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Abstract

This paper examines the market-wide effects of front-running and information-sharing by dealers in a quantitive microstructure model of Forex trading. Recent investigations by government regulators and court proceedings reveal that there has been widespread sharing of information among Forex dealers working at major banks, as well as the regular front-running of large customer orders. I use the model to study the effects of unilateral front-running, where individual dealers trade ahead of their own customer orders; and collusive front-running where individual dealers trade ahead of another dealer's customer order based on information that was shared among a group of dealers. I find that both forms of front-running create an information externality that significantly affects order flows and Forex prices by slowing down the process through which inter-dealer trading aggregates information from across the market. Font-running reduces dealers' liquidity provision costs by raising the price customers pay to purchase Forex, and lowering the price they receive when selling Forex. These cost reductions are substantial; they lower costs by more than 90 percent. Front-running also affects other market participants that are not directly involved in front-running trades. The information externality makes these participants less willing to speculate on their private information when trading with dealers. This indirect effect of front-running can reduce participants' expected returns by as much as 10 percent. My analysis also shows that collusive front-running has larger effects on order flows than unilateral front-running because information-sharing reduces the risks dealers face when trading ahead of customer orders. However, in other respects, the effects of collusive and unilateral front-running are quite similar. Greater collusion lowers the costs of providing liquidity and it reduces other participants' expected returns, but the effects are small.

Keywords: Forex Trading, Order Flows, Forex Price Fixes, Microstructure Trading Models.

JEL Codes: F3; F4; G1.

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Introduction

Foreign currency (Forex) trading appears to take place in a highly competitive environment. Since the mid-1990s, major currencies have traded almost continuously between large numbers of counterparties on multiple electronic platforms in high volumes and with very tight bid-ask spreads. However, in recent years, government regulators and enforcement authorities across the globe undertook investigations into whether many of the world's largest dealer-banks were acting anti-competitively in the Forex market. Between 2014 and 2015 reports issued by the U.S. Department of Justice, the Commodity Futures Trading Commission, New York Department of Financial Services, the U.K. Financial Conduct Authority, and the Swiss Financial Market Supervisory Authority all concluded that dealer-banks engaged in a range of collusive conduct aimed at manipulating the Forex benchmarks, specifically the ECB and WMR Fixes.[1] The investigators also found that the dealer-banks had engaged in other forms of anticompetitive conduct, including the collusive sharing of information and front-running. Following these reports, the U.S Department of Justice and the Federal Reserve Board indicted and placed lifetime bans on more than a dozen individual FX dealers and in 2017 a dealer was convicted of wire fraud for his part in a scheme to front-run a \$3.5 billion trade.[2] In addition, multiple law-suites have been brought before the courts in the United States and Canada alleging that dealer-banks engaged in anti-competitive behavior that harmed investors. By the end of 2018, these investigations and law-suites produced fines and settlements totaling over \$11 billion.

This paper examines how the anti-competitive behavior identified by government regulators and enforcement authorities affects the spot Forex market. In particular, I use a quantitative microstructure model to analyze how the collusive sharing of information and front-running by dealer-banks impacts the behavior of Forex prices, trading flows and the welfare of all market participants. According to the investigations and court proceedings, there has been wide-spread information-sharing among dealer-banks concerning their inventory positions and pending customer Forex orders (i.e., orders from non-dealers). I use the model to examine how such information-

[1] Information on these investigations can be found at: DOJ Press Release, Five Major Banks Agree to Parent Level Guilty Pleas (May 20, 2015), https://www.justice.gov/opa/pr/five-major-banks-agree-parent-level-guilty-pleas. CFTC Press Release No. 7056-14, CFTC Orders Five Banks to Pay over \$1.4 Billion in Penalties for Attempted Manipulation of Foreign Exchange Benchmark Rates (Nov. 12, 2014), http://www.cftc.gov/PressRoom/PressReleases/pr7056-14. DFS Press Release, NYDFS Announces Barclays to Pay \$2.4 Billion, Terminate Employees for Conspiring to Manipulate Spot FX Trading Market (May 20, 2015), http://www.dfs.ny.gov/about/press/pr1505201.htm. FCA Press Release, FCA fines five banks 1.1 billion for FX failings and announces industry-wide remediation program (Nov. 12, 2014), http://www.fca.org.uk/news/fca-fines-five-banks-for-fx-failings. FINMA Press Release, FINMA sanctions foreign exchange manipulation at UBS (Nov. 12, 2014), http://www.finma.ch/e/aktuell/pages/mm-ubs-devisenhandel-20141112.aspx.

