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Krouglov, Alexei

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Alexei Krouglov

alexkrouglov@gmail.com

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ABSTRACT

Presented here are simplified mathematical models to evaluate the long-term investment values. A framework of the single product economy is used, which clarifies conceptual explanation. The short-term effects are mostly discarded and focus is done on the long-term economic trends. Two models are examined. The first model estimates an equity value for the stable earnings. The second model assesses an equity value for the unstable earnings with instability caused by the capital investments.

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1 Introduction

The article continues a research to describe important fundamentals of economics through mathematical models presented in (Krouglov, 2006; 2017).

The first part of economic fundamentals considers supply, demand, and prices in equilibrium on a market and shows that changes in supply, demand, and prices can be described through a system of ordinary differential equations. Details are given in (Krouglov, 2006).

The second part of economic fundamentals examines how a systematic diversion from the equilibrium on market can serve as a mathematical cause of the economic growth. Details can be found in (Krouglov, 2017).

This article starts examining the third part of economic fundamentals that comprises mathematical models of the equity valuations.

Here, I utilize the previous results of (Krouglov, 2006; 2017) that an economic growth conceptually is caused by the (capital) investment. I use a mathematical model of the single product economy, which can give a good conceptual explanation. I try to estimate the investment as an objective value using the model. I mostly discard short-term economic effects and focus on the long-term economic trends.

I need to answer the question, how would one estimate an investment that produces particular numerical earnings. I would also try to choose an economic instrument with known value that produces similar earnings. The economic instrument that I use as a reference is debt, which market value can be knowingly devised from the interest payments and current interest rate. Thus, the stable earnings can be juxtaposed with a debt instrument. Unstable earnings (e.g., growing earnings) would require a more nuanced approach.

2 Interest Rate Return as Reference

The cost of debt is the interest a person pays on her borrowings. It is expressed as a percentage rate. The interest rate is applied to the principal, which is an amount of the loan. The interest rate is the cost of debt for a borrower and the rate of return for a lender. The money to be repaid exceeds the borrowed amount since lenders require compensation for the loss of use of the money during the loan period. The lender could have invested the money during that period instead of providing a loan, which would have generated income for the lender.

To calculate the cost of debt I will utilize concept of instantaneous interest payment obtained by the multiplication of a continuous interest rate by the principal (Krouglov, 2006). Now, I will look at the problem a bit differently. In (Krouglov, 2006) the continuous interest rate was defined as the ratio of instantaneous change in the price of money to the price of money at each point in time.

I assume $L_0\left(t_0\right)$ is a loan, which a lender provides to a borrower at time $t=t_0$. The difference between the total repayment sum and the principal is the interest charged $M_L\left(t\right)$. We denote the amount $M_L\left(t\right)$ as the earnings accumulated by lender over period $\left(t_0,t\right)$. We divide earnings $M_L\left(t+\Delta t\right)-M_L\left(t\right)$ that are accumulated over the period $\left(t,t+\Delta t\right)$ by Δt and obtain an average rate of earnings $\bar{m}_L\left(t\right)$, $\bar{m}_L(t)=\dfrac{M_L\left(t+\Delta t\right)-M_L\left(t\right)}{\Delta t}$. If we assume the period Δt to be infinitesimal, we get an instantaneous rate of earnings $m_L\left(t\right)=\dfrac{dM_L\left(t\right)}{dt}$ on the loan $L\left(t\right)$, i.e., $m_L\left(t\right)=\lim\limits_{\Delta t\to 0}\bar{m}_L(t)$.

I define an instantaneous interest rate $i_L\left(t\right)$ as a ratio of the instantaneous rate of earnings $m_L\left(t\right)$ on the loan $L\left(t\right)$ to the amount of loan,

$$i_L(t) = \frac{m_L(t)}{L(t)} \tag{1}$$

or equivalently,

$$m_L(t) = L(t) i_L(t) \tag{2}$$

I will use the instantaneous interest rate $i_L(t)$ defined in Eqs. (1) and (2) as a reference to estimate the market value of investments based on concept that investments producing equal earnings should be equally valued.[1]

3 Long-Term Equity Value for Stable Earnings

Now let me use results of the previous section to estimate the long-term value of equity when an investment produces stable earnings.

