p-value of 0.00**,** hence the reduced form evidence indirectly rejects the hypothesis that the relevance parameters are zero at the 5% level of significance.

Empirical results are summarized in Table A.2, which reproduces the same informations as Table 1 in the paper. Panel A of Table A.2, left-side, pertains to the estimates obtained with partial shocks identification strategy, while the right-side pertains to the estimates obtained with the full shocks identification strategy. Panel B of Table A.2 investigates the relevance conditions (partial shocks approach) and the relevance and orthogonality conditions (full shocks approach) and confirms that the two instruments (a,c) can be considered empirically 'valid'.

The IRFs are summarized in Figure A.1 along with 90%-bootstrap simultaneous 'sup-t' bands (the hypothesis $b_{F,a} = 0$ and $b_{M,a} = 0$, strongly supported by the data, is imposed in the estimation of the IRFs). The overall picture that emerges from Table A.2 and Figure A.1 confirms qualitatively the results obtained with the baseline AC-SVAR with external instruments (a,b).

A.12.3 Choleski-SVARs

In this section we resume the discussion in Section 4 of the paper and focus on the specification in eq. (15) which represents the starting point of our investigation of the exogeneity/endogeneity of uncertainty in small-scale SVARs.

The relationship between the reduced form disturbances and the structural shocks discussed in eq. (15) of the paper is here reported for convenience:

$$\underbrace{\begin{pmatrix} u_{a,t} \\ u_{F,t} \\ u_{M,t} \end{pmatrix}}_{u_t} = \underbrace{\begin{pmatrix} b_{a,a} & b_{a,F} & b_{a,M} \\ b_{F,a} & b_{F,F} & b_{F,M} \\ b_{M,a} & b_{M,F} & b_{M,M} \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} \varepsilon_{a,t} \\ \varepsilon_{F,t} \\ \varepsilon_{M,t} \end{pmatrix}}_{\varepsilon_t}. \tag{A.50}$$

In eq. (A.50), u_t:=$(u_{a,t}, u_{F,t}, u_{M,t})'$ is the vector of VAR reduced form disturbances and $\varepsilon_t := (\varepsilon_{a,t}, \varepsilon_{F,t}, \varepsilon_{M,t})'$ is the vector of structural shocks. Recall that we are interested in the identification of the parameters in the first column of the matrix B, in particular the parameters $b_{F,a}$ and $b_{M,a}$ which capture the instantaneous impact of the non-uncertainty shock of the system on financial and macroeconomic uncertainty, respectively. In principle, the three zero restrictions $b_{a,F} = b_{a,M} = b_{F,M} = 0$ permit to identify the SVAR as they imply a triangular structure for B. However, reverse causality between uncertainty and real economic activity can not be addressed in a so-specified Choleski-SVAR because, given the ordering of the variables in eq. (A.50), the restrictions $b_{a,F} = b_{a,M} = b_{F,M} = 0$ prevent financial and macroeconomic uncertainty to affect real economic activity within the month, and this appears in sharp contrast with a large empirical literature on uncertainty. In other words, and as observed in the paper, an empirical analysis based on a Choleski-SVAR with $b_{a,F} = b_{a,M} = b_{F,M} = 0$ would not consider that the 'endogeneity' of $U_{F,t}$ and $U_{M,t}$ should be inferred from the data, not postulated a-priori.

The empirical results discussed in the paper partly conflicts with the hypothesis that the matrix B has a lower triangular structure in which a_t is ordered first. In particular, Panel B of Table 1 of the paper, right-side, reports the ML estimates of the parameters of the matrix:

$$\widetilde{G} := \left(\widetilde{G}_1 \;\vdots\; \widetilde{G}_2 \right) = \left(\begin{array}{c:c:c} B_1 & B_2 & 0_{3\times 2} \\ \hdashline \Phi & 0_{2\times 2} & \Sigma_{\varpi}^{1/2} \end{array} \right) = \left(\begin{array}{c:cc:cc} b_{a,a} & b_{a,F} & b_{a,M} & 0 & 0 \\ b_{F,a} & b_{F,F} & 0 & 0 & 0 \\ b_{M,a} & b_{M,F} & b_{M,M} & 0 & 0 \\ \hdashline \varphi_{1,a} & 0 & 0 & \varpi_{1,1} & 0 \\ \varphi_{2,a} & 0 & 0 & 0 & \varpi_{2,2} \end{array} \right)$$

where it is seen that B is a block of $\widetilde{G}$. The empirical results in Table 1 of the paper suggest that while the estimated $b_{a,F}$ coefficient is not significant at the 5% level of significance, the coefficients $b_{a,M}$ is significant and the hypothesis $b_{a,M} = 0$ is rejected, meaning that macroeconomic uncertainty has a direct and instantaneous (within-the-month) contractionary effect on real economic activity. This effect would be ruled out by construction in a Choleski-SVAR for $Y_t := (U_{F,t}, U_{M,t}, a_t)'$.

