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Real Consequences of Open Market Operations: the Role of Limited Commitment*

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Abstract

We study how limited commitment in credit markets affects the implementation of open market operations and characterize when they result in real indeterminacies and when they have real effects. To do so, we consider a frictional and incomplete market framework where agents face stochastic trading opportunities and limited commitment in some markets. When limited commitment does not constraint agents' choices, we find necessary and sufficient conditions for the existence of a unique monetary equilibrium. However, real indeterminacies are possible when buyers face a binding no-default constraint. We also show that when the no-default constraint binds and bonds are not priced fundamentally, open market operations generically have real effects. A sale of government bonds can increase or decrease interest rates, depending on the nature of equilibria. The direction of the interest rate effects critically depend on the size of the liquidity premium on government bonds. Finally, government bonds purchases can be used to rule out real indeterminacies, thus finding another rationale for such policy.

JEL Codes: E40, E61, E62, H21.

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1 Introduction

Before the US Great Recession, open market operations (OMO) was the main tool used to implement monetary policy. These operations required the central bank to purchase/sale nominal government debt in order to change the money supply. Thus during *normal times*, one could view an easing of monetary policy as either a decline in short-term interest rates or as an expansionary OMO that increases the money supply.[1] In frictionless frameworks these operations do not alter long run real allocations, as shown by Wallace (1981) and Lucas (1984).[2] To capture both short and long run real effects, the economic environment needs to exhibit some imperfections. The literature has considered: (i) nominal rigidities, as in Woodford (1998) and Erceg et al. (2000), (ii) segmented markets, as in Andrés et al. (2004), Chen et al. (2012), among others, (iii) distortionary taxation, as in Gordon and Leeper (2006), (iv) distributional policies, as in Grossman and Weiss (1983), Rotemberg (1984), Alvarez and Lippi (2014) or Sterk and Tenreyro (2018), among others, or (v) different pledgeability among government bonds, as in Rocheteau et al. (2018) and Dong and Xiao (2019). Here we contribute to the literature by considering the role of limited commitment.

In this paper we study how limited commitment in credit markets affects the implementation of OMO and characterize when they result in real indeterminacies and when they have real effects. To do so, we consider a frictional and incomplete market framework. Agents face stochastic trading opportunities and limited commitment in credit markets.[3] This type of friction yields an endogenous borrowing constraint. Moreover, agents can also consume and produce an homogeneous good, rebalance their portfolio, and repay their private debt in a frictionless Walrasian market. Finally, there is a government that must finance a constant stream of exogenous expenditures by levying taxes and by issuing nominal bonds as well as fiat money. These two are the only durable assets in the economy. Other than implementing monetary policy and taxing agents, the government can exclude buyers from the credit market and impose additional punishments in case of default.

When limited commitment in credit markets does not constraint agents' choices, we find necessary and sufficient conditions for the existence of a unique monetary equilibrium. However, multiple stationary equilibria may exist, when buyers face a binding no-default constraint.[4] A different type of real indeterminacy can be observed, as there are regions of the parameter space where monetary equilibria with a binding and a non binding no-default constraints coexist. Multi-

[1] Taylor (1993) highlighted that the actual conduct of monetary policy is better captured by a short-term nominal interest rates rule rather than some measure of the nominal money supply. We refer the reader to Taylor (2007) for more on this topic.

[2] Peled (1985), Chamley and Polemarchakis (1984) and Sargent and Smith (1987), among others, extend Wallace's results in a variety of environments and find that OMO leave real allocations unaltered.

[3] In contrast to Carapella and Williamson (2015), limited commitment applies to just private credit.

[4] These real indeterminacies are fairly robust to different parameter configurations.

plicity of stationary equilibria relies on the dynamic complementarity between current and future bond liquidity premia. Specifically, if agents expect low (high) liquidity premia in the future, then their continuation value is relatively high (low). Thus their temptation to default today is relatively low (high). This mechanism relaxes (tightens) the borrowing constraint in the current period, making bonds relative less (more) scarce. This is consistent with observing a low (high) liquidity premium today. Thus, the original belief is self-fulfilling, and can deliver multiple stationary equilibria. These can be indexed by the corresponding liquidity premium. We also find that generically OMO have long run real effects when public debt has a liquidity premium. More precisely, we show that the off-equilibrium partial confiscation is a necessary condition for OMO to have real effects.[5] As a result, government debt plays an important role in disciplining credit market behavior.[6] Moreover, for economies with a unique monetary equilibrium consistent with a binding no-default constraint, we show that a sale of government bonds lowers the interest rate. When multiple equilibria consistent with a binding no-default constraint exist, OMO have differential effects on interest rates. In particular, they can increase or decrease the nominal interest rate, depending on the size of the bonds' liquidity premium. Finally, when the equilibrium with a binding borrowing constraint coexists with a slack one, a sale of government bonds lowers the interest rate when bonds are priced above their fundamental value. Such operation leaves interest rates unaltered when bonds are priced fundamentally. The mechanism underpinning the effects of OMO on interest rates is a direct consequence of taxes adjusting to satisfy the government budget constraint. This leaves buyers' equilibrium payoff unchanged. However, this operation changes buyers' off-equilibrium continuation value, tightening their endogenous borrowing constraint. Thus, the nominal interest rate needs to adjust to satisfy the binding no-default constraint. The direction of the interest rate adjustment depends on the size of the liquidity premium.

When real indeterminacies emerge, we show that purchases of government bonds can be used as an equilibrium selection mechanism. In particular, a sufficiently loose monetary policy with appropriate OMO can select the equilibrium with a slack no-default constraint. Finally, we further investigate the real effects of OMO by calibrating the model to the US economy from 1984 to 2007. By changing the off-equilibrium taxation, we can obtain multiple stationary equilibria. One monetary equilibrium is consistent with a slack and one with a binding no-default constraint. We also find that a tightening of monetary policy makes bond less scarce, thus reducing their liquidity premium. In the other stationary monetary equilibrium OMO are irrelevant.

[5]Partial confiscation is similar to Chapter 7 bankruptcy where, upon default, an agent can retain part of his exempt property.

[6]This function of public debt is similar to Carapella and Williamson (2015).

2 Literature

This paper connects with two different strands of literature. One that studies OMO in environments where financial markets are Walrasian. The other literature we relate to, is one where monetary policy is analyzed in an environment with flexible prices where some markets are incomplete, there are stochastic trading opportunities in some goods markets and government bonds can exhibit an endogenous liquidity premium.

In Wallace's (1981) seminal paper, when markets are complete, OMO do not have real effects. The author characterizes a class of government open market exchanges of capital for fiat currency that leave the equilibrium sequences for the price level and for real allocations unaffected.[7] When money is dominated in the rate of return, Sargent and Smith (1987) show that OMO do not affect real allocations.[8] Within the same spirit, Schreft and Smith (1998) consider a monetary growth model with nominal debt, where agents face spatial separation and limited communication. The authors show that contractionary monetary policy has differential effects on the nominal interest rate depending on the equilibrium where agents are trading nominal government liabilities. The authors also show that appropriate OMO can be used to rule out multiple steady states.

This paper also complements the literature that analyzes OMO in environments with frictions and bonds can exhibit a liquidity premium. Schabert (2004) considers nominal rigidities in an environment with a Taylor rule where Ricardian equivalence doesn't apply. This is the case as households can rebalance their portfolio through a financial intermediary that participates in repurchase agreement with the central bank, and government bonds are the only collateral in these procedures. Because of nominal rigidities, the author shows that expansionary fiscal policies tend to stimulate real activity. However, interest rate adjustments aimed at stabilizing the economy may hinder such stimulus. When flexible prices are considered in environments with incomplete markets and stochastic trading opportunities, Berensten and Waller (2011) show that traditional OMO have real effects. When private liquidity is issued by banks, Williamson (2012) shows that OMO are neutral when there are plentiful interest-bearing assets but not when they are scarce. In the latter case, a one time open market sale of government bonds reduces the liquidity premium on these government liabilities, increasing the nominal interest rate.

When long and short term real bonds are taken into account and these can be accepted as payment in some states of the world, Dong and Xiao (2019) show how OMO affects real allocations through a consumption channel. As in Williamson (2012), the authors find that open market sale

[7]Within the same spirit, Peled (1985) shows that open market operations between money and indexed bonds do not matter for allocations despite their different risk characteristics. When productive capital is considered, Chamley and Polemarchakis (1984) describe a government strategy of purchasing capital, financed by alterations in the stock of government issued currency, that leaves the real allocation unaltered.

[8]The authors show that so long as the government has the ability to issue a sufficiently rich set of state contingent liabilities, OMO do not alter real allocations.

of government bonds increase the nominal interest rate. Within ta similar framework, Rocheteau et al. (2018) show that when pledgeability is exogenous, the stationary monetary equilibrium is in general unique and OMO have real effects only if bonds are scarce. However, if pledgeability, is endogenous, multiple equilibria with different trading patterns can emerge. OMO always have real effects and have a differential impact on the nominal interest rate depending on which stationary equilibria agents are trading.

3 The Environment

The basic structure builds on the frictional and incomplete market framework of Lagos and Wright (2005) and Rocheteau and Wright (2005). Time is discrete and there is a continuum of infinitively lived buyers and sellers, both of measure one that discount the future at a rate $\beta \in (0, 1)$. These agents have access to fiat money and nominal government bonds. These are the only durable assets in the economy. Agents face technological shocks, have stochastic trading opportunities and trade sequentially in various markets that are characterized by different frictions. In particular, each period has two sub-periods. In the second sub-period, after the *technology* shocks are realized, agents have stochastic and bilateral trading opportunities in a decentralized frictional goods market (DM). Finally, in the first sub-period, agents trade in a frictionless centralized market (CM), where they can produce and consume a general good, re-adjust their portfolio, decide whether to pay their private debt and meet their tax obligations.

Preferences and Technologies

Agents have preferences over consumption of the general CM perishable good (X_t), effort to produce the CM good (H_t), consumption of the specialized DM perishable good (q_t) and effort to produce the DM good (h_t). More precisely buyers have the following preferences

$$\sum_{t=0}^{\infty} \beta^t \left[U(X_t) - H_t + u(q_t)\right]$$

where $u(x)$ and $U(\cdot)$ are concave and satisfy standard utility properties. Sellers, on the other hand, have the following preferences

$$\sum_{t=0}^{\infty} \beta^t \left[U(X_t) - H_t - h_t\right].$$

All perishable goods are produced according to a linear technology where labor is the only input. The production function is such that one unit of labor yields one unit of output.

All buyers are matched with a seller in DM. However, with probability n buyers are matched with sellers that have access to a technology that allows them to write contracts and issue unsecured credit. This technology lets sellers create a verifiable record of the DM trades that can be easily accessible.[9] In case of default, the buyer's type is permanent and is observed by everyone. We denote these buyers as C-buyers. With complementary probability, buyers are matched with sellers that do not have the technology that makes credit feasible. We refer these agents as N-buyers. Finally, we assume that all buyers face limited commitment in DM.

Timing and Assets

The only durable objects in this economy are fiat money and nominal bonds. As in Berentsen and Waller (2011), Martin (2011) and Dominguez and Gomis-Porqueras (2019), among others, government bonds are viewed as book-entries in the government's record.[10] Agents in DM do not have access to the financial record-keeping technology that keeps track of the identity of the owner of government bonds. This implies that households are not anonymous to the government in DM.

At the beginning of the period, buyers in DM are bilaterally matched with sellers and they exchange money for goods. C-buyers can also access credit, however they face a limited commitment problem. After trade has taken place in DM, agents enter the competitive frictionless CM. In this market there is a timing mismatch between the government's repayment of outstanding debt and buyers' decision to repay previous private loans. Specifically, first the government pays a fraction χ of its total debt, then buyers decide whether to repay the loan acquired in the previous DM market. Then, the government pays the remaining fraction of all outstanding debt to non-defaulters. Finally, agents decide on their portfolio and consumption profiles.

Government

The government must finance a constant stream of exogenous expenditures, G, through lump sum CM taxes and by issuing nominal bonds as well as fiat money. The corresponding per period government budget constraint is given by

$$\phi_{+1}\, B_{+1} + \phi_{+1}\tau_{+1} + \phi_{+1}\, M_{+1} = G + \phi_{+1}\, R\, B + \phi_{+1}\, M \tag{1}$$

[9] This type of technology is different from Carapella and Williamson (2015), where credit histories are perfect, but a would-be lender may not have access to credit histories. Moreover, the technology in our environment is also different from Gu et al. (2013), where their record-keeping technology is imperfect, and with some probability past trading histories are erased.

[10] Alternatively, this could be interpreted as a fraction of sellers where government bonds are not recognized as in Shi et al. (2014) or Rocheteau et al. (2018). This could be endogenized as in Lester et al. (2012), Li et al. (2012), or Gomis-Porqueras et al. (2017). This treatment is beyond the scope of this paper.

where M denotes money supply at time t, B represents nominal bonds, R is the gross nominal interest rate on bonds issued at t, τ_{+1} represents lump sum taxes levied to the buyer in CM and ϕ_{+1} is the real price of money in terms of the CM good at time $t+1$. The payment of interest on government debt is given throughout CM. The real value of all bond issues at every period is assumed to be bounded above by a sufficiently large constant as to avoid Ponzi schemes.

To implement monetary policy, the central bank follows a constant money growth rate rule such that $M_{+1} = \gamma M$ and also conducts OMO in CM as to keep a constant money to bond ratio, $M = \theta B$, where γ and θ are positive constants chosen by the government. Other than implementing monetary policy and levying CM taxes, the government can exclude buyers from the credit market and implement additional punishments in case of default. For instance, the government imposes an off equilibrium real tax $\phi\tilde{\tau}$.