I use a mathematical model of the single-product-economy market, which can give us a good conceptual view. The stable earnings here mean there are no disturbing economic forces acting on the market. The market is in equilibrium position, i.e., the supply of and demand for product is equal, the quantities of supply and demand are developing with a constant rate and a price of the product is fixed.

I assume the market had been in an equilibrium until time $t = t_0$, volumes of the product supply $V_S(t)$ and demand $V_D(t)$ on market were equal, and they both were developing with a constant rate r_D^0. The product price $P(t)$ at that time was fixed,

$$V_D(t) = r_D^0 (t - t_0) + V_D^0 \tag{3}$$

$$V_S(t) = V_D(t) \tag{4}$$

[1] I disregard the differences in credit risks here though one can make appropriate credit risk adjustments for the instantaneous interest rates.

$$P(t) = P^0 \quad (5)$$

where $V_D(t_0) = V_D^0$.

To estimate a long-term value of equity when investment produces the stable earnings I will use variable $E_D(t) \equiv P(t) \times r_D(t)$ where $r_D(t) \equiv \frac{dV_D(t)}{dt}$, i.e., a rate of demand in money terms for the product, which reflects investment's instantaneous rate of earnings on the market.

For investment producing the stable earnings described by Eqs. (3) – (5), it takes place $E_D(t) = P^0\, r_D^0$.

Taking into account the instantaneous interest rate $i_L(t)$ as a reference, we can estimate investment value $W_I(t)$ that corresponds to the stable earnings $E_D(t) = P^0\, r_D^0$ as

$$W_I(t) = \frac{E_D(t)}{i_L(t)} \quad (6)$$

or equivalently as

$$W_I(t) = \frac{P^0\, r_D^0}{i_L(t)} \quad (7)$$

Therefore, we can make the following observations based on Eqs. (6) – (7).

An investment value $W_I(t)$ is directly proportional to the instantaneous rate of earnings $E_D(t)$ provided the instantaneous interest rate $i_L(t)$ is fixed.

An investment value $W_I(t)$ is inversely proportional to the instantaneous interest rate $i_L(t)$ provided the instantaneous rate of earnings $E_D(t)$ is fixed.

Now, we will examine how the capital investments affect the price of equity investment.

4 Long-Term Values for Expanding Capital Investments

Now, let me use results of the previous section to estimate equity value, provided the capital investment produces an economic growth.

I deploy a mathematical model of the single-product-economy market, which can give a good conceptual explanation. Economic forces acting on the market reflect inherent market forces of the demand and supply complemented with forces caused by the investment. The market interactions are expressed through a system of ordinary differential equations.

When there are no disturbing economic forces, the market is in equilibrium position, i.e., the supply of and demand for product are equal, the quantities of supply and demand are developing with a constant rate and a price of the product is fixed.

I assume the market had been in an equilibrium until time $t = t_0$, volumes of the product supply $V_S(t)$ and demand $V_D(t)$ on market were equal, and they both were developing with a constant rate r_D^0. The product price $P(t)$ at that time was fixed,

$$V_D(t) = r_D^0 (t - t_0) + V_D^0 \tag{8}$$

$$V_S(t) = V_D(t) \tag{9}$$

$$P(t) = P^0 \tag{10}$$

where $V_D(t_0) = V_D^0$.

When the balance between the volumes of product supply and demand is broken, market is experiencing economic forces, which act to bring the market to a new equilibrium position.

In this section I use a model of the single-product economy where the investment rate is increasing with a constant acceleration (Krouglov, 2017).

Correspondingly, I assume the amount of capital investment $S_I(t)$ on market increases since time $t = t_0$ according to the following formula,

$$S_I(t) = \begin{cases} 0, & t < t_0 \\ \delta_I (t - t_0) + \frac{\varepsilon_I}{2}(t - t_0)^2, & t \geq t_0 \end{cases} \tag{11}$$

where $S_I(t) = 0$ for $t < t_0$, $\delta_I \geq 0$, and $\varepsilon_I > 0$.