[2] See, U.S. v. Johnson, No. 16-cr-457-NCG-1 (E.D.N.Y.).

sharing affects trading across the market. The investigations also revealed that the dealer-banks regularly front-run large customer orders. In this practice, a dealer establishes a speculative position before executing the customer's order so as to profit from its effect on prices. For example, a dealer will buy Forex before executing a large Forex purchase on behalf of a customer with the aim of making a capital gain from the rise in prices produced by the execution of the large purchase. I use the model to study the effects of unilateral front-running, where individual dealers "trade ahead" of their own customer orders; and collusive front-running where individual dealers trade ahead of another dealer's pending customer order based on information that was shared among a group of dealers.

The model I develop extends earlier multiple-dealer models of Forex trading in Lyons (1997), Evans and Lyons (2002) and Evans (2011). The model describes trading between a large number of dealer-banks (hereafter dealers), and two groups of customers called investors and hedgers. Trading takes place between dealers in the wholesale tier of the market, and between dealers and their customers in the retail tier. Dealers are risk-averse and choose their trades and price-quotes optimally in both tiers of the market. Investors are also risk averse and optimally determine the orders they place with dealers in the retail tier. In contrast, dealers receive orders from hedgers that are determined by an exogenous liquidity factor. The model provides a rich environment to study the market-wide effects of information-sharing and front-running. In particular, I analyze how the equilibrium behavior of prices and trading flows change when dealers share information about their customer orders, and when they front-run hedgers' orders, both unilaterally and collusively.

My analysis produces several noteworthy findings:

1. In the absence of front-running, the sharing of customer-order information among dealers increases the volatility of aggregate inter-dealer order flows but has little impact on equilibrium prices or the welfare of dealers and investors.

2. Risk-averse dealers have a strong incentive to unilaterally front-run their own customer orders, even when the execution of those orders has no impact on prices.

3. In an equilibrium where dealers have the opportunity to front-run their own customer orders, trading ahead of those orders creates an information externality that has significant effects on trading flows and prices. The externality slows down the process by which inter-dealer trading aggregates the information that is ultimately embedded into the prices, which in turn affects the trading decisions of both dealers and investors.

4. Front-running reduces the costs dealers incur from providing liquidity to hedgers. It raises the price hedgers pay when they are net purchasers of Forex, and reduces the price they receive

they are net sellers of Forex. These effects are substantial. They reduce dealers' costs of providing liquidity by more than 90 percent.

5. Front-running also affects the welfare of dealers and investors. The information externality makes risk-averse investors less willing to speculate on their private information when trading with dealers, so they make smaller trading profits when that information becomes embedded in future prices. This indirect effect of front-running can reduce investors' expected returns by as much as 10 percent. The reduction in investors' trading profits also benefits dealers, accounting for approximately half of the reduction in the total costs of providing liquidity across the market.

6. Collusive front-running has larger effects on aggregate inter-dealer order flows than unilateral front-running because information-sharing reduces the risks dealers face when trading ahead of customer orders. In other respects, the effects of collusive and unilateral front-running are quite similar. Greater collusion lowers the costs of providing liquidity to customers, and it reduces investors trading profits, but the effects are small.

It is worth emphasizing that these results address the impact of information-sharing and front-running across the entire market. In particular, my analysis looks beyond the direct impact of shared information or front-running by individual dealers to consider their equilibrium effects on the trading decisions of other dealers and investors. This perspective counters the widespread assumption that the effects of front-running are not market-wide (Kyle and Viswanathan, 2008). It is also empirically important because even though dealer information-sharing and front-running appear to have been widespread, it is unlikely to have directly involved more than a small fraction of all trades in the market. In my analysis, dealers do not front-run investors' trades or share information about those trades, so the impact of information-sharing and front-running on investors occurs indirectly via changes in the behavior of equilibrium prices. In this sense, the fall in investors' returns represents *collateral* damage from dealer front-running.

