New Essentials of Economic Theory III. Economic Applications

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Economic Applications

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Abstract

This paper presents applications of our theory to description of particular economic problems. We give all definitions and equations in Part I and II of our work. Here we argue propagation of small perturbations of economic variables and transactions on economic space. We show that small perturbations may follow wave equations that have parallels to propagation of sound waves and surface waves in fluids. We underline that nature of economic waves is completely different from waves in physical fluids but parallels between them may be useful for their studies. Wave generation, propagation and interactions are the most general properties of any complex system. Descriptions of economic waves on economic space fill existing gap in economic modeling. Usage of economic space allows distribute agents by their risk ratings as coordinates. Agents on economic space cover economic domain bounded by minimum and maximum risk grades. Change of risk ratings of agents due to their economic activity, economic processes or other factors induce flows of economic variables, transactions and expectations. Borders of economic domain cause fluctuations of economic flows and mean risks and these fluctuations describe business cycles. For example fluctuations of credit flows model credit cycles, investment flows model investment cycles and etc. Further we model assets price disturbances as consequences of relations between transactions and expectations. As last economic sample we argue classical Black-Scholes-Merton option pricing model and discuss problems those arise from modeling on economic space.

Keywords: Economic Theory, Economic Waves, Business Cycles, Assets Pricing, Option pricing

JEL: C00, C02, C5, E00, E3, E7, G12

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1. Introduction

In this Part III of our work we apply of our model equations to description of particular economic problems. We describe: wave propagation of economic disturbances on economic domain; business cycles induced by fluctuations of economic flows on economic domain; asset pricing perturbations as result of equations on economic transactions and expectations and argue some hidden complexities of classical Black-Scholes-Merton (BSM) option pricing. We explain definitions and model equations in Part I and II (Olkhov, 2019c; 2019d). Introduction of economic space gives ground for description of wave propagation of disturbances of density functions of economic variables, transactions and expectations over economic space. Wave propagation describes general properties of any complex system like macroeconomics and finance and may be responsible for "fast" fluctuations of economic and financial variables. In Sec. 2 we describe wave propagation of small economic disturbances in the simple approximations that give self-consistent model of mutual dependence for two variables and their flows. Let's consider economic agents with risk coordinates $\boldsymbol{x}=(x_1,\ldots x_n)$ on economic domain (I.1.1):

$$0 \leq x_i \leq 1\,, i = 1, \ldots n \tag{1.1}$$

Thus economic variables and transactions also are determined on economic domain with borders (1.1). Disturbances of economic variables or transactions near borders of economic domain induce waves that may propagate along borders and inside of economic domain. Wave propagation of disturbances of economic variables and transactions near borders of economic domain has parallels to surface wave propagation in fluids, but nature of economic waves has nothing common to waves in fluids. We describe surface-like economic waves in Sec.2. Borders of economic domain cause fluctuations of flows of economic variables and transactions on economic domain. These fluctuations describe change of direction of economic flows on economic domain (1.1) reduced by it's borders. Flows of economic variables and transactions impact change of mean risks of these variables and transactions. Thus fluctuations of economic and financial flows on economic domain induce fluctuations of mean risks. In Sec. 3 we describe credits cycles, investment cycles and etc., as fluctuations of mean risks of these economic variables on economic domain. Asset pricing is one of most important issues of macro finance. In Part II we argue how asset pricing dynamics and fluctuations can be described via economic equations on transactions and expectations. Here in Sec. 4 we study particular cases of asset pricing dynamics and model price and return disturbances. In Sec. 5 we argue classical BSM treatment of option pricing and study simple extensions of classical option equations induced by random motion of agents on economic domain. Conclusions are in Sec. 6. We use roman letters for scalars and bold for vectors.

2. Economic waves

Wave propagation of small disturbances is one of most general properties of any complex systems. In this Sec. we describe wave propagation of small disturbances of density functions of economic variables and transactions on economic domain (1.1) of economic space (Olkhov, 2016a-2017c).

2.1. Waves of economic variables

Any model of economic phenomena implies definite approximation. In this Sec we assume that equations (I.14; 17) on density functions of economic variables and their flows depend on other economic variables only. To simplify the problem we study mutual interactions between two economic variables and their flows. Such approximation permits describe self-consistent model of mutual dependence between two variables and describe wave propagation of small disturbances of economic variables. Let's study wave propagation of disturbances of economic variables on economic space (Olkhov, 2016a-2017a). As example let's take familiar demand-price relations that propose price growth with rise of demand and demand decline as price increases. Let's derive equations that describe wave propagation of perturbations of price and demand. Demand $A(t,\boldsymbol{x})$ is additive variable and price $p(t,\boldsymbol{x})$ is non-additive. Supply $S(t,\boldsymbol{x})$ of assets, commodities, service can be measured in physical units as cars, shares, tons et., and in currency units. For simplicity let's assume that supply $S(t,\boldsymbol{x})$ measured in physical units is constant $S(t,\boldsymbol{x})=S$ - const., and supply $B(t,\boldsymbol{x})$ measured in currency units equals product of $S(t,\boldsymbol{x})$ and price $p(t,\boldsymbol{x})$

$$B(t,\boldsymbol{x}) = S\,p(t,\boldsymbol{x}) \;\; ; \;\; S - const \tag{1.2}$$

For such simplified assumptions demand $A(t,\boldsymbol{x})$ and supply $B(t,\boldsymbol{x})$ are additive variables and follow equations (I.14;17). We define flows of variables $A(t,\boldsymbol{x})$ and $B(t,\boldsymbol{x})$ in (I.6-10). Let's take equations (I.14; 17) on economic variables $A(t,\boldsymbol{x})$ and $B(t,\boldsymbol{x})$ and their flows $\boldsymbol{P}_A(t,\boldsymbol{x})$ and $\boldsymbol{P}_B(t,\boldsymbol{x})$:

$$\frac{\partial}{\partial t}A(t,\boldsymbol{x}) + \nabla\cdot\big(A(t,\boldsymbol{x})\,\boldsymbol{v}(t,\boldsymbol{x})\big) = F_A(t,\boldsymbol{x}) \tag{2.1}$$

$$\frac{\partial}{\partial t}B(t,\boldsymbol{x}) + \nabla\cdot\big(B(t,\boldsymbol{x})\,\boldsymbol{u}(t,\boldsymbol{x})\big) = F_B(t,\boldsymbol{x}) \tag{2.2}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_A(t,\boldsymbol{x}) + \nabla\cdot\big(\boldsymbol{P}_A(t,\boldsymbol{x})\,\boldsymbol{v}(t,\boldsymbol{x})\big) = \boldsymbol{G}_A(t,\boldsymbol{x}) \tag{2.3}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_B(t,\boldsymbol{x}) + \nabla\cdot\big(\boldsymbol{P}_B(t,\boldsymbol{x})\,\boldsymbol{u}(t,\boldsymbol{x})\big) = \boldsymbol{G}_B(t,\boldsymbol{x}) \tag{2.4}$$

$$\boldsymbol{P}_A(t,\boldsymbol{x}) = A(t,\boldsymbol{x})\,\boldsymbol{v}(t,\boldsymbol{x}) \;;\; \boldsymbol{P}_B(t,\boldsymbol{x}) = B(t,\boldsymbol{x})\,\boldsymbol{u}(t,\boldsymbol{x}) \tag{2.5}$$

To describe Demand-Price model (2.1-2.5) let's define functions $F_A(t,\boldsymbol{x})$ and $F_B(t,\boldsymbol{x})$. Let's remind that

$$\nabla - represents\ gradient; \quad \nabla\cdot\ - represents\ divirgence \tag{2.6}$$

Let's assume that function $F_A(t,\boldsymbol{x})$ is proportional to time derivative of supply $B(t,\boldsymbol{x})$:

$$F_A(t,\boldsymbol{x}) = \alpha_1 \frac{\partial}{\partial t} B(t,\boldsymbol{x}) \;;\; F_B(t,\boldsymbol{x}) = \alpha_2 \frac{\partial}{\partial t} A(t,\boldsymbol{x}) \;;\; \alpha_1 < 0\,;\; \alpha_2 > 0 \tag{3.1}$$

and function $F_B(t,\boldsymbol{x})$ is proportional to time derivative of demand $A(t,\boldsymbol{x})$. These assumptions for $\alpha_1 < 0$ give simple model of demand decline with price growth and price growth with demand increase for $\alpha_2 > 0$. Indeed, due to assumption (1.2) supply $B\ (t,\boldsymbol{x})$ measured in currency units is proportional to price $p(t,\boldsymbol{x})$ and hence time derivative of supply $B(t,\boldsymbol{x})$ equals time derivative of price $p(t,\boldsymbol{x})$. To define functions $\boldsymbol{G}_A(t,\boldsymbol{x})$ and $\boldsymbol{G}_B(t,\boldsymbol{x})$ in equations (2.3; 2.4) let's take

$$\boldsymbol{G}_A(t,\boldsymbol{x}) = \beta_1 \nabla B(t,\boldsymbol{x}) \;;\; \boldsymbol{G}_B(t,\boldsymbol{x}) = \beta_2 \nabla A(t,\boldsymbol{x}) \;;\; \beta_1 < 0 \;;\; \beta_2 > 0 \tag{3.2}$$

Relations (3.2) propose that demand velocity $\boldsymbol{v}(t,\boldsymbol{x})$ decrease in the direction of economic domain with high supply prices (3.3) with

$$\nabla B(t,\boldsymbol{x}) > 0 \tag{3.3}$$

and (3.2) represents that supply velocity $\boldsymbol{u}(t,\boldsymbol{x})$ grows up in the direction of economic domain with high demand (3.4):

$$\nabla A(t,\boldsymbol{x}) > 0 \tag{3.4}$$

Thus equations (2.1-2.4) take form:

$$\frac{\partial}{\partial t} A(t,\boldsymbol{x}) + \nabla\cdot\big(A(t,\boldsymbol{x})\,\boldsymbol{v}(t,\boldsymbol{x})\big) = \alpha_1 \frac{\partial}{\partial t} B(t,\boldsymbol{x}) \tag{4.1}$$

$$\frac{\partial}{\partial t} B(t,\boldsymbol{x}) + \nabla\cdot\big(B(t,\boldsymbol{x})\,\boldsymbol{u}(t,\boldsymbol{x})\big) = \alpha_2 \frac{\partial}{\partial t} A(t,\boldsymbol{x}) \tag{4.2}$$

$$\frac{\partial}{\partial t} \boldsymbol{P}_A(t,\boldsymbol{x}) + \nabla\cdot\big(\boldsymbol{P}_A(t,\boldsymbol{x})\,\boldsymbol{v}(t,\boldsymbol{x})\big) = \beta_1 \nabla B(t,\boldsymbol{x}) \tag{4.3}$$

$$\frac{\partial}{\partial t} \boldsymbol{P}_B(t,\boldsymbol{x}) + \nabla\cdot\big(\boldsymbol{P}_B(t,\boldsymbol{x})\,\boldsymbol{u}(t,\boldsymbol{x})\big) = \beta_2 \nabla A(t,\boldsymbol{x}) \tag{4.4}$$

$$\alpha_1 < 0\,;\; \alpha_2 > 0 \;;\; \beta_1 < 0 \;;\; \beta_2 > 0 \tag{4.5}$$

To derive equations that describe wave propagation of disturbances of demand and price let's take linear approximation for equations (4.1-4.4) for disturbances of demand $A(t,\boldsymbol{x})$ and price $p(t,\boldsymbol{x})$. Let's take disturbances as follows:

$$A(t,\boldsymbol{x}) = A_0\big(1+\ \varphi(t,\boldsymbol{x})\big);\;\; B(t,\boldsymbol{x}) = Sp_0\big(1+\ \pi(t,\boldsymbol{x})\big) \tag{5.1}$$

Relations (5.1) define dimensionless disturbances of demand $\varphi(t,\boldsymbol{x})$ and price $\pi(t,\boldsymbol{x})$. Let's take that velocities $\boldsymbol{v}(t,\boldsymbol{x})$ and $\boldsymbol{u}(t,\boldsymbol{x})$ are small and in linear approximation equations (4.1-4.4) take form:

$$\frac{\partial}{\partial t}\varphi(t,\boldsymbol{x}) + \nabla\cdot\boldsymbol{v}(t,\boldsymbol{x}) = \alpha_1 C\frac{\partial}{\partial t}\pi(t,\boldsymbol{x}) \;\;;\;\; C = \frac{Sp_0}{A_0} \tag{5.2}$$

$$C\left(\frac{\partial}{\partial t}\pi(t,\boldsymbol{x}) + \nabla\cdot\boldsymbol{u}(t,\boldsymbol{x})\right) = \alpha_2\frac{\partial}{\partial t}\varphi(t,\boldsymbol{x}) \tag{5.3}$$

$$\frac{\partial}{\partial t}\boldsymbol{v}(t,\boldsymbol{x}) = \beta_1 C\nabla\pi(t,\boldsymbol{x}) \;\;;\;\; C\frac{\partial}{\partial t}\boldsymbol{u}(t,\boldsymbol{x}) = \beta_2\nabla\varphi(t,\boldsymbol{x}) \tag{5.4}$$

In Appendix A we show that equations (5.2-5.4) can take form of equations (5.5) on disturbances of demand $\varphi(t,\boldsymbol{x})$ and price $\pi(t,\boldsymbol{x})$:

$$\left[(1-\alpha_1\alpha_2)\frac{\partial^4}{\partial t^4} + (\alpha_1\beta_2 + \beta_1\alpha_2)\Delta\frac{\partial^2}{\partial t^2} - \beta_1\beta_2\Delta^2\right]\varphi(t,\boldsymbol{x}) = 0 \tag{5.5}$$

As we show in Appendix A for $\alpha_1\alpha_2<0$ for any negative $\beta_1<0$ there exist domain with positive $\beta_2>0$ for which equations on disturbances of demand $\varphi(t,\boldsymbol{x})$ and price $\pi(t,\boldsymbol{x})$ take form of bi-wave equation (5.6):

$$\left(\frac{\partial^2}{\partial t^2} - c_1^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - c_2^2\Delta\right)\varphi(t,\boldsymbol{x}) = 0 \tag{5.6}$$

with different values of wave speed c_1 and c_2 determined by α_1, α_2, β_1, β_2 (A.5; 6). Bi-wave equations (5.6) describe more complex wave propagation than common second order wave equations. In Appendix A we show that equations (5.6) allow wave propagation of price disturbances $\pi(t,\boldsymbol{x})$ (A.8) with exponential growth of amplitude as $exp(\gamma t)$. Thus exponential growth of small price disturbances $\pi(t,\boldsymbol{x})$ may disturb sustainable economic evolution.