3.1 CM Problem

In period t, buyers enter CM with m_{-1} units of money, b_{-1} units of nominal bonds, and with the promise to repay l_{-1} from the previous DM market. Given that the loan is repaid in the frictionless market, the loan can be paid using labor, cash, or assets.

Having decided to repay the loan, the value function of a buyer is given by

$$W_t^B(m_{-1}, b_{-1}, l_{-1}) = \max_{\{X,H,m,b\}} \Big\{ U(X) - H + V_t^B(m,b) \Big\} \qquad (P^B)$$
$$s.t. \quad X - H + \phi\, m + \phi\, b + l_{-1} \leq \phi\, m_{-1} + R_{-1}\, \phi\, b_{-1} - \phi\, \tau$$

where $V_t^B(m,b)$ is the buyers' DM expected value.

Sellers enter CM market with m_{-1}^S units of money, b_{-1}^S nominal bonds, and the payment in terms of CM good of the previously issued unsecured credit l_{-1}^s. Their corresponding value function is given by

$$W_t^S(m_{-1}^S, b_{-1}^S, l_{-1}^s) = \max_{\{X^S,H^S,m^S,b^S,l_{-1}^S\}} \Big\{ U(X^S) - H^S + V_t^S(m^S, b^S) \Big\} \qquad (P^S)$$
$$s.t. \quad X^S - H^S + \phi\, m^S + \phi\, b^S \leq \phi\, m_{-1}^S + R\, \phi\, b_{-1}^S + l_{-1}^s$$

where $V_t^S(m^S, b^S)$ is the sellers' expected value function in DM.

The first order conditions to these problems are given by

$$X: \; U'(X) = 1 \qquad (2)$$

$$m: \quad -\phi + \frac{\partial\, V_t^B(m,b)}{\partial m} \leq 0 \;\; = \;\text{if } m > 0 \qquad (3)$$

$$b: \quad -\phi + \frac{\partial\, V_t^B(m,b)}{\partial b} \leq 0 \;\; = \;\text{if } b > 0 \qquad (4)$$

$$X^S: \; U'(X^S) = 1 \tag{5}$$

$$m^S: \; -\phi + \frac{\partial\, V_t^S(m^S, b^S)}{\partial m^S} \leq 0 \;\; = \;\text{if } m^S > 0 \tag{6}$$

$$b^S: \; -\phi + \frac{\partial\, V_t^S(m^S, b^S)}{\partial b^S} \leq 0 \;\; = \;\text{if } b^S > 0 \tag{7}$$

with the following envelope conditions for the CM value function

$$\frac{\partial\, W_t^B(m_{-1}, b_{-1}, l_{-1})}{\partial m_{-1}} = \frac{\partial\, W_t^S(m_{-1}^S, b_{-1}^S, l_{-1}^s)}{\partial m_{-1}^S} = \phi \tag{8}$$

$$\frac{\partial\, W_t^B(m_{-1}, b_{-1}, l_{-1})}{\partial b_{-1}} = \frac{\partial\, W_t^S(m_{-1}^S, b_{-1}^S, l_{-1}^s)}{\partial b_{-1}^S} = \phi\, R_{-1}. \tag{9}$$

3.2 DM Problem

A buyer enters DM with a portfolio of money and bonds (m, b). With probability n buyers can purchase the specialized good using fiat money and unsecured credit. With probability $1-n$ buyers can only finance DM consumption with fiat money. Thus, the value function of a buyer that enters DM, before the shock is realized, is given by

$$V_t^B(m, b) = n\, V_t^{B,C}(m, b) + (1-n)\, V_t^{B,N}(m, b)$$

where $V_t^{B,C}(m, b)$ is the value function of a buyer who has access to credit and $V_t^{B,N}(m, b)$ is the value function of a buyer that only uses fiat money.

We assume that the terms of trade in DM are given by a buyer take-it-or-leave-it offer. The buyer who does not have access to unsecured credit solves the following problem

$$\begin{aligned} V_t^{B,N}(m, b) = &\max_{\{q^{nl} d_m^{nl}\}} \left\{u(q^{nl}) + \beta\, W_{t+1}(m - d_m^{nl}, b, 0)\right\} \\ s.t. \quad & -q^{nl} + \beta\, W_{t+1}^S(m^S + d_m^{nl}, b^S, 0) \geq \beta\, W_{t+1}^S(m^S, b^S, 0) \\ & d_m^{nl} \leq m \end{aligned}$$

where the first constraint is the seller's participation constraint with threat point corresponding to no-trade, q^{nl} is the DM-quantity traded and d_m^{nl} is the corresponding cash payment. The second constraint is the feasibility over money balances, whereby a buyer cannot hand in more money balances than the ones that he has brought into the match. Using the linearity of $W_{t+1}^S(m^S, b^S, l^s)$, we can rewrite the previous problem as

$$V_t^{B,N}(m, b) = \max_{\{q^{nl}, d_m^{nl}\}} \left\{u(q^{nl}) - \beta\, \phi_{+1}\, d_m^{nl} + \beta\, W_{t+1}^B(m, b, 0)\right\} \tag{10}$$

$$s.t. \quad -q^{nl} + \beta\, \phi_{+1}\, d_m^{nl} \geq 0 \tag{11}$$

$$d_m^{nl} \le m. \tag{12}$$

The optimal terms of trade are such that constraint (11) always binds, and either i) the payment constraint (12) is slack, which implies $q^{nl} = q^*$, $d_m^{nl} = \frac{q^*}{\phi_{+1}}$, where q^* solves $u'(q^*) = 1$, or ii) the payment constraint (12) binds, which results in $d_m^{nl} = m$, $q^{nl} = \beta\, \phi_{+1}\, m \le q^*$. From now on, we assume positive nominal interest rates, so the cash constraint (12) holds with equality. Thus, the terms of trade are given by

$$d_m^{nl} = m \tag{13}$$

$$q^{nl} = \beta\, \phi_{+1} d_m^{nl} \tag{14}$$

which imply the following envelope condition for the N-buyer

$$\frac{\partial\, V_t^{B,N}(m,b)}{\partial\, m} = u'(q^{nl})\, \beta\, \phi_{+1} \tag{15}$$

$$\frac{\partial\, V_t^{B,N}(m,b)}{\partial\, b} = \beta\, \frac{\partial\, W_{t+1}^B(m,b,0)}{\partial\, b} = \beta\, R\, \phi_{+1}. \tag{16}$$

Similarly, for buyers that can use fiat money and unsecured credit to purchase DM goods, the optimal terms of trade $\{d_m^l, q^l, l\}$ solve the following problem

$$V_t^{B,C}(m,b) = \max_{\{q^l, d_m^l, l\}} \left\{ u(q^{nl}) - \beta \phi_{+1} d_m^l - \beta l + \beta\, W_{t+1}^B(m,b,0) \right\} \tag{17}$$

$$s.t. \qquad -q^l + \beta\, \phi_{+1}\, d_m^l + \beta\, l \ge 0 \tag{18}$$

$$d_m^l \le m \tag{19}$$

$$-l + R\, \phi_{+1}\, b + W_{t+1}^B(0,0,0) \ge \chi\, R\, \phi_{+1} b + \tilde{W}_{t+1}^B(0,0) \tag{20}$$

where (18) is the seller's participation constraint, (19) represents the feasibility condition on money holdings, (20) denotes the no-default constraint that captures the limited commitment problem, and $\tilde{W}_{t+1}^B(m,b)$ is the continuation payoff of a defaulting buyer in CM.

It is easy to check that constraint (18) always binds, such that

$$q^l = \beta\, \phi_{+1}\, d_m^l + \beta\, l. \tag{21}$$

Moreover, optimality requires that

$$\begin{aligned} d_m^l : &\qquad \beta\, \phi_{+1}[u'(q^l) - 1 - \lambda_m] = 0 \\ l : &\qquad \beta[u'(q^l) - 1 - \lambda_l] = 0 \end{aligned} \tag{22}$$

where $\beta\phi_{+1}\lambda_m$ is the multiplier associated to constraint (19) and $\beta\lambda_l$ represents the multiplier associated with (20). Notice that the optimal terms of trade are such that either: i) $q^l = 1$, in which case $\lambda_m = \lambda_l = 0$, or ii) $q^l < 1$, in which case $\lambda_m = \lambda_l = [u'(q^l) - 1]$.

Finally the envelope conditions for the value function of a buyer that has access to unsecured credit are given by

$$\frac{\partial V_t^{B,C}(m,b)}{\partial m} = \beta \frac{\partial W_{t+1}^B(m-d,b,l)}{\partial m} + \beta\,\phi_{+1}\lambda_m = \beta\,\phi_{+1}[1+\lambda_m]$$
$$\frac{\partial V_t^{B,C}(m,b)}{\partial b} = \beta \frac{\partial W_{t+1}^B(m,b,l)}{\partial b} + \beta\lambda_l\,\phi_{+1}R(1-\chi) = \beta\,R\,\phi_{+1}[1+\lambda_l(1-\chi)].$$

Having characterized the envelope conditions, we can rewrite the intertemporal conditions (3) and (4) associated with a buyer CM problem as follows

$$-\phi + \beta\phi_{+1}\left[n\Big(1+\lambda_m\Big) + (1-n)u'(q^{nl})\right] = 0 \tag{23}$$
$$-\phi + \beta R\phi_{+1}\Big[1 + n\,\lambda_l\,(1-\chi)\Big] = 0. \tag{24}$$

Similarly, for sellers we can rewrite equations (6) and (7) as follows

$$-\phi + \beta\,\phi_{+1} < 0 \tag{25}$$
$$-\phi + \beta\,R\,\phi_{+1} \leq 0. \tag{26}$$

From (25) we can conclude that $m^S = 0$. From now on, when bonds are priced fundamentally we consider equilibria where $b^s = 0$.

3.3 Deviating Buyer's Problem

Finally, we have to determine the value function $\tilde{W}_t(\tilde{m}_{-1}, \tilde{b}_{-1})$ of a buyer that defaulted on a previous loan, and therefore he can only use money in DM. Let $\tilde{V}_t(\tilde{m})$ be his DM value function, which is given by

$$\begin{aligned} \tilde{V}_t(\tilde{m},\tilde{b}) = \max_{\{\tilde{q},\tilde{d}_m\}} \quad & u(\tilde{q}) + \beta\,\tilde{W}_{t+1}(\tilde{m}-\tilde{d}_m,\tilde{b}) \\ s.t. \quad & -\tilde{q} + \beta\phi_{+1}\tilde{d}_m \geq 0 \\ & \tilde{m} - \tilde{d}_m \geq 0. \end{aligned} \tag{$\tilde{P}^{B,N}$}$$

In CM, a buyer that has previously defaulted on a loan solves the following problem

$$\tilde{W}_t(\tilde{m}_{-1},\tilde{b}_{-1}) = \phi\tilde{m}_{-1} + \phi\,R\,\tilde{b}_{-1} - \phi\,\tilde{\tau} + \max_{\{\tilde{X},\tilde{m},\tilde{b}\}} \{U(\tilde{X}) - \tilde{X} - \phi\,\tilde{m} - \phi\tilde{b} + \tilde{V}_t(\tilde{m},\tilde{b})\} \tag{$\tilde{P}^B$}$$

where $\phi\tilde{\tau}$ is the off-equilibrium tax.

Throughout the rest of the paper we focus on stationary monetary equilibria.

4 Monetary Equilibrium

Given exogenous government expenditures and operating procedures for monetary and fiscal policy (G, γ, θ, χ, $\phi\tilde{\tau}$), a stationary monetary equilibrium consists of a list of constant quantities and prices $\{X, X^S, \phi\, m, \phi\, b, \phi\, m^S, \phi\, b^S, q^{nl}, \phi\, d_m^{nl}, q^l, \phi\, d_m^l, \phi\, M, \phi\, B, \phi\, \tau, R, \beta\lambda_l, \frac{\beta}{\gamma}\phi\lambda_m\}$ that satisfy buyers' and sellers' optimality conditions, and markets clear.

Lemma 4.1 *At a stationary monetary equilibrium, it must be that* $R \in \left[\underline{R}, \overline{R}\right]$, *where* $\underline{R} = \frac{\gamma}{\beta + n(1-\chi)(\gamma-\beta)}$ *and* $\overline{R} = \frac{\gamma}{\beta}$.

All Proofs can be found in the Appendix.

It is worth highlighting that $\underline{R} > 1$, which precludes liquidity trap equilibria. This is the case as fiat money can always be used as a medium of exchange in DM. Moreover, note that the interval $\left[\underline{R}, \overline{R}\right]$ becomes smaller when χ increases. For $\chi = 1$, the only interest rate consistent with a stationary monetary equilibrium is $\overline{R}$.

Corollary 4.2 *If* $\chi = 1$ *bonds are priced fundamentally,* $R = \frac{\gamma}{\beta}$.

When $\chi = 1$, borrowers can retain the entire bond proceeds upon default. Therefore, government bonds have the same payoff on and off-equilibrium. They do not expand buyers' consumption possibilities. As a result, buyers are willing to hold bonds only if these are priced fundamentally.