The model shows that front-running by individual dealers has a market-wide impact because it creates an externality that affects how the information contained in external Forex orders becomes embedded in the prices dealers quote. In the absence of front-running, dealers trade in the wholesale tier to replenish their inventories after filling investor orders from earlier in the day. As in other models (e.g., Lyons, 1997, and Evans, 2011), the aggregate order flow produced by this inter-dealer trading contains information on the market-wide imbalance in investors' orders, which dealers then use to revise their price quotes. Front-running disrupts this process. When individual dealers learn about their future orders from hedgers, the information becomes an additional factor determining the trades they initiate with other dealers. This means that the aggregate order flow produced by inter-dealer trading now contains information on the imbalance in past external orders from investors and future orders from hedgers. Because dealers draw inferences from order flow about the price they should quote to share risk efficiently across the market, when the information conveyed by order flow changes, so too do dealer's inferences and the prices they quote *based on a given order flow*. Thus, front-running affects the determinants of aggregate order flow *and* its price-impact. These equilibrium effects feedback on dealers' decisions to front-run in the wholesale tier, and they affect how investors trade in the retail tier.

This paper contributes to the literature on the manipulation of securities prices; originating with Hart (1977), Vila (1989), and Allen and Gale (1992). Its closest antecedents in that literature appear in the work on dual and predatory trading.[3] Rochet and Vila (1994) examine a static dual trader model in which a monopolist trades on his own account and processes all the liquidity trades from retail customers. They show that the equilibrium does not depend on whether the monopolist sees the liquidity trades because the price-impact of order flow endogenously adjusts to changes in the monopolists' information. This irrelevance result counters the widespread intuition that dual trading must harm liquidity traders, as a monopolist will exploit information about liquidity trades that drive prices away from true asset values. However, the irrelevance result breaks down in a dynamic setting. Bernhardt and Taub (2008) show that a monopolist with knowledge of current and future liquidity trades can gain by front-running future liquidity trades. Predatory trading models examine situations where some traders become aware of another trader's need to liquidate a position. In Brunnermeier and Pedersen (2005), so-called predator traders sell ahead of or alongside the liquidating trader, before reversing their positions. Bessembinder et al. (2016) examine how this particular type of strategic trading depends on the price-impact of trades and on competition between the predators (see, also: Admati and Pfleiderer, 1991, Carlin, Lobo, and Viswanathan, 2007, and Schied and Schöneborn, 2009). My analysis contains elements of both the dual and predatory theories. Forex dealers act as dual traders insofar as they fill the external orders from customers in the retail tier and initiate trades with other dealers in the wholesale tier of the market. Dealers also could be viewed as engaging in predatory trading when they use the information on the pending external orders they receive from hedgers to trade strategically in the wholesale tier.

This paper is also related to research on Forex benchmarks, such as the WMR and ECB Fixes. Melvin and Prins (2015) drew attention to the fact that global portfolio managers have a strong

[3]Models of security price manipulation cover a wide range of topics; including manipulation in futures via corners and squeezes (see, e.g., Kumar and Seppi, 1992) and the manipulation of closing equity prices (see, e.g., Cushing and Madhavan, 2000, Hillion and Suominen, 2004, and Comerton-Forde and Putnיnš, 2011), but these models have limited applicability to Forex market. Manipulation via corners and squeezes is impractical for major currencies, while pump-and-dump schemes requiring the release of credible but false information that moves Forex prices are implausible. Evans (2018) discusses the differences between the manipulation of closing equity prices and Forex benchmarks.