2.2 Waves of transactions

Transactions and their flows are determined on economic domain (II.1.1; 1.2):

$$\boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \;\;;\;\; \boldsymbol{x} = (x_1 \dots x_n) \;\;;\;\; \boldsymbol{y} = (y_1 \dots y_n) \tag{6.1}$$

$$0 \le x_i \le 1\,, i = 1,\dots n \;\;;\;\; 0 \le y_j \le 1\,, j = 1,\dots n \tag{6.2}$$

and are described by (II.5.9; 5.10). Let's take transactions $S(t,z)$ at $z=(\boldsymbol{x},\boldsymbol{y})$ that describe supply of goods, commodities or assets from point $\boldsymbol{x}$ to $\boldsymbol{y}$ and may depend on macroeconomic variables, other transactions and expectations (Olkhov, 2017b; 2019d). Self-consistent description of transactions, expectation, variables and other transaction is a too complex problem. Let's study simple self-consistent model of mutual interaction between two transactions and their flows. Let's assume that transaction $S(t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ supply goods or commodities from point $\boldsymbol{x}$ to point $\boldsymbol{y}$ as respond to demand $D(t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ for these commodities from point $\boldsymbol{y}$ to point $\boldsymbol{x}$. Let's assume that interactions between transactions $S(t,z)$ and $D(t,z)$ and their flows $\boldsymbol{P}(t,z)$ and $\boldsymbol{Q}(t,z)$ are described by functions $F_1(t,z)$, $F_2(t,z)$ and $\boldsymbol{G}_1(t,z)$, $\boldsymbol{G}_2(t,z)$ and depend only on each other and their flows. Both transactions follow equations alike to (II.5.9; 5.10). Let's define functions $F_1(t,z)$, $F_2(t,z)$ and $\boldsymbol{G}_1(t,z)$, $\boldsymbol{G}_2(t,z)$ for equations on $S(t,z)$ and $D(t,z)$ and flows $\boldsymbol{P}(t,z)$ and $\boldsymbol{Q}(t,z)$ respectively as (see 2.5):

$$F_1(t,\boldsymbol{z}) = \alpha_1 \nabla\cdot \boldsymbol{Q}(t,\boldsymbol{z}) \;\; ; \; F_2(t,\boldsymbol{z}) = \alpha_2 \nabla\cdot \boldsymbol{P}(t,\boldsymbol{z}) \tag{6.3}$$

$$\boldsymbol{G}_1(t,\boldsymbol{z}) = \beta_1 \nabla D(t,\boldsymbol{z}) \;\; ; \; \boldsymbol{G}_2(t,\boldsymbol{z}) = \beta_2 \nabla S(t,\boldsymbol{z}) \tag{6.4}$$

Economic meaning of (6.3; 6.4) is follows. Due to (II.5.6) flows $\boldsymbol{P}(t,z)$ and $\boldsymbol{Q}(t,z)$ looks as:

$$\boldsymbol{P}(t,\boldsymbol{z}) = S(t,\boldsymbol{z})\boldsymbol{v}(t,\boldsymbol{z}) \;\; ; \; \boldsymbol{v}(t,\boldsymbol{z}) = (\boldsymbol{v_x}(t,\boldsymbol{z}); \boldsymbol{v_y}(t,\boldsymbol{z})) \tag{6.5}$$

$$\boldsymbol{Q}(t,\boldsymbol{z}) = D(t,\boldsymbol{z})\boldsymbol{u}(t,\boldsymbol{z}) \;\; ; \; \boldsymbol{u}(t,\boldsymbol{z}) = (\boldsymbol{u_x}(t,\boldsymbol{z}); \boldsymbol{u_y}(t,\boldsymbol{z})) \tag{6.6}$$

Velocity $\boldsymbol{v_x}$ of supply flow $\boldsymbol{P}(t,z)$ describes motion of suppliers at and velocity $\boldsymbol{v_y}$ describe motion of consumers on economic domain. Divergence in (6.3) describes sources and run-off of flows in a unit volume

$$dV = dV_x dV_y$$

Volume dV_x describes a unit volume of variable $\boldsymbol{x}$ and dV_y describes a unit volume near variable $\boldsymbol{y}$. Transactions $S(t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ supply goods from a unit volume dV_x near point $\boldsymbol{x}$ to a unit volume dV_y near $\boldsymbol{y}$. Transactions $D(t,z)$ describe demand of goods from a unit volume dV_y near $\boldsymbol{y}$ to a unit volume dV_x near $\boldsymbol{x}$. Divergence in (6.3) equals:

$$\nabla\cdot \boldsymbol{Q}(t,\boldsymbol{z}) = \nabla_x\cdot \boldsymbol{Q}(t,\boldsymbol{x},\boldsymbol{y}) + \nabla_y\cdot \boldsymbol{Q}(t,\boldsymbol{x},\boldsymbol{y}) \tag{6.7}$$

Here $\boldsymbol{x}$-divergence $\nabla_x\cdot Q(t,\boldsymbol{x},\boldsymbol{y})$ describes sources and sinks of demand flow $\boldsymbol{Q}(t,z)$ of suppliers at point $\boldsymbol{x}$ in a unit volume dV_x . Divergence $\nabla_y\cdot Q(t,\boldsymbol{x},\boldsymbol{y})$ – describes sources and sinks of demand flow $\boldsymbol{Q}(t,z)$ of consumers of goods, those who generate demand at point $\boldsymbol{y}$ in a unit volume dV_y. Let's treat

$$\nabla_x\cdot \boldsymbol{Q}(t,x,y) < 0 \tag{6.8}$$

as sinks of demand flow into point $\boldsymbol{x}$ that is met by supply $S(t,z)$ from point $\boldsymbol{x}$. Let's present divergence of supply flow $\boldsymbol{P}(t,z)$ (6.9) similar to (6.7):

$$\nabla\cdot \boldsymbol{P}(t,\boldsymbol{z}) = \nabla_x\cdot \boldsymbol{P}(t,\boldsymbol{x},\boldsymbol{y}) + \nabla_y\cdot \boldsymbol{P}(t,\boldsymbol{x},\boldsymbol{y}) \tag{6.9}$$

Here $\boldsymbol{x}$-divergence $\nabla_x\cdot P(t,\boldsymbol{x},\boldsymbol{y})$ describes sources and sinks of supply flow $\boldsymbol{P}(t,z)$ of from $\boldsymbol{x}$ in a unit volume dV_x. Relations (6.10)

$$\nabla_x\cdot \boldsymbol{P}(t,x,y) > 0 \tag{6.10}$$

describe sources of supply flow $\boldsymbol{P}(t,z)$ from point $\boldsymbol{x}$ to $\boldsymbol{y}$. Due to (6.3; 6.4) equations on transactions $S(t,z)$ and $D(t,z)$ take form similar to (II.5.9):

$$\frac{\partial}{\partial t}S + \nabla\cdot(S\,\boldsymbol{v}) = \alpha_1 \nabla\cdot \boldsymbol{Q}(t,\boldsymbol{z}) \tag{7.1}$$

$$\frac{\partial}{\partial t}D + \nabla\cdot(D\,\boldsymbol{u}) = \alpha_2 \nabla\cdot \boldsymbol{P}(t,\boldsymbol{z}) \tag{7.2}$$

and equations on flows $\boldsymbol{P}(t,z)$ and $\boldsymbol{Q}(t,z)$

$$\boldsymbol{P}(t,\boldsymbol{z}) = S(t,\boldsymbol{z})\boldsymbol{v}(t,\boldsymbol{z}) \;\; ; \;\; \boldsymbol{Q}(t,\boldsymbol{z}) = D(t,\boldsymbol{z})\boldsymbol{u}(t,\boldsymbol{z}) \tag{7.3}$$

on $2n$-dimensional economic domain $z=(\boldsymbol{x},\boldsymbol{y})$ take form similar to (II.5.10):

$$\frac{\partial}{\partial t}\boldsymbol{P}(t,\boldsymbol{z}) + \nabla\cdot\big(\boldsymbol{P}(t,\boldsymbol{z})\,\boldsymbol{v}(t,\boldsymbol{z})\big) = \beta_1\,\nabla D(t,\boldsymbol{z}) \tag{7.4}$$

$$\frac{\partial}{\partial t}\boldsymbol{Q}(t,\boldsymbol{z}) + \nabla\cdot\big(\boldsymbol{Q}(t,\boldsymbol{z})\,\boldsymbol{u}(t,\boldsymbol{z})\big) = \beta_2\,\nabla S(t,\boldsymbol{z}) \tag{7.5}$$

Equations (7.1; 7.2; 7.3; 7.4) cause equations on macroeconomic supply $S(t)$ and demand $D(t)$ (II.4.1). Functions $S(t)$ and $D(t)$ (7.6) describe macroeconomic supply and demand of selected goods, commodities etc.

$$S(t) = \int d\boldsymbol{x}d\boldsymbol{y}\; S(t,\boldsymbol{x},\boldsymbol{y}) \;\; ; \;\; D(t) = \int d\boldsymbol{x}d\boldsymbol{y}\, D(t,\boldsymbol{x},\boldsymbol{y}) \tag{7.6}$$

$$\frac{d}{dt}S(t) = 0 \;\; ; \;\; \frac{d}{dt}D(t) = 0 \; ; \; \frac{d}{dt}\boldsymbol{P}(t) = 0 \;\; ; \;\; \frac{d}{dt}\boldsymbol{Q}(t) = 0 \tag{7.7}$$

Relations (7.7) valid as integral of divergence over economic space equals zero due to divergence theorem (Gauss' Theorem) (Strauss, 2008, p.179) because no flows exist outside of economic domain and because transactions are equal zero outside of economic domain. Thus model interactions (6.3; 6.4) and equations (7.1-7.5) describe constant or slow-changing macroeconomic supply and demand, but allow model wave propagation of small disturbances of supply and demand. To derive wave equations let's study small perturbations of transactions $S(t,z)$ and $D(t,z)$ and assume that velocities $\boldsymbol{v}(t,z)$ and $\boldsymbol{u}(t,z)$ of supply and demand flows are small. Let's take:

$$S(t,\boldsymbol{z}) = S_0(1 + s(t,\boldsymbol{z}))\,;\; D(t,\boldsymbol{z}) = D_0(1 + d(t,\boldsymbol{z})) \tag{7.8}$$

$$\boldsymbol{P}(t,\boldsymbol{z}) = S_0\boldsymbol{v}(t,\boldsymbol{z}) \;\; ; \; \boldsymbol{Q}(t,\boldsymbol{z}) = D_0\boldsymbol{u}(t,\boldsymbol{z}) \tag{7.9}$$

and let's assume that velocities $\boldsymbol{v}(t,z)$ and $\boldsymbol{u}(t,z)$ in (7.9) are small. Relations (7.7) model S_0 and D_0 that are constant or slow-changing to compare with small disturbances $s(t,z)$ and $d(t,z)$. Let's take equations (7.1; 7.2; 7.4; 7.5) in linear approximation by perturbations $s(t,z)$, $d(t,z)$ (7.8) and $\boldsymbol{v}(t,z)$ and $\boldsymbol{u}(t,z)$.

$$S_0\frac{\partial}{\partial t}s(t,\boldsymbol{z}) + S_0\nabla\cdot\boldsymbol{v} = \alpha_1 D_0\nabla\cdot\boldsymbol{u} \;\; ; \; D_0\frac{\partial}{\partial t}d(t,\boldsymbol{z}) + D_0\nabla\cdot\boldsymbol{u} = \alpha_2 S_0\nabla\cdot\boldsymbol{v} \tag{8.1}$$

$$S_0\frac{\partial}{\partial t}\boldsymbol{v}(t,\boldsymbol{z}) = \beta_1 D_0\nabla\, d(t,\boldsymbol{z})\,;\;\; D_0\frac{\partial}{\partial t}\boldsymbol{u}(t,\boldsymbol{z}) = \beta_2 S_0\nabla\, s(t,\boldsymbol{z}) \tag{8.2}$$

Equations (8.1; 8.2) cause (see Appendix B, B.5) equations on $s(t,z)$, $d(t,z)$ (8.3):

$$\Big[\frac{\partial^4}{\partial t^4} - a\Delta\,\frac{\partial^2}{\partial t^2} + b\Delta^2\,\Big]s(t,\boldsymbol{z}) = 0 \tag{8.3}$$

Equations (8.3) may take form of bi-wave equation (B.7):

$$\Big(\frac{\partial^2}{\partial t^2} - c_1^2\Delta\Big)\,\Big(\frac{\partial^2}{\partial t^2} - c_2^2\Delta\Big)s(t,\boldsymbol{z}) = 0 \tag{8.4}$$

Wave propagation of small disturbances of supply $s(t,z)$ and demand $d(t,z)$ transactions induces wave propagation of disturbances of economic variables (B.14.1-B.16.5) determined by transactions $S(t,\boldsymbol{x},\boldsymbol{y})$ and $D(t,\boldsymbol{x},\boldsymbol{y})$. Bi-wave equations describe wave propagation of disturbances of economic variables induced by transactions and take form (B.17.3) similar to (8.4). Wave propagation of small disturbances of transactions induces fluctuations (B.18.1; 18.2) of macroeconomic variables *S(t)* and *D(t)* (7.6). As we show in Appendix B disturbances *s(t)* of macroeconomic supply *S(t)* at moment *t* may grow up as $exp(\gamma t)$ for $\gamma>0$ or dissipate to constant rate S_0 for $\gamma<0$ and fluctuate with frequency ω.

2.3 Economic surface-like waves

In sections 2.1 and 2.2 we study wave propagation of small disturbances of densities functions of economic variables and transactions. These waves have parallels to sound waves in continuous media. Now let's show that disturbances of velocities of transactions flows may be origin of waves alike to surface waves in fluids (Olkhov, 2017c). Let's study simple model of economics under action of a single risk on *1*-dimensional economic space. Hence economic transactions are determined on *2*-dimensional economic domain (6.1; 6.2). Borders of economic domain establish bound lines for economic transactions. Disturbances of transactions near these bound lines may disturb bound lines and induce surface-like waves of along borders of economic domain. On other hand disturbances of transactions at bound lines may induce surface-like waves of perturbations that propagate inside economic domain and cause disturbances of transactions and economic variables far from borders of economic domain. Such surface-like waves may propagate along with growth of wave amplitude and thus impact of such waves of small perturbations may grow up in time. Thus description of economic surface-like waves may explain propagation and amplification of transactions disturbances near borders of economic domain. Let's remind that borders of economic domain are areas with maximum or minimum risk ratings. Thus, for example, perturbations of transactions near maximum risk ratings may propagate inside economic domain to areas with low risk ratings and growth of amplitudes of such perturbation may hardly disturb economic processes with low risk ratings.

For simplicity let's consider same example as in sec. 2.2 and Appendix B. Let's take model relations between supply transactions $S(t,z)$ and Demand transactions $D(t,z)$ on economic domain (6.1; 6.2), $z=(x,y)$ and study small disturbances of transactions and flows similar to (7.8; 7.9) and equations (8.1; 8.2). Velocities of transactions on *2*-dimensinal economic domain take form:

$$\boldsymbol{v}(t,x,y) = \left(v_x(t,x,y); v_y(t,x,y)\right); \boldsymbol{u}(t,x,y) = \left(u_x(t,x,y); u_y(t,x,y)\right) \qquad (9.1)$$

Let's take that transactions $D(t,z)$, $z=(x,y)$ transfer demand request from consumes at *y* to suppliers at *x*. Hence velocities v_x and u_x along axis *X* describe motion of suppliers and velocities υ_y and u_y along *Y* describe motion of consumers of goods and services provided by suppliers. Let's study possible waves that can be generated by disturbances (7.8; 7.9) near border *y=1* of economic domain (6.1; 6.2). Border *y=1* describes consumers with maximum risks. Let's define perturbations of the border as *y=ξ(t,x)*. Interactions between transactions *S(t,z)* and *D(t,z)* require that border *y= ξ(t,x)* should be common for both. Otherwise interaction between them will be violated. Time derivations of function *y=ξ(t,x)* define *y*-velocities υ_y and u_y at *y= ξ(t,x)* as:

$$\frac{\partial}{\partial t}\xi(t,x) = \upsilon_y(t,x,y = \ \xi(t,x)) = u_y(t,x,y = \ \xi(t,x)) \tag{9.2}$$

Time derivation (9.2) describes velocities υ_y of consumers with maximum risks and velocities u_y of demanders of goods. Let's modify equations (8.2) and assume that near border *y=1*

$$S_0\frac{\partial}{\partial t}\boldsymbol{v}(t,\boldsymbol{z}) = D_0(\beta_1\nabla\, d(t,\boldsymbol{z}) + \boldsymbol{g})\ ;\ \ D_0\frac{\partial}{\partial t}\boldsymbol{u}(t,\boldsymbol{z}) = S_0(\beta_2\nabla\, s(t,\boldsymbol{z}) + \boldsymbol{h}) \tag{9.3}$$

As ***g*** and ***h*** we introduce constant economic or financial *"accelerations*" ***h***=(h_x,h_y) and ***g***=(g_x, g_y) that act on economic agents, supply *S(t,z)* and demand *D(t,z)* transactions along axes *X* and *Y* and prevent agents from taking excess risk. Let's introduce functions *G* and *H*:

$$G(x,y) = g_x\, x + g_y y\ \ ;\ H(x,y) = h_x\, x + h_y y\ ;\ g_x\,, g_y, h_x, h_y - const \tag{9.4}$$

Let's assume that potentials *φ* and *ψ* determine velocities ***υ*** and ***u*** as:

$$\boldsymbol{v} = \nabla\,\varphi\ \ ;\ \ \boldsymbol{u} = \nabla\,\psi \tag{9.5}$$

Thus equations (8.2) on velocities take form:

$$S_0\frac{\partial}{\partial t}v_x = D_0(\beta_1\frac{\partial}{\partial x}d - g_x)\ ;\ S_0\frac{\partial}{\partial t}v_y = D_0(\beta_1\frac{\partial}{\partial y}d - g_y) \tag{9.6}$$

$$D_0\frac{\partial}{\partial t}u_x = S_0\left(\beta_2\frac{\partial}{\partial x}s - h_x\right);\ B_0\ \frac{\partial}{\partial t}u_y = S_0(\ \beta_2\frac{\partial}{\partial y}\ s - h_y) \tag{9.7}$$

Relations (9.5) allow present (9.6; 9.7) as

$$S_0\frac{\partial}{\partial t}\frac{\partial}{\partial x}\varphi = D_0(\beta_1\frac{\partial}{\partial x}d - g_x)\ \ \ ;\ S_0\frac{\partial}{\partial t}\frac{\partial}{\partial y}\varphi = D_0(\beta_1\frac{\partial}{\partial y}d - g_y) \tag{9.8}$$

$$D_0\frac{\partial}{\partial t}\frac{\partial}{\partial x}\psi = S_0\left(\beta_2\frac{\partial}{\partial x}s - h_x\right)\ \ ;\ D_0\ \frac{\partial}{\partial t}\frac{\partial}{\partial y}\psi = S_0(\ \beta_2\frac{\partial}{\partial y}\ s - h_y) \tag{9.9}$$

Then (9.4) supply *s(t,x,y)* and demand *d(t,x,y)* transactions can be written as:

$$\beta_2 S_0 s(t,x,y) = S_0\big[h_x(x-1) + h_y(y-1)\big] + D_0\frac{\partial}{\partial t}\ \psi(t,x,y) \tag{10.1}$$

$$\beta_1 D_0 d(t,x,y) = D_0\big[g_x(x-1) + g_y(y-1)\big] + S_0\frac{\partial}{\partial t}\ \varphi(t,x,y) \tag{10.2}$$