For $R \in [\underline{R}, \overline{R}]$, using equations (22), (23), and (24), define the functions

$$q^l(R) = u^{'^{-1}}\left(1 + \frac{\frac{\gamma}{\beta R} - 1}{n(1-\chi)}\right), \qquad q^{nl}(R) = u^{'^{-1}}\left(\frac{\frac{\gamma}{\beta} - \frac{n(1-\chi) + \frac{\gamma}{\beta R} - 1}{1-\chi}}{1-n}\right). \tag{27}$$

Notice that the functions in equation (27) are well-defined if $n \in (0,1)$ and $\chi < 1$. Using equations (27), define the borrowing limit

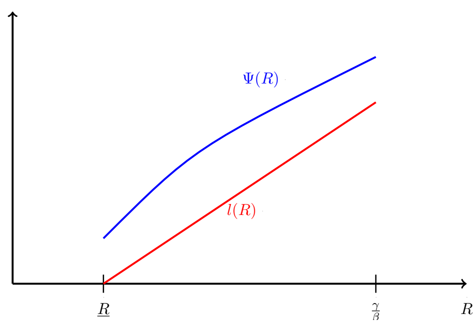
$$\begin{aligned}\Psi(R) = &-\frac{q^{nl}(R)}{\beta} + \frac{n\left[u\left(q^l(R)\right) - q^l(R)\right] + (1-n)\left[u\left(q^{nl}(R)\right) - q^{nl}(R)\right]}{1-\beta} - \chi R \frac{q^{nl}(R)}{\beta\theta} \\ &- \frac{G + u\left(u^{'^{-1}}\left(\frac{\gamma}{\beta}\right)\right) - \frac{\gamma}{\beta} u^{'^{-1}}\left(\frac{\gamma}{\beta}\right) - \phi\tilde{\tau}}{1-\beta}.\end{aligned} \tag{28}$$

Equation (28) gives the largest loan in DM that buyers can credibly promise to repay. We can then characterize stationary monetary equilibria by considering the loan schedule $l(R) = \frac{q^l(R) - q^{nl}(R)}{\beta}$ and the borrowing limit $\Psi(R)$ over the interval $[\underline{R}, \overline{R}]$.

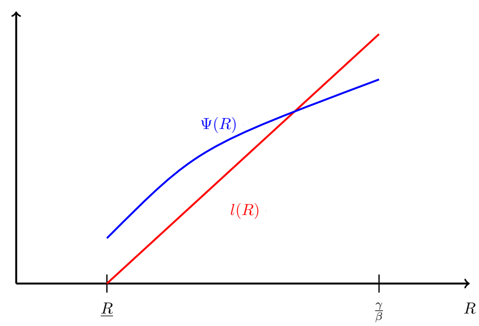
Lemma 4.3 *If $n \in (0,1)$ and $\chi < 1$, a stationary monetary equilibrium is characterized by a nominal interest rate R that satisfies*

$$R = \begin{cases} \hat{R} \in [\underline{R}, \frac{\gamma}{\beta}) & \text{and } \frac{q^l(\hat{R}) - q^{nl}(\hat{R})}{\beta} = \Psi(\hat{R}) \\ \frac{\gamma}{\beta} & \text{and } \frac{q^l(\frac{\gamma}{\beta}) - q^{nl}(\frac{\gamma}{\beta})}{\beta} \leq \Psi(\frac{\gamma}{\beta}). \end{cases} \tag{29}$$

An interest rate $\hat{R}$ at which $l(\hat{R}) = \Psi(\hat{R})$ defines a stationary monetary equilibrium with a binding no-default constraint. Also, the borrowing constraint can hold as a weak inequality if bonds are priced fundamentally, i.e. $R = \frac{\gamma}{\beta}$. Generically, there can be a unique or multiple steady states. Figure 1 shows the different type of unique monetary equilibria that can emerge in our environment.[11]



(a) Unique slack.



(b) Unique binding.

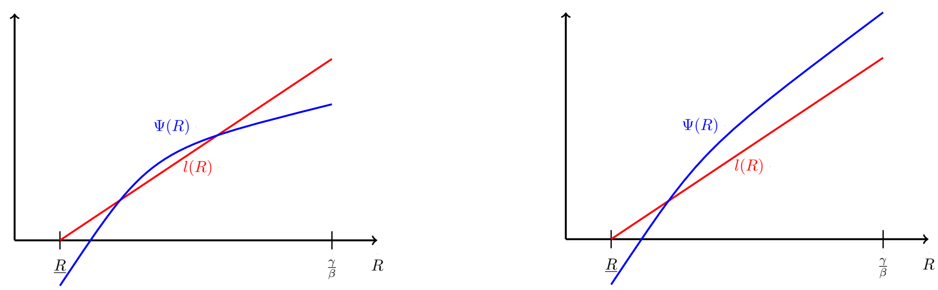
Figure 1: Loan in DM, borrowing limit, and unique stationary monetary equilibrium.

Figure 1a describes a situation where the no-default constraint is slack. This corresponds to a situation where nominal bonds are priced fundamentally. Uniqueness can also occur when bonds are not priced fundamentally and the no-default constraint binds. This type of equilibrium is illustrated in Figure 1b.

Our environment can also yield multiple steady states, thus allowing for real indeterminacies. These are possible under different circumstances, which are illustrated in Figure 2.

Figure 2a corresponds to an economy where there exist two steady states with binding borrowing constraints. Thus, nominal bonds are not priced fundamentally in neither of the two equilibria. Finally, a different type of multiplicity can emerge, which is depicted in Figure 2b. More specifically, a steady state with a binding borrowing constraint coexists with a steady state where the no-default constraint is slack.

[11]In Figure 1 the function $\Psi(R)$ is always monotone increasing, but this doesn't have to be necessarily the case.



(a) Multiple binding.

(b) Coexistence binding and slack.

Figure 2: Loan in DM, borrowing limit, and multiple stationary monetary equilibria.

4.1 Unique Stationary Monetary Equilibrium

The possibility of multiple equilibria hinges on the punishment upon default, as well as on the existence of two states of the world that have differential use for unsecured credit. Indeed, a stationary monetary equilibrium is unique when government bonds have the same use on and off-equilibrium. A stationary monetary equilibrium is also unique when bonds have the same use in all states of the world.

Lemma 4.4 *When $n = 1$ or $n = 0$ there may exist at most one monetary equilibrium.*

When $n = 0$, government bonds do not indirectly expand buyers' consumption opportunities. As a result, buyers are willing to hold bonds only if these are priced fundamentally. In contrast, when $n = 1$ (and $\chi < 1$), government bonds may help relax the limited commitment problem. However, there exists a unique nominal interest rate consistent with a stationary monetary equilibrium, as buyers can not substitute consumption across states of the world. Since $\gamma > \beta$, the unique stationary monetary equilibrium must be consistent with a binding no-default constraint. This is the case as when $n = 1$, at an equilibrium consistent with a slack no-default constraint, buyers consume the efficient level q^*. Since $\gamma > \beta$, their money demand would have to be $\phi m = 0$.

Throughout the rest of this section, we consider economies where $n \in (0, 1)$ and $\chi < 1$. We provide sufficient conditions for uniqueness of a stationary monetary equilibrium.

4.1.1 Slack No-default Constraint

In this monetary equilibrium, buyers that are able to use unsecured credit in DM consume the efficient quantity, and nominal bonds are priced fundamentally. This is the case as they do not help expand the DM consumption possibilities. This implies that the multipliers are $\lambda_l = \lambda_m = 0$.

Proposition 4.5 *When $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$, $n < \frac{1-\beta}{\gamma-\beta}$, and $\sigma < \frac{1}{(1-\chi)\left(\frac{\gamma}{\beta}-n\right)}$, there exists a unique stationary monetary equilibrium, where the no-default constraint is slack, if we satisfy*

$$
G-\phi\tilde{\tau}<\min\left\{\frac{\sigma}{1-\sigma}\left[n-\left(\frac{\gamma}{\beta}\right)^{-\frac{1-\sigma}{\sigma}}\right]+\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}\left[\frac{\frac{\gamma}{\beta}-n}{1-\sigma}-(1-n)-\frac{\chi\gamma(1-\beta)}{\theta\beta^2}\right]-\frac{1-\beta}{\beta},\right.
$$
$$
\left.\left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}\left[\frac{\gamma-1}{\beta}-\frac{\chi}{\beta\theta}\frac{\gamma(1-\beta)}{\beta+n(1-\chi)(\gamma-\beta)}\right]\right\}. \tag{30}
$$

From (30), note that, other than the curvature of the utility function (that reflects the value of insurance), off-equilibrium taxes and government spending are also relevant for the existence of a unique stationary monetary equilibrium.

4.1.2 Binding No-default Constraint

In this monetary equilibrium buyers that are able to use unsecured credit in DM cannot consume the first best quantity. In particular, we have that $\frac{\beta}{\gamma}\phi\lambda_m=\beta\lambda_l=u'(q^l)-1>0$. In this equilibrium buyers are willing to buy nominal bonds above their fundamental value. This is the case as by acquiring more bonds, buyers are able to increase their borrowing limit. This allows them to increase their DM consumption when unsecured credit is available.

Proposition 4.6 *When $u(x)=\frac{x^{1-\sigma}}{1-\sigma}$, $n<\frac{1-\beta}{\gamma-\beta}$, and $\sigma<\frac{1}{(1-\chi)\left(\frac{\gamma}{\beta}-n\right)}$, there exists a unique stationary monetary equilibrium, where the no-default constraint is binding, if we satisfy the following condition*

$$
\left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}\left[\frac{\gamma-1}{\beta}-\frac{\chi}{\beta\theta}\frac{\gamma(1-\beta)}{\beta+n(1-\chi)(\gamma-\beta)}\right]>G-\phi\tilde{\tau}
$$
$$
>\frac{\sigma}{1-\sigma}\left[n-\left(\frac{\gamma}{\beta}\right)^{-\frac{1-\sigma}{\sigma}}\right]+\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}\left[\frac{\frac{\gamma}{\beta}-n}{1-\sigma}-(1-n)-\frac{\chi\gamma(1-\beta)}{\theta\beta^2}\right]-\frac{1-\beta}{\beta} \tag{31}
$$

The parameter restriction and the specification of the utility function in Proposition 4.6 guarantee that the difference between the borrowing limit and the loan schedule is a concave function. This ensures that, generically, there exist at most two interest rates that make this difference equal to zero. Moreover, equation (31) guarantees that this difference can be equal to zero for at most one interest rate.

4.2 Multiple Stationary Equilibria

4.2.1 Binding No-default Constraint

The possibility of multiple equilibria hinges on the future continuation value, which critically depends on the nominal interest rate on government bonds. If the buyer expects a high interest rate tomorrow, the temptation to default becomes smaller. This is the case as the interest rate foregone by defaulting and the future expected utility are also larger. Thus, today the seller is willing to extend the buyer a larger loan. As a result, bonds prices are lower, which is consistent with an increase in the interest rate. Thus, expecting a higher interest rate is self-fulfilling.

Proposition 4.7 *When* $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$, $n < \frac{1-\beta}{\gamma-\beta}$, $\sigma < \frac{1}{(1-\chi)\left(\frac{\gamma}{\beta}-n\right)}$, *and fundamentals are such that*

$$G - \phi\tilde{\tau} > \max\left\{ \frac{\sigma}{1-\sigma}\left[n - \left(\frac{\gamma}{\beta}\right)^{-\frac{1-\sigma}{\sigma}}\right] + \left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}\left[\frac{\frac{\gamma}{\beta}-n}{1-\sigma} - (1-n) - \frac{\chi\gamma(1-\beta)}{\theta\beta^2}\right] - \frac{1-\beta}{\beta}, \right.$$
$$\left. \left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}\left[\frac{\gamma-1}{\beta} - \frac{\chi}{\beta\theta}\frac{\gamma(1-\beta)}{\beta+n(1-\chi)(\gamma-\beta)}\right]\right\}, \tag{32}$$

and

$$\Psi(R^o) - \frac{q^l(R^o) - q^{nl}(R^o)}{\beta} > 0 \tag{33}$$

where

$$R^o = \underset{R\in[\underline{R},\overline{R})}{argmax}\left\{\Psi(R) - \frac{q^l(R) - q^{nl}(R)}{\beta}\right\},$$

then there exist two stationary monetary equilibria, both consistent with a binding no-default constraint.

Proposition 4.7 provides sufficient conditions for the existence of two equilibria where the no-default constraint binds. One is consistent with a low and one with a high interest rate. From now on, we will refer to the equilibrium with a low (high) interest rate as the low (high) monetary equilibrium. The low monetary equilibrium delivers larger real money balances and lower credit. Instead, the high monetary equilibrium is associated with lower money holdings and larger credit. The coexistence of these equilibrium allocations is a direct result of the complementarity between the interest rate and DM credit.

4.2.2 Binding and non-Binding Equilibria

We analyze under what conditions equilibria with a binding no-default constraint co-exist with an equilibrium where (20) is slack.

Proposition 4.8 *When $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$ and we satisfy the following condition*

$$\begin{aligned}\frac{\sigma}{1-\sigma}\left[n-\left(\frac{\gamma}{\beta}\right)^{-\frac{1-\sigma}{\sigma}}\right]+\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}\left[\frac{\frac{\gamma}{\beta}-n}{1-\sigma}-(1-n)-\frac{\chi\gamma(1-\beta)}{\theta\beta^2}\right]-\frac{1-\beta}{\beta} \\ > G-\phi\tilde{\tau} > \left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}\left[\frac{\gamma-1}{\beta}-\frac{\chi}{\beta\theta}\frac{\gamma(1-\beta)}{\beta+n(1-\chi)(\gamma-\beta)}\right],\end{aligned} \tag{34}$$

then there exist a unique stationary equilibrium with a slack no-default constraint and at least one stationary equilibrium with a binding no-default constraint. Moreover, if $n < \frac{1-\beta}{\gamma-\beta}$ and $\sigma < \frac{1}{(1-\chi)\left(\frac{\gamma}{\beta}-n\right)}$ the equilibrium with a binding no-default constraint is unique.