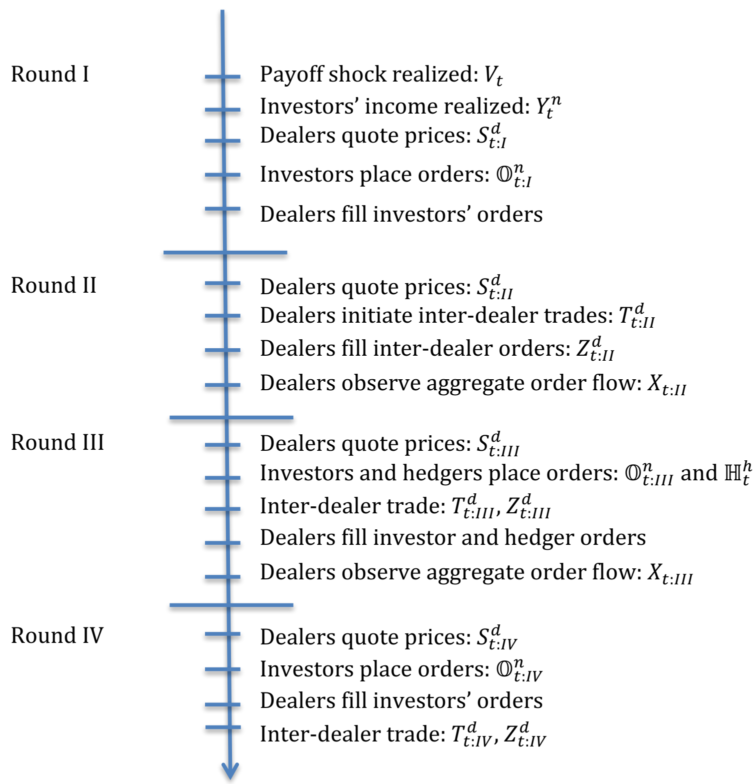
hedging incentive to submit orders that execute at the Fix price, particularly at the end of each month. According to the reports issued by government regulators, transcripts from electronic chatrooms show that a number of dealers collusively front-ran the Fix orders they received from portfolio managers and others. In Evans (2018), I found that Forex prices were unusually volatile around WMR 4:00 pm Fix, and that Forex returns were negatively correlated either side of 4:00 pm (see, also Ito and Yamada, 2015). Furthermore, these empirical findings were inconsistent with a competitive model of Fix trading. In that model, dealers do not front-run their Fix orders because the associated gain is offset by an endogenous change in the composition of the orders. In this paper, dealers can distinguish between customers who have price-insensitive reasons for trading (i.e., hedgers) and price-sensitive reasons (i.e., investors), and can front-run the price-insensitive orders. The front-running of selected customer orders is consistent with the evidence provided by government regulators and trial testimony. Osler and Turnbull (2017) also study dealer trading around the Fix in a model where trades are assumed to have a permanent price-impact that is proportional to the size of order flow. They emphasize that dealers trade strategically; front-running their own Fix orders and anticipating how the front-running by other dealers affects prices. Strategic trading also plays an important role in my model because it determines how aggregate order flows convey information that dealers use to quote prices. Unlike Osler and Turnbull (2017) and other strategic trading models, the price-impact of trade is determined endogenously in my model from dealers' optimal quotes.

The remainder of the paper is structured as follows: Section 1 presents the model. The next section examines the equilibrium where dealers have no opportunity to share information about or front-run their customer orders. This serves as a benchmark for the rest of the analysis. Section 3 introduces front-running. I first examine why dealers have an incentive to unilaterally front-run their own customer orders in the benchmark equilibrium. Next, I analyze the equilibrium effects of unilateral front-running by all dealers. Section 4 considers the effects of collusive front-running where dealers share information about pending external orders. Section 5 examines how information-sharing and front-running affect dealers' and investors' welfare. Section 6 concludes.

1 The Benchmark Model

I study the effects of dealer collusion and front-running in a standard microstructure model of OTC Forex trading. The overall structure of the model extends Evans (2011) with an additional round of trading and the inclusion of two types of Forex customers. This section presents the benchmark version of the model in which collusion and front-running are absent. In the following sections, I introduce dealer front-running and collusion and study how these activities affect the equilibrium.

Figure 1: Benchmark Model Timing



1.1 Overview

The model describes trading between a large number of dealers, investors, and hedgers over a trading day. There is one risky asset that represents Forex and one risk-free asset with a daily return of $1+r$. Market participants comprise a continuum of investors, a continuum of hedgers, and a finite number of Forex dealers. Both investors and dealers are risk-averse and choose their trades optimally, while hedgers trade for exogenous reasons. There are four rounds of trading each day; denoted as I, II, III and IV. When necessary, I use the notation $X_{t:j}$ to identify variable X in round $j = \{$I, II, III IV$\}$ on day t. Figure 1 shows the sequence of events.