For *φ=ψ=0* (10.1; 10.2) describe steady state of supply *s(t,x,y)* and demand *d(t,x,y)* perturbations and on border *y=1 s(t,x,y)* and *d(t,x,y)* take form (10.3):

$$\beta_2 s(t,x,1) = h_x(x-1)\ ;\ \ \beta_1 d(t,x,1) = g_x(x-1) \tag{10.3}$$

On surface *y= ξ(t,x)* disturbances *s(t,x,y)* and *d(t,x,y)* take form:

$$\beta_2 S_0 s(t,x,y)|_{y=\xi(t,x)} = S_0\left[h_x(x-1) + h_y(\xi(t,x)-1)\right] + D_0 \frac{\partial}{\partial t}\psi\big(t,x,\xi(t,x)\big) \quad (10.4)$$

$$\beta_1 D_0 d(t,x,y)|_{y=\xi(t,x)} = D_0\left[g_x(x-1) + g_y(\xi(t,x)-1)\right] + S_0 \frac{\partial}{\partial t}\varphi\big(t,x,\xi(t,x)\big) \quad (10.5)$$

Let's propose that perturbations $y=\xi(t,x)$ near $y=1$ are small and assume that $s(t,x,y)$ and $d(t,x,y)$ take values $s(t,x,1)$ and $d(t,x,1)$ in a steady state for $\varphi=\psi=0$ on $y=1$ (10.3). Hence from (10.4; 10.5) obtain:

$$S_0 h_y(\xi(t,x)-1) = -D_0 \frac{\partial}{\partial t}\psi\big(t,x,\xi(t,x)\big) \quad (10.6)$$

$$D_0 g_y(\xi(t,x)-1) = -S_0 \frac{\partial}{\partial t}\varphi\big(t,x,\xi(t,x)\big) \quad (10.7)$$

Hence obtain:

$$\xi(t,x) - 1 = -\frac{D_0}{S_0 h_y}\frac{\partial}{\partial t}\psi\big(t,x,\xi(t,x)\big) = -\frac{S_0}{D_0 g_y}\frac{\partial}{\partial t}\varphi\big(t,x,\xi(t,x)\big) \quad (10.8)$$

Equations (10.8) determine relations between h_y and g_y

$$S_0{}^2 h_y = D_0{}^2 g_y$$

$$\frac{\partial}{\partial t}\xi(t,x) = \frac{\partial}{\partial y}\psi = \frac{\partial}{\partial y}\varphi = -\frac{S_0}{D_0 g_y}\frac{\partial^2}{\partial t^2}\varphi\big(t,x,y=\xi(t,x)\big) \quad (10.9)$$

Equation (10.9) describes constraints on potentials φ and ψ at $y=\xi(t,x)$. To derive equations on potentials φ and ψ let's substitute (10.1; 10.2) into (8.1) and neglect all non-linear terms with potentials and financial "*accelerations*". Equations on φ and ψ take form:

$$S_0\left(\frac{\partial^2}{\partial t^2} - \alpha_2\beta_1\Delta\right)\varphi = -\beta_1 D_0\Delta\psi \;;\; D_0\left(\frac{\partial^2}{\partial t^2} - \alpha_1\beta_2\Delta\right)\psi = -\beta_2 S_0\Delta\varphi \;;\; \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (11.1)$$

From (11.1) obtain:

$$[\left(\frac{\partial^2}{\partial t^2} - \alpha_2\beta_1\,\Delta\right)\left(\frac{\partial^2}{\partial t^2} - \alpha_1\beta_2\,\Delta\right) - \beta_1\beta_2\Delta^2\,]\,\varphi = 0 \quad (11.2)$$

Let's take functions φ and ψ *as*:

$$\varphi = \psi = cos(kx - \omega t)\, f(y-1) \;\;;\;\; f(0) = 1 \quad (11.3)$$

Let's take into account that perturbations $\xi(t,x)$ near steady boundary $y=X$ are small and hence relations (10.9) for (11.3) at $y=1$ give:

$$\frac{\partial}{\partial y} f(0) = \frac{S_0\omega^2}{D_0 g_y} > 0 \quad (11.4)$$

and substitute (11.3) into (11.2). Then (B.17.2) obtain equation on function $f(y)$ as ordinary differential equation of forth order :

$$\left(q_4\frac{\partial^4}{\partial y^4} + q_2\frac{\partial^2}{\partial y^2} + q_0\right) f(y) = 0 \quad (11.5)$$

$$q_4 = b \;\;;\;\; q_2 = a\omega^2 - 2bk^2 \;\;;\;\; q_0 = \omega^4 - a\omega^2 k^2 + bk^4 \quad (11.6)$$

Characteristic equation (11.7) of equation (11.5)

$$q_4\gamma^4 + q_2\gamma^2 + q_0 = 0 \quad (11.7)$$

defines roots γ^2:

$$\gamma_{1,2}^2 = \frac{-q_2 +/- \sqrt{q_2^2 - 4q_0 q_4}}{2q_4} = \frac{-q_2 +/- \omega^2 \sqrt{a^2 - 4b}}{2b} \tag{11.8}$$

For single positive root $\gamma > 0$ obtain simplest potentials φ and ψ as:

$$\varphi = \psi = cos(kx - \omega t)\, exp(\gamma(y-1)) \;\; ; \;\; \gamma = \frac{S_0 \omega^2}{D_0 g_y} > 0 \tag{12.1}$$

Function $y=\xi(t,x)$ (10.8) takes form:

$$\xi(t,x) = 1 - \frac{S_0 \omega}{D_0 g_y} sin(kx - \omega t) = 1 - \sqrt{\frac{S_0 \gamma}{D_0 g_y}} sin(kx - \omega t) \tag{12.2}$$

Border $y=1$ define position of consumers for supply transactions $s(t,x,y)$ and consumers as origin of demand for demand transactions $d(t,x,y)$. Supply $s(t,x,y)$ and demand $d(t,x,y)$ waves at stationary border $y=1$ take form:

$$\beta_2 S_0 s(t,x,1) = S_0 h_x (x-1) + D_0 \omega\, sin(kx - \omega t) \tag{12.3}$$

$$\beta_1 D_0 d(t,x,1) = D_0 g_x (x-1) + S_0 \omega\, sin(kx - \omega t) \tag{12.4}$$

Surface-like waves of supply transactions $s(t,x,1)$ (12.3) reflect change of supply for consumers at $y=1$ from suppliers at x. Relations (12.4) describe change of demand from consumers at $y=1$ to suppliers at x. Integral of supply transactions $s(t,x,1)$ by dx (12.3) along border $y=1$ over $(0,1)$ define supply $s(t,1)$ at risk border $y=1$ as function of time:

$$\beta_2 S_0 s(t,1) = \; S_0 [1 - \tfrac{h_x}{2}] + 2 \frac{D_0 \omega}{k} sin\left(\omega t - \frac{k}{2}\right) sin\left(\frac{k}{2}\right) \tag{12.5}$$

Function $s(t,1)$ (12.5) describes fluctuations of supply to consumers at $y=1$ with frequency ω from all suppliers of the economy. Simplest solution (12.1) with $\gamma>0$ describe exponential dissipation of disturbances induced by surface-like waves inside macro domain $y<1$.

Actually there might be surface-like waves that describe amplification of disturbances at $y=1$ inside economic domain along axis Y for $y<<1$. For root $\gamma^2>0$ (11.8) let's take two roots:

$$\gamma_{1,2} = +/- \sqrt{\gamma^2}$$

Then from (11.3; 11.4) obtain:

$$f(0) = \lambda_1 + \lambda_2 = 1 \;\; ; \;\; \frac{\partial}{\partial y} f(0) = \gamma(\lambda_1 - \lambda_2) = \frac{S_0 \omega^2}{D_0 g_y} > 0$$

$$\lambda_1 = \frac{1}{2} + \frac{S_0 \omega^2}{2\gamma D_0 g_y} \;\; ; \;\; \lambda_2 = \frac{1}{2} - \frac{S_0 \omega^2}{2\gamma D_0 g_y}$$

$$\varphi = \psi = cos(kx - \omega t)\, [\lambda_1 \exp(\gamma(y-1)) + \lambda_2 \exp(-\gamma(y-1))]$$

$$\beta_2 S_0 s(t,x,y) = S_0 \big[h_x(x-1) + h_y(y-1)\big] + \; \omega D_0 \; sin(kx - \omega t)\, [\lambda_1\, exp(\gamma(y-1)) + \lambda_2\, exp(-\gamma(y-1))]$$

$$\beta_1 D_0 d(t,x,y) = D_0\left[g_x(x-1) + g_y(y-1)\right] + \ \omega S_0 \, sin(kx-\omega t)\,[\lambda_1 \, exp(\gamma(y-1)) + \lambda_2 \, exp(-\gamma(y-1))]$$

and supply $s(t,x,y)$ and demand $d(t,x,y)$ transactions grow up as exponent for $(y-1)<0$

$$s(t,x,y) \sim d(t,x,y) \sim \lambda_2 \, exp(-\gamma(y-1)) \tag{12.6}$$

This example shows that small disturbances of supply to consumers at $y=1$ may induce exponentially growing (12.6) disturbances of supply and demand at $y<1$ far from risk border. Suppliers at x may stop provide goods to consumers at y with high risks at border $y=1$ and redirect their supply to more secure consumers with $y<1$.

3 Business cycles

In Sec 2 we show that waves of small disturbances of economic variables or transactions on economic domain (6.1; 6.2) induce time fluctuations of small perturbations of macroeconomic variables. Velocities of these waves define time scales of such fluctuations. Let's call these economic fluctuations as "fast" contrary to "slow" fluctuations of economic variables noted as business cycles. In this section we show that "slow" fluctuations of flows of variables and transactions can cause oscillations of credits, investment, demand and economic growth noted as business cycles. Business cycles as slow fluctuations of macroeconomic and financial variables as GDP, investment, demand and etc., for decades are under permanent research (Tinbergen, 1935, Schumpeter, 1939, Lucas, 1980, Kydland & Prescott, 1991, Zarnowitz, 1992, Diebold & Rudebusch, 1999; Kiyotaki, 2011; Jorda, Schularick & Taylor, 2016). Below we present approximation of the business cycles induced by flows of economic transactions (Olkhov, 2017b; 2019a). For simplicity let's take same supply $S(t,z)$ and demand $D(t,z)$ transactions as in Sec.2 and let's describe business cycles of supply and demand. Let's take equations on $S(t,z)$ and $D(t,z)$ similar to (II. 5.9; 5.10) as:

$$\frac{\partial}{\partial t}S + \nabla\cdot(S\boldsymbol{v}) = F_S(t,\boldsymbol{z}) \ ; \ \ \frac{\partial}{\partial t}D + \nabla\cdot(D\boldsymbol{u}) = F_D(t,\boldsymbol{z}) \tag{13.1}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_S + \nabla\cdot(\boldsymbol{P}_S\,\boldsymbol{v}) = \boldsymbol{G}_S(t,\boldsymbol{z}) \ \ ; \ \ \frac{\partial}{\partial t}\boldsymbol{P}_D + \nabla\cdot(\boldsymbol{P}_D\,\boldsymbol{u}) = \boldsymbol{G}_D(t,\boldsymbol{z}) \tag{13.2}$$

For simplicity let's study economic evolution under action of a single risk similar to sec.2.3 and study business cycles on 2-dimensional economic domain (6.1; 6.2). Thus coordinates x describe evolution of suppliers with economic variable E and y evolution of consumers of variable E, $z=(x,y)$. As variable E one may study any goods, commodities, credits, service, shares, assets and etc. To simplify model calculations let's assume that supply transactions $S(t,z)$ and their flows $\boldsymbol{P}_S(t,z)$ depend on demand $D(t,z)$ transactions and their flows $\boldsymbol{P}_D(t,z)$

only. We propose that demand transactions *D(t,z)* describe demand from consumers of variable *E* at *y* to suppliers at *x*. Let's take F_S and F_D for (13.1) as (*a* and *b* – const):

$$F_S(t,\boldsymbol{z}) = a\,\boldsymbol{z}\cdot\boldsymbol{P}_D(t,\boldsymbol{z}) = a(\,x\cdot P_{Dx}(t,\boldsymbol{z}) + y\cdot P_{Dy}(t,\boldsymbol{z})) \tag{13.3}$$

$$F_D(t,\boldsymbol{z}) = b\,\boldsymbol{z}\cdot\boldsymbol{P}_S(t,\boldsymbol{z}) = b(\,x\cdot P_{Sx}(t,\boldsymbol{z}) + y\cdot P_{Sy}(t,\boldsymbol{z})) \tag{13.4}$$

Relations (13.3-13.4) describe model with supply *S(t,z)* growth up if F_S is positive and hence (13.3) for $a>0$ is positive if at least one component of demand velocities

$$\boldsymbol{u}(t,\boldsymbol{z}) = \left(u_x(t,\boldsymbol{z});\ u_y(t,\boldsymbol{z})\right) \tag{13.5}$$

direct from safer to risky direction. In other words: if demand transactions *D(t,z)* flew into risky direction that can increase supply *S(t,z).* As well negative value of (13.3) models demand flows from risky to secure domain and cause decrease supply *S(t,z)* as suppliers may prefer more secure consumers. Such assumptions simplify relations between suppliers and consumers and neglect time gaps between providing supply from *x* to consumers at *y* and receiving demand from consumers at *y* to suppliers at *x* and neglect other factors that impact supply. Actually we neglect direct dependence of economic variables and transactions on risk coordinates of economic domain. This assumption simplifies the model and allows outline impact of mutual interactions between transactions *S(t,z)* and *D(t,z)* and their flows on the business cycle fluctuations of variable *E*. Let's take $\boldsymbol{G}_S(t,z)$ and $\boldsymbol{G}_D(t,z)$ for (13.2) as:

$$\boldsymbol{G}_{Sx}(t,\boldsymbol{z}) = c_x P_{Dx}(t,\boldsymbol{z})\ ;\ \ \boldsymbol{G}_{Sy}(t,\boldsymbol{z}) = c_y P_{Dy}(t,\boldsymbol{z}) \tag{13.6}$$

$$\boldsymbol{G}_{Dx}(t,\boldsymbol{z}) = d_x P_{Sx}(t,\boldsymbol{z})\ ;\ \ \boldsymbol{G}_{Dy}(t,\boldsymbol{z}) = d_y P_{Sy}(t,\boldsymbol{z}) \tag{13.7}$$

Equations (13.2; 13.6; 13.7) describe simple linear dependence between transaction flows $\boldsymbol{P}_S(t,z)$ and $\boldsymbol{P}_D(t,z)$. Integrals by *dz* over economic domain (6.1; 6.2) for components of flows due to (II. 4.1; 5.6; 5.7; 5.8) equal:

$$\boldsymbol{P}_S(t) = \int d\boldsymbol{z}\ \boldsymbol{P}_S(t,\boldsymbol{z}) = \int dxdy\ S(t,\boldsymbol{z})\boldsymbol{v}(t,\boldsymbol{z}) = S(t)\boldsymbol{v}(t)\ ;\ \boldsymbol{v} = (v_x; v_y) \tag{13.8}$$

$$\boldsymbol{P}_D(t) = \int d\boldsymbol{z}\ \boldsymbol{P}_D(t,\boldsymbol{z}) = \int dxdy\ D(t,\boldsymbol{z})\boldsymbol{u}(t,\boldsymbol{z}) = D(t)\boldsymbol{u}(t)\ ;\ \boldsymbol{u} = (u_x; u_y) \tag{13.9}$$

$$S(t) = \int dxdy\ S(t,x,y)\ ;\ \ D(t) = \int dxdy\ D(t,x,y) \tag{13.10}$$

As we show in Appendix C, distributions of economic agents by their risk ratings as coordinates on economic domain permit derive mean risk coordinates for each economic variable of transactions (Olkhov, 2017d; 2019a). Relations (C.2.3) define mean risk $X_S(t)$ of suppliers *S(t)* with economic variable *E* and mean risk $Y_C(t)$ of consumers of variable *E*:

$$S(t)X_S(t) = \int dxdy\ x\,S(t,x,y)\ \ ;\ \ S(t)Y_C(t) = \int dxdy\ y\,S(t,x,y) \tag{14.1}$$

We argue the business cycles of economic variables *E* (credit, investment, assets, commodities and etc.,) as processes induced and correlated with fluctuations of mean risks $X_S(t)$ of suppliers and mean risk $Y_C(t)$ of consumers of variable *E.* Flows of economic transactions of supply $\boldsymbol{P}_S(t)$ and action (13.3, 13.4) of demand flows $\boldsymbol{P}_D(t)$ cause fluctuations of mean risks $X_S(t)$ of suppliers and consumers $Y_C(t)$ as well as mean risks of demanders $Y_D(t)$ and $X_D(t)$ (14.2, 13.10):

$$D(t)X_D(t) = \int dxdy\ x\,D(t,x,y)\ \ ;\ \ D(t)Y_D(t) = \int dxdy\ \ y\,D(t,x,y) \tag{14.2}$$

We show in Appendix C (C.2.5-2.7) mean risk $X_S(t)$ (14.1) moves as

$$\frac{d}{dt}X_S(t) = v_x(t) + w_x(t) \tag{14.3}$$

$$w_x(t) = [X_{SF}(t) - X_S(t)]\frac{d}{dt}lnS(t) \tag{14.4}$$

$$F_S(t) = \int dxdy\,F_S(t,x,y)\quad ;\quad X_{SF}(t)F_S(t) = \int dxdy\ \ x\,F_S(t,x,y) \tag{14.5}$$

Borders of economic domain (6.1, 6.2) reduce motion of mean risks (14.1,14.3) and thus velocities $\upsilon_x(t)$ (13.8) and $w_x(t)$ (14.4) should fluctuate and cause oscillations of mean risks. Frequencies of $\upsilon_x(t)$ describe impact of flow fluctuations and frequencies of $w_x(t)$ describe oscillations induced by interactions between supply and demand transactions. In Appendix C we study model equations (C.2.1-2.2) that describe fluctuations of macro supply $S(t)$ (C.1.4) with variable E determined by flows $\boldsymbol{P}_S(t)$, $\boldsymbol{P}_D(t)$ (C.3.4-3.5) and derive relations for $S(t)$ (C.5.6) in simple form as:

$$S(t) = S(0) + a\left[S_x(1)\sin\omega t + S_y(1)\sin\nu t\right] + a\,S_x(3)exp\,\gamma t \tag{14.6}$$

Relations (14.6) model the business cycles with frequencies ω and ν of macro supply $S(t)$ with variable E accompanied by exponential growth as $exp(\gamma t)$ due to economic growth of $S(t)$. Hence (14.6) may model credit cycles determined by fluctuations of creditors with frequencies ω and borrowers with frequencies ν with exponential growth as $exp(\gamma t)$ of credits provided in economy due to economic growth. The same approach may model investment cycles, consumption cycles and etc.