Overall, the empirical analyses presented in the paper support the specification of a 'standard' SVARs with ordering $Y_t^\circ := (U_{F,t}, U_{M,t}, a_t)'$ and matrix B°:

$$B^\circ := \begin{pmatrix} b_{F,F} & 0 & 0 \\ b_{M,F} & b_{M,M} & 0 \\ b_{a,F} & b_{a,M} & b_{a,a} \end{pmatrix}. \tag{A.51}$$

We call this model the 'ex-post Cholesi-SVAR' as we have inferred its recursive structure from the estimation of a non-recursive specification. The estimated parameters of the matrix B° are reported in Table A.3 with associated standard errors. It can be noticed that the estimated parameter $b_{a,F}$ is not significant as it happens in Table 1 of the paper. The implied IRFs with 90%-bootstrap simultaneous ('sup-t') confidence bands are plotted in Figure A.2 along with the IRFs obtained from the baseline proxy-SVAR estimated in the paper (i.e. the ones in Figure 1 of the paper). As expected, it is seen that the estimated dynamics causal effects are substantially similar.

A.12.4 An invalid external instrument

In this section we still address the empirical assessment of the exogeneity/endogeneity of uncertainty considered in Section 8 of the paper by changing the estimated AC-SVAR model along two directions. We consider (i) the 'one shock-one instrument' case, $r = g = 1$, and (ii) a scenario in which the external instrument selected for the non-uncertainty shock $\varepsilon_{a,t}$ is not orthogonal to the non-instrumented structural shocks (which are the financial uncertainty shock $\varepsilon_{F,t}$ and the macroeconomic uncertainty shock $\varepsilon_{M,t}$, respectively). We do so in order to check the empirical performance of our approach in the presence of an invalid (non-orthogonal) external instruments. The results of the Monte Carlo experiments discussed in Section A.8 suggest that one should reject true parametric restrictions if the external instruments are not orthogonal to the non-instrumented structural shocks. We investigate to what extent our approach detects situations like these in practice.

Let $Z_t(v_{Z,t})$ be a scalar external instrument for $\varepsilon_{a,t}$, so that $r = g = 1$. In this case, the structural matrix $\widetilde{G}$ that characterizes the AC-SVAR model (see eq. (11) in the paper) reads:

$$\widetilde{G} := \left(\begin{array}{c:c} \widetilde{G}_1 & \widetilde{G}_2 \end{array} \right) = \left(\begin{array}{c:c:c} B_1 & B_2 & 0_{3\times 1} \\ \hdashline \phi & 0_{2\times 1} & \sigma_{\varpi} \end{array} \right) = \left(\begin{array}{c:cc:c} b_{a,a} & b_{a,F} & b_{a,M} & 0 \\ b_{F,a} & b_{F,F} & b_{F,M} & 0 \\ b_{M,a} & b_{M,F} & b_{M,M} & 0 \\ \hdashline \varphi_{1,a} & 0 & 0 & \sigma_{\varpi} \end{array} \right) \tag{A.52}$$

where $\Phi = \phi = \varphi_{1,a}$ is the relevance parameter which captures the correlation between $v_{Z,t}$ and $\varepsilon_{a,t}$, and $\sigma_{\varpi}^2 := E(v_{Z,t}^2)$. Recall that we are interested in testing the restrictions $b_{F,a} = 0$ and $b_{M,a} = 0$ which make financial and macroeconomic uncertainty 'contemporaneously exogenous' with respect to business cycle fluctuations.

The results in Proposition 1 of the paper suggest that the column $\widetilde{G}_1 := (b_{a,a}, b_{F,a}, b_{M,a}, \varphi_{1,a})'$ of the matrix $\widetilde{G}$ is identified and that $\phi := \varphi_{1,a} \neq 0$ is a necessary and sufficient condition for identification. Instead, the results in Proposition 2 of the paper suggests that if one adds the restrictions $b_{F,M} = 0$ (discussed in the paper) in the third column of $\widetilde{G}$ in eq. (A.52), the three structural shocks of the system can be identified from the AC-SVAR model despite only $\varepsilon_{a,t}$ is instrumented. Identification, however, holds conditional to the validity of the orthogonality restriction $E(v_{Z,t}(\varepsilon_{F,t}, \varepsilon_{M,t})) = (0, 0)$ which is reflected in the zeros in the positions (4,2) and (4.3) of the matrix $\widetilde{G}$. The Monte Carlo experiments discussed in Section A.8 point out that regardless of whether we work under a partial shocks or full shocks identification strategy, the restrictions $b_{F,a} = 0$ and $b_{M,a} = 0$ should be rejected by the overidentification restrictions tests if $E(v_{Z,t}(\varepsilon_{F,t}, \varepsilon_{M,t})) \neq (0, 0)$.

We consider the following external instruments for $\varepsilon_{a,t}$: changes in the log of real personal consumption expenditure, denoted Δc_t (source: Fred). Let $v_{Z,t} := \Delta c_t - E(\Delta c_t \mid \mathcal{F}_{t-1})$ be the innovation associated with a dynamic auxiliary reduced form equation specified for real consumption expenditure growth. It is reasonable to assume that $v_{Z,t}$ is correlated with the non-uncertainty shock, but it is also reasonable to conjecture that the orthogonality condition $E(v_{Z,t}\varepsilon_{M,t}) = 0$ is likely not to hold because of the precautionary saving channel through which macroeconomic uncertainty induces a decline in activity.