At the start of round I on day t, public information arrives in the form of a payoff, $\mathcal{P}_t$, paid to the current holders of Forex. Each investor n also receives foreign income, Y_t^n. This is private information to each investor and provides their motive for trading Forex. Next, each dealer d simultaneously and independently quotes a scalar price, $S_{t:\text{I}}^d$, at which they will fill investors' orders to buy or sell Forex. Prices are observed by all dealers and investors and are good for orders of any size. Each investor n then places their optimally chosen order, $\mathbb{O}_{t:\text{I}}^n$, where positive (negative) values denote purchases (sales) of Forex. Orders may be placed with more than one dealer. If two or more dealers quote the same price, the order is randomly assigned among them. Dealers then fill the investors' orders they receive.

Round II is a round of inter-dealer trading. As above, each dealer d simultaneously and independently quotes a scalar price $S_{t:\text{II}}^d$ at which they will trade Forex with other dealers. Quoted prices are observed by all and are good for inter-dealer trades of any size. Each dealer then simultaneously and independently trades on the quotes. I denote the Forex orders made by dealer d as $T_{t:\text{II}}^d$ and orders received by dealer d as $Z_{t:\text{II}}^d$. When dealer d initiates a purchases (sale) of Forex, $T_{t:\text{II}}^d$ is positive (negative). Positive (negative) values of $Z_{t:\text{II}}^d$ denote purchases (sales) of Forex initiated by another dealer. Once again, trading with multiple dealers is feasible. If multiple dealers quote the same price, trades are allocated equally between them. At the end of round-II trading all dealers observe aggregate inter-dealer order flow, $X_{t:\text{II}} = \sum_{d=1}^{\text{D}} T_{t:\text{II}}^d$.

At the start of round III, dealers quote prices $S_{t:\text{III}}^d$ at which they will fill Forex orders from investors, hedgers and other dealers. Each investor n and hedger h then place their orders, $\mathbb{O}_{t:\text{III}}^n$ and $\mathbb{H}_{t:\text{III}}^h$, with dealers following the protocol in round I. Dealers then engage in another round of inter-dealer trading (as in round II) and fill their orders from investors and hedgers. At the end of the round, all dealers observe aggregate inter-dealer order flow, $X_{t:\text{III}} = \sum_{d=1}^{\text{D}} T_{t:\text{III}}^d$.

Finally, to begin round IV, all dealers quote prices $S_{t:\text{IV}}^d$ for investor and inter-dealer trades. Investors then place their orders $\mathbb{O}_{t:\text{IV}}^n$ with dealers. After dealers have filled these orders, they engage in inter-dealer trade as in rounds II and III.

Before describing the details of the model, a few comments concerning its structure are in order. In this benchmark configuration, the model describes the process through which inter-dealer trading aggregates the information contained in investor and hedgers' orders, and dealers use their observations on aggregate order flow to embed that information into their price quotes. Investors' orders in rounds I, III and IV convey information about foreign income, while hedgers' orders convey information about exogenous shocks to aggregate hedging demand. Inter-dealer trading in rounds II and III aggregates the information from earlier customer orders. In Section 3, I make one small modification to accommodate front-running: hedgers must now place their orders with dealers immediately after they observe round-II prices. This means that dealers have advanced information on the hedgers' orders they must fill in round III when they trade with other dealers in round II. One focal point of my analysis is on how dealers use this information. Notice, also, that dealers use their observations on aggregate orders flows from inter-dealer trading in rounds II and III to determine the prices they quote at the start of the next trading round. Thus, the price-impact of order flow is determined endogenously as part of the model's equilibrium.