4 Expectations, price and return

Assets pricing is the key issue of modern finance. Assets pricing research account thousands studies and we chose (Campbell, 1985; Campbell and Cochrane, 1995; Heaton and Lucas, 2000; Cochrane, 2001; Cochrane and Culp, 2003; Cochrane, 2017) for clear, precise and general treatment of the problem. Expectations as factors that impact assets pricing are studied at least since Muth (1961) and (Fama, 1965; Lucas, 1972; Sargent and Wallace, 1976; Hansen and Sargent, 1979; Blume and Easley, 1984; Brunnermeier and Parker, 2005; Dominitz and Manski, 2005; Greenwood and Shleifer, 2014; Lof, 2014; Manski, 2017). Assets pricing and return are studied by (Keim and Stambaugh, 1986; Mandelbrot, Fisher and Calvet, 1997; Brock and Hommes, 1998; Fama, 1998; Plerou et.al., 1999; Andersen et.al., 2001; Gabaix et.al., 2003; Stanley et.al., 2008; Hansen, 2013; Greenwald, Lettau and Ludvigson, 2014; Gontis et.al., 2016; van Binsbergen and Koijen, 2017) and present only small part of publications. Below we study a simple case and describe possible impact of expectations on transactions, assets pricing and return (Olkhov, 2018; 2019b).

Let's study transactions with particular assets *E* at Exchange. Let's assume that agents perform different parts of transactions with assets *E* at Exchange under different expectations. Each transaction defines quantity *Q* of assets *E* (for example number of shares) and cost or value *C* of the deal. Obvious relations define assets price *p* of this transaction:

$$C = pQ$$

Transactions performed under different expectations may have different quantity, cost and asset price. Let's assume that agent *i* at point $\boldsymbol{x}$ have *k,l=1,..K* different expectations $\boldsymbol{ex}_i(k,l;t,\boldsymbol{x})$ that approve transactions $\boldsymbol{bs}_i(k,l;t,\boldsymbol{x})$ of asset *E* with Exchange:

$$\boldsymbol{bs}_i(k,l;t,\boldsymbol{x}) = \big(Q_i(k;t,\boldsymbol{x});C_i(l;t,\boldsymbol{x})\big) \tag{15.1}$$

Here $Q_i(k;t,\boldsymbol{x})$ and $C_i(l;t,\boldsymbol{x})$– quantity and cost of transaction performed by agent *i* under expectation *k,l*. We propose that decision on quantity $Q_i(k;t,\boldsymbol{x})$ of transaction is taken under expectation of type *k* and decision on cost $C_i(l;t,\boldsymbol{x})$ of transaction is taken under expectation of type *l*. Let's define expectations $\boldsymbol{ex}_i(k,l;t,\boldsymbol{x})$ as:

$$\boldsymbol{ex}_i(k,l;t,\boldsymbol{x}) = \Big(ex_{Qik}(k;t,\boldsymbol{x}), ex_{Ci}(l;t,\boldsymbol{x})\Big);\ k,l = 1,\dots K \tag{15.2}$$

Expectations $ex_{Qi}(k;t,\boldsymbol{x})$ and $ex_{Ci}(l;t,\boldsymbol{x})$ approve quantity *Q* and cost *C* of the transaction $\boldsymbol{bs}_i(k,l;t,\boldsymbol{x})$. Relations (II, 2.1, 2.2, 7.2) for define macro transaction $\boldsymbol{BS}(k,l;t,\boldsymbol{x})$ under expectation of type *k,l=1,…K* as

$$\boldsymbol{BS}(k,l;t,\boldsymbol{x}) = \big(Q(k;t,\boldsymbol{x});C(l;t,\boldsymbol{x})\big) = \textstyle\sum_{i\in dV(\boldsymbol{x});\,\Delta} \boldsymbol{bs}_i(k,l;t,\boldsymbol{x}) \tag{15.3}$$

$$Q(k;t,\boldsymbol{x}) = \sum_{i\in dV(\boldsymbol{x});\Delta} Q_i(k;t,\boldsymbol{x}) \quad ; \quad C(t,\boldsymbol{x}) = \sum_{i\in dV(\boldsymbol{x});\Delta} C_i(l;t,\boldsymbol{x})$$

Similar to (II, 7.5-7.7) let's introduce expected transactions $\boldsymbol{Et}(k,l;t,\boldsymbol{x})$at point $\boldsymbol{x}$ as

$$\boldsymbol{Et}(k,l;t,\boldsymbol{x}) = (Et_Q(k;t,\boldsymbol{x});Et_C(l;t,\boldsymbol{x})) \tag{15.4}$$

$$Et_Q(t,\boldsymbol{x}) = \sum_{i\in dV(\boldsymbol{x});\Delta} ex_{Qi}(k;t,\boldsymbol{x})Q_i(k;t,\boldsymbol{x})$$

$$Et_C(l;t,\boldsymbol{x}) = \sum_{i\in dV(\boldsymbol{x});\Delta} ex_{Ci}(l;t,\boldsymbol{x})C_i(l;t,\boldsymbol{x})$$

Let's study relations between transactions $\boldsymbol{BS}(k,l;t)$ (15.3) and expected transactions $\boldsymbol{Et}(k,l;t)$ (15.4) of entire economics as functions of time *t* only:

$$\boldsymbol{BS}(k,l;t) = \int d\boldsymbol{x}\ \ \boldsymbol{BS}(k,l;t,\boldsymbol{x})\ ;\ \ \boldsymbol{Et}(k,l;t) = \int d\boldsymbol{x}\ \ \boldsymbol{Et}(k,l;t,\boldsymbol{x})\,;k,l = 1,..K \tag{15.5}$$

Integrals in (15.5) define ***BS**(k,l;t)* all transactions with asset *E* made by all agents of entire economics at Exchange under expected transactions ***Et**(k,l;t)*. Due to equations (5.1-5.3), (8.1, 8.2) and (9.1, 9.2) equations on (15.5) take form:

$$\frac{d}{dt}Q(k;t) = F_Q(k;t)\ ;\ \frac{d}{dt}C(l;t) = F_C(l;t) \tag{15.6}$$

$$\boldsymbol{F}(k;t) = (F_Q;F_C);\ \ F_Q(k;t) = \int d\boldsymbol{x}\ F_Q(k;t,\boldsymbol{x})\ ;\ F_C(l;t) = \int d\boldsymbol{x}\, F_C(l;t,\boldsymbol{x}) \tag{15.7}$$

$$\frac{d}{dt}Et_Q(k;t) = Fe_Q(k;t)\ ;\ \frac{d}{dt}Et_C(l;t) = Fe_C(l;t) \tag{15.8}$$

$$\boldsymbol{Fe}(k,l;t) = \left(Fe_Q;Fe_C\right);Fe_Q(k;t) = \int d\boldsymbol{x}\ Fe_Q(k;t,\boldsymbol{x});Fe_C(l;t) = \int d\boldsymbol{x}\,Fe_C(l;t,\boldsymbol{x}) \tag{15.9}$$

Relations (15.1-15.3) define expectations $Ex_{kl}(t)$ of entire economics as:

$$\boldsymbol{Ex}(k,l;t) = \left(Ex_Q;Ex_C\right)$$

$$Et_Q(k;t) = Ex_Q(k;t)Q(k;t) \quad ; \quad Et_C(l;t) = Ex_C(l;t)C(l;t) \tag{15.10}$$

Equations (15.6-9) describe transactions ***BS**(k,l;t)* (15.5) with assets *E* of the entire economics under expectations ***Ex**(k,l;t)* (15.10). Let's describe a model of mutual action between small disturbances of transactions and expectations in a linear approximation. Let's consider (15.6-9) and assume that mean transactions $\boldsymbol{BS}_0(k,l;t)$ and $\boldsymbol{Et}_0(k,l;t)$ are slow to compare with small dimensionless disturbances ***bs**(k,l;t)* and ***et**(k,l;t)* and let's take $\boldsymbol{BS}_0(k,l)$ and $\boldsymbol{Et}_0(k,l)$ as const. Due to (15.3-5):

$$\boldsymbol{BS}(k,l;t) = (Q;C);\ Q(k;t) = Q_{0k}\big(1+q(k;t)\big);\ C(l;t) = C_{0l}\big(1+c(l;t)\big) \tag{16.1}$$

$$\boldsymbol{Et}(k,l;t) = \left(Et_Q(k;t);Et_C(l;t)\right) \tag{16.2}$$

$$Et_Q(k;t) = Et_{Q0k}\left(1+et_q(k;t)\right);\ Et_C(l;t) = Et_{C0l}\big(1+et_c(l;t)\big) \tag{16.3}$$

Equations on small disturbances ***bs**(k,l;t)* and ***et**(k,l;t)* take form:

$$Q_{0k}\ \frac{d}{dt}q(k;t) = f_q(k;t)\ ;\ C_{0l}\frac{d}{dt}c(l;t) = f_c(l;t) \tag{16.2}$$

$$Et_{Q0k}\ \frac{d}{dt}et_q(k;t) = fe_q(k;t)\ ;\ Et_{C0l}\ \frac{d}{dt}et_c(l;t) = fe_c(l;t) \tag{16.3}$$

$$Fe_{Qk} = Fe_{Q0k} + fe_Q(k;t)\ ;Fe_{Cl} = Fe_{C0l} + fe_c(l;t\ ) \tag{16.4}$$

Let's assume that factors $f_q(k;t)$ and $f_c(l;t)$ in (16.2) depend on disturbances of expected transactions $et_q(k;t)$ and $et_c(l;t)$ and $fe_q(k;t)$ and $fe_c(l;t)$ in (16.3) depend on disturbances of *q(k;t)* and *c(l;t)*. For linear approximation by disturbances let's take (16.2-3) as:

$$Q_{0k}\ \frac{d}{dt}q(k;t) = a_{qk}Et_{Q0k}et_q(k;t)\ \ ;\ C_{0l}\frac{d}{dt}c(l;t) = a_{cl}Et_{C0l}\ et_c(l;t) \tag{16.5}$$

$$Et_{Q0k}\ \frac{d}{dt}et_q(k;t) = be_{qk}Q_{0k}q(k;t)\ \ ;\ Et_{C0l}\frac{d}{dt}et_c(l;t) = be_{cl}C_{0l}\ c(l;t) \tag{16.6}$$

$$\omega_{qk}^2 = -a_{qk}be_{qk} > 0\ \ ;\ \omega_{cl}^2 = -\ a_{cl}be_{cl} > 0 \tag{16.7}$$

If relations (16.7) are valid, then (16.5-6) are equations for harmonic oscillators:

$$\left(\frac{d^2}{dt^2}+\omega_{qk}^2\right)q(k;t)=0\ ;\ \left(\frac{d^2}{dt^2}+\omega_{cl}^2\right)c(l;t)=0 \tag{16.8}$$

$$\left(\frac{d^2}{dt^2}+\omega_{qk}^2\right)et_q(k;t)=0\ ;\ \left(\frac{d^2}{dt^2}+\omega_{cl}^2\right)et_c(l;t)=0\ ;k,l=1,..K \tag{16.9}$$

Simple solutions of (16.8) for dimensionless disturbances $q_k(t)$ and $c_l(t)$:

$$q(k;t)=g_{qk}sin\omega_{qk}t+\ d_{qk}cos\omega_{qk}t \tag{17.1}$$

$$c(l;t)=g_{cl}sin\omega_{cl}t+d_{cl}cos\omega_{cl}t \tag{17.2}$$

$$g_{qk},d_{qk}\ ,g_{cl},d_{cl}\ll 1 \tag{17.3}$$

Relations (17.1-3) describe simple harmonic fluctuations of disturbances of volume *Q(k;t)* and cost *C(l;t)* of transactions ***BS****(k,l;t)* performed under different expectations ***Ex****(k,l;t)* (16.10).

Price fluctuations. Let's note price of transaction made by all agents of entire economics under expectations of type *k, l* as *p(k,l;t)*

$$C(k,l;t)=p(k,l;t)Q(k,l;t) \tag{18.1}$$

Now for convenience let's call *C(k,l;t)* as cost of transaction made under expectation of type *l* for volume *Q(k,l;t)* of transaction made under expectation of type *k*. Thus transaction ***BS****(k,l;t)* has cost *C(k,l;t)* made under expectation of type *l* and volume *Q(k,l;t)* of transaction made under expectation of type *k*. Double indexes *(k,l)* determine transaction with cost under expectation *l* and volume under expectation *k*. Sum of transactions ***BS****(k,l;t)* (16.1) by all expectations *k,l=1,…K* define transactions ***BS****(t)* in the entire economics:

$$\boldsymbol{BS}(t)=(Q(t);C(t))\ ;\ Q(t)=\sum_{kl}Q(k,l;t);\ \ C(t)=\sum_{k,l}C(k.l;t) \tag{18.2}$$

Price *p(t)* of transactions ***BS****(t)* (18.2) equals:

$$C(t)=p(t)Q(t) \tag{18.3}$$

Let's study disturbances of cost *C(t)*, volume *Q(t)* and price *p(t)* for (18.3) as:

$$Q(t)=\sum_{k,l}Q_{0kl}\big(1+q(k,l;t)\big)=Q_0\sum_{k,l}\lambda_{kl}\big(1+q(k,l;t)\big) \tag{18.4}$$

$$C(t)=\sum_{k,l}C_{0kl}\big(1+c(k,l;t)\big)=C_0\sum_{k,l}\mu_{kl}\big(1+c(k,l;t)\big) \tag{18.5}$$

Relations (18.4) describe impact of dimensionless disturbances *q(k,l;t)* on volume *Q(t)* and (18.5) describe impact of dimensionless disturbances *c(k,l;t)* on cost C(t) of transactions.

$$Q_0=\sum_{k,l}Q_{0kl}\ ;\ \lambda_{kl}=\frac{Q_{0kl}}{Q_0}\ ;\ C_0=\sum_{k,l}C_{0kl}\ ;\ \mu_{kl}=\frac{C_{0kl}}{C_0}\ ;\ \sum\lambda_{kl}=\sum\mu_{kl}=1 \tag{18.6}$$

Relations (18.3) define price *p(t)* for *Q(t)* (18.4) and *C(t)* (18.5):

$$p(t)=\frac{C(t)}{Q(t)}=\frac{\sum_{k,l}C(k,l;t)}{\sum_{k,l}Q(k,l;t)}\quad ;\quad p_0=\frac{C_0}{Q_0}=\frac{\sum_{k,l}C_{0kl}}{\sum_{k,l}Q_{0kl}} \tag{18.7}$$

In linear approximation by disturbances *q(k,l;t)* and *c(k,l;t)* price *p(t)* (18.7) take form:

$$p(t) = \frac{C(t)}{Q(t)} = \frac{C_0 \sum_{k,l} \mu_{kl}\big(1 + c(k,l;t)\big)}{Q_0 \sum_{k,l} \lambda_{kl}\big(1 + q(k,l;t)\big)} = p_0 \left[1 + \sum\nolimits_{k,l} \mu_{kl} c(k,l;t) - \sum\nolimits_{k,l} \lambda_{kl} q(k,l;t)\right]$$

$$p(t) = p_0[1 + \pi(t)] = p_0\big[1 + \textstyle\sum_{k,l}(\mu_{kl} c(k,l;t) - \lambda_{kl} q(k,l;t))\big] \qquad (18.8)$$

Dimensionless fluctuations of price $\pi(t)$ (18.8) equals weighted sum of disturbances $q(k,l;t)$ and $c(,lk;t)$ as (18.9):

$$\pi(t) = \textstyle\sum_{k,l} \mu_{kl}\, c(k,l;t) - \lambda_{kl} q(k,l;t) \qquad (18.9)$$

Now let's take (18.1) and present $\pi(t)$ in other form:

$$C(k,l;t) = C_{0kl}[1 + c(k,l;t)] = p_{0kl}[1 + \pi(k,l;t)]Q_{0kl}[1 + q(k,l;t)] \qquad (19.1)$$

From (18.6-7) and (19.1) in linear approximation by $c(k,l;t)$, $\pi(k,l;t)$ and $q(k,l;t)$ obtain:

$$C_{0kl} = p_{0kl} Q_{0kl} \quad ; \quad c(k,l;t) = \pi(k,l;t) + q(k,l;t) \qquad (19.2)$$

Let's substitute (19.2) into (18.9):

$$\pi(t) = \textstyle\sum_{k,l} \mu_{kl} \pi(k,l;t) + \sum_{k,l}(\mu_{kl} - \lambda_{kl})\, q(k,l;t) \qquad (19.3)$$

Relations (19.3) describe price perturbations $\pi(t)$ as weighted sum of partial price disturbances $\pi(k,l;t)$ and volume disturbances $q(k,l;t)$. Thus statistics of price disturbances $\pi(t)$ is defined by statistics of partial price disturbances $\pi(k,l;t)$ and statistics of volume disturbances $q_k(k,l;t)$.