When conditions in Proposition 4.8 are satisfied, an equilibrium where bonds are scarce coexists with one in which bonds are abundant and are priced fundamentally. In general, the rational behind the existence of multiple equilibria is different from Rocheteau et al. (2018), as it relies on the dynamic complementarity of today's and tomorrow's payoff. This mechanism is common in limited commitment problems.[12]

Proposition 4.9 *When $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$, agents face pledgeability rather then limited commitment, and the following condition holds*

$$\sigma < \frac{1}{(1-\chi)\left[\frac{\gamma}{\beta}-n\right]},$$

then there exists a unique monetary equilibrium, where the borrowing constraint is either slack or binding.

When buyers face limited pledgeability, the possibility of multiple self-fulling interest rates supporting the same allocation can not be consistent with a monetary equilibrium. However, when the underlying friction is limited commitment, multiplicity can occur. From Lemma 4.4 and the last proposition we see that the possibility of multiple equilibria hinges on the existence of two states of the world that have differential use for unsecured credit. This allows for self-fulfilling prophecies.

5 Open Market Operations

Having characterized the existence of monetary equilibria, we now analyze the effect of open market operations (OMO) on resulting allocations. As Wallace (1981) and Lucas (1984) highlight, one way to interpret Ricardian equivalence in a monetary economy is as an irrelevance proposition

[12]See Gu, Mattesini, Monnet, and Wright (2013) and Bethune et al. (2018) for more on the multiplicity of equilibria in limited commitment environments.

about OMO. However, when agents trade in incomplete markets and face distortionary taxes, OMO can have real effects.

Proposition 5.1 *When agents face a slack no-default constraint, open market operations have no real effects.*

Even though agents face some market incompleteness, our results are consistent with Wallace (1981), among others, as assets are priced fundamentally. In frictional and incomplete environments, Dong and Xiao (2019), Berentsen et al. (2016) Rocheteau et al. (2018), find similar results.

Lemma 5.2 *Open market operations have real effects when $n \in (0, 1)$, $\chi \in (0, 1)$, and the equilibrium with a binding no-default constraint exists.*

In our environment there is a differential treatment between taxes on and off the equilibrium. This can have consequences for OMO as long as there is partial repayment of government bonds upon an agent's private default. Since on-equilibrium taxes are lump-sum, they will adjust to offset the fiscal pressures of changing the money-to-bond ratio. Moreover, the off-equilibrium tax is such that the money-to-bond ratio does not alter the deviator's portfolio decision. Thus, on and off-equilibrium, lump-sum taxes alone cannot have real effects. However, if upon default the deviator can retain part of the returns on previous bond holdings, OMO have a direct impact on the endogenous borrowing constraint. This is the case even if the distortionary taxation is in the off-equilibrium and for one period. These features make our economy non-Ricardian. Then, it is not surprising that OMO can have real effects.

5.1 Interest Rates

In frictionless and flexible price environments, a one-time purchase of public debt does not change nominal interest rates. However, when nominal rigidities are present, the same procedure, in the short run, delivers a decrease in the nominal interest rate. Moreover, when liquidity considerations are taken into account and government bonds are scarce, a one time open market sale of government bonds have long-run real effects. In particular, Williamson (2012) shows that these operations reduce the liquidity premium on these government liabilities, increasing the nominal interest rate. Similarly, in an environment where agents face pledgeability problems, Rocheteau et al. (2018) find the same type of results.

In contrast to these previous results, in our model the effects of a one-time purchase of public debt depends on the equilibrium that emerges. This is consistent with Schreft and Smith (1998) and Rocheteau et al. (2018) with endogenous acceptability of payments in frictional markets.

Proposition 5.3 *When $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$, $n < \frac{1-\beta}{\gamma-\beta}$, and $\sigma < \frac{1}{(1-\chi)\left(\frac{\gamma}{\beta}-n\right)}$ hold, we can establish the following*

1. *If (31) is satisfied, then a decrease in the money-to-bond ratio lowers the unique equilibrium nominal interest rate.*

2. *If (32) and (33) are satisfied, a decrease in the money-to-bond ratio lowers the nominal interest rate associated with the high monetary equilibrium, while increases the interest rate for the low monetary equilibrium.*

3. *If (34) is satisfied, then a decrease in the money-to-bond ratio increases the nominal interest rate associated with the low monetary equilibrium, and has no effect on the high monetary equilibrium because bonds are priced fundamentally.*

Case 1 of Proposition 5.3 is in sharp contrast to what Williamson (2012) finds. This is the case as taxes adjust to satisfy the government budget constraint, leaving buyers' equilibrium payoff unchanged. Instead, in the off-equilibrium buyers face a different continuation value, which tightens their endogenous borrowing constraint. Thus, the nominal interest rate needs to adjust to satisfy the binding no-default constraint.

In Case 2 OMO can have differential real effects. This is the case as both monetary equilibria have a binding no-default constraint. Thus public debt is scarce, exhibiting a liquidity premium. As in Case 1, the only possible adjustment is through the interest rate. However, there are two nominal interest rates that are consistent with self-fulling beliefs. As in Williamson (2012) and Rocheteau et al. (2018) with exogenous pledgeability, at the high monetary equilibrium, an open market sale of government bonds increases the nominal interest rate. However, for the low monetary equilibrium, such operating procedure has the opposite effect.

Finally, when the equilibrium in Case 3 emerges, OMO do not always have real effects. When bonds are priced fundamentally these operations are irrelevant. Instead, when bonds are scarce the impact of OMO on interest rates is as in Williamson (2012) and Rocheteau et al. (2018) with exogenous pledgeability.

Proposition 5.4 *If agents face limited pledgeability rather then limited commitment, and the following condition holds*

$$\sigma < \frac{1}{(1-\chi)\left[\frac{\gamma}{\beta} - n\right]},$$

then a decrease in the money-to-bond ratio is irrelevant if the limited pledgeability constraint is slack, and increases the nominal interest rate if the limited pledgeability constraint binds.

The result in Proposition 5.4 highlights the importance of the dynamic complementarity in shaping the effects of OMO on nominal interest rates. In particular, in environments with limited pledgeability, a sale of government bonds always relaxes the borrowing constraint. Such operation has real effects when the pledgeability constraint binds, increasing the nominal interest rate.

5.2 Equilibrium Selection

There is a long tradition in economics that emphasizes that government policies can be used as an equilibrium selection device.[13] Thus, it is important to think how policies can help rule out real indeterminacies. In our context, OMO can be used to select an equilibrium and rule out indeterminacies. In particular, consider economies where an equilibrium with a slack no-default constraint coexists with one consistent with a binding no-default constraint. In such scenario, an appropriate increase in the money to bond ratio selects the equilibrium with a slack no-default constraint.

Proposition 5.5 *Suppose that $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$, $n < \frac{1-\beta}{\gamma-\beta}$, $\sigma < \frac{1}{(1-\chi)\left(\frac{\gamma}{\beta}-n\right)}$, and θ is such that an equilibrium with a binding no-default constraint coexists with one where the no-default constraint is slack. Then, there exists a policy $\theta' > \theta$ that selects the equilibrium with a slack no default constraint whenever*

$$G - \phi\tilde{\tau} < \min\left\{\frac{\sigma}{1-\sigma}\left[n - \left(\frac{\gamma}{\beta}\right)^{-\frac{1-\sigma}{\sigma}}\right] + \left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}\left[\frac{\frac{\gamma}{\beta}-n}{1-\sigma} - (1-n)\right] - \frac{1-\beta}{\beta}, \left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}\frac{\gamma-1}{\beta}\right\}. \tag{35}$$

When parameter configurations are such that the multiplicity of equilibria described in Proposition 5.5 is observed, then an open market purchase of government bonds that results in a money to bond ratio equal to θ' is inconsistent with self-fulfilling beliefs that deliver an equilibrium with a binding no-default constraint. Government bonds purchases can be used to rule out real indeterminacies, thus finding another rationale for such policy.

6 Numerical Examples

In this section we quantify some of the model equilibrium properties. In our benchmark calibration, we assume that there exists a unique equilibrium with a slack no-default constraint. Thus the frictional consumption when unsecured credit is feasible is efficient, $q^l = 1 = q^*$, and bonds are priced fundamentally. We then explore the resulting monetary equilibria as χ and $\phi\tilde{\tau}$ change.

Our analysis uses US quarterly data from 1984 to 2007.[14] To pin down the fraction of DM trades where unsecured credit is not available, we follow Aruoba et al. (2011) and set $n = 0.15$. We then consider the average inflation and nominal interest rates and match the average money to bond and government expenditure. In our environment, nominal GDP is $Y = Y^{CM} + Y^{DM}$. Assuming the specific functional form, $U(X) = \Omega\frac{X^\eta}{\eta}$ for CM preferences, nominal output in the

[13] For instance, see Schreft and Smith (1998), Ennis and Keister (2005), Antinolfi et al. (2007) among others, for more on this topic.

[14] In our analysis M corresponds to the monetary base (BOGMBASE), B denotes total public debt (GFDEGDQ188S); for the long-run inflation rate we consider CPI inflation (CPIAUCSL), and for the nominal interest rate we take the 1-year treasury constant maturity rate (DGS1).

centralized market is $Y^{CM} = (G + X^B + X^S)/\phi = (2\Omega^{\frac{1}{1-\eta}} + G)/\phi$, and nominal output in the decentralized market is $Y^{DM} = \frac{n\,q^l + (1-n)q^{nl}}{\phi}$. In the data, the previous two ratios are given by

$$\theta = \frac{M}{B} = 0.0975$$

and

$$\frac{G}{Y} = \frac{G}{2\Omega^{\frac{1}{1-\eta}} + G + n + (1-n)q^{nl}} = 0.20. \tag{36}$$

To pin down the rest of the parameters, we use the methodology from Lucas (2000) and examine the relationship between the nominal rate, i, and liquidity services, L.[15] In equilibrium we have that liquidity services are given by

$$L = \frac{\phi\, M}{Y} = \frac{q^{nl}(1+i)}{(2\Omega^{\frac{1}{1-\eta}} + G) + n + (1-n)q^{nl}}. \tag{37}$$

To determine the values of σ, Ω, and G, we minimize the distance between the observed liquidity services at three different nominal interest rates and their implied equilibrium counterpart and the government expenditure to GDP ratio.[16] Table 1 summarizes the parameters in the benchmark calibration.

θ	0.0975	γ	1.0025
β	0.95	σ	0.278
n	0.15	Ω	2.564
g	0.2	η	0.46

Table 1: Benchmark Calibration

As we can see, the calibrated parameters n, γ, β, and σ satisfy both conditions $n < \frac{1-\beta}{\gamma-\beta}$ and $\sigma < \frac{1}{(1-\chi)\left(\frac{\gamma}{\beta}-n\right)}$ that are assumed in Propositions 4.5, 4.6, 4.7, and 4.8.

The only two parameters that are left to determine are χ and $\phi\tilde{\tau}$. In particular, we choose $\chi = 0.8$ to guarantee that the following condition is satisfied

$$\frac{\sigma}{1-\sigma}\left[n - \left(\frac{\gamma}{\beta}\right)^{-\frac{1-\sigma}{\sigma}}\right] + \left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}\left[\frac{\frac{\gamma}{\beta}-n}{1-\sigma} - (1-n) - \frac{\chi\gamma(1-\beta)}{\theta\beta^2}\right] + \\ -\frac{1-\beta}{\beta} > \left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}\left[\frac{\gamma-1}{\beta} - \frac{\chi}{\beta\theta}\frac{\gamma(1-\beta)}{\beta + n(1-\chi)(\gamma-\beta)}\right] \tag{38}$$

[15] This relationship represents "money demand" in the sense that "desired" real balances M/P are proportional to Y, with a factor of proportionality L that depends on the cost of holding cash, i.

[16] The assumption $\sigma \leq 1$ guarantees that the surplus from trading in DM is non-negative.

which is necessary (but not sufficient) for the coexistence of equilibria with a binding and a slack no-default constraint is possible. Finally, to ensure the existence of a unique equilibrium consistent with a slack no-default constraint, we choose $\phi\tilde{\tau}$ large enough to satisfy condition (30). Specifically, we restrict off-equilibrium taxes to be

$$\phi\tilde{\tau} = \frac{1}{2}[U(\tilde{X}) - \tilde{(}X)] + u(\tilde{q}) - \phi\tilde{m}.$$

Under the benchmark calibration, we obtain a unique stationary monetary equilibrium where the no-default constraint is slack and the resulting endogenous observables are summarized in Table 2.

l	0.212
q^{nl}	0.799
q^{l}	1
$\phi\, m$	0.843
$\phi\, b$	8.64
X	5.72
R	1.055

Table 2: Equilibrium Outcomes.

Since the no-default constraint is slack, consumption in the state where credit is feasible equals the efficient level, $q^l = 1$, and bonds are priced fundamentally, $R = \frac{\gamma}{\beta}$. Finally buyers consume less in states where credit is not feasible, $q^{nl} < q^l$, as carrying real balances across periods is costly.

Note that multiple equilibria can exist when the off-equilibrium taxation changes. In particular, when

$$\phi\tilde{\tau} = 0.4624[U(\tilde{X}) - \tilde{(}X)] + u(\tilde{q}) - \phi\tilde{m},$$

an equilibrium with a slack no-default constraint coexists with an equilibrium consistent with a binding no-default constraint. Table 3 reports these monetary stationary equilibria.