1.2 Market Participants

1.2.1 Investors

There are a continuum of investors indexed by $n \in [0,1]$. Each investor chooses Forex orders on day t to maximize expected utility defined over wealth on day $t+1$:

$$\mathcal{U}^n_{t:j} = \mathrm{E}\big[-\omega \exp(-\omega W^n_{t+1:\mathrm{I}})|\Omega^n_{t:j}\big]\,, \tag{1}$$

with $\omega > 0$, where $W^n_{t+1:\mathrm{I}}$ is the wealth of investor n at the start of round I on day $t+1$, and $\Omega^n_{t:j}$ is the information available to the investor when making their round j trading decision. Investors receive two pieces of information in round I: public information on the Forex payoff and private information on their foreign income. The payoff follows a random walk with daily increments,

$$\mathcal{P}_t = \mathcal{P}_{t-1} + V_t, \qquad V_t \sim i.i.d.N(0, \sigma^2_{\mathrm{V}}), \tag{2}$$

while foreign income comprises an aggregate component Y_t and an idiosyncratic component ε^n_t,

$$Y^n_t = Y_t + \varepsilon^n_t \qquad \varepsilon^n_t \sim i.i.d.N(0, \sigma^2_{\varepsilon}). \tag{3}$$

Investors do not initially observe either income component but the value of Y_t is inferred from their observation of dealers' price quotes over the trading day. I denote the set of $d = 1, 2..\mathrm{D}$ price quotes in round j by $\{S^d_{t:j}\}$. Investors' information evolves as $\Omega^n_{t:\mathrm{I}} = \big\{\{S^d_{t:\mathrm{I}}\}, Y^n_t, \mathcal{P}_t, \Omega^n_{t-1:\mathrm{IV}}\big\}$ and $\Omega^n_{t:j} = \{\{S^d_{t:j}\}, \Omega^n_{t:j-1}\}$ for rounds $j = \{\mathrm{II}, \mathrm{III}, \mathrm{IV}\}$.

Investors place Forex orders with dealers at the start of rounds I, III and IV that maximize expected utility subject to the following sequence of budget constraints:

$$\begin{aligned} W^n_{t:\mathrm{IV}} &= A^n_{t:\mathrm{III}}\Delta S_{t:\mathrm{IV}} + A^n_{t:\mathrm{I}}\Delta S_{t:\mathrm{III}} + W^n_{t:\mathrm{I}} + S_{t:\mathrm{I}}Y^n_t, & \text{(4a)} \\ W^n_{t+1:\mathrm{I}} &= A^n_{t:\mathrm{IV}}R_{t+1} + (1+r)W^n_{t:\mathrm{IV}}, & \text{(4b)} \end{aligned}$$

where $\Delta S_{t:j} = S_{t:j} - S_{t:j-1}$ is the change in price between rounds $j-1$ and j, $R_{t+1} = S_{t+1:\mathrm{I}} + \mathcal{P}_{t+1} -$

$(1+r)S_{t:\text{IV}}$ is the overnight excess return on Forex, and $A^n_{t:j}$ is investor n's holding of Forex after trading in round j. In round I, investor n chooses an order of $\mathbb{O}^n_{t:\text{I}} = A^n_{t:\text{I}} - A^n_{t-1:\text{IV}} - Y^n_t$, where $A^n_{t:\text{I}}$ is their desired round-I position that maximizes $\mathcal{U}^n_{t:\text{I}}$ subject to (4) with information $\Omega^n_{t:\text{I}}$. In round III, the investor's order is $\mathbb{O}^n_{t:\text{III}} = A^n_{t:\text{III}} - A^n_{t:\text{I}}$, where their choice for $A^n_{t:\text{III}}$ maximizes $\mathcal{U}^n_{t:\text{III}}$ subject to (4) with information $\Omega^n_{t:\text{III}}$. Similarly, their round IV order is $\mathbb{O}^n_{t:\text{IV}} = A^n_{t:\text{IV}} - A^n_{t:\text{III}}$, where the choice for $A^n_{t:\text{IV}}$ maximizes $\mathcal{U}^n_{t:\text{IV}}$ subject to (4b) with information $\Omega^n_{t:\text{IV}}$. Thus, positive (negative) values for $\mathbb{O}^n_{t:j}$ represent investors' purchase (sale) orders of Forex because they facilitate an increase (decrease) in their desired Forex position. Notice, also, that the Forex orders placed in any round will generally differ across investors because they are determined by different information and existing positions. Despite this heterogeneity in investors' orders, it turns out that in equilibrium investors hold the entire stock of Forex overnight. I denote this stock by $A_t = \int_0^1 A^n_{t:\text{IV}} dn$.