Economic forces trying to bring the market into a new equilibrium position are described by the following ordinary differential equations with regard to the volumes of product supply $V_S(t)$, demand $V_D(t)$, and price $P(t)$ given the accumulated amount of capital investment $S_I(t)$ on market,

$$\frac{dP(t)}{dt} = -\lambda_P \left(V_S(t) - V_D(t) - S_I(t)\right) \tag{12}$$

$$\frac{d^2 V_S(t)}{dt^2} = \lambda_S \frac{dP(t)}{dt} \tag{13}$$

$$\frac{d^2 V_D(t)}{dt^2} = -\lambda_D \frac{d^2 P(t)}{dt^2} \tag{14}$$

In Eqs. (12) – (14) above the values $\lambda_P, \lambda_S, \lambda_D \geq 0$ are constants and they reflect the price inertness, supply inducement, and demand amortization correspondingly.

One may observe that a difference between the volumes of product demand and supply on the market is increasing, in the sense:

$$V_D(t) - V_S(t) + S_I(t) \geq V_D(t) - V_S(t)$$

when the amount of accumulated capital investment $S_I(t)$ is positive.

Let me use a new variable $D(t) \equiv (V_S(t) - V_D(t) - S_I(t))$ representing the volume of product surplus (or shortage) on the market. Therefore, behavior of $D(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2 D(t)}{dt^2} + \lambda_P \lambda_D \frac{dD(t)}{dt} + \lambda_P \lambda_S D(t) + \varepsilon_I = 0 \tag{15}$$

with the initial conditions, $D(t_0) = 0$, $\frac{dD(t_0)}{dt} = -\delta_I$.

If one introduces another variable $D_1(t) \equiv D(t) + \frac{\varepsilon_I}{\lambda_P \lambda_S}$, then Eq. (15) becomes,

$$\frac{d^2 D_1(t)}{dt^2} + \lambda_P \lambda_D \frac{dD_1(t)}{dt} + \lambda_P \lambda_S D_1(t) = 0 \tag{16}$$

with the initial conditions, $D_1(t_0) = \frac{\varepsilon_I}{\lambda_P \lambda_S}$, $\frac{dD_1(t_0)}{dt} = -\delta_I$.

Similar to Eq. (15), the product price $P(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2 P(t)}{dt^2} + \lambda_P \lambda_D \frac{dP(t)}{dt} + \lambda_P \lambda_S \left(P(t) - P^0 - \frac{\delta_I}{\lambda_S} - \frac{\varepsilon_I}{\lambda_S}(t - t_0) \right) = 0 \tag{17}$$

with the initial conditions, $P(t_0) = P^0$, $\frac{dP(t_0)}{dt} = 0$.

Let me use variable $P_1(t) \equiv P(t) - P^0 - \frac{\delta_I}{\lambda_S} - \frac{\varepsilon_I}{\lambda_S}(t - t_0) + \frac{\lambda_D}{\lambda_S^2}\varepsilon_I$ to simplify analysis of the price behavior. The behavior of variable $P_1(t)$ is described by following equation for $t > t_0$,

$$\frac{d^2 P_1(t)}{dt^2} + \lambda_P \lambda_D \frac{dP_1(t)}{dt} + \lambda_P \lambda_S P_1(t) = 0 \qquad (18)$$

with the initial conditions, $P_1(t_0) = -\frac{\delta_I}{\lambda_S} + \frac{\lambda_D}{\lambda_S^2}\varepsilon_I$, $\frac{dP_1(t_0)}{dt} = -\frac{\varepsilon_I}{\lambda_S}$.

The behavior of solutions for $D_1(t)$ and $P_1(t)$ described by Eqs. (16) and (18) depends on the roots of the corresponding characteristic equations (Piskunov, 1965; Petrovski, 1966). Also Eqs. (16) and (18) have the same characteristic equations.

When the roots of characteristic equation are complex-valued (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} < \lambda_P \lambda_S$) both variables $D_1(t)$ and $P_1(t)$ experience damped oscillations for time $t \geq t_0$. If the roots of characteristic equation are real and different (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} > \lambda_P \lambda_S$) both variables $D_1(t)$ and $P_1(t)$ don't oscillate for time $t \geq t_0$. If the roots of characteristic equation are real and equal (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} = \lambda_P \lambda_S$) both variables $D_1(t)$ and $P_1(t)$ don't oscillate for time $t \geq t_0$ as well.