In the equilibrium with a binding no-default constraint, we observe a liquidity premium on bonds. Moreover, consumption inequality in the frictional market is smaller. Furthermore, the demand for real government bonds is larger. This is a direct consequence of agents facing different prices. Such real indeterminacy critically depends on the off-equilibrium punishment.

For the previous economy, we show numerically how the equilibrium allocation consistent with a binding no-default constraint responds to an open market operation.[17] The results are presented in Table 4.

[17] Recall that open market operations do not have real effects when agents face a slack no-default constraint.

	Slack no-default	Binding no-default
l	0.212	0.006
q^{nl}	0.799	0.825
q^{l}	1	0.83
$\phi\, m$	0.843	0.87
$\phi\, b$	8.64	8.92
X	5.72	5.72
R	1.055	1.0532

Table 3: Coexistence of stationary equilibria with binding and slack no-default constraint.

	$\theta = 0.0975$	$\theta = 0.0974$
l	0.006	0.0283
q^{nl}	0.825	0.821
q^{l}	0.83	0.848
m	0.87	0.867
b	8.92	8.897
X	5.725	5.725
R	1.0532	1.0534

Table 4: The effects of Open Market Operations

Even a small decrease in the money to bond ratio has significant effects on unsecured credit. More precisely, a 0.1% decrease in θ induces a fivefold increase in real loans. This is the case as a decrease in the money to bond ratio makes bonds more plentiful, thus relaxing the borrowing constraint, lowering the liquidity premium on bonds, and expanding buyers' credit limit. From Table 4, we see that the liquidity premium on bonds is lower as the nominal interest rate is increased by 0.02%. Moreover, because sellers are willing to extend more credit in DM, buyers substitute consumption across states in DM. Thus they need to acquire fewer cash balances in CM. More specifically, consumption in the states where credit is possible increased by 2%, whereas consumption in the states where credit is not feasible decreased by 0.5%. The increase in consumption-gap across the different states of nature is consistent with findings in Coibion et al. (2017), who document that consumption inequality increases after monetary contraction. However, the mechanism in our model does not rely on prices being sluggish.

7 Conclusions

In this paper we study the impact of limited commitment in credit markets on open market operations. To do so, we consider a monetary, frictional, and incomplete market framework. Agents face stochastic trading opportunities, anonymity, and limited commitment in frictional markets. These give rise to an essential medium of exchange and an endogenous borrowing constraint. Finally, there is a government that must finance a constant stream of exogenous expenditures by levying taxes and by issuing nominal bonds as well as fiat money. Other than implementing monetary policy and taxing agents, the government can exclude buyers from the credit market and impose additional punishments in case of default.

Within this environment, we characterize the existence of unique and multiple stationary monetary equilibria. We find conditions for the existence of a unique monetary equilibrium: i) with a slack no-default constraint and ii) with a binding no-default constraint. This environment can also yield multiple monetary equilibria where at least one of the stationary equilibria has a binding no-default constraint. In particular, there is a type of real indeterminacy where agents face a binding and a non binding no-default constraints. All of these real indeterminacies are a direct consequence of the dynamic complementarities arising from the limited commitment problem. This is in sharp contrast to an environment with exogenous limited pledgeability, where generically the monetary equilibrium is unique, even when the pledgeability constraint binds.

In this paper we also study the real effects of open market operations. Generically, these operations have real effects when public debt exhibits a liquidity premia. These are different depending on the monetary equilibrium that emerges. More precisely, a sale of government bonds can decrease or increase the nominal interest rate, depending on the nature of the equilibrium. This is the case as after such sale, taxes adjust to satisfy the government budget constraint, leaving buyers' equilibrium payoff unchanged. Instead, in the off-equilibrium buyers face a different continuation value, which tightens their endogenous borrowing constraint. Thus, the nominal interest rate needs to adjust to satisfy the binding no-default constraint. As a result, the impact of these operations on the nominal interest rate depend on the nature of the equilibrium that emerges. Finally, when real indeterminacies emerge, we show that purchases of government bonds can be used as an equilibrium selection mechanism.

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Appendix

Deviation value function

Finally, we have to determine the value function $\tilde{W}_t(\tilde{m}_{-1}, \tilde{b}_{-1})$ of a buyer that defaulted on a previous loan, and therefore he can only use money in the DM. Let $\tilde{V}_t(\tilde{m})$ be his value function in DM:

$$\tilde{V}_t(\tilde{m}, \tilde{b}) = \max_{\{\tilde{q}, \tilde{d}_m\}} \quad u(\tilde{q}) + \beta\, \tilde{W}_{t+1}(\tilde{m} - \tilde{d}_m, \tilde{b}) \qquad (\tilde{P}^{B,N})$$
$$s.t. \quad -\tilde{q} + \beta\phi_{+1}\tilde{d}_m \geq 0$$
$$\tilde{m} - \tilde{d}_m \geq 0$$

A solution to this problem is such that $\tilde{q} = \beta\phi_{+1}\tilde{d}_m$ and $\tilde{d}_m = \tilde{m}$ and $\tilde{q} = \beta\phi_{+1}\tilde{m}$.

In the CM, a buyer that previously defaulted on a loan solves

$$\tilde{W}_t(\tilde{m}_{-1}, \tilde{b}_{-1}) = \quad \phi\tilde{m}_{-1} + \phi\, R\, \tilde{b}_{-1} - \phi\, \tilde{\tau} + \max_{\{\tilde{X}, \tilde{m}, \tilde{b}\}} \left\{ U(\tilde{X}) - \tilde{X} - \phi\, \tilde{m} - \phi\tilde{b} + \tilde{V}_t(\tilde{m}, \tilde{b}) \right\} \qquad (\tilde{P}^{B})$$

The buyer optimal decisions solve

$$\tilde{X}: \quad -U'(\tilde{X}) - 1 = 0$$
$$\tilde{m}: \quad -\phi + \frac{\partial \tilde{V}_t(\tilde{m}, \tilde{b})}{\partial \tilde{m}} \leq 0$$
$$\tilde{b}: \quad -\phi + \frac{\partial \tilde{V}_t(\tilde{m}, \tilde{b})}{\partial \tilde{b}} \leq 0$$

where the second and third condition must hold with equality if $\tilde{m} > 0$ and if $\tilde{b} > 0$. The envelope conditions are

$$\frac{\partial \tilde{W}_t(\tilde{m}_{-1}, \tilde{b}_{-1})}{\partial \tilde{m}_{-1}} = \phi$$
$$\frac{\partial \tilde{W}_t(\tilde{m}_{-1}, \tilde{b}_{-1})}{\partial \tilde{b}_{-1}} = \phi R$$

Upon default, a buyer will choose $\tilde{b} = 0$ given that $R \leq \frac{\gamma}{\beta}$. Thus, the envelope condition in the DM gives us

$$\frac{\partial \tilde{V}_t(\tilde{m}, \tilde{b})}{\partial \tilde{m}} = \beta\phi_{+1}u'(\tilde{q})$$

which combined with the first order condition for money holdings gives:

$$-\phi + \beta\, \phi_{+1}\, u'(\tilde{q}) = 0$$
$$\tilde{q} = \beta\, \phi_{+1}\, \tilde{m}$$

In a stationary equilibrium we can rewrite

$$-1 + \frac{\beta}{\gamma}\, u'(\tilde{q}) = 0 \tag{39}$$
$$\tilde{q} = \frac{\beta}{\gamma}\, \phi\, \tilde{m} \tag{40}$$

Replacing this value in ($\tilde{P}^B$) obtain

$$\tilde{W}(\tilde{m}_{-1}, \tilde{b}_{-1}) = \phi \tilde{m}_{-1} + \phi\, R\, \tilde{b}_{-1} + \frac{U\Big(U'^{-1}(1)\Big) - U'^{-1}(1) + u\left(u'^{-1}\left(\frac{\gamma}{\beta}\right)\right) - \frac{\gamma}{\beta} u'^{-1}\left(\frac{\gamma}{\beta}\right) - \phi\tilde{\tau}}{1-\beta} \tag{41}$$

Proof of Lemma 4.1

From equation (24), we have that

$$R = \frac{\gamma}{\beta} \frac{1}{1 + n\lambda_l(1-\chi)} \leq \frac{\gamma}{\beta} = \overline{R}$$

where we used stationarity of the equilibrium, thus $\frac{\phi}{\phi_{+1}} = \gamma$, and the fact that $\lambda_l \geq 0$.

Next, from equations (13) and (14) we have that $q^{nl} = \frac{\beta}{\gamma}\phi m$, whereas from equation (22), when $\lambda_l = \lambda_m = 0$, then $q^l = q^* > q^{nl}$, otherwise when $\lambda_l = \lambda_m > 0$, from (21) $q^l = \frac{\beta}{\gamma}\phi m + \beta l = q^{nl} + \beta l \geq q^{nl}$. Thus, in general $q^l \geq q^{nl}$. From equation (23) we have

$$\lambda_l = \lambda_m = \frac{\frac{\gamma}{\beta} - (1-n)u'(q^{nl})}{n} - 1.$$

Combining this equation with $\lambda_l = u'(q^l) - 1$, we have $nu'(q^l) + (1-n)u'(q^{nl}) = \frac{\gamma}{\beta}$. Because $q^l \geq q^{nl}$, it must be that $u'(q^{nl}) \geq \frac{\gamma}{\beta}$. Replacing this in the expression for λ^l we obtain $\lambda_l \leq \frac{\gamma}{\beta} - 1$. Then, in (24) this last inequality gives us

$$R = \frac{\gamma}{\beta} \frac{1}{1 + n\lambda_l(1-\chi)} \geq \frac{\gamma}{\beta} \frac{1}{1 + n\left(\frac{\gamma}{\beta} - 1\right)(1-\chi)} = \frac{\gamma}{\beta + n(1-\chi)(\gamma - \beta)} = \underline{R}.$$

which concludes the proof.

Proof of Corollary 4.2

The conclusion follows directly from $\underline{R} = \overline{R} = \frac{\gamma}{\beta}$ when $\chi = 1$.

Proof of Lemma 4.3

From equations (2) and (5), $X = X^S = U^{-1}(1)$; also, sellers optimal decision in CM is $\phi m^S = \phi b^S = 0$. From equations (13), (14), stationarity of equilibrium, and money market clearing, $\phi\, d_m^{nl} = \phi\, d_m^l = \phi m = \phi M = \frac{\gamma q^{nl}}{\beta}$.[18] Also, from the bond market clering condition $\phi b = \phi B = \frac{\gamma q^{nl}}{\beta\theta}$. From (1) $\phi\tau = G + \frac{\gamma}{\beta} q^{nl}\left[\frac{R}{\gamma\theta} + \frac{1}{\gamma} - \frac{1}{\theta} - 1\right]$. Finally, from (21) we have that $l = \frac{q^l - q^{nl}}{\beta}$. We prove the rest of the lemma by looking separately at an equilibrium with a slack no-default constraint and then equilibria with a binding no-default constraint.

Equilibrium with a slack no-default constraint

Consider first an equilibrium with a slack no-default constraint: in (20) it must be

$$l \leq (1-\chi) R \phi_{+1} b + W^B(0,0,0) - \tilde{W}(0,0).$$

From (22) we have $\lambda^l = 0$, $q^l = q^* = u^{'^{-1}}(1)$, and $\lambda_m = 0$. Also, from (24),

$$R = \frac{\gamma}{\beta},$$

and from (23),

$$q^{nl} = u^{'^{-1}}\left(\frac{\frac{\gamma}{\beta} - n}{1-n}\right)$$

Replacing these values in (21) we have that $l = \frac{q^* - q^{nl}}{\beta}$. Thus, we only have left to check that the no-default constraint holds. The value function $W(0,0,0)$ is

$$W(0,0,0) = \frac{U(X) - X - \phi m - \phi b - \phi\tau}{1-\beta} + \frac{n\, u(q^l) + (1-n)\, u(q^{nl})}{1-\beta} + \frac{\beta \frac{R\,\phi b}{\gamma} - \beta n l}{1-\beta} \tag{42}$$

Using equation (41) we can rewrite the no-default constraint (20) as follows:

$$l \leq \frac{n\, u(q^l) + (1-n)\, u(q^{nl}) - \beta n l}{1-\beta} + \frac{\frac{R\,\phi b}{\gamma} - \phi m - \phi b - \phi\tau}{1-\beta}$$

[18] We assume that $\phi\, d_m^l = \phi m$, thus buyers transfer all their money balances also in states when the no-default constraint is slack. This is without loss of generality, as the composition of payments (money vs. credit) is indeterminate in this case.

$$
-\chi\frac{R\,\phi b}{\gamma}-\frac{u\left(u^{'-1}\left(\frac{\gamma}{\beta}\right)\right)-\frac{\gamma}{\beta}u^{'-1}\left(\frac{\gamma}{\beta}\right)-\phi\tilde{\tau}}{1-\beta}
$$

Replacing the values of l, $\phi\tau$, R, ϕm, ϕb, and q^{nl} we can rewrite this equation as

$$
\begin{aligned}
\frac{q^*-u^{'-1}\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)}{\beta}\leq&-\frac{u^{'-1}\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)}{\beta}+\frac{n[u(q^*)-q^*]+(1-n)[u\left(u^{'-1}\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)\right)-u^{'-1}\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)]-G}{1-\beta}\\
&-\frac{\chi}{\theta}\frac{\gamma}{\beta^2}u^{'-1}\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)-\frac{u\left(u^{'-1}\left(\frac{\gamma}{\beta}\right)\right)-\frac{\gamma}{\beta}u^{'-1}\left(\frac{\gamma}{\beta}\right)-\phi\tilde{\tau}}{1-\beta}.
\end{aligned}
\tag{43}
$$

It is easy to check that, for $q^l(R)$ and $q^{nl}(R)$ defined in (27), and $\Psi(R)$ in (28), $q^l(\frac{\gamma}{\beta})=q^*$, $q^{nl}(\frac{\gamma}{\beta})=u^{'-1}\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)$, and $\Psi(\frac{\gamma}{\beta})$ equals to the right-hand side of equation (43). Thus, an equilibrium with a slack no-default constraint exists if and only if $\frac{q^l(\frac{\gamma}{\beta})-q^{nl}(\frac{\gamma}{\beta})}{\beta}\leq\Psi(\frac{\gamma}{\beta})$, for $q^l(R)$ and $q^{nl}(R)$ defined in (27) and $\Psi(R)$ in (28).