In this setup, the AC-SVAR model is given by the VAR for $Y_t := (a_t, U_{F,t}, U_{M,t})'$ and a dynamic auxiliary reduced form model for Δc_t, given respectively by:

$$Y_t = \Pi_1 Y_{t-1} + \Pi_2 Y_{t-2} + \Pi_3 Y_{t-3} + \Pi_4 Y_{t-4} + \Upsilon_y + u_t \tag{A.53}$$

$$\Delta c_t = \gamma_1 U_{F,t-1} + \gamma_2 U_{F,t-2} + \gamma_z + v_{Z,t} \tag{A.54}$$

where $m = n + r = 4$ and the VAR lag order is $\ell^{op} = k = 4$. The reduced form disturbances $\eta_t := (u_t', v_{Z,t}')'$ of system (A.53)-(A.54) are linked to $\xi_t := (\varepsilon_t', \omega_t^{\circ\prime})'$ by the relationship $\eta_t = \widetilde{G}\xi_t$, where the form of the matrix $\widetilde{G}$ is in eq. (A.52). System (A.53)-(A.54) is estimated on the period 2008M1-2015M4. A battery of diagnostic tests (nor reported to save space) show that the estimated AC-SVAR model displays 'well-behaved' residuals.

Panel A of Table A.4 summarizes the estimation and testing results. The left-side refers to the partial shocks identification approach and the CMD estimates, and the right-side to the full shocks identification strategy and the ML estimates.

The left-side of Panel A of Table A.4 shows that the estimated relevance parameter $\varphi_{1,a} = E(v_{Z,t}\varepsilon_{a,t})$ is positive and strongly significant. The overidentification restrictions test $TQ(\hat{\vartheta}_T)_{exog}$ rejects the hypothesis of 'contemporaneous exogeneity' of financial and macroeconomic uncertainty, $b_{F,a} = 0$ and $b_{M,a} = 0$, with a p-value slightly inferior to 0.04. It can be noticed that the estimated parameter $b_{F,a}$, which captures the instantaneous impact of the non-uncertainty shock on financial uncertainty, is negative and strongly significant. The estimated correlation between $v_{Z,t}$ and $\varepsilon_{a,t}$ in Panel B of Table A.4, left-side, is 0.34 and is significant at the 5% level of significance. Based on these results, one might be tempted to conclude that financial uncertainty responds endogenously to the identified non-uncertainty shock. However, the results of the Monte Carlo experiments discussed in Section A.8 suggest that the rejection of the restrictions $b_{F,a} = 0$ and $b_{M,a} = 0$ might be due to the failure of the orthogonality condition between $v_{Z,t}$ and the non-instrumented shocks $\varepsilon_{2,t} := (\varepsilon_{F,t}, \varepsilon_{M,t})'$. We investigate this issue next.

Panel A of Table A.4 summarizes the ML estimate of the parameters of the entire matrix $\widetilde{G}$ in eq. (A.52) along with the LR test for the restrictions $b_{F,a} = 0$ and $b_{M,a} = 0$. Also in this case, the tests rejects the null hypothesis with a p-value of 0.03. The estimates in the first column of $\widetilde{G}$ confirm the results obtained with the partial shock identification approach. In addition, we note that the identified financial and macroeconomic uncertainty shocks do not affect significantly the industrial production growth on-impact (and the sign of the instantaneous impact is positive). The novelty, however, is that we can now evaluate the quality of the identification by computing the correlation between $v_{Z,t}$ and the estimated shocks $\varepsilon_t := (\varepsilon_{1,t}, \varepsilon_{2,t})'$. Results in Panel B of Table A.4 show that $v_{Z,t}$ and $\varepsilon_{a,t}$ are significantly correlated, but also $v_{Z,t}$ and $\varepsilon_{M,t}$ are correlated at the 5% level of significance.

We have documented the fact that the innovations built from the model for the changes in real personal consumption growth do not represent valid external instrument for the non-uncertainty shock.

A.12.5 Controlling for the stance of monetary policy

In this section we check the robustness of the results obtained with the baseline AC-SVAR model estimated in the paper to the stance of monetary policy. Since the estimation sample

2008M1-2015M4 broadly coincides with the zero lower bound constraint, we prefer not to consider explicitly the identification of a monetary policy shock. As is known, assessing the impact of unconventional policy is more challenging than it is for conventional policy, see, among many others, Gertler and Karadi (2015) and Roger *et al.* (2018). In this section we check whether and to what extends the results obtained with the baseline specification discussed in Section 8 of the paper change once we include Wu and Xia's (2016) 'shadow rate' and the inflation rate as control variables in the proxy-SVAR. Related to this, recently, Pellegrino (2017) has shown that expansionary monetary policy shocks tend to reduce the uncertainty in tranquil and turbulent periods, but are significantly less powerful during uncertain times.