It takes place $D_1(t) \to 0$ and $P_1(t) \to 0$ for $t \to +\infty$ if roots of characteristic equations are complex-valued ($\frac{\lambda_P^2 \lambda_D^2}{4} < \lambda_P \lambda_S$), real and different ($\frac{\lambda_P^2 \lambda_D^2}{4} > \lambda_P \lambda_S$), or real and equal ($\frac{\lambda_P^2 \lambda_D^2}{4} = \lambda_P \lambda_S$).

We observe the following behavior for the product surplus (shortage) $D(t)$, the product price $P(t)$, the product demand $V_D(t)$, the product supply $V_S(t)$, the amount of capital investment $S_I(t)$ when $t \to +\infty$,

$$D(t) \to -\frac{\varepsilon_I}{\lambda_P \lambda_S} \tag{19}$$

$$P(t) \to \frac{\varepsilon_I}{\lambda_S}(t-t_0) + P^0 + \frac{\delta_I}{\lambda_S} - \frac{\lambda_D}{\lambda_S^2}\varepsilon_I \tag{20}$$

$$V_D(t) \to \left(r_D^0 - \frac{\lambda_D}{\lambda_S}\varepsilon_I \right)(t-t_0) + V_D^0 - \frac{\lambda_D}{\lambda_S}\delta_I + \frac{\lambda_D^2}{\lambda_S^2}\varepsilon_I \tag{21}$$

$$V_S(t) \to \left(r_D^0 + \delta_I - \frac{\lambda_D}{\lambda_S}\varepsilon_I \right)(t-t_0) + \frac{\varepsilon_I}{2}(t-t_0)^2 + V_D^0 - \frac{\lambda_D}{\lambda_S}\delta_I - \frac{\varepsilon_I}{\lambda_P \lambda_S} + \frac{\lambda_D^2}{\lambda_S^2}\varepsilon_I \tag{22}$$

$$S_I(t) = \delta_I(t-t_0) + \frac{\varepsilon_I}{2}(t-t_0)^2 \tag{23}$$

To analyze an economic growth we use variable $E_D(t) \equiv P(t) \times r_D(t)$ where $r_D(t) \equiv \dfrac{dV_D(t)}{dt}$, i.e., the instantaneous rate of earnings for product on the market.

The variable $E_D(t)$ converges toward $E_D(t) \to \left(\dfrac{\varepsilon_I}{\lambda_S}(t-t_0) + P^0 + \dfrac{\delta_I}{\lambda_S} - \dfrac{\lambda_D}{\lambda_S^2}\varepsilon_I \right)\left(r_D^0 - \dfrac{\lambda_D}{\lambda_S}\varepsilon_I \right)$ for $t \to +\infty$.

If $0 < \varepsilon_I < \dfrac{\lambda_S}{\lambda_D} r_D^0$ then it takes place $r_D^0 - \dfrac{\lambda_D}{\lambda_S}\varepsilon_I > 0$. Thus, it brings unrestricted increase of the rate of earnings $E_D(t)$ for product on the market with the passage of time, i.e., $E_D(t) \to +\infty$ for $t \to +\infty$.

We can estimate a change $e_D(t)$ of the rate of earnings $E_D(t)$ for product where $e_D(t) \equiv \frac{dE_D(t)}{dt}$.

It takes place $e_D(t) \to \frac{\varepsilon_I}{\lambda_S}\left(r_D^0 - \frac{\lambda_D}{\lambda_S}\varepsilon_I\right) > 0$ when $0 < \varepsilon_I < \frac{\lambda_S}{\lambda_D} r_D^0$ for $t \to +\infty$.

If $\frac{\lambda_S}{\lambda_D} r_D^0 < \varepsilon_I < +\infty$, then it takes place $r_D^0 - \frac{\lambda_D}{\lambda_S}\varepsilon_I < 0$. Hence, it brings unrestricted decrease of the rate of earnings $E_D(t)$ for product on market with the passage of time, i.e., $E_D(t) \to -\infty$ for $t \to +\infty$.