Return perturbations. Price disturbances (19.3) cause perturbations of return $r(t,d)$:

$$r(t,d) = \frac{p(t)}{p(t-d)} - 1 \qquad (20.1)$$

Let's introduce partial returns $r(k,l;t,d)$ for price $p(k,l;t)$ (18.1) and "returns" $w(k,l;t,d)$ for volumes $Q(k,l;t)$ (18.2):

$$r(k,l;t,d) = \frac{p(k,l;t)}{p(k,l;t-d)} - 1 \quad ; \quad w(k,l;t,d) = \frac{Q(k,l;t)}{Q(k,l;t-d)} - 1 \qquad (20.2)$$

Let's assume for simplicity that mean price p_{0kl} and trade volumes Q_{0kl} are constant during time term d and (18.7; 19.3) present (20.1, 20.2) as

$$r(t,d) = \frac{\pi(t)-\pi(t-d)}{1+\pi(t-d)} \;\; ; \;\; w(k,l;t,d) = \frac{q(k,l;t)-q(k,l;t-d)}{1+q(k,l;t-d)} \qquad (20.3)$$

$$r(t,d) = \textstyle\sum \mu_{kl} \frac{1+\pi(k,l;t-d)}{1+\pi(t-d)} r(r,l;t,d) + \sum(\mu_{kl} - \lambda_{kl}) \frac{1+q(k;t-d)}{1+\pi(t-d)} w(k,l;t,d) \qquad (20.4)$$

Let's define

$$\varepsilon_{kl}(t-d) = \mu_{kl} \frac{1+\pi(k;t-d)}{1+\pi(t-d)} \quad ; \quad \eta_{kl}(t-d) = (\mu_{kl} - \lambda_{kl}) \frac{1+q(k;t-d)}{1+\pi(t-d)} \qquad (20.5)$$

$$\textstyle\sum_{k,l}[\varepsilon_{kl}(t-d) + \eta_{kl}(t-d)] = 1 \qquad (20.6)$$

$$r(t,d) = \textstyle\sum_{k,l} \varepsilon_{kl}\,(t-d) r(k,l;t,d) + \sum_{k,l} \eta_{kl}\,(t-d) w(k,l;t,d) \qquad (20.7)$$

Relations (20.6-7) describe return (20.1) as sum of partial returns and volume "returns" $w(k,l;t,d)$ (20.2, 20.3). Sum for coefficients μ_{kl} and μ_{kl}-λ_{kl} for price $p(t)$ (18.7; 19.3) and $\varepsilon_{kl}(t)$

and $\eta_{kl}(t)$ for return $r(t,d)$ (20.1) equals unit but (19.3) and (20.7) can't be treated as averaging procedure as some coefficients μ_{kl}-λ_{kl} and $\eta_{kl}(t)$ should be negative. If mean price (19.2) $p_{0kl}=p_0$ for all pairs of expectations (k,l) then from (18.6, 18.7) obtain

$$p_{0kl} = p_0 = const \;\rightarrow\; \lambda_{kl} = \mu_{kl} \;\;;\;\; \eta_{kl}(t) = 0 \text{ for all } k,l \qquad (20.8)$$

and relations (19.3; 20.7) take simple form

$$\pi(t) = \sum_{k,l} \mu_{kl}\, \pi(k,l;t) \qquad (20.9)$$

$$r(t,d) = \sum_{k,l} \mu_{kl} \frac{1+\pi(k,l;t-d)}{1+\pi(t-d)} r(k,l;t,d) = \sum_{k,l} \mu_{kl} \frac{\pi(k,l;t)-\pi(k,l;t-d)}{1+\pi(t-d)} \qquad (20.10)$$

Thus assumption (20.8) on prices (19.2) for all pairs of expectations (k,l) cause representation (20.9, 20.10) of price disturbances $\pi(t)$ as weighted sum of partial price disturbances $\pi(k,l;t)$ for different pairs of expectations (k,l). Otherwise price disturbances $\pi(t)$ should take (19.3) and depend on volume perturbations $q(k,l;t)$. Assumption (20.8) cause returns as (20.10), otherwise returns take (20.7). Actually expectations are key factors for market competition and different expectations (k,l) should cause different mean partial prices p_{0kl}. That should cause complex representation of price (19.3) and return (20.7) disturbances as well as impact volatility and statistic distributions of price and return disturbances.

5 Option pricing

Option pricing accounts thousands articles published since classical Black, Scholes (1973) and Merton (1973) (BSM) studies (Hull and White, 1987; Hansen, Heaton, and Luttmer, 1995; Hull, 2009). Current observations of market data show that option pricing don't follow Brownian motion and classical BSM model (Fortune, 1996). Stochastic volatility is only one of factors that cause BSM model violation (Heston, 1993, Bates, 1995). Studies of economic origin of price stochasticity are important for correct modeling asset and option pricing. We propose that economic space modeling may give new look on description of asset stochasticity and option pricing. Indeed, economic space establishes ground for description of density functions of economic variables and transactions. On other hand economic space allows describe price evolution of assets for selected agent in a random economic environment. Random evolution of risk coordinates of selected assets impact assets and option pricing. Nevertheless it is clear that Brownian motion models don't fit real market option pricing, simple Brownian considerations allow argue some hidden complexities of option pricing problem. Below we discuss classical BSM treatment of option pricing based on assumption of price Brownian motion (Hull, 2009). We start with classical BSM approximation and describe model for option price caused by Brownian motion of economic agent on economic space that gives generalization of the classical BSM equations (Olkhov, 2016a-2016c). Further we argue BSM assumptions and restrictions that arise from previous Section and may impact assets and option pricing models.

Let's start with classical derivation of the BSM (Hull, 2009) based on assumption that price p of selected agent's assets obeys Brownian motion $dW(t)$ with volatility σ and linear trend υ:

$$dp(t) = p\,\upsilon\,dt + p\sigma dW(t)\ \ ;\ \ < dW(t) >= 0;\ < dW(t)dW(t) > = dt \tag{21.1}$$

Assumptions (21.1) give the classical BSM equation for the option price $V(p;t)$ for risk-free rate r (Hull, 2009):

$$\frac{\partial V}{\partial t} + rp\frac{\partial V}{\partial p} + \frac{1}{2}\sigma^2 p^2 \frac{\partial^2}{\partial p^2} V\ =\ rV \tag{21.2}$$

In Sec.4 we use coordinates $\boldsymbol{x}$ to define positions of agents those involved in transactions at Exchange with assets of selected agent $\boldsymbol{A}$. Let's note $\boldsymbol{y}$ as coordinates of selected agent $A(t,\boldsymbol{y})$. Let's assume that price p of assets of selected agent $A(t,\boldsymbol{y})$ depends on time t and on risk coordinates $\boldsymbol{y}$ as $p(t,\boldsymbol{y})$. Let's propose that disturbances of risk coordinates $\boldsymbol{y}$ of selected agent $A(t,\boldsymbol{y})$ follow Brownian motion $d\boldsymbol{Y}(t)$ on n-dimensional economic space:

$$d\boldsymbol{y}\ =\ \boldsymbol{v}dt\ +\ d\boldsymbol{Y}(t)\ \ ;\ d\boldsymbol{Y}(t) = (dY_1, ..dY_n)\ \ ;\ \ < dY_i(t) > = 0 \tag{21.3}$$

$$< dY_i(t)dY_j(t) > = \eta_{ij}\,dt\quad ;\ \ < dW(t)\,dY_i(t) > = b_i$$

Factors η_{ii} describe volatility of Brownian motion dY_i along axis i and η_{ij} for $i{\neq}j$ describe correlations between Brownian motions dY_i along axes i and dY_j along axes j. Factors b_i – describe correlations between Brownian motion dW and dY_i along axes i. Now let's extend assumption (21.1) and let's propose (21.4) that price $p(t,\boldsymbol{y})$ depend on time t and on Brownian motion $d\boldsymbol{Y}(t)$ (21.3) of selected agent $A(t,\boldsymbol{y})$ on economic space:

$$dp(t,\boldsymbol{y}) = p\,\upsilon\,dt + p\sigma dW(t)\ + p\,\boldsymbol{k}\cdot d\boldsymbol{Y}\quad ;\ \ \boldsymbol{k} = (k_1, \dots k_n) -\ const \tag{21.4}$$

Similar to (Hall, 2009) for risk-free rate r from (21.4) obtain extension of the classical BSM equation (21.2) for the option price $V(p;t,\boldsymbol{y})$ on n-dimensional economic space (Olkhov, 2016c) :

$$\frac{\partial V}{\partial t} + rp\frac{\partial V}{\partial p} + ry_i\frac{\partial V}{\partial y_i} + \frac{1}{2}p^2 q^2\frac{\partial^2 V}{\partial p^2} + p\big(\sigma b_i + k_j\,\eta_{ji}\big)\frac{\partial^2 V}{\partial p\partial y_i} + \frac{\eta_{ij}}{2}\frac{\partial^2 V}{\partial y_i\partial y_j} = rV \tag{21.5}$$

$$q^2 = \big(\sigma^2 + k_i\,k_j\,\eta_{ij} + 2\sigma k_i\,b_i\big)\,;\ \ i,j = 1, \dots n$$

Additional parameters k_i, b_i, η_{ij}, $i,j=1,\dots n$, define volatility q^2 and coefficients for additional terms of equation (21.5) and impact option price $V(p;t,\boldsymbol{y})$. Extension (21.5) of the classical BSM equations (21.2) may uncover hidden complexities of option pricing that have origin in the random motion of agents $A(t,\boldsymbol{y})$ on economic space. As special case for (21.5) one can study equation on option price $V(p;t,\boldsymbol{y})$ on 1-dimensional economic space for $\sigma=0$ without classical BSM assumptions (21.1):

$$\frac{\partial V}{\partial t}+rp\frac{\partial V}{\partial p}+ry\frac{\partial V}{\partial y}+\frac{1}{2}p^2k^2\eta\frac{\partial^2 V}{\partial p^2}+pk\eta\frac{\partial^2 V}{\partial p\partial y}+\frac{\eta}{2}\frac{\partial^2 V}{\partial y^2}=rV \tag{21.6}$$

Equations (21.6) describe option price $V(p;t,y)$ of assets which price $p(t,y)$ depends only on Brownian motion $dY(t)$ (21.3) of agents coordinates y on 1-dimensional economic space. Let's mention that assumptions (21.3, 21.4) simplify assets pricing model that we argue in Sec.4. Indeed, in Sec.4 we discuss that asset price and its disturbances should depend on relations between transactions and expectations. Thus assumptions on Brownian motion (21.3) of coordinates of selected agent $A(t,\boldsymbol{y})$ on economic space should impact transactions with assets of particular agent $A(t,\boldsymbol{y})$ and corresponding expectations. Let's take relations (19.3) for price disturbances $\pi(t,\boldsymbol{y})$ of assets of selected agent $A(t,\boldsymbol{y})$ with coordinates $\boldsymbol{y}$

$$\pi(t,\boldsymbol{y})=\sum_{k,l}\mu_{kl}\,\pi(k,l;t,\boldsymbol{y})+\sum_{k,l}(\mu_{kl}-\lambda_{kl})\,q(k,l;t,\boldsymbol{y}) \tag{22.1}$$

Let's remind that $\pi(k,l;t,\boldsymbol{y})$ describe partial price disturbances of assets of agent $A(t,\boldsymbol{y})$ for transactions of all economic agents with Exchange made under expectations of type k for decisions on trading volume $Q(k,l;t,\boldsymbol{y})$ and expectations of type l for decisions on cost $C(k,l;t,\boldsymbol{y})$ of transaction. As we mention in Sec.4, if partial price p_{0kl} (19.2) is constant for all type of expectations k,l then price disturbances $\pi(t,\boldsymbol{y})$ take form (20.9) and equal weighted sum of partial prices $\pi(k,l;t,\boldsymbol{y})$. Otherwise price disturbances $\pi(t,\boldsymbol{y})$ should depend on disturbances of partial prices $\pi(k,l;t,\boldsymbol{y})$ and on perturbations of trading volumes $q(k,l;t,\boldsymbol{y})$. Let's mention that statistic distribution of price disturbances $\pi(t,\boldsymbol{y})$ (22.1) may depend also on coefficients λ_{kl} and μ_{kl} (18.6) that can fluctuate due to random change of coordinates of selected agent $A(t,\boldsymbol{y})$. Possible impact of these numerous factors on option pricing should be studied further.

6. Conclusions

There are endless economic and financial problems that should be described. In this paper we present only few to demonstrate advantages of our approach to economic theory. We develop economic theory on base of well known economic notions – economic agents, economic and financial variables and transactions, expectations of economic agents and risk ratings of economic agents. Economic modeling for decades use these notions. Our contribution to economic theory is follows. First, we propose distribute economic agents by their risk ratings as their coordinates on economic space. Second, we propose move from description of separate agents, their variables, transactions and expectations on economic space to description of aggregated, averaged density functions of variables, transactions and expectations on economic space. To make this transition we introduce two scales: dV and Δ . Scale dV define averaging over economic space and scale Δ define averaging over the time.

Thus different scales *Δ = 1 day, 1 month, 1 year* describe different approximation of economy. All other considerations are consequences of these two steps.

We regard risks as main drivers of macroeconomic evolution and development. Any beneficial economic activity is related with risks and no risk-free financial success is possible. We propose that risk-free treatments of economic problems have not too much economic sense. Change of risk rating of economic agents due to their economic activity, their financial transactions with other agents, their economic and financial expectations, market trends, regulatory or technology changes, political, climate and other reasons induce change of risk ratings that cause motion of mean macroeconomic risks and flows of economic and financial variables and transactions on economic space. Motion of mean risks and economic flows impact evolution of macroeconomic states and cycles. We regard description of mean risks and economic flows as one of major problems of economic theory.

Any economic motions and flows are accompanied by generation of small perturbations of economic variables, transactions and expectations. Description of propagation of small economic and financial disturbances on economic space reflect most general problem of evolution of any complex system. Economic and financial dynamics are accompanied by generation, propagation and interactions of numerous economic waves of variables, transactions and expectations on economic domain. Wave propagation of small perturbations on economic space may explain interactions between different markets, industries, countries and describe transfer of economic and financial influence over macroeconomics. Total distinction of economic processes from physical problems cause room for amplification of small economic and financial perturbations during wave propagation over economic domain. Growth of wave amplitudes of economic disturbances during propagation on economic space may impact huge perturbations and shocks of entire macroeconomics. In Sec. 2 we describe cases of economic wave propagation of perturbations of variables and transactions. We describe economic waves that have parallels to sound waves and to surface waves. Economic sound-like waves describe propagation of variables and transactions density perturbations through economic domain. Economic surface-like waves describe propagation of perturbations along borders of economic domain. Such diversity has analogy in hydrodynamics but nature and properties of economic waves are completely different.

Borders of economic domain reduce area for economic agents by minimum and maximum risk grades. Thus borders reduce flows of economic variables and transactions on economic domain and cause fluctuations of these economic flows. Fluctuations of economic flows of variables and transactions induce fluctuations of corresponding mean risks. In Sec 3 we regard fluctuations of mean risks and fluctuations of economic flows as characters of business cycles. Fluctuations of credit mean risks reflect credit cycles, fluctuations of investment mean risks reflect investment cycles and so on. Interactions between major economic and financial variables cause correlations of corresponding cycles. Description of these fluctuations requires relatively complex economic equations.