1.2.2 Hedgers

Hedgers provide a source of external Forex orders that do not depend on Forex prices or other market conditions. There is a continuum of hedgers indexed by $h \in [0,1]$ who place Forex orders $\mathbb{H}^h_t$ at the start of round III. Individual hedger's orders comprise a common component $\mathbb{H}_t$ and an idiosyncratic component ξ^h_t. The common component depends on the stock of Forex held by investors overnight, and a random shock:

$$\mathbb{H}_t = (1-\psi)A_{t-1} + H_t \qquad H_t \sim i.i.d.N(0, \sigma^2_{\text{H}}), \tag{5}$$

with $0 < \psi < 1$. This specification implies that hedgers Forex orders on day t are exogenous with respect to Forex trading on day t. I assume that $\mathbb{H}_t$ depends on A_{t-1} for analytical convenience - it ensures that A_t follows a stationary AR(1) process in equilibrium which simplifies the analysis.

1.2.3 Dealers

Dealers play a central role in the model. Unlike investors and hedgers, there are D dealers in the market (indexed by d) that act strategically when choosing their price quotes and engaging in inter-dealer trade. These quote and trading decisions take the form of a multi-stage simultaneous move game. One stage of the game occurs at the start of each round when dealers must simultaneously quote prices, $\{S^d_{t:j}\}$. The other stage occurs in rounds II, III and IV when each dealer simultaneously initiates trades against other dealers' quotes. At each decision point on day t, dealer d chooses a quote or trade that maximizes expected utility defined over wealth on day $t+1$, given the equilibrium decisions of other dealers. The resulting quotes and trades identify a Bayesian-Nash Equilibrium (BNE).

Dealers receive three types of information during the trading day: (i) public information on the Forex payoff in round I, (ii) market-wide information on prices and aggregate inter-dealer order flows, and (iii) private information on the Forex orders they receive from investors, hedgers, and other dealers. It proves useful to distinguish between the information available to dealer d at the start of each round j, $\mathcal{I}^d_{t:j}$, and the information available when trading decisions are made, $\mathcal{F}^d_{t:j}$. During day t, these information sets evolve according to

$$\mathcal{I}^d_{t:\text{I}} = \left\{\mathcal{P}_t, \mathcal{F}^d_{t-1:\text{IV}}\right\}, \qquad \mathcal{F}^d_{t:\text{I}} = \{\{S^d_{t:\text{I}}\}, \mathcal{I}^d_{t:\text{I}}\}, \tag{6a}$$

$$\mathcal{I}^d_{t:\text{II}} = \left\{\mathbb{O}^d_{t:\text{I}}, \mathcal{F}^d_{t:\text{I}}\right\}, \qquad \mathcal{F}^d_{t:\text{II}} = \left\{\{S^d_{t:\text{II}}\}, \mathcal{I}^d_{t:\text{II}}\right\}, \tag{6b}$$

$$\mathcal{I}^d_{t:\text{III}} = \left\{X_{t:\text{II}}, Z^d_{t:\text{II}}, \mathcal{F}^d_{t:\text{II}}\right\}, \qquad \mathcal{F}^d_{t:\text{III}} = \left\{\mathbb{O}^d_{t:\text{III}}, \mathbb{H}^d_t, \{S^d_{t:\text{III}}\}, \mathcal{I}^d_{t:\text{III}}\right\}, \tag{6c}$$

$$\mathcal{I}^d_{t:\text{IV}} = \left\{X_{t:\text{III}}, Z^d_{t:\text{III}}, \mathcal{F}^d_{t:\text{III}}\right\}, \qquad \mathcal{F}^d_{t:\text{IV}} = \left\{\mathbb{O}^d_{t:\text{IV}}, \{S^d_{t:\text{IV}}\}, \mathcal{I}^d_{t:\text{IV}}\right\}. \tag{6d}$$