We can estimate a change $e_D(t)$ of the rate of earnings $E_D(t)$ for product.

It takes place $e_D(t) \to \frac{\varepsilon_I}{\lambda_S}\left(r_D^0 - \frac{\lambda_D}{\lambda_S}\varepsilon_I\right) < 0$ when $\frac{\lambda_S}{\lambda_D} r_D^0 < \varepsilon_I < +\infty$ for $t \to +\infty$.

In this section we presented a single-product-economy model. We have also observed how a capital investment affects the product price, the product demand, the product supply, and the product earnings in the long run.

In the next section we will examine how the capital investment affects the price of equity.

5 Impact of Capital Investment on Long-Term Equity Price

Now, let me use results of the previous section to estimate an equity value provided the capital investment produces an economic growth.

I deploy a mathematical model of the single-product-economy market, which can give us a good conceptual explanation. Economic forces acting on the product market reflect inherent market forces of the demand and supply complemented with forces caused by the capital investment. The product market interactions are expressed through a system of ordinary differential equations.

When there are no disturbing economic forces, the product market is in equilibrium position, i.e., the supply of and demand for product are equal, the quantities of supply and demand are developing with a constant rate and a price of the product is fixed.

Likewise, I consider the market of equity corresponding to the entity operating in a single product economy. Economic forces acting on the equity market reflect inherent market forces of the demand and supply complemented with the forces operating on both product and equity markets (Krouglov, 2006). The equity market interactions are expressed through the system of non-homogeneous ordinary differential equations. When there are no disturbing economic forces, both the product market and equity market is in equilibrium positions, i.e., the supply of and demand for product and the supply of and demand for equity are equal, the quantities of supply and demand are developing with a constant rate and a price of the product and equity is fixed.

I assume the product market and equity market had been in an equilibrium until time $t = t_0$. Volumes of the product supply $V_S(t)$ and demand $V_D(t)$ on the product market were equal, and they both were developing with a constant rate r_D^0. The product price $P(t)$ at that time was fixed,

$$V_D(t) = r_D^0 (t - t_0) + V_D^0 \tag{24}$$

$$V_S(t) = V_D(t) \tag{25}$$

$$P(t) = P^0 \tag{26}$$

where $V_D(t_0) = V_D^0$.

Likewise, volumes of the equity supply $V_{SE}(t)$ and demand $V_{DE}(t)$ on the equity market were equal, and they both were developing with a constant rate r_{DE}^0. The equity price $P_E(t)$ at that time was fixed,

$$V_{DE}(t) = r_{DE}^0 (t - t_0) + V_{DE}^0 \tag{27}$$

$$V_{SE}(t) = V_{DE}(t) \tag{28}$$

$$P_E(t) = P_E^0 \tag{29}$$

where $V_{DE}(t_0) = V_{DE}^0$.

When balance between the volumes of product supply and demand is broken, the product market is experiencing economic forces, which act to bring the product market to a new equilibrium position. Likewise, when balance between the volumes of equity supply and demand is broken, the equity market is experiencing economic forces, which act to bring the equity market to a new equilibrium position. Moreover, there are economic forces operating on both the product and equity markets (Krouglov, 2006).

In this section I assume balances on both the product market and equity market are broken by the capital investment with rate that increases with a constant acceleration (Krouglov, 2017).

I assume the amount of capital investment $S_I(t)$ on the product market increases since time $t = t_0$ according to following formula,

$$S_I(t) = \begin{cases} 0, & t < t_0 \\ \delta_I (t - t_0) + \frac{\varepsilon_I}{2}(t - t_0)^2, & t \geq t_0 \end{cases} \tag{30}$$

where $S_I(t) = 0$ for $t < t_0$, $\delta_I \geq 0$, and $\varepsilon_I > 0$.