We consider the same specification as in eq.s (29)-(30) of the paper, i.e. the model based on Y_t:=$(a_t, U_{F,t}, U_{M,t})'$ and $Z_t := (Z_{1,t}, Z_{2,t})' = (\Delta house_t,\, oil_t)'$ (i.e. the couple (a,b)), with an important difference: we include Wu and Xia (2016)'s 'shadow rate', sr_t, and the inflation rate, π_t, as 'exogenous-X' variables as follows:

$$\begin{aligned} Y_t &= \Pi_1 Y_{t-1} + \Pi_2 Y_{t-2} + \Pi_3 Y_{t-3} + \Pi_4 Y_{t-4} + \varkappa L_t + \Upsilon_y + u_t \\ Z_t &= \Theta_1 Z_{t-1} + \Gamma_1 Y_{t-1} + \Gamma_2 Y_{t-2} + \Upsilon_z + v_{Z,t} \end{aligned}$$

where $L_t := (sr_t, \pi_t)'$ and the 3×2 matrix $\varkappa$ captures the instantaneous impact of the shadow rate and the inflation rate on Y_t.[9] The structural specification is still given by:

$$(u_t', v_{Z,t}')' =: \eta_t = \widetilde{G}\xi_t$$

and the matrix $\widetilde{G}$ if given by:

$$\widetilde{G} := \left(\begin{array}{c:cc:cc} b_{a,a} & b_{a,F} & b_{a,M} & 0 & 0 \\ b_{F,a} & b_{F,F} & 0 & 0 & 0 \\ b_{M,a} & b_{M,F} & b_{M,M} & 0 & 0 \\ \hdashline \varphi_{1,a} & 0 & 0 & \varpi_{1,1} & 0 \\ \varphi_{2,a} & 0 & 0 & 0 & \varpi_{2,2} \end{array}\right)$$

as in the baseline specification of the paper. The novelty in this model is the appearance of the control variables L_t on the right-hand-side of the Y_t-equations.

The estimation results are summarized in Table A.5 which reproduces the same structure as Table 1 in the paper. The implied IRFs with 90%-bootstrap simultaneous confidence intervals

[9]Thus, from this version of the AC-SVAR model one can potentially track the dynamic impact of L_t on Y_{t+h}, $h = 0, 1, 2, ...$ by computing the multipliers

$$(J_n \left(A_y\right)^h J_n')\varkappa\ ,\ \ h = 0, 1, 2, ...$$

where J_n and A_y are the same selection matrix and companion matrix used for the IRFs.

are summarized in Figure A.3 (the hypothesis $b_{F,a} = 0$ and $b_{M,a} = 0$ is imposed in the estimation of the IRFs). It can be noticed that the results do not change substantially relative to what obtained with the baseline AC-SVAR model discussed in the paper.

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Table A.1. Monte Carlo experiments

Panel A			
Partial Shocks Identification			
T	$v_1 = 0, v_2 = 0$	$v_1 = 0, v_2 = -0.1$	$v_1 = -0.75, v_2 = -0.25$
100	0.080	0.229	1
250	0.058	0.470	1
500	0.053	0.758	1
Panel B			
Full Shocks Identification			
	case (i)	case (ii)	case (iii)
T	$v_1 = 0, v_2 = 0$	$v_1 = 0, v_2 = -0.1$	$v_1 = -0.75, v_2 = -0.25$
100	0.082	0.187	1
250	0.056	0.362	1
500	0.054	0.616	1

NOTES: Rejection frequencies (computed over $M = 1000$ simulations) of tests for the null hypothesis that the shock $\varepsilon_{A,t}$ does not affect the variables $(Y_{B,t}, Y_{C,t})'$ instantaneously, corresponding to the two over-identification restrictions $b_{B,A} = b_{C,A} = 0$, under three different scenarios about the exogeneity of the external instruments. Panel A: rejection frequencies of the CMD-based test for the overidentification restrictions in the partial shocks identification approach, see Section 5 of the paper. Panel B: rejection frequencies of the LR test for the overidentification restrictions in the full shocks identification approach, see Section 6 of the paper. All tests are computed at the 5% nominal level of significance.

Table A.2. Estimated AC-SVAR model, $Z_t := (\Delta house_t, \Delta hours_t)'$

Panel A

Partial Shocks Identification

$$\widehat{\widetilde{G}}_1 = \left(\begin{array}{c} \underset{0.0980}{0.6218} \\ \underset{0.0009}{-0.0009} \\ \underset{0.0037}{-0.0003} \\ \hdashline \underset{0.0114}{0.0239} \\ \underset{0.0009}{0.0027} \end{array}\right)$$

$$TQ(\vartheta)_{exog} = 0.07[0.97]$$

Full Shocks Identification

$$\widehat{\widetilde{G}} = \left(\begin{array}{ccc:cc} \underset{0.0603}{0.6197} & \underset{0.0806}{-0.0710} & \underset{0.0794}{-0.2064} & 0 & 0 \\ \underset{0.0037}{-0.0009} & \underset{0.0017}{0.0222} & 0 & 0 & 0 \\ \underset{0.0016}{-0.0002} & \underset{0.0010}{0.0034} & \underset{0.0007}{0.0094} & 0 & 0 \\ \hdashline \underset{0.0089}{0.0240} & 0 & 0 & \underset{0.0062}{0.0819} & 0 \\ \underset{0.0004}{0.0027} & 0 & 0 & 0 & \underset{0.0003}{0.0034} \end{array}\right)$$

$$LR_T = 5.72[0.13] \quad LR_{exog} = 0.06[0.97]$$

Panel B

Correlations (relevance)