Equilibrium with a binding no-default constraint

When that the no-default constraint binds, we have $\lambda_m=\lambda_l=u'(q^l)-1\geq 0$ and $q^l\leq q^*$. From equations (23) and (24), we can solve for q^l and q^{nl} to obtain (27):

$$
q^l(R)=u^{'-1}\left(1+\frac{\frac{\gamma}{\beta R}-1}{n(1-\chi)}\right),\qquad q^{nl}(R)=u^{'-1}\left(\frac{\frac{\gamma}{\beta}-\frac{n(1-\chi)+\frac{\gamma}{\beta R}-1}{1-\chi}}{1-n}\right).
$$

Notice that $\lambda^l\geq 0$ $q^l\leq q^*$ if and only if $R\leq\overline{R}=\frac{\gamma}{\beta}$. Also, from the definition of l in (21) and q^l and q^{nl} from (27) we have that $l\geq 0$ if and only if $R\geq\underline{R}=\frac{\gamma}{\beta+n(1-\chi)(\gamma-\beta)}$. Then, an equilibrium with a binding no-default constraint exists if for there is an $R\in[\underline{R},\overline{R}]$ such that the no-default constraint binds and the government budget constraint is satisfied, together with the constant money growth and the constant money-to-bond ratio. Equations (42) and (41) define the value functions on and off the equilibrium. Then, replacing $W(0,0,0)$ and $\tilde{W}(0,0)$ from (42) and (41), q^l and q^{nl} from (27), and $\phi\tau$ from (1), we obtain that an equilibrium with a binding no-default constraint must satisfy

$$
\begin{aligned}
\frac{q^l(R)-q^{nl}(R)}{\beta}=&-\frac{q^{nl}(R)}{\beta}+\frac{n\Big[u\Big(q^l(R)\Big)-q^l(R)\Big]+(1-n)\Big[u\Big(q^{nl}(R)\Big)-q^{nl}(R)\Big]}{1-\beta}-\chi R\frac{q^{nl}(R)}{\beta\theta}\\
&-\frac{G+u\left(u^{'-1}\left(\frac{\gamma}{\beta}\right)\right)-\frac{\gamma}{\beta}u^{'-1}\left(\frac{\gamma}{\beta}\right)-\phi\tilde{\tau}}{1-\beta}.
\end{aligned}
$$

Thus, an equilibrium with a binding no-default constraint is characterized by a $\hat{R} \in (\underline{R}, \overline{R})$ such that $\frac{q^l(\hat{R})-q^{nl}(\hat{R})}{\beta} = \Psi(\hat{R})$, for $q^l(R)$ and $q^{nl}(R)$ defined in (27) and $\Psi(R)$ in (28).

Proof of Lemma 4.4

Proof. Consider first the case of $n = 1$. In such a case a monetary equilibrium with a slack no-default constraint does not exist as long as $\gamma > \beta$. Indeed, when the no-default constraint is slack, replacing $\lambda_m = 0$ in equation (23) we obtain

$$-1 + \frac{\beta}{\gamma} = 0,$$

which can never hold for $\gamma > \beta$. Consider then the equilibrium with a binding no-default constraint, in which $\lambda_m = \lambda_l = [u'(q^l) - 1]$. Equations (23) and (24) become

$$\begin{aligned} -1 + \frac{\beta}{\gamma} u'(q^l) &= 0 \\ -1 + \frac{\beta\, R}{\gamma} \Big[1 + (1-\chi)\Big(u'(q^l) - 1\Big)\Big] &= 0 \end{aligned}$$

and equations (18) and (20) give

$$l = \frac{q^l}{\beta} - \frac{\phi m}{\gamma} \tag{44}$$

$$\frac{q^l}{\beta} = -\chi \frac{R\,\phi m}{\theta\gamma} - \frac{G}{1-\beta} + \frac{u(q^l) - q^l}{1-\beta} \tag{45}$$

Notice that the first two equations pin down uniquely q^l and R, then (45) pins down uniquely ϕm, and finally (44) pins down uniquely l.

Consider now the case of $n = 0$. Equation (23) gives us $R = \frac{\gamma}{\beta}$, and the model collapses to the standard Lagos and Wright (2005). ■

To prove the remaining results, it is easier to transform the problem and characterize the solution for the transformed problem. To do this, we state an equivalence result in the following lemma.

Lemma 7.1 *Let $q^l(R)$, $q^{nl}(R)$ be defined in (27), and $\Psi(R)$ in (28), and define the function*

$g : [\underline{R}, \overline{R}] \to \Re_+$ as follows:

$$g(R) = u^{'^{-1}} \left(\frac{\frac{\gamma}{\beta} - n - \frac{\frac{\gamma}{\beta R} - 1}{1-\chi}}{1-n} \right). \tag{46}$$

Note that $g(\cdot)$ is a bijection, that $g'(R) < 0$, and $[g(\overline{R}), g(\underline{R})] = \left[u^{'^{-1}} \left(\frac{\frac{\gamma}{\beta} - n}{1-n} \right), u^{'^{-1}} \left(\frac{\gamma}{\beta} \right) \right]$. Define the function $\mathcal{F} : \left[u^{'^{-1}} \left(\frac{\frac{\gamma}{\beta} - n}{1-n} \right), u^{'^{-1}} \left(\frac{\gamma}{\beta} \right) \right] \to \Re$ as

$$\mathcal{F}(x) = -\frac{u^{'^{-1}} \left(\frac{\frac{\gamma}{\beta} - (1-n)(x)^{-\sigma}}{n} \right)}{\beta} + \frac{n \left[u \left(u^{'^{-1}} \left(\frac{\frac{\gamma}{\beta} - (1-n)(x)^{-\sigma}}{n} \right) \right) - u^{'^{-1}} \left(\frac{\frac{\gamma}{\beta} - (1-n)(x)^{-\sigma}}{n} \right) \right] + (1-n) \left[u(x) - x \right]}{1-\beta}$$
$$- \frac{\chi}{\theta} \frac{\gamma}{\beta^2} \frac{x}{1 + (1-\chi) n \left[\frac{\frac{\gamma}{\beta} - (1-n) u'(x)}{n} - 1 \right]} - \frac{G + u \left(u^{'^{-1}} \left(\frac{\gamma}{\beta} \right) \right) - \frac{\gamma}{\beta} u^{'^{-1}} \left(\frac{\gamma}{\beta} \right) - \phi \tilde{\tau}}{1-\beta} \tag{47}$$

Then, for $x \in \left[u^{'^{-1}} \left(\frac{\frac{\gamma}{\beta} - n}{1-n} \right), u^{'^{-1}} \left(\frac{\gamma}{\beta} \right) \right]$,

$$\mathcal{F}(x) \begin{cases} > 0 & \text{iff } \Psi\big(g^{-1}(x)\big) > \frac{q^l\left(g^{-1}(x)\right) - q^{nl}\left(g^{-1}(x)\right)}{\beta} \\ = 0 & \text{iff } \Psi\big(g^{-1}(x)\big) = \frac{q^l\left(g^{-1}(x)\right) - q^{nl}\left(g^{-1}(x)\right)}{\beta} \\ < 0 & \text{iff } \Psi\big(g^{-1}(x)\big) < \frac{q^l\left(g^{-1}(x)\right) - q^{nl}\left(g^{-1}(x)\right)}{\beta} \end{cases}$$

Proof. *Replacing the function $g^{-1}(x)$ into (47) and rearranging terms we obtain the result.* ■

From Lemma 7.1 and Lemma 4.3, an equilibrium with a slack no-default constraint exists if ad only if $\mathcal{F} \left(u^{'^{-1}} \left(\frac{\frac{\gamma}{\beta} - n}{1-n} \right) \right) \geq 0$, and an equilibrium with a binding no-default constraint can be characterized by a $\hat{x} \in \left[u^{'^{-1}} \left(\frac{\frac{\gamma}{\beta} - n}{1-n} \right), u^{'^{-1}} \left(\frac{\gamma}{\beta} \right) \right]$ such that $\mathcal{F}(\hat{x}) = 0$. Once we have such $\hat{x}$, then the equilibrium interest rate is $\hat{R} = g^{-1}(\hat{x})$.

Lemma 7.2 *Assume that $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$, $n < \frac{1-\beta}{\gamma-\beta}$, $\sigma < \frac{1}{(1-\chi)\left(\frac{\gamma}{\beta} - n\right)}$. Then, the function $\mathcal{F}(x)$ is concave in the interval $\left[u^{'^{-1}} \left(\frac{\frac{\gamma}{\beta} - n}{1-n} \right), u^{'^{-1}} \left(\frac{\gamma}{\beta} \right) \right] = \left[\left(\frac{\frac{\gamma}{\beta} - n}{1-n} \right)^{-\frac{1}{\sigma}}, \left(\frac{\gamma}{\beta} \right)^{-\frac{1}{\sigma}} \right]$.*

Proof. *Consider $\mathcal{F}'(x)$: after some manipulations, we obtain*

$$\mathcal{F}'(x) = \frac{\beta(1-n)\left[(x)^{-\sigma} - 1\right] - \left(\gamma - 1 - \beta(1-n)\left[(x)^{-\sigma} - 1\right]\right) \left(\frac{1-n}{n}\right) \left(\frac{\frac{\gamma}{\beta}(x)^{\sigma} - (1-n)}{n} \right)^{-\frac{1+\sigma}{\sigma}}}{\beta(1-\beta)}$$
$$- \frac{\chi}{\theta} \frac{\gamma}{\beta^2} \frac{1 + (1-\chi)\left(\frac{\gamma}{\beta} - n\right) - (1-n)(1-\chi)(x)^{-\sigma}(1+\sigma)}{\left\{ 1 + (1-\chi)\left[\frac{\gamma}{\beta} - n - (1-n)(x)^{-\sigma}\right] \right\}^2}$$

Consider now the second derivative $\mathcal{F}''(x)$:

$$\begin{aligned}\mathcal{F}''(x) = &-\frac{\sigma\beta(1-n)(x)^{-\sigma-1}}{\beta(1-\beta)} - \frac{\beta\sigma(1-n)(x)^{-\sigma-1}\left(\frac{1-n}{n}\right)\left(\frac{\frac{\gamma}{\beta}(x)^{\sigma}-(1-n)}{n}\right)^{-\frac{1+\sigma}{\sigma}}}{\beta(1-\beta)} \\ &+ \frac{\left(\gamma-1-\beta(1-n)\Big[(x)^{-\sigma}-1\Big]\right)}{\beta(1-\beta)}\left(\frac{1-n}{n}\right)(1+\sigma)(x)^{\sigma-1}\left(\frac{\frac{\gamma}{\beta}(x)^{\sigma}-(1-n)}{n}\right)^{-\frac{1+\sigma}{\sigma}-1} \\ &- \frac{\chi\gamma}{\theta\beta^2}\frac{\sigma(x)^{-(1+\sigma)}(1-n)(1-\chi)}{\left\{1+(1-\chi)\left[\frac{\gamma}{\beta}-n-(1-n)(x)^{-\sigma}\right]\right\}^3}\times\Bigg\{ \\ &\left[1+(1-\chi)\left(\frac{\gamma}{\beta}-n-(1-n)(x)^{-\sigma}\right)\right](1+\sigma)+2\left[1+(1-\chi)\left(\frac{\gamma}{\beta}-n-(1-n)(x)^{-\sigma}\right)\right]\Bigg\}\end{aligned}$$

Notice that sufficient conditions for $\mathcal{F}''(x)<0$ for all x is that

$$\gamma-1-\beta(1-n)\Big[(x)^{-\sigma}-1\Big]<0 \ \textit{ for all } x$$

and

$$1+(1-\chi)\left(\frac{\gamma}{\beta}-n-(1-n)(x)^{-\sigma}\right)>0 \ \textit{ for all } x$$

The first condition holds if $n<\frac{1-\beta}{\gamma-\beta}$ and the second condition holds for $\sigma<\frac{1}{(1-\chi)\left(\frac{\gamma}{\beta}-n\right)}$. Therefore, if $n<\frac{1-\beta}{\gamma-\beta}$ and $\sigma<\frac{1}{(1-\chi)\left(\frac{\gamma}{\beta}-n\right)}$ the function $\mathcal{F}(x)$ is concave.