Economic forces trying to bring the product market into a new equilibrium position are described by the following ordinary differential equations with regard to the volumes of product supply $V_S(t)$, product demand $V_D(t)$, and product price $P(t)$ given the accumulated amount of capital investment $S_I(t)$ on the product market,

$$\frac{dP(t)}{dt} = -\lambda_P \left(V_S(t) - V_D(t) - S_I(t) \right) \tag{31}$$

$$\frac{d^2 V_S(t)}{dt^2} = \lambda_S \frac{dP(t)}{dt} \tag{32}$$

$$\frac{d^2 V_D(t)}{dt^2} = -\lambda_D \frac{d^2 P(t)}{dt^2} \tag{33}$$

In Eqs. (31) – (33) above the values $\lambda_P, \lambda_S, \lambda_D \geq 0$ are constants and they reflect the product price inertness, supply inducement, and demand amortization correspondingly.

Economic forces trying to bring the equity market into a new equilibrium position are described by the following non-homogeneous ordinary differential equations with regard to the volumes of equity supply $V_{SE}(t)$, equity demand $V_{DE}(t)$, equity price $P_E(t)$ and product price $P(t)$ on the equity market,

$$\frac{dP_E(t)}{dt} = -\lambda_{PE} \left(V_{SE}(t) - V_{DE}(t) \right) \tag{34}$$

$$\frac{d^2 V_{SE}(t)}{dt^2} = \lambda_{SE} \frac{dP_E(t)}{dt} \tag{35}$$

$$\frac{d^2 V_{DE}(t)}{dt^2} = -\lambda_{DE} \frac{d^2 P_E(t)}{dt^2} + \lambda_E \frac{dP(t)}{dt} \tag{36}$$

In Eqs. (34) – (36) above the values $\lambda_{PE}, \lambda_{SE}, \lambda_{DE}, \lambda_E \geq 0$ are constants.

Thus, the equity price $P_E(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2 P_E(t)}{dt^2} + \lambda_{PE} \lambda_{DE} \frac{dP_E(t)}{dt} + \lambda_{PE} \lambda_{SE} \left(P_E(t) - P_E^0 \right) - \lambda_{PE} \lambda_E \left(P(t) - P^0 \right) = 0 \tag{37}$$

with the initial conditions, $P_E\left(t_0\right) = P_E^0$, $P\left(t_0\right) = P^0$, $\frac{dP_E\left(t_0\right)}{dt} = 0$, $\frac{dP\left(t_0\right)}{dt} = 0$.

We perform in Eq. (37) the passage to the limit for $t \to +\infty$ based on results of Eq. (20),

$$\frac{d^2 P_E\left(t\right)}{dt^2} + \lambda_{PE}\,\lambda_{DE}\,\frac{dP_E\left(t\right)}{dt} + \lambda_{PE}\,\lambda_{SE}\left(P_E\left(t\right) - P_E^0\right) - \lambda_{PE}\,\lambda_E\left(\frac{\varepsilon_I}{\lambda_S}\left(t - t_0\right) + \frac{\delta_I}{\lambda_S} - \frac{\lambda_D}{\lambda_S^2}\,\varepsilon_I\right) = 0 \quad (38)$$

I use a variable $P_2\left(t\right) \equiv P_E\left(t\right) - P_E^0 - \frac{\lambda_E}{\lambda_{SE}^2}\delta_I - \frac{\lambda_E}{\lambda_{SE}^2}\varepsilon_I\left(t - t_0\right) + \frac{2\lambda_D\,\lambda_E}{\lambda_{SE}^3}\varepsilon_I$ to simplify the analysis of equity price behavior. The behavior of variable $P_2\left(t\right)$ is described by following equation for $t \to +\infty$,

$$\frac{d^2 P_2\left(t\right)}{dt^2} + \lambda_{PE}\,\lambda_{DE}\,\frac{dP_2\left(t\right)}{dt} + \lambda_{PE}\,\lambda_{SE}\,P_2\left(t\right) = 0 \quad (39)$$

with the initial conditions, $P_2\left(t_0\right) = -\frac{\lambda_E}{\lambda_{SE}^2}\delta_I + \frac{2\lambda_D\,\lambda_E}{\lambda_{SE}^3}\varepsilon_I$, $\frac{dP_2\left(t_0\right)}{dt} = -\frac{\lambda_E}{\lambda_{SE}^2}\varepsilon_I$.