Evolution of economic variables is performed by transactions between agents. Agents take decisions on economic and financial transactions under numerous expectations. Agents form their expectations on base of macroeconomic and financial variables, transactions, market regulatory and technology trends, expectations of other agents and etc. Relations between economic and financial variables, transactions and expectations establish a really complex system. Assets pricing problem is only one that is determined by relations between transactions and expectations. In Sec. 4 we describe simple relations between transactions and expectations and model assets price disturbances as consequences of perturbations of transactions made under numerous expectations. As last economic example in Sec.5 we argue classical Black-Scholes-Merton (BSM) option price model. We show that economic space uncovers hidden complexities of classical BSM model and discuss relations between modeling price disturbances and option pricing.

As sample of items that differs our approach from general equilibrium let's outline factors dV and Δ (I. 2-4) that determine densities of economic variables, transactions and expectations. Factors dV are responsible for averaging over scales of economic space and Δ define averaging over time scales. For example $\Delta=1$ day, *1* month or *1* year determine different economic models with time averaging during *1* day, *1* month or *1* year. Thus each particular economic model describes processes with approximation determined by factors dV and Δ. That seems important for comparison of model predictions with economic observations. As we know there are no similar scales in general equilibrium models.

Let's underline that we present only essentials of economic theory and many problems should be studied further. Econometric problems and observation of economic and financial variables, transactions and expectations of agents and agents risk assessment are among the central. Up now there are no sufficient econometric data required to establish distribution of economic agents by their risk ratings as coordinates on economic space. Nevertheless we hope that our model may be useful for better understanding and description of economic and financial processes.

Wave equations for economic variables

Let's start with equations (5.2) and take time derivative. We obtain with help of (5.4):

$$\frac{\partial^2}{\partial t^2}\varphi(t,\boldsymbol{x}) = \alpha_1 C\frac{\partial^2}{\partial t^2}\pi(t,\boldsymbol{x}) - \beta_1 C\Delta\pi(t,\boldsymbol{x}) \quad (A.1)$$

We have the similar equation from (5.3) and (5.4):

$$C\frac{\partial^2}{\partial t^2}\pi(t,\boldsymbol{x}) = \alpha_2\frac{\partial^2}{\partial t^2}\varphi(t,\boldsymbol{x}) - \beta_2\Delta\varphi(t,\boldsymbol{x}) \quad (A.2)$$

Thus for (A.1) and (A.2) obtain:

$$(1-\alpha_1\alpha_2)\frac{\partial^2}{\partial t^2}\varphi(t,\boldsymbol{x}) = -\alpha_1\beta_2\Delta\varphi(t,\boldsymbol{x}) - \beta_1 C\Delta\pi(t,\boldsymbol{x}) \quad (A.3)$$

Let's take second time derivative from (A.3) and with (A.1; A.2) obtain for $\varphi(t,\boldsymbol{x})$ and $\pi(t,\boldsymbol{x})$:

$$\left[(1-\alpha_1\alpha_2)\frac{\partial^4}{\partial t^4} + (\alpha_1\beta_2+\beta_1\alpha_2)\Delta\frac{\partial^2}{\partial t^2} - \beta_1\beta_2\Delta^2\right]\varphi(t,\boldsymbol{x}) = 0 \quad (A.4)$$

To derive wave equations let's take Fourier transform by time and coordinates or let's substitute the wave type solution $\varphi(t,\boldsymbol{x}) = \varphi(\boldsymbol{x}\text{-}\boldsymbol{c}t)$. Than (A.4) takes form

$$(1-\alpha_1\alpha_2)c^4 + (\alpha_1\beta_2+\alpha_2\beta_1)c^2 - \beta_1\beta_2 = 0 \quad (A.5)$$

$$a = 1-\ \alpha_1\alpha_2 > 1\ ;\ b = \alpha_1\beta_2\ +\alpha_2\beta_1\ < 0\ ;\ d =\ \beta_1\beta_2 < 0$$

For positive roots c^2

$$c_{1,2}^2 = \frac{-b+/-\sqrt{b^4+4ad}}{2a} \quad (A.6)$$

equation (A.4) takes form of bi-wave equation (A.7) for $\varphi(t,\boldsymbol{x})$ and $\pi(t,\boldsymbol{x})$:

$$\left(\frac{\partial^2}{\partial t^2} - c_1^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - c_2^2\Delta\right)\varphi(t,\boldsymbol{x}) = 0 \quad (A.7)$$

Bi-wave equations (A.7) describe propagation of waves with two different speeds c_1 and c_2. If α_1 and α_2 equals zero, there are no wave equations and (A.4) take form

$$\left[\frac{\partial^4}{\partial t^4} - d\Delta^2\right]\varphi(t,\boldsymbol{x}) = 0;\ \ d < 0$$

Due to (1) supply $B(t,\boldsymbol{x})$ is proportional to price $p(t,\boldsymbol{x})$ and supply disturbances are proportional to price disturbances $\pi(t,\boldsymbol{x})$ (5.1). Let's take $\pi(t,\boldsymbol{x})$ as:

$$\pi(t,\boldsymbol{x}) = \pi_0\cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\exp(\gamma t + \boldsymbol{p}\cdot\boldsymbol{x})\ \ ;\ \ \pi_0 \ll 1 \quad (A.8)$$

Here $\boldsymbol{k}\cdot\boldsymbol{x}$ is scalar product of vectors $\boldsymbol{k}$ and $\boldsymbol{x}$. For price disturbances $\pi(t,\boldsymbol{x})$ (A.8) equation (A.4) becomes a system of two equations:

$$a[(\gamma^2-\omega^2)^2 - 4\gamma^2\omega^2] + b\,[(p^2-k^2)(\gamma^2-\omega^2) + 4\gamma\omega\,\boldsymbol{k}\cdot\boldsymbol{p}] - d[(p^2-k^2)^2 - 4(\,\boldsymbol{k}\cdot\boldsymbol{p}\,)^2] = 0 \quad (A.9)$$

$$4a\omega\gamma(\gamma^2-\omega^2) + b[\,2\omega\gamma\,(p^2-k^2) - 2(\gamma^2-\omega^2)\,\boldsymbol{k}\cdot\boldsymbol{p}\,] + 4d(p^2-k^2)\,\boldsymbol{k}\cdot\boldsymbol{p}\ = 0$$

Let's study simple case. Let's $\boldsymbol{p}=0$. Then (A.9) takes form:

$$a[(\gamma^2-\omega^2)^2-4\gamma^2\omega^2] - bk^2(\gamma^2-\omega^2) - dk^4 = 0$$

$$\gamma^2-\omega^2 = \frac{bk^2}{2a} \ ; \ 4ad+b^2<0 \tag{A.10}$$

Thus due to (A.10) roots $c^2_{1,2}$ (A.6) of equations (A.5) become complex numbers.

$$\gamma^4 - \frac{bk^2}{2a}\gamma^2 + \frac{k^4(b^2+4ad)}{16a^2} = 0 \ ; \ \gamma^2{}_{1,2} = \frac{k^2}{4a}\left(b +/- \sqrt{-4ad}\ \right)$$

Thus $\gamma^2 > 0$ for

$$\gamma^2 = \frac{k^2}{4a}\left(b+\sqrt{-4ad}\ \right) > 0 \quad ; \quad \omega^2 = \frac{k^2}{4a}\left(-b+\sqrt{-4ad}\ \right) > 0$$

For $\gamma > 0$ wave amplitude (A.8) grows up as $exp(\gamma t)$. Thus waves of small price disturbances $\pi(t,\boldsymbol{x})$ can propagate on economic domain with exponential growth of amplitude in time and that may disturb sustainable economic evolution.

Wave equations for perturbations of economic transactions

Let's start with equation for perturbations of supply *s(t,z)* (8.1) and take time derivative $\partial/\partial t$:

$$S_0 \frac{\partial^2}{\partial t^2} s(t,\boldsymbol{z}) + S_0 \nabla \cdot \frac{\partial}{\partial t} \boldsymbol{v} = \alpha_1 D_0 \nabla \cdot \frac{\partial}{\partial t} \boldsymbol{u} \quad \text{(B.1)}$$

and substitute equations on velocity $\boldsymbol{v}(t,z)$ and $\boldsymbol{u}(t,z)$ (8.2):

$$S_0 \frac{\partial^2}{\partial t^2} s(t,\boldsymbol{z}) - \alpha_1 \beta_2 S_0 \Delta\, s(t,\boldsymbol{z}) = -\beta_1 D_0 \Delta\, d(t,\boldsymbol{z}) \quad \text{(B.2)}$$

The same obtain for equation for perturbations of demand *d(t,z):*

$$D_0 \frac{\partial^2}{\partial t^2} d(t,\boldsymbol{z}) = \alpha_2 \beta_1 D_0 \Delta\, d(t,\boldsymbol{z}) - \beta_2 S_0 \Delta\, s(t,\boldsymbol{z}) \quad \text{(B.3)}$$

Let's take second derivative by time $\partial^2/\partial t^2$ of (B.2):

$$S_0 \frac{\partial^4}{\partial t^4} s(t,\boldsymbol{z}) - S_0 \alpha_1 \beta_2 \Delta \frac{\partial^2}{\partial t^2} s(t,\boldsymbol{z}) = -D_0 \beta_1 \Delta \frac{\partial^2}{\partial t^2} d(t,\boldsymbol{z})$$

and substitute (B.3):

$$S_0 \left[\frac{\partial^4}{\partial t^4} s(t,\boldsymbol{z}) - \alpha_1 \beta_2 \Delta \frac{\partial^2}{\partial t^2} s(t,\boldsymbol{z}) - \beta_1 \beta_2 \Delta^2\, s(t,\boldsymbol{z})\right] = -D_0 \alpha_2 \beta_1 \beta_1 \Delta^2 d(t,\boldsymbol{z}) \quad \text{(B.4)}$$

Now take operator Δ of (B.2) and obtain:

$$S_0 \frac{\partial^2}{\partial t^2} \Delta s(t,\boldsymbol{z}) - S_0 \alpha_1 \beta_2 \Delta^2 s(t,\boldsymbol{z}) = -D_0 \beta_1 \Delta^2\, d(t,\boldsymbol{z})$$

and substitute into (B.4) obtain equations for perturbations of supply *s(t,z)* and demand *d(t,z)*:

$$\left[\frac{\partial^4}{\partial t^4} - (\alpha_1 \beta_2 + \alpha_2 \beta_1) \Delta \frac{\partial^2}{\partial t^2} + \beta_1 \beta_2 (\alpha_1 \alpha_2 - 1) \Delta^2\right] s(t,\boldsymbol{z}) = 0 \quad \text{(B.5)}$$

Let's define

$$a = (\alpha_1 \beta_2 + \alpha_2 \beta_1) \quad ; \quad b = \beta_1 \beta_2 (\alpha_1 \alpha_2 - 1) \quad \text{(B.6)}$$

Let's take

$$s(t,\boldsymbol{z}) = s(\boldsymbol{z} - \boldsymbol{c}t)$$

and (B.5) takes form of bi-wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - c_1^2 \Delta\right)\left(\frac{\partial^2}{\partial t^2} - c_2^2 \Delta\right) s(t,\boldsymbol{z}) = 0\,; \quad \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \quad \text{(B.7)}$$

$$c_{1,2}^4 - a c_{1,2}^2 + b = 0$$

1. For $a>0$; $b>0$ there are two positive roots for squares of velocities c^2

$$c_{1,2}^2 = \frac{a +/- \sqrt{a^2 - 4b}}{2} > 0 \quad \text{(B.8)}$$

2. For $a>0$; $b<0$ or for $a<0$; $b<0$ there is one positive root for speed square

$$c_1^2 = \frac{a + \sqrt{a^2 - 4b}}{2} > 0 \quad \text{(B.9)}$$

3. For $a<0$; $b>0$ there are no positive roots and thus no wave regime.

For each positive square of speed c^2

$$c^2 = c_x^2 + c_y^2 > 0 \tag{B.10}$$

Here $c_x{}^2$ – describes wave speed of suppliers along axes $\boldsymbol{x}$ and $c_y{}^2$ – describes wave speed of consumers of goods along axes $\boldsymbol{y}$. Thus single positive value of c^2 means that there can be a lot of different waves of supply perturbations with different wave speed $\boldsymbol{c}_x$ along axes $\boldsymbol{x}$ and speed $\boldsymbol{c}_y$ along axes $\boldsymbol{y}$. The same value c^2 (B.8) or (B.9) may induce waves of supply $s(t,z)$ and demand $d(t,z)$ perturbations with different waves speed $\boldsymbol{c}_s$ of supply and $\boldsymbol{c}_d$ of demand that fulfill the conditions (B.10):

$$\boldsymbol{c_s} = (\boldsymbol{c_{sx}}\,;\,\boldsymbol{c_{sy}}) \quad c_s{}^2 = c_{sx}^2 + c_{sy}^2 > 0 \tag{B.11}$$

$$\boldsymbol{c_d} = (\boldsymbol{c_{dx}}\,;\,\boldsymbol{c_{dy}}) \quad c_d{}^2 = c_{dx}^2 + c_{dy}^2 > 0 \tag{B.12}$$

$$\boldsymbol{c_s} = (\boldsymbol{c_{sx}}\,;\,\boldsymbol{c_{sy}}) \neq \boldsymbol{c_d} = (\boldsymbol{c_{dx}}\,;\,\boldsymbol{c_{dy}}) \;\; but \;\; c_s{}^2 = c_d^2 > 0$$

Let show that equations (B.5) allow propagation of supply disturbances waves with amplitudes growing as exponent. Let take $s(t,z)$ as:

$$s(t,\boldsymbol{z}) = \cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{z})\exp(\gamma t) \;\; ; \quad \boldsymbol{k} = (\boldsymbol{k}_x, \boldsymbol{k}_x) \tag{B.13}$$

Function (B.13) satisfies equations (B.5) if:

$$\omega^2 = \gamma^2 + \frac{ak^2}{2} \qquad 4\gamma^2\omega^2 = k^4\left(b - \frac{a^2}{4}\right) > 0 \;;\; 4b > a^2$$

$$\gamma^2 = k^2\frac{\sqrt{4b+3a^2}-2a}{8} > 0 \qquad \omega^2 = k^2\frac{\sqrt{4b+3a^2}+2a}{8} > 0$$

For $\gamma > 0$ wave amplitude grows up as $exp(\gamma t)$. Let's show that equations (8.1; 8.2) on disturbances of supply transactions from $\boldsymbol{x}$ to $\boldsymbol{y}$ and demand transactions from $\boldsymbol{y}$ to $\boldsymbol{x}$ induce equations on perturbations of economic variables – densities of supply $S_{out}(t,\boldsymbol{x})$ from point $\boldsymbol{x}$, supply $S_{in}(t,\boldsymbol{y})$ to point $\boldsymbol{y}$, demand $D_{out}(t,\boldsymbol{y})$ from point $\boldsymbol{y}$ and demand $D_{in}(t,\boldsymbol{x})$ at point $\boldsymbol{x}$ and their flows. To do that let's take integral by $d\boldsymbol{y}$ over economic domain (II.1.1; 1.2). Due to (II.3) supply $S_{out}(t,\boldsymbol{x})$ from point $\boldsymbol{x}$ and supply $S_{in}(t,\boldsymbol{y})$ to point $\boldsymbol{y}$ are defined as:

$$S_{out}(t,\boldsymbol{x}) = \int d\boldsymbol{y}\; S(t,\boldsymbol{x},\boldsymbol{y}) \;\;;\;\; S_{in}(t,\boldsymbol{y}) = \int d\boldsymbol{x}\; S(t,\boldsymbol{x},\boldsymbol{y}) \tag{B.14.1}$$

and use (7.3) to define their flows $\boldsymbol{P}_{out}(t,\boldsymbol{x})$ and $\boldsymbol{P}_{in}(t,\boldsymbol{y})$:

$$\boldsymbol{P}_{out}(t,\boldsymbol{x}) = \int d\boldsymbol{y}\; \boldsymbol{P}(t,\boldsymbol{x},\boldsymbol{y}) \;\;;\;\; \boldsymbol{P}_{in}(t,\boldsymbol{y}) = \int d\boldsymbol{x}\; \boldsymbol{P}(t,\boldsymbol{x},\boldsymbol{y}) \tag{B.14.2}$$

The similar relations define demand $D_{out}(t,\boldsymbol{y})$ from point $\boldsymbol{y}$ and demand $D_{in}(t,\boldsymbol{x})$ at point $\boldsymbol{x}$ and their flows:

$$D_{out}(t,\boldsymbol{y}) = \int d\boldsymbol{x}\; D(t,\boldsymbol{x},\boldsymbol{y}) \;\;;\;\; D_{in}(t,\boldsymbol{x}) = \int d\boldsymbol{y}\; D(t,\boldsymbol{x},\boldsymbol{y}) \tag{B.14.3}$$

$$\boldsymbol{Q}_{out}(t,\boldsymbol{y}) = \int d\boldsymbol{x}\; \boldsymbol{Q}(t,\boldsymbol{x},\boldsymbol{y}) \;\;;\;\; \boldsymbol{Q}_{in}(t,\boldsymbol{x}) = \int d\boldsymbol{y}\; \boldsymbol{Q}(t,\boldsymbol{x},\boldsymbol{y}) \tag{B.14.4}$$