	$\hat{\varepsilon}_a$
$\hat{v}_{Z_1}$	0.28[0.01]
$\hat{v}_{Z_2}$	0.63[0.00]

Correlations (relevance, orthogonality)

	$\hat{\varepsilon}_a$	$\hat{\varepsilon}_F$	$\hat{\varepsilon}_M$
$\hat{v}_{Z_1}$	0.28[0.01]	−0.02[0.87]	0.04[0.67]
$\hat{v}_{Z_2}$	0.63[0.00]	−0.02[0.83]	−0.02[0.86]

NOTES: Estimated AC-SVAR model for $Y_t := (a_t, U_{Ft}, U_{Mt})'$ and external instrument $Z_t := (\Delta house_t, \Delta hours_t)'$, period 2008:M1-2015:M4 (T=88). Panel A: estimates. Left side, CMD estimates of $\widetilde{G}_1$ with associated standard errors, '$TQ(\vartheta)_{exog}$' is the overidentification restriction test for the null $b_{F,a} = 0$ and $b_{M,a} = 0$. Right side, ML estimates of $\widetilde{G}$ with associated standard errors, 'LR_T' is a test for the 3 overidenfication restrictions featured by the estimated model, 'LR_{exog}' is the overidentification test for the null $b_{F,a} = 0$ and $b_{M,a} = 0$. Numbers in brackets are p-values. Panel B: ex-post correlations. Left side, ex-post correlations between the structural shock $\hat{\varepsilon}_a$ and the reduced form shocks $\hat{v}_{Z_1}$ and $\hat{v}_{Z_2}$ (relevance). Right side, ex-post correlations between the structural shocks $\hat{\varepsilon}_t := (\hat{\varepsilon}_{at}, \hat{\varepsilon}_{Ft}, \hat{\varepsilon}_{Mt})'$ and the reduced form shocks $\hat{v}_{Z_1}$ and $\hat{v}_{Z_2}$ (relevance and orthogonality).

Table A.3. Estimated Choleski-SVAR

$$\widehat{B}^{\circ} = \begin{pmatrix} \underset{0.0017}{0.0223} & 0 & 0 \\ \underset{0.0010}{0.0034} & \underset{0.0007}{0.0094} & 0 \\ \underset{0.0676}{-0.0944} & \underset{0.0653}{-0.2103} & \underset{0.0448}{0.5937} \end{pmatrix}$$

NOTES: Estimated Choleski-SVAR for $Y_t^{\circ} := (U_{Ft}, U_{Mt}, a_t)'$, period 2008:M1-2015:M4 (T=88), with associated standard errors.

Table A.4. Estimated AC-SVAR model with an invalid external instrument

Panel A

Partial Shocks Identification

$$\widehat{\widetilde{G}}_1 = \left(\begin{array}{c} \underset{0.1194}{0.5488} \\ \underset{0.0022}{-0.0091} \\ \underset{0.0065}{-0.0078} \\ \hdashline \underset{0.0004}{0.0009} \end{array}\right)$$

$TQ(\vartheta)_{exog} = 6.90[0.03]$

Full Shocks Identification

$$\widehat{\widetilde{G}} = \left(\begin{array}{ccc:c} \underset{0.1074}{0.5488} & \underset{0.1696}{0.1393} & \underset{0.1715}{0.2986} & 0 \\ \underset{0.0065}{-0.0091} & \underset{0.0032}{0.0203} & 0 & 0 \\ \underset{0.0020}{-0.0078} & \underset{0.0025}{0.0003} & \underset{0.0022}{0.0063} & 0 \\ \hdashline \underset{0.0003}{0.0009} & 0 & 0 & \underset{0.0002}{0.0025} \end{array}\right)$$

$LR_{exog} = 6.52[0.04]$

Panel B

Correlations (relevance)

	$\hat{\varepsilon}_a$
$\hat{\upsilon}_{Z_1}$	0.34[0.00]

Correlations (relevance, orthogonality)

	$\hat{\varepsilon}_a$	$\hat{\varepsilon}_F$	$\hat{\varepsilon}_M$
$\hat{\upsilon}_{Z_1}$	0.22[0.04]	−0.14[0.19]	−0.23[0.03]