■

Proof of Proposition 4.5

Proof. Existence of an equilibrium with a slack no-default constraint follows from the fact that condition (30) guarantees that $\frac{q^l(\frac{\gamma}{\beta})-q^{nl}(\frac{\gamma}{\beta})}{\beta}<\Psi(\frac{\gamma}{\beta})$: indeed, from (27) and (28) we obtain

$$q^l(\frac{\gamma}{\beta})=q^*=1, \qquad q^{nl}(\frac{\gamma}{\beta})=\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}$$

and

$$\begin{aligned}\Psi(\frac{\gamma}{\beta}) = &-\frac{\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}}{\beta} + \frac{n\Big[\frac{1}{1-\sigma}-1\Big]+(1-n)\left[\frac{\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1-\sigma}{\sigma}}}{1-\sigma}-\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}\right]}{1-\beta} - \chi\frac{\gamma}{\beta}\frac{\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}}{\beta\theta} \\ &- \frac{G+\frac{\left(\frac{\gamma}{\beta}\right)^{-\frac{1-\sigma}{\sigma}}}{1-\sigma}-\frac{\gamma}{\beta}\left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}-\phi\tilde{\tau}}{1-\beta}\end{aligned} \tag{48}$$

After rearranging, using condition (30) we obtain

$$(1-\beta)\Psi(\frac{\gamma}{\beta}) = \phi\tilde{\tau} - G + \frac{\sigma}{1-\sigma}\left[n - \left(\frac{\gamma}{\beta}\right)^{-\frac{1-\sigma}{\sigma}}\right]$$
$$+ \left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}\left[-\frac{1-\beta}{\beta} - \frac{\chi\gamma}{\beta^2\theta}(1-\beta) + \frac{\frac{\gamma}{\beta}-n}{1-\sigma} - (1-n)\right]$$
$$> \frac{1-\beta}{\beta}\left[1 - \left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}\right] = \frac{1-\beta}{\beta}\left[q^l(\frac{\gamma}{\beta}) - q^{nl}(\frac{\gamma}{\beta})\right]$$

from which we conclude $\Psi(\frac{\gamma}{\beta}) > \frac{q^l(\frac{\gamma}{\beta})-q^{nl}(\frac{\gamma}{\beta})}{\beta}$.

Uniqueness follows from the fact that condition (30) guarantees also that $\frac{q^l(\underline{R})-q^{nl}(\underline{R})}{\beta} < \Psi(\underline{R})$ and concavity of $\mathcal{F}(\cdot)$. Indeed, from (27) and (28) we obtain

$$q^l(\underline{R}) = q^{nl}(\underline{R}) = \left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}$$

and

$$\Psi(\underline{R}) = -\frac{\left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}}{\beta} - \frac{\left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}}{1-\beta} - \chi\frac{\gamma}{\beta+n(1-\chi)(\gamma-\beta)}\frac{\left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}}{\beta\theta} - \frac{G-\frac{\gamma}{\beta}\left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}-\phi\tilde{\tau}}{1-\beta} \tag{49}$$

After rearranging, using condition (30) we obtain

$$(1-\beta)\Psi(\underline{R}) = \phi\tilde{\tau} - G + \left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}\left[-\frac{1}{\beta} + \frac{\gamma}{\beta} - \chi\frac{\gamma}{\beta+n(1-\chi)(\gamma-\beta)}\frac{1-\beta}{\beta\theta}\right] > 0$$

Thus, given that $q^l(\underline{R}) = q^{nl}(\underline{R})$, we conclude that $\Psi(\underline{R}) > \frac{q^l(\underline{R})-q^{nl}(\underline{R})}{\beta}$.

Suppose then by contradiction that there exists an equilibrium with a binding no-default constraint. Then there should exist a $\hat{R} \in (\underline{R}, \overline{R})$ such that $\frac{q^l(\hat{R})-q^{nl}(\hat{R})}{\beta} = \Psi(\hat{R})$. From Lemma 7.1, then $\mathcal{F}(g(\hat{R})) = 0$. But since from condition (30) we have $\frac{q^l(\frac{\gamma}{\beta})-q^{nl}(\frac{\gamma}{\beta})}{\beta} < \Psi(\frac{\gamma}{\beta})$, Lemma 7.1 implies that $\mathcal{F}(g(\overline{R})) > 0$. Since the function $\mathcal{F}$ is concave, it must be that $\mathcal{F}'(x) < 0$ for $x \geq g(\hat{R})$. Thus, $\mathcal{F}(g(\underline{R})) < 0$. But this implies that $\frac{q^l(\underline{R})-q^{nl}(\underline{R})}{\beta} > \Psi(\underline{R})$, which contradicts condition (30). ■

Proof of Proposition 4.6

Proof. The proof is similar to the one of Proposition 4.5. From condition (31) and equation (48) we obtain

$$
\begin{aligned}
(1-\beta)\Psi(\frac{\gamma}{\beta}) &= \phi\tilde{\tau} - G + \frac{\sigma}{1-\sigma}\left[n - \left(\frac{\gamma}{\beta}\right)^{-\frac{1-\sigma}{\sigma}}\right] \\
&+ \left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}\left[-\frac{1-\beta}{\beta} - \frac{\chi\gamma}{\beta^2\theta}(1-\beta) + \frac{\frac{\gamma}{\beta}-n}{1-\sigma} - (1-n)\right] \\
&< \frac{1-\beta}{\beta}\left[1 - \left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}\right] = \frac{1-\beta}{\beta}\left[q^l(\frac{\gamma}{\beta}) - q^{nl}(\frac{\gamma}{\beta})\right]
\end{aligned}
$$

from which we conclude $\Psi(\frac{\gamma}{\beta}) < \frac{q^l(\frac{\gamma}{\beta})-q^{nl}(\frac{\gamma}{\beta})}{\beta}$. From condition (31) and equation (49) we obtain

$$
(1-\beta)\Psi(\underline{R}) = \phi\tilde{\tau} - G + \left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}\left[-\frac{1}{\beta} + \frac{\gamma}{\beta} - \chi\frac{\gamma}{\beta+n(1-\chi)(\gamma-\beta)}\frac{1-\beta}{\beta\theta}\right] > 0
$$

thus $\Psi(\underline{R}) > \frac{q^l(\underline{R})-q^{nl}(\underline{R})}{\beta} = 0$. Therefore, from the intermediate value theorem, it should exist a $\hat{R} \in (\underline{R}, \frac{\gamma}{\beta})$ such that $\Psi(\hat{R}) = \frac{q^l(\hat{R})-q^{nl}(\hat{R})}{\beta}$, which proves the existence of a stationary equilibrium consistent with a binding no-default constraint.

Uniqueness follows from concavity of the function $\mathcal{F}(x)$: since $\Psi(\frac{\gamma}{\beta}) < \frac{q^l(\frac{\gamma}{\beta})-q^{nl}(\frac{\gamma}{\beta})}{\beta}$, from (46) and (47) we have $\mathcal{F}(g(\frac{\gamma}{\beta})) < 0$; also, since $\Psi(\underline{R}) > \frac{q^l(\underline{R})-q^{nl}(\underline{R})}{\beta}$, $\mathcal{F}(g(\underline{R})) > 0$. Finally, notice that $\mathcal{F}(g(\hat{R})) = 0$. Then, by concavity of $\mathcal{F}$, we have that $\mathcal{F}(x) < 0$ if $x \in \left[g(\frac{\gamma}{\beta}), g(\hat{R})\right)$ and $\mathcal{F}(x) > 0$ if $x \in \left(\hat{R}, g(\underline{R})\right]$. Thus $\Psi(R) > \frac{q^l(R)-q^{nl}(R)}{\beta}$ if $R \in [\underline{R}, \hat{R})$ and $\Psi(R) < \frac{q^l(R)-q^{nl}(R)}{\beta}$ if $R >\in (\hat{R}, \frac{\gamma}{\beta}]$. ■

Proof of Propostion 4.7

Proof. Following similar arguments to the ones in the proof of Proposition 4.5, using condition (32) and equations (48) and (49) we can show that $\Psi(\frac{\gamma}{\beta}) < \frac{q^l(\frac{\gamma}{\beta})-q^{nl}(\frac{\gamma}{\beta})}{\beta}$ and $\Psi(\underline{R}) < \frac{q^l(\underline{R})-q^{nl}(\underline{R})}{\beta}$. Thus, from (46) and (47) we have $\mathcal{F}(g(\frac{\gamma}{\beta})) < 0$ and $\mathcal{F}(g(\underline{R})) < 0$. Next, from (33) there exists a R^o such that $\Psi(R^o) > \frac{q^l(R^o)-q^{nl}(R^o)}{\beta}$, thus $\mathcal{F}(g(R^o)) > 0$. From the intermediate value theorem, there exist R^a and R^b such that $\mathcal{F}(g(R^a)) = \mathcal{F}(g(R^b)) = 0$, and thus thus $\Psi(R^a) = \frac{q^l(R^a)-q^{nl}(R^a)}{\beta}$ and $\Psi(R^b) = \frac{q^l(R^b)-q^{nl}(R^b)}{\beta}$. Moreover, from concavity of $\mathcal{F}(\cdot)$ we have $\mathcal{F}(x) > 0$ if $x \in (g(R^a), g(R^b))$, and $\mathcal{F}(x) < 0$ if $x < g(R^a)$ or $x > g(R^b)$, which proves that there are exactly two stationary equilibria, corresponding to the nominal interest rate R^a and R^b. ■

Proof of Proposition 4.8

Proof. Following similar arguments to the ones in the proof of Proposition 4.5, using condition (34) and equations (48) we can show that $\Psi(\frac{\gamma}{\beta}) > \frac{q^l(\frac{\gamma}{\beta})-q^{nl}(\frac{\gamma}{\beta})}{\beta}$, then an equilibrium with a slack no-default constraint exists. Also from condition (34) and (49) and we can show that $\Psi(\underline{R}) < \frac{q^l(\underline{R})-q^{nl}(\underline{R})}{\beta}$. Thus, from (46) and (47) we have $\mathcal{F}(g(\frac{\gamma}{\beta})) > 0$ and $\mathcal{F}(g(\underline{R})) < 0$. From the intermediate value theorem, there exists a $\hat{R}$ such that $\mathcal{F}(g(\hat{R})) = 0$, and thus $\Psi(\hat{R}) = \frac{q^l(\hat{R})-q^{nl}(\hat{R})}{\beta}$. Thus, there esists an equilibrium consistent with a binding no-default constraint corresponding to the nominal interest rate $\hat{R}$. Finally, from concavity of $\mathcal{F}(\cdot)$ we have $\mathcal{F}(x) > 0$ if $x \in (g(\frac{\gamma}{\beta}), g(\hat{R}))$, and $\mathcal{F}(x) < 0$ if $x \in (g(\hat{R}), g(\underline{R}))$. Thus there exists no other equilibrium consistent with a binding no-default constraint. ■

Proof of Proposition 4.9

Proof. If lending in DM was subject to limited pledgeability, a stationary monetary equilibrium with a slack no-default constraint would be characterized by $(\lambda_l, \lambda_m, q^l, q^{nl}, R)$ satisfying equations (22), (23), (24), $\lambda_l = \lambda_m$, and

$$\frac{q^l - q^{nl}}{\beta} < (1-\chi)\frac{R\, q^{nl}}{\beta\, \theta}$$

Thus, it is easy to show that a unique stationary monetary equilibrium with a slack pledgeability constraint exists if the following parameter restriction is satisfied:[19]

$$\left[\frac{\frac{\gamma}{\beta} - n}{1-n}\right]^{-\frac{1}{\sigma}} \left[1 + \frac{(1-\chi)}{\theta}\frac{\gamma}{\beta}\right] > 1 \tag{50}$$

Similarly, an equilibrium with a binding pledgeability constraint would be characterized by $(\lambda_l, \lambda_m, q^l, q^{nl}, R)$ satisfying equations (22), (23), (24), $\lambda_l = \lambda_m$

$$\frac{q^l - q^{nl}}{\beta} = (1-\chi)\frac{R\, q^{nl}}{\beta\, \theta}$$

[19]This condition comes from the equilibrium conditions $q^l = q^* = 1$, $q^{nl} = \frac{\beta}{\gamma}\phi m$, $R = \frac{\gamma}{\beta}$, and

$$-1 + \frac{\beta}{\gamma}[nu'(q^l) + (1-n)u'(q^{nl})] = 0$$

$$\frac{q^l - q^{nl}}{\beta} = l < (1-\chi)\frac{R\,\phi b}{\gamma} = (1-\chi)\frac{R\,\phi m}{\gamma\,\theta} = (1-\chi)\frac{R\, q^{nl}}{\beta\,\theta}$$

Thus, it is easy to show that a stationary monetary equilibrium with a binding pledgeability constraint exists if there exists a $q_{nl} \in \left[\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}, \left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}\right]$ satisfying the following condition:[20]

$$\mathcal{H}(q_{nl}) = q_{nl}\left[1 + \frac{(1-\chi)\gamma}{\beta\theta}\frac{1}{1+(1-\chi)\left[\frac{\gamma}{\beta}-n-(1-n)q_{nl}^{-\sigma}\right]}\right] - \left[\frac{\frac{\gamma}{\beta}-(1-n)q_{nl}^{-\sigma}}{n}\right]^{-\frac{1}{\sigma}} = 0 \quad (51)$$

Notice that

$$\mathcal{H}\left(\left[\frac{\frac{\gamma}{\beta}-n}{1-n}\right]^{-\frac{1}{\sigma}}\right) = \left[\frac{\frac{\gamma}{\beta}-n}{1-n}\right]^{-\frac{1}{\sigma}}\left[1+\frac{(1-\chi)}{\theta}\frac{\gamma}{\beta}\right] - 1$$