Economic meaning of supply $S_{out}(t,\boldsymbol{x})$ - it is total supply of selected goods, commodities etc., from point $\boldsymbol{x}$. Function $S_{in}(t,\boldsymbol{y})$ describes total supply of selected goods to point $\boldsymbol{y}$. Economic density function $D_{out}(t,\boldsymbol{y})$ describes total demand from point $\boldsymbol{y}$ and $D_{in}(t,\boldsymbol{x})$ – total demand at point $\boldsymbol{x}$ from entire economy. Equations on density functions $S_{out}(t,\boldsymbol{x})$, $S_{in}(t,\boldsymbol{y})$, $D_{in}(t,\boldsymbol{x})$, $D_{out}(t,\boldsymbol{y})$ and their flows can be derived from (7.1; 7.2; 7.4; 7.5). Let's take integrals by $d\boldsymbol{x}$ or $d\boldsymbol{y}$ over economic space:

$$\frac{\partial}{\partial t}S_{out}(t,\boldsymbol{x}) + \nabla\cdot(S_{out}\,\boldsymbol{v_{out}}) = \alpha_1\,\nabla\cdot\;\boldsymbol{Q}_{in}(t,\boldsymbol{x}) \tag{B.15.1}$$

$$\frac{\partial}{\partial t}D_{in}(t,\boldsymbol{x}) + \nabla\cdot(D_{in}\,\boldsymbol{u_{in}}) = \alpha_2\,\nabla\cdot\;\boldsymbol{P}_{out}(t,\boldsymbol{x}) \tag{B.15.2}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_{out}(t,\boldsymbol{x}) + \nabla\cdot(\boldsymbol{P}_{out}\,\boldsymbol{v}_{out}) = \beta_1\,\nabla D_{in}(t,\boldsymbol{x}) \tag{B.15.3}$$

$$\frac{\partial}{\partial t}\boldsymbol{Q}_{in}(t,\boldsymbol{x}) + \nabla\cdot(\boldsymbol{Q}_{in}\,\boldsymbol{u}_{in}) = \beta_2\,\nabla S_{out}(t,\boldsymbol{x}) \tag{B.15.4}$$

$$\boldsymbol{P}_{out}(t,\boldsymbol{x}) = S_{out}(t,\boldsymbol{x})\boldsymbol{v}_{out}(t,\boldsymbol{x})\;\;;\;\;\boldsymbol{Q}_{in}(t,\boldsymbol{x}) = D_{in}(t,\boldsymbol{x})\boldsymbol{u}_{in}(t,\boldsymbol{x}) \tag{B.15.5}$$

Similar equations are valid for $S_{in}(t,\boldsymbol{y})$, $D_{out}(t,\boldsymbol{y})$ and their flows $\boldsymbol{P}_{in}(t,\boldsymbol{y})$, $\boldsymbol{Q}_{out}(t,\boldsymbol{y})$. To derive wave equations on disturbances of $S_{out}(t,\boldsymbol{x})$, $D_{in}(t,\boldsymbol{x})$ and their flows let's take integrals by $d\boldsymbol{y}$ of (7.8; 7.9):

$$S_{out}(t,\boldsymbol{x}) = S_{0out}(1+s_{out}(t,\boldsymbol{x}))\;;\;D_{in}(t,\boldsymbol{x}) = D_{0in}(1+d_{in}(t,\boldsymbol{x})) \tag{B.16.4}$$

$$\boldsymbol{P}_{out}(t,\boldsymbol{x}) = S_{0out}\boldsymbol{v}_{out}(t,\boldsymbol{x})\;\;;\;\boldsymbol{Q}_{in}(t,\boldsymbol{x}) = D_{0in}\boldsymbol{u}_{in}(t,\boldsymbol{x}) \tag{B.16.5}$$

Equations on disturbances $s_{out}(t,\boldsymbol{x})$, $d_{in}(t,\boldsymbol{x})$ and their flows are similar to (8.1; 8.2) but perturbations depend on $\boldsymbol{x}$ only:

$$\frac{\partial}{\partial t}s_{out}(t,\boldsymbol{x}) + S_0\nabla\cdot\boldsymbol{v}_{out} = \alpha_1 D_0\nabla\cdot\boldsymbol{u}_{in}(t,\boldsymbol{x}) \tag{B.16.6}$$

$$\frac{\partial}{\partial t}d_{in}(t,\boldsymbol{x}) + D_0\nabla\cdot\boldsymbol{u}_{in} = \alpha_2 S_0\nabla\cdot\boldsymbol{v}_{out}(t,\boldsymbol{x}) \tag{B.16.7}$$

$$S_0\frac{\partial}{\partial t}\boldsymbol{v}_{out}(t,\boldsymbol{z}) = \beta_1\nabla\,d(t,\boldsymbol{x})\;;\;\;D_0\frac{\partial}{\partial t}\boldsymbol{u}_{in}(t,\boldsymbol{x}) = \beta_2\nabla\,s(t,\boldsymbol{x}) \tag{B.16.8}$$

Equations on disturbances $s_{out}(t,\boldsymbol{x})$ and $d_{in}(t,\boldsymbol{x})$ as well on $s_{in}(t,\boldsymbol{x})$ and $d_{out}(t,\boldsymbol{x})$ take form similar to (B.5; B.6):

$$\left[\frac{\partial^4}{\partial t^4} - a\Delta\,\frac{\partial^2}{\partial t^2} + b\Delta^2\right]\;s_{out}(t,\boldsymbol{x}) = 0 \tag{B.17.1}$$

Let's argue signs of α_1, α_2, β_1, β_2. Positive divergence $D_0\nabla\cdot\boldsymbol{u}_{in}(t,\boldsymbol{x}) > 0$ for disturbances of demand flow means that demand flows out of a unit volume dV at point $\boldsymbol{x}$ and thus reduce amount of demand at $\boldsymbol{x}$. Decline of demand may decline supply $s_{out}(t,\boldsymbol{x})$ and hence we take $\alpha_1<0$. As well positive divergence $S_0\nabla\cdot\boldsymbol{v}_{out}(t,\boldsymbol{x}) > 0$ for disturbances of supply flow means that supply flows out of a unit volume dV at point $\boldsymbol{x}$ and hence decline supply at $\boldsymbol{x}$. Reduction of supply at $\boldsymbol{x}$ may increase demand at this point and we take $\alpha_2>0$. Equations (B.16.8) model relations between supply flows $S_0v(t,\boldsymbol{x})$ and gradient of demand perturbations. We propose that supply flows $S_0\boldsymbol{v}(t,\boldsymbol{x})$ grow up in the direction of higher demand determined by gradient of demand perturbations $\nabla d(t,\boldsymbol{x})$ and thus take $\beta_1>0$. As well demand flows $D_0\boldsymbol{u}(t,\boldsymbol{x})$ decline in the direction of higher supply determined by gradient of supply perturbations $\nabla s(t,\boldsymbol{x})$ and thus take $\beta_2<0$. Hence we obtain:

$$\alpha_1 < 0 \ ; \ \ \alpha_2 > 0 \ ; \ \beta_1 > 0 \ ; \ \ \beta_2 < 0 \tag{B.17.2}$$

$$a = (\alpha_1\beta_2 + \alpha_2\beta_1) > 0 \ ; \ b = \beta_1\beta_2(\alpha_1\alpha_2 - 1) > 0$$

and due to (B.8) there are two positive roots for c^2 of (B.7). Same considerations are valid for equations on $s_{in}(t,\boldsymbol{x})$ and $d_{out}(t,\boldsymbol{x})$. Thus disturbances of economic variables $s_{out}(t,\boldsymbol{x})$ and $d_{in}(t,\boldsymbol{x})$ follow bi-wave equations

$$\left(\frac{\partial^2}{\partial t^2} - c_1^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - c_2^2\Delta\right)s(t,\boldsymbol{x}) = 0 \tag{B.17.3}$$

Wave equations (B.7) on transactions disturbances induce similar wave equations on disturbances of *–in* and *–out* economic variables that are determined by transactions. Let's show that these waves induce small fluctuations of macroeconomic variables. Let's study economics under action of a single risk. Due to (II.1.1; 1.2) transactions are defined on *2*-dimensional economic domain. For (7.8) and (B.13) macroeconomic supply $S(t)$ at moment t (II.4.1; 4.2)

$$S(t) = S_0\big(1 + s(t)\big) \ ; \quad s(t) = \int_0^1 dxdy \ s(t,x,y) \tag{B.18.1}$$

$$s(t) = \frac{4\exp(\gamma t)}{k_x k_y}\cos\left(\frac{k_x+k_y}{2} - \omega t\right) sin\frac{k_x}{2} \ sin\frac{k_y}{2} \tag{B.18.2}$$

Hence disturbances $s(t)$ of macroeconomic supply $S(t)$ at moment t may grow up as $exp(\gamma t)$ for $\gamma>0$ or dissipate to constant rate S_0 for $\gamma<0$ and fluctuate with frequency ω.

Appendix C

The business cycle equations

Let's show that macroeconomic supply *S(t)* and demand *D(t)* follow fluctuations that can be treated as business cycles. To derive equations on *S(t)* and *D(t)* as (II.4.1) let's take integral by $dz=dxdy$ of (13.1; 13.3):

$$\frac{d}{dt}S(t) = \frac{d}{dt}\int d\boldsymbol{z}\, S(t,\boldsymbol{z}) = -\int d\boldsymbol{z}\ \nabla\cdot\big(\boldsymbol{v}(t,\boldsymbol{z})S(t,\boldsymbol{z})\big) + a\ \int d\boldsymbol{z}\ \boldsymbol{z}\cdot\boldsymbol{P}_D(t,\boldsymbol{z}) \qquad \text{(C.1.1)}$$

First integral in the right side (C.1.1) is integral of divergence over 2-dimensional economic domain (6.1; 6.2) and due to divergence theorem (Strauss 2008, p.179) it equals integral of flux through surface of economic domain and hence equals zero as no economic fluxes exist outside of economic domain (6.1; 6.2). Let's define $Pz(t)$ and $Dz(t)$ as:

$$P_S z(t) = \int dxdy\, xP_{Sx}(t,x,y) + yP_{Sy}(t,x,y) = P_S x(t) + P_S y(t) \qquad \text{(C.1.2)}$$

$$P_D z(t) = \int dxdy\, xP_{Dx}(t,x,y) + yP_{Dy}(t,x,y) = P_D x(t) + P_D y(t) \qquad \text{(C.1.3)}$$

Due to (C.1.1-1.3) equations on *S(t)* and *D(t)* take form:

$$\frac{d}{dt}S(t) = a\,[P_D x(t) + P_D y(t)\,] \quad ; \qquad \frac{d}{dt}D(t) = b\,[P_S x(t) + P_S y(t)] \qquad \text{(C.1.4)}$$

To derive equations on $Pz(t)$ and $Dz(t)$ let's use equations (13.2; 13.4) on flows $\boldsymbol{P}_S(t)$, $\boldsymbol{P}_S(t)$ and matrix operators as (13.6; 13.7).

$$P_{Sx}(t) = \int dxdy\, P_{Sx}(t,x,y) = S(t)v_x(t) \qquad \text{(C.1.5)}$$

$$P_{Sy}(t) = \int dxdy\, P_{Sy}(t,x,y) = S(t)v_y(t) \qquad \text{(C.1.6)}$$

$$P_{Dx}(t) = \int dxdy\, P_{Dx}(t,x,y) = D(t)u_x(t) \qquad \text{(C.1.7)}$$

$$P_{Dy}(t) = \int dxdy\, P_{Dy}(t,x,y) = D(t)u_y(t) \qquad \text{(C.1.8)}$$

Similar to (C.1.1) from (13.2; 13.6; 13.7) for (C.1.5- C.1.8) obtain:

$$\frac{d}{dt}P_{Sx}(t) = c_1 P_{Dx}(t) \ \ ; \ \ \frac{d}{dt}P_{Dx}(t) = d_1 P_{Sx}(t) \qquad \text{(C.2.1)}$$

$$\frac{d}{dt}P_{Sy}(t) = c_2 P_{Sy}(t) \ \ ; \ \ \frac{d}{dt}P_{Dy}(t) = d_2 P_{Sy}(t) \qquad \text{(C.2.2)}$$

As we mentioned before, flows (C.1.5-1.8) can't have constant sign of velocities (C.1.5-1.8). Indeed, let's define mean risk $X_S(t)$ of suppliers with variable *E* and mean risk $Y_C(t)$ of consumers of variable *E* as:

$$S(t)X_S(t) = \int dxdy\, x\, S(t,x,y) \ ; \ S(t)Y_C(t) = \int dxdy\, y\, S(t,x,y) \qquad \text{(C.2.3)}$$

It is easy to show that for $F_S(t,x,y)=0$ one derive from (13.1; 13.8):

$$\frac{d}{dt}S(t) = 0\,;\ S(t) = S_0 = const;\ \ \frac{d}{dt}X_S(t) = v_x(t)\,;\ \frac{d}{dt}Y_C(t) = v_y(t) \qquad \text{(C.2.4)}$$

Thus in the absence of interaction $F_S(t,x,y)=0$ mean risk $X_S(t)$ of suppliers of variable *E* moves along axis *X* with velocity $\upsilon_x(t)$ (C.2.4) and mean risk $Y_C(t)$ of consumers of variable *E* moves along axis Y with velocity $\upsilon_y(t)$ (C.2.4). Borders of economic domain reduce motion of mean risks. Hence velocities $\upsilon_x(t)$ and $\upsilon_y(t)$ must change sign and should fluctuate. Let's underline that relations (C.2.3, 2.4) simplify real economic processes as we neglect interactions between transactions $F_S(t,x,y)$ and neglect direct dependence of economic variables and transactions on risk coordinates $z=(x,y)$ on economic domain. Indeed, risks impact on economic performance and activity of economic agents. Thus change of risk coordinates should change value of density functions of economic variables and transactions. Starting with (13.1) it is easy to show that in the presence of interactions between supply $S(t,x,y)$ and demand $D(t,x,y)$ transactions mean risks $X_S(t)$ of suppliers of variable E change due to two factors as:

$$\frac{d}{dt}X_S(t) = \upsilon_x(t) + w_x(t) \tag{C2.5}$$

$$w_x(t) = [X_{SF}(t) - X_S(t)]\frac{d}{dt}lnS(t) \tag{C.2.6}$$

$$F_S(t) = \int dxdy\, F_S(t,x,y) \;\; ; \;\; X_{SF}(t)F_S(t) = \int dxdy\;\, x\, F_S(t,x,y) \tag{C.2.7}$$

Here $\upsilon_x(t)$ is determined by (13.8) and velocity $w_x(t)$ (C.2.6, 2.7) describes motion (C.2.5) of mean risk $X_S(t)$ (C.2.3) of suppliers along axis X due to interaction $F_S(t,x,y)$ (13.1) of supply and demand transactions. Mean risk $X_S(t)$ of suppliers and mean risk $Y_C(t)$ of consumers (C.2.3) of variable E on economic domain (6.1; 6.2) are reduced by borders of economic domain (C.2.8):

$$0 \le X_S(t) \le 1 \;\; ; \;\; 0 \le Y_C(t) \le 1 \tag{C.2.8}$$

Hence velocities $\upsilon_x(t)$ (C.1.5-1.8) and $w_x(t)$ (C.2.6-7) should fluctuate as (C.2.8) reduce motion of mean risks (C.2.3, 2.5). Thus (C.2.5) describes two sources of fluctuations caused by velocities $\upsilon_x(t)$ (C.1.5-1.8) and $w_x(t)$ (C.2.6-7). Let's model fluctuations of flows $\boldsymbol{P}_S(t)$ and $\boldsymbol{P}_D(t)$ by equations (C.2.1-2) that describe harmonique oscillations with frequencies ω, ν:

$$\omega^2 = -c_1 d_1 > 0 \;\; ; \;\; \nu^2 = -c_2 d_2 > 0 \tag{C.3.1}$$

$$\left[\frac{d^2}{dt^2} + \omega^2\right]P_{Sx}(t) = 0 \;\; ; \;\; \left[\frac{d^2}{dt^2} + \omega^2\right]P_{Dx}(t) = 0 \tag{C.3.2}$$

$$\left[\frac{d^2}{dt^2} + \nu^2\right]P_{Sy}(t) = 0 \;\; ; \;\; \left[\frac{d^2}{dt^2} + \nu^2\right]P_{Dy}(t) = 0 \tag{C.3.3}$$