NOTES: Estimated AC-SVAR model for $Y_t := (a_t, U_{Ft}, U_{Mt})'$ and external instrument $Z_t := (\Delta c_t)$, period 2008:M1-2015:M4 (T=88). Panel A: estimates. Left side, CMD estimates of $\widetilde{G}_1$ with associated standard errors, '$TQ(\vartheta)_{exog}$' is the overidentification restriction test for the null $b_{F,a} = 0$ and $b_{M,a} = 0$. Right side, ML estimates of $\widetilde{G}$ with associated standard errors, 'LR_{exog}' is the overidentification test for the null $b_{F,a} = 0$ and $b_{M,a} = 0$. Numbers in brackets are p-values. Panel B: ex-post correlations. Left side, ex-post correlations between the structural shock $\hat{\varepsilon}_a$ and the reduced form shock $\hat{\upsilon}_{Z_1}$ (relevance). Right side, ex-post correlations between structural shocks $\hat{\varepsilon}_t := (\hat{\varepsilon}_{at}, \hat{\varepsilon}_{Ft}, \hat{\varepsilon}_{Mt})'$ and the reduced form shock $\hat{\upsilon}_{Z_1}$ (relevance and orthogonality).

Table A.5. Estimated AC-SVAR model with control variable $X_t := (\pi_t, sr_t)'$.

Panel A

Partial Shocks Identification

$$\widehat{\widetilde{G}}_1 = \begin{pmatrix} \underset{0.1085}{0.5395} \\ \underset{0.0020}{-0.0051} \\ \underset{0.0029}{0.0014} \\ \underset{0.0110}{0.0256} \\ \underset{0.0706}{0.1489} \end{pmatrix}$$

$TQ(\vartheta)_{exog} = 1.48[0.48]$

Full Shocks Identification

$$\widehat{\widetilde{G}} = \left(\begin{array}{ccc:cc} \underset{0.0944}{0.5396} & \underset{0.1369}{0.0794} & \underset{0.1332}{-0.3274} & 0 & 0 \\ \underset{0.0052}{-0.0050} & \underset{0.0019}{0.0197} & 0 & 0 & 0 \\ \underset{0.0025}{0.0014} & \underset{0.0012}{0.0028} & \underset{0.0009}{0.0091} & 0 & 0 \\ \hdashline \underset{0.0084}{0.0256} & 0 & 0 & \underset{0.0058}{0.0757} & 0 \\ \underset{0.0575}{0.1488} & 0 & 0 & 0 & \underset{0.0397}{0.5222} \end{array}\right)$$

$LR_T = 0.42[0.94]$ $LR_{exog} = 1.52[0.47]$

Panel B

Correlations (relevance)

	$\hat{\varepsilon}_a$
$\hat{v}_{Z_1}$	0.32[0.00]
$\hat{v}_{Z_2}$	0.27[0.01]

Correlations (relevance, orthogonality)

	$\hat{\varepsilon}_a$	$\hat{\varepsilon}_F$	$\hat{\varepsilon}_M$
$\hat{v}_{Z_1}$	0.31[0.00]	−0.04[0.71]	0.09[0.40]
$\hat{v}_{Z_2}$	0.26[0.01]	−0.11[0.30]	0.03[0.79]

NOTES: Estimated AC-SVAR for $Y_t := (a_t, U_{Ft}, U_{Mt})'$, external instrument $Z_t := (\Delta house_t, oil_t)'$ and exogenous control variable $X_t := (\pi_t, sr_t)'$. Panel A: estimates. Left side, CMD estimates of $\widetilde{G}_1$ with associated standard errors, '$TQ(\vartheta)_{exog}$' is the overidentification restriction test for the null $b_{F,a} = 0$ and $b_{M,a} = 0$. Right side, ML estimates of $\widetilde{G}$ with associated standard errors, 'LR_T' is a test for the 3 overidenfication restrictions featured by the estimated model, 'LR_{exog}' is the overidentification test for the null $b_{F,a} = 0$ and $b_{M,a} = 0$. Numbers in brackets are p-values. Panel B: ex-post correlations. Left side, ex-post correlations between the structural shock $\hat{\varepsilon}_a$ and the reduced form shocks $\hat{v}_{Z_1}$ and $\hat{v}_{Z_2}$ (relevance). Right side, ex-post correlations between the structural shocks $\hat{\varepsilon}_t := (\hat{\varepsilon}_{at}, \hat{\varepsilon}_{Ft}, \hat{\varepsilon}_{Mt})'$ and the reduced form shocks $\hat{v}_{Z_1}$ and $\hat{v}_{Z_2}$ (relevance and orthogonality).