The behavior of solutions for $P_2\left(t\right)$ described by Eq. (39) depends on the roots of the corresponding characteristic equation (Piskunov, 1965; Petrovski, 1966).

When the roots of characteristic equation are complex-valued (i.e., $\frac{\lambda_P^2\,\lambda_D^2}{4} < \lambda_P\,\lambda_S$) variable $P_2\left(t\right)$ experiences damped oscillations for time $t \geq t_0$. If the roots of characteristic equation are real and different (i.e., $\frac{\lambda_P^2\,\lambda_D^2}{4} > \lambda_P\,\lambda_S$) variable $P_2\left(t\right)$ doesn't oscillate for time $t \geq t_0$. If the roots of characteristic equation are real and equal (i.e., $\frac{\lambda_P^2\,\lambda_D^2}{4} = \lambda_P\,\lambda_S$) variable $P_2\left(t\right)$ doesn't oscillate for time $t \geq t_0$.

It may be observed that $P_2(t) \to 0$ for $t \to +\infty$ if the roots of characteristic equations are complex-valued ($\frac{\lambda_P^2 \lambda_D^2}{4} < \lambda_P \lambda_S$), real and different ($\frac{\lambda_P^2 \lambda_D^2}{4} > \lambda_P \lambda_S$), or real and equal ($\frac{\lambda_P^2 \lambda_D^2}{4} = \lambda_P \lambda_S$).

We observe the following behavior for the equity price $P_E(t)$ given the amount of capital investment $S_I(t)$ on the product market when $t \to +\infty$,

$$P_E(t) \to \frac{\lambda_E}{\lambda_{SE}^2} \varepsilon_I (t - t_0) + P_E^0 + \frac{\lambda_E}{\lambda_{SE}^2} \delta_I - \frac{2\lambda_{DE} \lambda_E}{\lambda_{SE}^3} \varepsilon_I \qquad (40)$$

$$S_I(t) = \delta_I (t - t_0) + \frac{\varepsilon_I}{2} (t - t_0)^2 \qquad (41)$$

We can compare the rate $\frac{dP(t)}{dt}$ of product price $P(t)$ change on the product market from Eq. (20) with the rate $\frac{dP_E(t)}{dt}$ of equity price $P_E(t)$ change on the equity market from Eq. (40) when $t \to +\infty$,

$$\frac{d\,P(t)}{dt} \to \frac{\varepsilon_I}{\lambda_S} \qquad (42)$$

$$\frac{d\,P_E(t)}{dt} \to \frac{\lambda_E}{\lambda_{SE}} \cdot \frac{\varepsilon_I}{\lambda_{SE}} \qquad (43)$$

We can use the coefficient $\lambda_E \geq 0$ to reconcile the rate $\frac{dP(t)}{dt}$ of the product price change with the rate $\frac{dP_E(t)}{dt}$ of the equity price change for $t \to +\infty$.

Thus, we have determined for a model of single product economy that capital investment on the product market performed with a constant acceleration induces both (a) an unrestricted increase of the product price on the product market, and (b) an unrestricted increase of the equity price on the equity market.

The rates at which both product price and equity price are changing with time are directly proportional to an acceleration of the capital investment in the long run.

6 Conclusions

The article presents and examines the mathematical models of equity valuations in the long run.

First, I examine the equity value of an economic entity delivering the stable earnings. I determine two investments that produce similar stable earnings. I choose an economic instrument with known value that produces the stable earnings. The economic instrument I use as a reference is debt, which market value can be estimated from the interest payments and interest rate. Thus, stable earnings are contrasted with the debt instrument value.

Second, unstable earnings (e.g., growing earnings) are associated with the capital investments in a model of the single product economy, which gives good conceptual explanation. I mostly neglect short-term effects and focus on the long-term economic trends. The capital investments considered here are the ones accruing with a constant acceleration.

Third, I determine that capital investments on the product market accrued with a constant acceleration induce both (a) an unrestricted increase of the product price on the product market, and (b) an unrestricted increase of the equity price on the equity market. The rates at which both the product price and the equity price are changing with time are in the long run directly proportional to an acceleration of the capital investments.

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