Frequencies ω describe oscillations of mean risk $X_S(t)$ (C.2.3-2.4) of suppliers along axis X and ν describe oscillations of consumers mean risk $Y_C(t)$ along axis Y. Solutions (C.3.1-3.3):

$$P_{Sx}(t) = P_{Sx}(1)\sin\omega t + P_{Sx}(2)\cos\omega t\, ; P_{Sy}(t) = P_{Sy}(1)\sin\nu t + P_{Sy}(2)\cos\nu t \tag{C.3.4}$$

$$P_{Dx}(t) = P_{Dx}(1)\sin\omega t + P_{Dx}(2)\cos\omega t\, ; P_{Dy}(t) = P_{Dy}(1)\sin\nu t + P_{Dy}(2)\cos\nu t \tag{C.3.5}$$

To derive equations on *Pz(t)* and *Dz(t)* let's derive equations on their components $P_Sx(t)$, $P_Sy(t)$, $P_Dx(t)$, $P_Dy(t)$ (C.1.2;1.3) and use equations (13.2; 13.6). Let's multiply equations (13.2) by $z=(x,0)$ and take integral by *dxdy*

$$\frac{d}{dt}P_S x(t) = \frac{d}{dt}\int dxdy\, xP_{Sx}(t,x,y) = \int dxdy\,[-x\frac{\partial}{\partial x}(v_x\, P_{Sx}) + c_1 xP_{Dx}(t,x,y)]$$

$$-\int dxdy\, x\frac{\partial}{\partial x}(v_x\, P_{Sx}) = \int dxdy\, v_x^2(t,x,y)S(t,x,y)$$

For $P_Sx(t)$, $P_Sy(t)$, $P_Dx(t)$, $P_Dy(t)$ (C.1.2;1.3) obtain equations:

$$\frac{d}{dt}P_S x(t) = ESx(t) + c_1 P_D x(t) \;\; ; \;\; \frac{d}{dt}P_D x(t) = EDx(t) + d_1 P_S x(t)$$

$$\frac{d}{dt}P_S y(t) = ESy(t) + c_2 P_D x(t) \;\; ; \;\; \frac{d}{dt}P_D y(t) = EDy(t) + d_2 P_S y(t)$$

Let's use (13.10) and denote *ESx(t,x,y), ESy(t,x,y), EDx(t,x,y) EDy(t,x,y)* and *ESx(t), ESy(t), EDx(t) EDy(t)* as:

$$ESx(t) = \int dxdy\; ESx(t,x,y) = \int dxdy\, v_x^2(t,x,y)S(t,x,y) = S(t)v_x^2(t) \qquad \text{(C.4.1)}$$

$$ESy(t) = \int dxdy\; ESy(t,x,y) = \int dxdy\, v_y^2(t,x,y)S(t,x,y) = S(t)v_y^2(t) \qquad \text{(C.4.2)}$$

$$EDx(t) = \int dxdy\; EDx(t,x,y) = \int dxdy\, u_x^2(t,x,y)D(t,x,y) = D(t)u_x^2(t) \qquad \text{(C.4.3)}$$

$$EDy(t) = \int dxdy\; EDy(t,x,y) = \int dxdy\, u_y^2(t,x,y)D(t,x,y) = D(t)u_y^2(t) \qquad \text{(C.4.4)}$$

Equations on $P_Sx(t)$, $P_Sy(t)$, $P_Dx(t)$, $P_Dy(t)$ take form:

$$\left[\frac{d^2}{dt^2} + \omega^2\right] P_S x(t) = \frac{d}{dt}ESx(t) + c_1 EDx(t)\,; \left[\frac{d^2}{dt^2} + \omega^2\right] P_D x(t) = \frac{d}{dt}EDx(t) + d_1 ESx(t) \;\text{(C.4.5)}$$

$$\left[\frac{d^2}{dt^2} + \nu^2\right] P_S y(t) = \frac{d}{dt}ESy(t) + c_2 EDy(t)\,; \left[\frac{d^2}{dt^2} + \nu^2\right] P_D y(t) = \frac{d}{dt}EDy(t) + d_2 ESy(t) \;\text{(C.4.6)}$$

Equations (C.4.5-4.6) describe fluctuations of $P_Sx(t)$, $P_Sy(t)$, $P_Dx(t)$, $P_Dy(t)$ with frequencies ω and ν under action of *ESx, ESy, EDx, EDy* (C.4.1-4.4). To close system of ordinary differential equations (C.4.5-4.6) let's define equations on *ESx, ESy, EDx, EDy*. Let's outline that relations (C.4.1-4.4) are proportional to product of supply *S(t)* and velocity square $\upsilon^2(t)$ and *looks alike to* energy of a particle with mass *S(t)* and velocity square velocity $\upsilon^2(t)$. We underline that this is only *similarity* between (C.4.1-4.5) and energy of a particle and have no further analogies. To define equations on (C.4.1-4.5) let's propose that:

$$\frac{\partial}{\partial t}ESx(t,x,y) + \frac{\partial}{\partial x}(v_x ESx) = \mu_1 EDx \;\; ; \;\; \frac{\partial}{\partial t}EDx(t,x,y) + \frac{\partial}{\partial x}(u_x EDx) = \eta_1 ESx \quad \text{(C.5.1)}$$

$$\frac{\partial}{\partial t}ESy(t,x,y) + \frac{\partial}{\partial y}\left(v_y ESy\right) = \mu_2 EDy \;\; ; \;\; \frac{\partial}{\partial t}EDy(t,x,y) + \frac{\partial}{\partial y}\left(u_y EDy\right) = \eta_2 ESx \quad \text{(C.5.2)}$$

$$\gamma_1^2 = \mu_1\eta_1 > 0 \;\; ; \;\; \gamma_2^2 = \mu_2\eta_2 > 0 \qquad \text{(C.5.3)}$$

Equations (C.5.1-3) give equations on *ESx(t), ESy(t), EDx(t), EDy(t)*

$$\left[\frac{d^2}{dt^2} - \gamma_1^2\right] ESx(t) = 0 \;\; ; \;\; \left[\frac{d^2}{dt^2} - \gamma_1^2\right] EDx(t) = 0 \qquad \text{(C.5.4)}$$

$$\left[\frac{d^2}{dt^2} - \gamma_2^2\right] ESy(t) = 0 \ ; \ \left[\frac{d^2}{dt^2} - \gamma_2^2\right] EDy(t) = 0 \qquad \text{(C.5.5)}$$

Let's explain economic meaning of (C.5.1-5.5): "energies" *ESx(t), ESy(t), EDx(t), EDy(t)* grow up or decay in time by exponent $exp(\gamma_1 t)$ and $exp(\gamma_2 t)$ that can be different for each risk axis. Here γ_1 define exponential growth or decay in time of *ESx(t)* induced by motion of suppliers along axis *X* and γ_2 describe exponential growth or decrease in time of *ESy(t),* induced by motion of consumers along axis *Y*. The same valid for *EDx(t)* and *EDy(t)* respectively. Solutions of (C.5.4-5.5; C.4.5-4.6) with exponential growth have form:

$$ESx(t) = ESx(1)\, exp\, \gamma_1 t \ \ ; \ ESy(t) = ESy(1)\, exp\, \gamma_2 t$$

$$EDx(t) = EDx(1)\, exp\, \gamma_1 t \ ; \ \ EDy(t) = EDy(1)\, exp\, \gamma_2 t$$

$$P_S x(t) = P_S x(1) \sin \omega t + P_S x(2) \cos \omega t + P_S x(3)\, exp\, \gamma_1 t$$

$$P_S y(t) = P_S y(1) \sin \nu_i t + P_S y(2) \cos \nu_i t + P_S y(3)\, exp\, \gamma_2 t$$

$$P_D x(t) = P_D x(1) \sin \omega t + P_D x(2) \cos \omega t + P_D x(3)\, exp\, \gamma_1 t$$

$$P_D y(t) = P_D y(1) \sin \nu_i t + P_D y(2) \cos \nu_i t + P_D y(3)\, exp\, \gamma_2 t$$

Macroeconomic supply *S(t)* of variable *E* as solution of (C.1.4) takes form:

$$S(t) = S(0) + a\big[S_x(1) \sin \omega t + S_x(2) \cos \omega t + S_y(1) \sin \nu t + S_y(2) \cos \nu t\big] + a\big[S_x(3) exp\, \gamma_1 t + S_y(3)\, exp\, \gamma_2 t\big] \qquad \text{(C.5.6)}$$

Initial values and equations (C.1.4-C.5.5) define simple but long relations on constants $S_x(j)$, $S_y(j)$, $j=0,..3$ and we omit them here. Similar relations valid for demand *D(t).*

References

Andersen, T., Bollerslev, T., Diebold, F.X. and Ebens, H., (2001). The Distribution of Realized Stock Return Volatility, Journal of Financial Economics, 61, 43-76

Bates, D.S., (1995). Testing Option Pricing Models, NBER, WP 5129, 1-75

van Binsbergen, J. H., Koijen, R., (2017). The term structure of returns: Facts and theory, NBER WP 21234, Cambridge, MA

Black, F. and M. Scholes, (1973). The Pricing of Options and Corporate Liabilities. *The Journal of Political Economy,* **81**, 637-65

Blume, L.E., Easley, D., (1984). Rational Expectations Equilibrium: An Alternative Approach, Journal Of Economic Theory, 34, 116-129

Brock, W.A., Hommes, C.H., (1998). Heterogeneous beliefs and routes to chaos in a simple asset pricing model. Journal of Economic Dynamics and Control, 22, 1235-1274

Brunnermeier, M.K., Parker, J.A., (2005). Optimal Expectations. American Economic Review, 95 (4), 1092-1118

Campbell, J.Y., (1985). Stock Returns and The Term Structure, WP1626, NBER, Cambridge

Campbell, J.Y., Cochrane, J.H., (1995). By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior, NBER WP 4995, Cambridge, US

Cochrane, J.H., (2001). Asset pricing. Princeton University Press, Princeton, N. Jersey, US

Cochrane, J.H., Culp, C.L., (2003). Equilibrium Asset Pricing and Discount Factors: Overview and Implications for Derivatives Valuation and Risk Management. In Modern Risk Management. A History, Ed. S.Jenkins, 57-92.

Cochrane, J.H., (2017). Macro-Finance. Review of Finance, European Finance Association, 21(3), 945-985.

Cohen, S.N., Tegner, M., (2018). European Option Pricing with Stochastic Volatility models under Parameter Uncertainty. pp. 1-44, arXiv:1807.03882v1

Diebold, F.X. and Rudebusch, G.D. (1999). Business Cycles Durations, Dynamics, and Forecasting. Princeton Univ.Press.

Dominitz, J., Manski, C.F., (2005). Measuring And Interpreting Expectations Of Equity Returns. NBER, WP 11313, Cambridge, MA

Fama, E.F., (1965). The Behavior of Stock-Market Prices. Jour. of Business, 38 (1), 34-105.

Fama, E.F., (1998). Market efficiency, long-term returns, and behavioral finance, Journal of Financial Economics 49, 283-306

Fortune, P., (1996). Anomalies in Option Pricing: the Black-Scholes Model Revisited. New

England Economic Review, March/April, 17-40
Gabaix, X., Gopikrishnan, P., Plerou, V., Stanley, H,E., (2003). A theory of power-law distributions in financial market fluctuations, Nature, 423, 267-270
Gontis, V., Havlin, S., Kononovicius, A., Podobnik, B., Stanley, H.E., (2016). Stochastic model of financial markets reproducing scaling and memory in volatility return intervals. Physica A, 462, 1091–1102
Greenwald, D.L., Lettau, M., Ludvigson, S., (2014). Origins of Stock Market Fluctuations. NBER, WP 19818, Cambridge, MA
Greenwood, R., Shleifer, A., (2014). Expectations of Returns and Expected Returns. The Review of Financial Studies, 27 (3), 714–746
Hansen, L.P., Sargent, T.J., (1979). Formulating and Estimating Dynamic Rational Expectations Models. NBER, WP 127
Hansen, L.P., Heaton, J., Luttmer, E.G.J., (1995). Econometric Evaluation of Asset Pricing Models. Review of Financial Studies, 8, (2), 237-274
Hansen, L. P., (2013). Uncertainty outside and inside economic models. Nobel lecture.
Heaton, J., Lucas, D., (2000). Stock Prices and Fundamentals, in Ed. Bernanke, B.S., Rotemberg, J.J. NBER Macroeconomics Annual 1999, Volume 14, 213 - 264
Heston, S.L., (1993). A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. The Review of Financial Studies, 6, (2), 327-343
Hull, J., White,A., (1987). The Pricing of Options on Assets with Stochastic Volatilities. Journal of Finance, 17, (2), 281-300
Hull, J.C. (2009). Options, Futures and other Derivatives, 7th.ed. Englewood Cliffs, NJ, Pearson Prentice Hall
Jorda, O., Schularick, M., Taylor, A.M., (2016). Macrofinancial History and the New Business Cycle Facts. FRB San Francisco, WP 2016-23
Keim, D.B., Stambaugh, R.F., (1986). Predicting Returns In The Stock And Bond Markets, Journal of Financial Economics, 17, 357-390
Kiyotaki, N., (2011). A Perspective on Modern Business Cycle Theory. Economic Quarterly, 97, (3), 195–208
Kydland, F.E. and Prescott, E.C., (1991). The Econometrics of the General Equilibrium Approach to Business Cycles. Scand. J. of Economics **93**(2): 161-178
Lof, M., (2014). Essays on Expectations and the Econometrics of Asset Pricing, MPRA WP 59064, http://mpra.ub.uni-muenchen.de/59064/

Lucas, R.E., (1972). Expectations and the Neutrality of Money. Jour. of Economic Theory 4, 103-124

Lucas, R.E., (1980). Methods and Problems in Business Cycle Theory, Jour.of Money, Credit and Banking, 12, (4) Part 2: Rational Expectations, 696-715

Mandelbrot, B., Fisher, A., Calvet, L., (1997). A Multifractal Model of Asset Returns. Cowles Foundation Discussion Paper #1164

Manski, C.F., (2017). Survey Measurement Of Probabilistic Macroeconomic Expectations: Progress and Promise. NBER Macro Annual Conference, 1-76

Merton, R. (1973). Theory of Rational Option Pricing. The Bell Journal of Economic and management Sci, 4, 141-183

Muth, J.F., (1961). Rational Expectations and the Theory of Price Movements. Econometrica, 29 (3), 315-335

Olkhov, V., (2016a). On Economic space Notion, Int. Rev. Financial Analysis, 47, 372-381

Olkhov, V., (2016b). Finance, Risk and Economic space, ACRN Oxford J. of Finance and Risk Perspectives, Special Issue of Finance Risk and Accounting Perspectives, 5, 209-221

Olkhov, V., (2016c). On Hidden Problems of Option Pricing. https://ssrn.com/abstract=2788108

Olkhov, V., (2017a). Quantitative Wave Model of Macro-Finance. Int. Rev. Financial Analysis, 50, 143-150

Olkhov, V., (2017b). Quantitative Description of Financial Transactions and Risks" ACRN Oxford Journal of Finance and Risk Perspectives 6, (2), 41-54.

Olkhov, V., (2017c). Credit-Loans Non-Local Transactions and Surface-Like Waves. https://ssrn.com/abstract=2971417

Olkhov, V., (2017d). Econophysics of Business Cycles: Aggregate Economic Fluctuations, Mean Risks and Mean Square Risks. arxiv.org/abs/1709.00282, q-fin.EC

Olkhov, V., (2018). Expectations, Price Fluctuations and Lorenz Attractor. MPRA WP89105, https://mpra.ub.uni-muenchen.de/89105/

Olkhov, V., (2019a). Economic and Financial Transactions Govern Business Cycles. ACRN Oxford Journal of Finance and Risk Perspectives 7, (1/2), 102-122.

Olkhov, V., (2019b). Econophysics of asset price, transactions and expectations. http://arxiv.org/abs/1901.05024

Olkhov, V., (2019c). New Essentials of Economic Theory I. Assumptions, Economic Space and Variables. MPRA, WP 93085

Olkhov, V., (2019d). New Essentials of Economic Theory II. Economic Transactions, Expectations and Asset Pricing. MPRA, WP 93428

Plerou, V., Gopikrishnan, P., Amaral, L., Meyer, M., Stanley, H.E., (1999). Scaling of the distribution of price fluctuations of individual companies. Phys. Rev. E, 60 (6), 6519-6529

Sargent, T.J., Wallace, N., (1976). Rational Expectations And The Theory Of Economic Policy, Journal of Monetary Economics 2 (1976) 169-183

Schumpeter, J.A., (1939). Business Cycles. A Theoretical, Historical and Statistical Analysis of the Capitalist Process. NY McGraw-Hill Book Company

Stanley, H.E., Pleroua, V., Gabaixb, X., (2008). A statistical physics view of financial fluctuations: Evidence for scaling and universality. Physica A, 387, 3967–3981

Strauss, W.A., (2008). Partial Differential Equations. An Introduction. John Wiley&Sons, NJ, US. p.179

Tinbergen, J., (1935). Annual Survey: Suggestions on Quantitative Business Cycle Theory, Econometrica, 3 (3), 241-308

Zarnowitz, V., (1992). Business Cycles: Theory, History, Indicators, and Forecasting. NBER