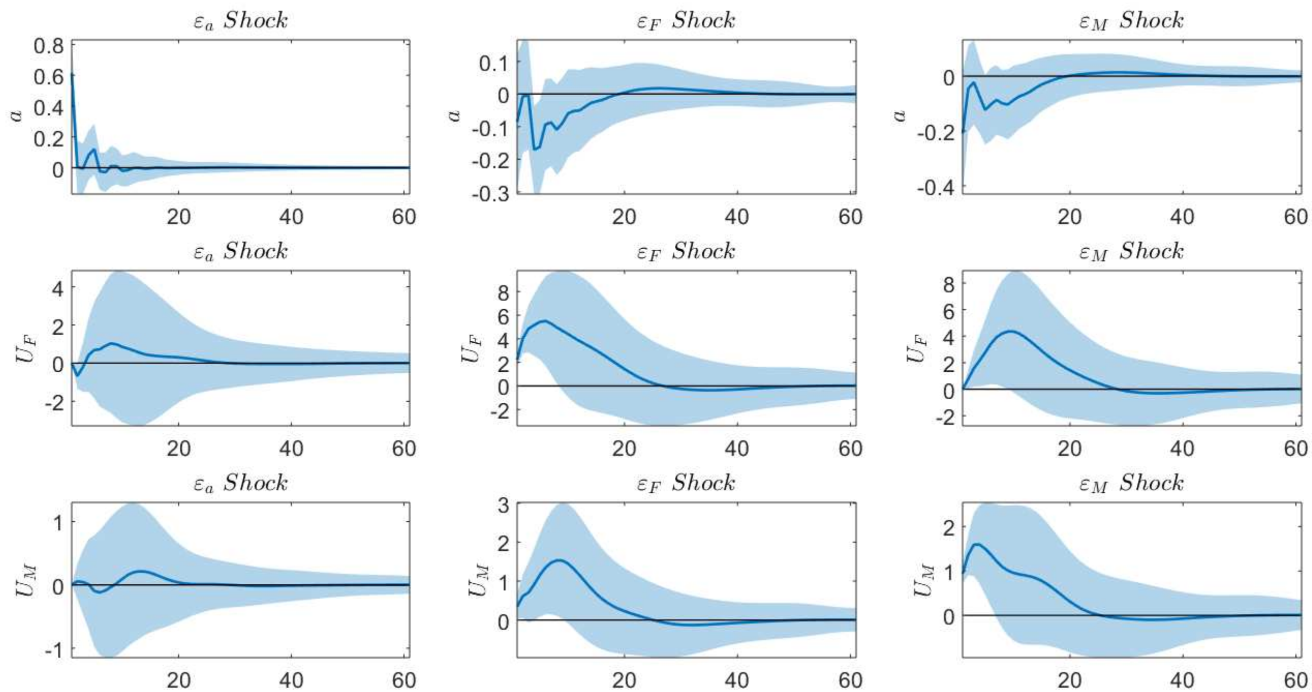

FIGURE A.1: *IRFs obtained from the AC-SVAR model for* $Y_t := (a_t, U_{Ft}, U_{Mt})'$ *and external instrument* $Z_t := (\Delta house_t, \Delta hours_t)'$, *period 2008:M1-2015:M4 (T=88). Blue shaded areas denote 90%-bootstrap simultaneous 'sup-t' confidence bands (Algorithm 3 in Montiel Olea and Plagborg-Møller, 2019). Responses are measured with respect to one standard deviation changes in the structural shocks. The on-impact coefficients are estimated by imposing the null hypothesis* $b_{F,a} = 0$ *and* $b_{M,a} = 0$ *of exogenous financial and macro uncertainty.*