Thus, a sufficient condition for the equilibrium with a pledgeability constraint to be unique is $\mathcal{H}'(q_{nl}) > 0$. Computing $\mathcal{H}'(q_{nl})$ we obtain

$$\begin{aligned}
\mathcal{H}'(q_{nl}) = 1 &+ \frac{(1-\chi)\gamma}{\beta\theta}\frac{1}{1+(1-\chi)\left[\frac{\gamma}{\beta}-n-(1-n)q_{nl}^{-\sigma}\right]} \\
&- q_{nl}\,\frac{(1-\chi)\gamma}{\beta\theta}\frac{(1-\chi)(1-n)\sigma q_{nl}^{-(1+\sigma)}}{\left\{1+(1-\chi)\left[\frac{\gamma}{\beta}-n-(1-n)q_{nl}^{-\sigma}\right]\right\}^2} + \frac{1-n}{n}q_{nl}^{-(1+\sigma)} \\
&= 1 + \frac{1-n}{n}q_{nl}^{-(1+\sigma)} \\
&+ \frac{(1-\chi)\gamma}{\beta\theta}\frac{1}{1+(1-\chi)\left[\frac{\gamma}{\beta}-n-(1-n)q_{nl}^{-\sigma}\right]}\left[1-\frac{(1-\chi)(1-n)\sigma q_{nl}^{-\sigma}}{1+(1-\chi)\left[\frac{\gamma}{\beta}-n-(1-n)q_{nl}^{-\sigma}\right]}\right]
\end{aligned}$$

Thus, a sufficient condition for $\mathcal{H}'(q_{nl}) > 0$ (in the interval for q_{nl} we are interested in) is that the term in the last parenthesis is positive. After rearranging, this requires the following condition to hold:

$$1+(1-\chi)\left[\frac{\gamma}{\beta}-n\right] > (1-\chi)(1-n)q_{nl}^{-\sigma}\left[1+\sigma\right]$$

Notice that this condition is more difficult to satisfy when $q_{nl} = \left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}$. Replacing such a value for q_{nl} in the last expression, we obtain that a sufficient condition for $\mathcal{H}'(q_{nl}) > 0$ for all

[20] This condition comes from the equilibrium conditions $q^{nl} = \frac{\beta}{\gamma}\phi m$, $R = \frac{\gamma}{\beta[1+n(1-\chi)(u'(q^l)-1)]}$, and

$$-1+\frac{\beta}{\gamma}[nu'(q^l)+(1-n)u'(q^{nl})] = 0$$

$$\frac{q^l - q^{nl}}{\beta} = l = (1-\chi)\frac{R\,\phi b}{\gamma} = (1-\chi)\frac{R\,\phi m}{\gamma\,\theta} = (1-\chi)\frac{R\,q^{nl}}{\beta\,\theta}$$

$q_{nl} \in \left[\left(\frac{\frac{\gamma}{\beta} - n}{1-n} \right)^{-\frac{1}{\sigma}}, \left(\frac{\gamma}{\beta} \right)^{-\frac{1}{\sigma}} \right]$ is

$$\sigma < \frac{1}{(1-\chi)\left[\frac{\gamma}{\beta} - n\right]}$$

which concludes the proof. ■

Proof of Proposition 5.3

Proof. Consider the decrease in the money to bond ratio from θ to θ', where $\theta' < \theta$.

1. If (31) is satisfied both at θ and θ', from Proposition 4.6 in both cases there exists a unique equilibrium consistent with a biding no-default constraint. Specifically, in the first case the equilibrium with a binding no-default constraint is characterized by $\hat{R}$ such that $\frac{q^l(\hat{R}) - q^{nl}(\hat{R})}{\beta} = \Psi(\hat{R}; \theta)$, and in the second case by $\hat{R}'$ such that $\frac{q^l(\hat{R}') - q^{nl}(\hat{R}')}{\beta} = \Psi(\hat{R}'; \theta')$, where $q^l(R)$ and $q^{nl}(R)$ are defined in (27) and $\Psi(R;\theta)$ in (28). Also, from (28) observe that $\Psi(R;\theta) > \Psi(R, \theta')$ for all $R \in [\underline{R}, \frac{\gamma}{\beta}]$. Then, we obtain

$$\frac{q^l(\hat{R}) - q^{nl}(\hat{R})}{\beta} = \Psi(\hat{R}; \theta) > \Psi(\hat{R}; \theta').$$

   Also, since (31) is satisfied both at θ and θ', we know that

$$\frac{q^l(\underline{R}) - q^{nl}(\underline{R})}{\beta} < \Psi(\underline{R}; \theta')$$

   Thus, by the intermediate value theorem, there exists a $\hat{R}'' \in (\underline{R}, \hat{R})$ such that $\frac{q^l(\hat{R}'') - q^{nl}(\hat{R}'')}{\beta} = \Psi(\hat{R}''; \theta')$. Because, also with θ', there exists a unique equilibrium consistent with a binding no-default constraint, it must be that $\hat{R}' = \hat{R}''$, so that $\hat{R}' < \hat{R}$. Thus, a decrease in the money to bond ratio lowers equilibrium interest rate.

2. If (32) and (33) are satisfied both at θ and θ', from Proposition 4.7 there exist two stationary equilibria consistent with a binding no-default constraint, $\hat{R}_1$ and $\hat{R}_2$ when the money to bond ratio is θ, as well as two stationary equilibria $\hat{R}_1'$ and $\hat{R}_2'$ consistent with a biding no-default constraint when the money to bond ratio is θ'. Specifically, $\frac{q^l(\hat{R}_1) - q^{nl}(\hat{R}_1)}{\beta} = \Psi(\hat{R}_1; \theta)$, $\frac{q^l(\hat{R}_2) - q^{nl}(\hat{R}_2)}{\beta} = \Psi(\hat{R}_2; \theta)$, $\frac{q^l(\hat{R}_1') - q^{nl}(\hat{R}_1')}{\beta} = \Psi(\hat{R}_1'; \theta)$, $\frac{q^l(\hat{R}_2') - q^{nl}(\hat{R}_2')}{\beta} = \Psi(\hat{R}_2'; \theta)$. Also, from the proof of Proposition 4.7, observe that $\frac{q^l(R) - q^{nl}(R)}{\beta} > \Psi(R;\theta)$ if $R \in (\hat{R}_1, \hat{R}_2)$, and $\frac{q^l(R) - q^{nl}(R)}{\beta} < \Psi(R;\theta)$ if $R \in (\underline{R}, \hat{R}_1)$ or $R \in (\hat{R}_2, \frac{\gamma}{\beta})$. Finally, from (28) observe that $\Psi(R;\theta) > \Psi(R, \theta')$ for all $R \in [\underline{R}, \frac{\gamma}{\beta}]$.

Let then solve

$$R^{o'} = \underset{R\in[\underline{R},\overline{R})}{\operatorname{argmax}} \left\{ \Psi(R;\theta') - \frac{q^l(R) - q^{nl}(R)}{\beta} \right\}.$$

We know that $\Psi(R^{o'};\theta') - \frac{q^l(R^{o'})-q^{nl}(R^{o'})}{\beta} > 0$ because (33) is satisfied at θ'. Observe then that $R^{o'} \in (\hat{R}_1, \hat{R}_2)$. Indeed, for $R \in (\underline{R}, \hat{R}_1]$ or $R \in [\hat{R}_2, \frac{\gamma}{\beta})$, we have $\frac{q^l(R)-q^{nl}(R)}{\beta} \leq \Psi(R;\theta) < \Psi(R;\theta')$. In summary, we have $\Psi(\hat{R}_1;\theta') < \Psi(\hat{R}_1;\theta) = \frac{q^l(\hat{R}_1)-q^{nl}(\hat{R}_1)}{\beta}$, $\Psi(\hat{R}_2;\theta') < \Psi(\hat{R}_2;\theta) = \frac{q^l(\hat{R}_2)-q^{nl}(\hat{R}_2)}{\beta}$, and $\Psi(R^{o'};\theta') > \frac{q^l(R^{o'})-q^{nl}(R^{o'})}{\beta}$. Then by the intermediate value theorem there exist at least two interest rates $\hat{R}''_1 < \hat{R}''_2 \in (\hat{R}_1, \hat{R}_2)$ such that $\frac{q^l(\hat{R}''_1)-q^{nl}(\hat{R}''_1)}{\beta} = \Psi(\hat{R}''_1;\theta')$ and $\frac{q^l(\hat{R}''_2)-q^{nl}(\hat{R}''_2)}{\beta} = \Psi(\hat{R}''_2;\theta')$. Because, also with θ', there exists two equilibria consistent with a binding no-default constraint, it must be that $\hat{R}'_1 = \hat{R}''_1$, and $\hat{R}'_2 = \hat{R}''_2$ so that $\hat{R}'_1 > \hat{R}_1$ and $\hat{R}'_2 > \hat{R}_2$. Thus, a decrease in the money to bond ratio increases equilibrium interest rate at the low equilibrium and decreases equilibrium interest rate at the high equilibrium.

3. If (34) is satisfied both at θ and θ' from Proposition 4.8 in both cases an equilibrium consistent with a biding no-default constraint coexists with an equilibrium consistent with a slack no-default constraint. At the equilibrium consistent with the slack no-default constraint, open-market operations have no effects from Proposition 5.1. The equilibrium consistent with a binding no-default constraint is characterized by $\hat{R}$ such that $\frac{q^l(\hat{R})-q^{nl}(\hat{R})}{\beta} = \Psi(\hat{R};\theta)$ when the money to bond ratio is θ and by $\hat{R}'$ such that $\frac{q^l(\hat{R}')-q^{nl}(\hat{R}')}{\beta} = \Psi(\hat{R}';\theta')$ when the money to bond ratio is θ'. Also, from (28) observe that $\Psi(R;\theta) > \Psi(R,\theta')$ for all $R \in [\underline{R}, \frac{\gamma}{\beta}]$. Then, we obtain
$$\frac{q^l(\hat{R}) - q^{nl}(\hat{R})}{\beta} = \Psi(\hat{R};\theta) > \Psi(\hat{R};\theta').$$
Also, since (34) is satisfied both at θ and θ', we know that
$$\frac{q^l(\frac{\gamma}{\beta}) - q^{nl}(\frac{\gamma}{\beta})}{\beta} < \Psi(\frac{\gamma}{\beta};\theta')$$
Thus, by the intermediate value theorem, there exists a $\hat{R}'' \in (\hat{R}, \frac{\gamma}{\beta})$ such that $\frac{q^l(\hat{R}'')-q^{nl}(\hat{R}'')}{\beta} = \Psi(\hat{R}'';\theta')$. Because, also with θ', the equilibrium consistent with a binding no-default constraint is unique, it must be that $\hat{R}' = \hat{R}''$, so that $\hat{R}' > \hat{R}$. Thus, a decrease in the money to bond ratio increases equilibrium interest rate.

■

Proof of Proposition 5.4

Proof. From the proof of Proposition 4.9, an equilibrium with a binding pledge-abiligy constraint is given by a $q_{nl} \in \left[\left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}, \left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}\right]$ such that $\mathcal{H}(q^{nl};\theta) = 0$, where $\mathcal{H}(q;\theta)$ is defined in (51).

Let then q^{nl} be the equilibrium consumption in states where credit is not feasible associated to the money to bond ratio θ, and $q^{nl'}$ be equilibrium consumption in states where credit is not feasible when the money to bond ratio θ'. Thus

$$\mathcal{H}(q^{nl};\theta) = \mathcal{H}(q^{nl'};\theta') = 0.$$

Notice that, for $\theta' < \theta$, equation (51) implies that $\mathcal{H}(q;\theta) < \mathcal{H}(q;\theta')$. Since from the assumption that $\sigma < \frac{1}{(1-\chi)\left[\frac{\gamma}{\beta}-n\right]}$ we know from the proof of Proposition 4.9 that $\mathcal{H}(q;\theta)$ is increasing in q, it must be that $q^{nl'} > q^{nl}$. Thus, since q^{nl} is inversely related to R from equation (27), we can conclude that $R' > R$. Therefore a decrease in the money to bond ratio increases equilibrium interest rate (at the equilibrium associated with a binding limited pledgeability constraint). ■

Proof of Proposition 5.5

Proof. Suppose that equation (34) is satisfied, $n < \frac{1-\beta}{\gamma-\beta}$, $\sigma < \frac{1}{(1-\chi)\left(\frac{\gamma}{\beta}-n\right)}$. Let

$$A(\theta) = \frac{\sigma}{1-\sigma}\left[n - \left(\frac{\gamma}{\beta}\right)^{-\frac{1-\sigma}{\sigma}}\right] + \left(\frac{\frac{\gamma}{\beta}-n}{1-n}\right)^{-\frac{1}{\sigma}}\left[\frac{\frac{\gamma}{\beta}-n}{1-\sigma} - (1-n) - \frac{\chi\gamma(1-\beta)}{\theta\beta^2}\right] - \frac{1-\beta}{\beta}$$

$$B(\theta) = \left(\frac{\gamma}{\beta}\right)^{-\frac{1}{\sigma}}\left[\frac{\gamma-1}{\beta} - \frac{\chi}{\beta\theta}\frac{\gamma(1-\beta)}{\beta+n(1-\chi)(\gamma-\beta)}\right],$$

Notice that we can rewrite (34) as

$$A(\theta) > G - \phi\tilde{\tau} > B(\theta)$$

Also, observe that $A'(\theta) > 0$ and $B'(\theta) > 0$. Finally, notice that we can rewrite (35) as

$$G - \phi\tilde{\tau} < \min\left\{\overline{A}, \overline{B}\right\}$$

where $\overline{A} = \lim_{\theta\to\infty} A(\theta)$ and $\overline{B} = \lim_{\theta\to\infty} B(\theta)$. Then, by continuity of $A(\theta)$ and $B(\theta)$ if (35) holds there exists a θ' such that (30) is satisfied for all $\theta > \theta'$. On the other hand, if (35) is violated, then (30) is violated for all $\theta \in \Re_+$. ■