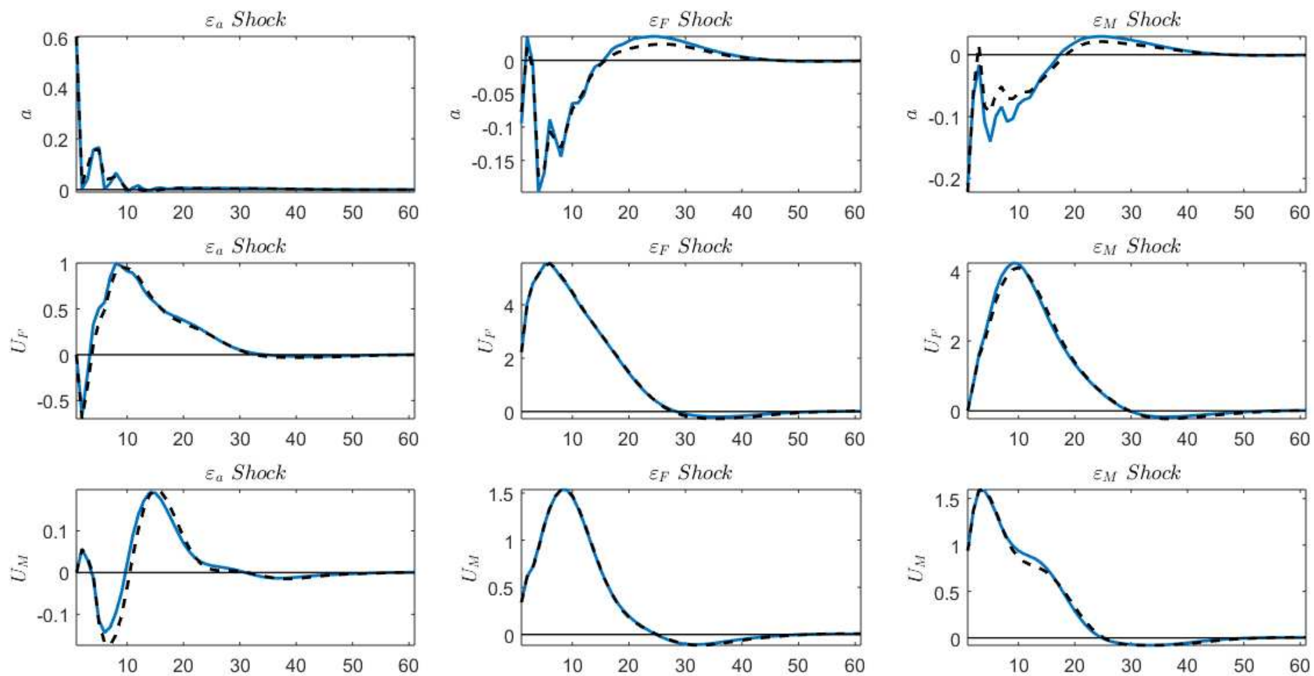

FIGURE A.2: *IRFs obtained from the Choleski-SVAR for* $Y_t := (U_{Ft}, U_{Mt}, a_t)'$*, period 2008:M1-2015:M4 (T=88). Blue lines denote the IRF associated with the Choleski decomposition and dashed black lines denote the IRF obtained with the baseline case propose in the Figure 1 of the paper. Responses are measured with respect to one standard deviation changes in the structural shocks. The on-impact coefficients of the AC-SVAR model are estimated by imposing the null hypothesis* $b_{F,a} = 0$ *and* $b_{M,a} = 0$ *of exogenous financial and macro uncertainty.*

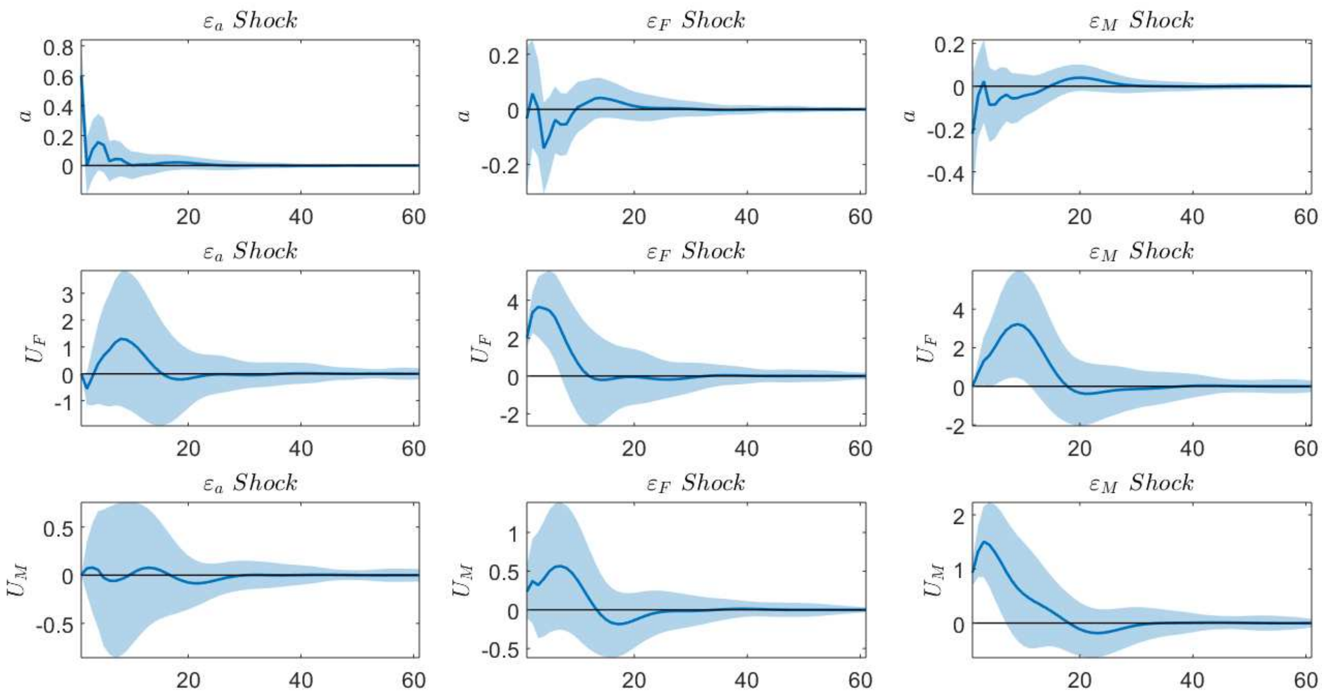

FIGURE A.3: *IRFs obtained from the AC-SVAR model for $Y_t := (a_t, U_{Ft}, U_{Mt})'$, external instrument $Z_t := (\Delta house_t, oil_t)'$ and exogenous control variable $X_t := (\pi_t, sr_t)'$, period 2008:M1-2015:M4 (T=88). Blue shaded areas denote 90%-bootstrap simultaneous 'sup-t' confidence bands (Algorithm 3 in Montiel Olea and Plagborg-Møller, 2019). Responses are measured with respect to one standard deviation changes in the structural shocks. The on-impact coefficients are estimated by imposing the null hypothesis $b_{F,a} = 0$ and $b_{M,a} = 0$ of exogenous financial and macro uncertainty.*