Each dealer d enters day t with information $\mathcal{F}^d_{t-1:\text{IV}}$ and observes the new Forex payoff $\mathcal{P}_t$, so they begin with information $\mathcal{I}^d_{t:\text{I}} = \{\mathcal{P}_t, \mathcal{F}^d_{t-1:\text{IV}}\}$ when quoting round-I prices. After round-I prices have been quoted, dealer d's information is $\mathcal{F}^d_{t:\text{I}} = \{\{S^d_{t:\text{I}}\}, \mathcal{I}^d_{t:\text{I}}\}$. By the start of round II, the dealer has additional information on the investors' orders received in round I, which are denoted by $\mathbb{O}^d_{t:\text{I}}$, so $\mathcal{I}^d_{t:\text{II}} = \{\mathbb{O}^d_{t:\text{I}}, \mathcal{F}^d_{t:\text{I}}\}$. This information is supplemented by dealers' round-II quotes $\{S^d_{t:\text{II}}\}$ before the dealer trades in round II using $\mathcal{F}^d_{t:\text{II}} = \{\{S^d_{t:\text{II}}\}, \mathcal{I}^d_{t:\text{II}}\}$. By the start of round III, the dealer knows the orders he received from other dealers during round II, $Z^d_{t:\text{II}}$, and aggregate order flow, $X_{t:\text{II}}$, so these variables appear in $\mathcal{I}^d_{t:\text{III}}$. Next, dealers quote round III prices and receive orders from investors and hedgers, $\mathbb{O}^d_{t:\text{III}}$ and $\mathbb{H}^d_t$, so these variables add to the information used in making round-III trading decisions $\mathcal{F}^d_{t:\text{III}}$. By the start of round IV, each dealer knows the orders they received from other dealers and aggregate order flow from round III, so these variables appear in $\mathcal{I}^d_{t:\text{IV}}$. Finally, by the time dealers trade in round IV, they know prices $\{S^d_{t:\text{IV}}\}$ and investors' orders, $\mathbb{O}^d_{t:\text{IV}}$.

Each dealer d quotes prices and makes trading decisions to maximize expected utility

$$U^d_{t:j} = \mathrm{E}\Big[-\omega \exp(-\omega W^d_{t+1:\text{I}})|\Omega^d_{t:j}\Big], \tag{7}$$

where $W^d_{t+1:\text{I}}$ is the wealth of dealer d at the start of round I on day $t+1$, and $\Omega^d_{t:j}$ is the information available to the dealer when making the decision (i.e., $\Omega^d_{t:j} = \mathcal{I}^d_{t:j}$ when the dealer quotes prices and $\Omega^d_{t:j} = \mathcal{F}^d_{t:j}$ when the dealer initiates inter-dealer trades). Because all price quotes are observable to market participants and are good for any amount, the BNE strategy for each dealer is to quote a common price at the start of each trading round (i.e., $S^d_{t:j} = S_{t:j}$ for $d = 1, 2, ..\text{D}$). Below I show how this price depends on dealers' common information, $\mathcal{I}_{t:j} = \cap_d \mathcal{I}^d_{t:j}$. The BNE strategy for dealers' inter-dealer trades in rounds II, III and IV take the form:

$$T^d_{t:\text{II}} = \wp^d_{t:\text{II}} + \mathrm{E}\left[Z^d_{t:\text{II}}|\mathcal{F}^d_{t:\text{II}}\right] - A^d_{t:\text{I}}, \tag{8a}$$

$$T^d_{t:\text{III}} = \wp^d_{t:\text{III}} + \mathrm{E}\left[Z^d_{t:\text{III}}|\mathcal{F}^d_{t:\text{III}}\right] + \mathbb{H}^d_t + \mathbb{O}^d_{t:\text{III}} - A^d_{t:\text{II}}, \qquad \text{and} \tag{8b}$$

$$T^d_{t:\text{IV}} = \wp^d_{t:\text{IV}} + \mathrm{E}\left[Z^d_{t:\text{IV}}|\mathcal{F}^d_{t:\text{IV}}\right] + \mathbb{O}^d_{t:\text{IV}} - A^d_{t:\text{III}}, \tag{8c}$$

where $\wp^d_{t:j}$, denotes the dealer's desired position in round j, and $A^d_{t:j}$ is the dealer's actual Forex holding at the end of round j. In words, the dealer's BNE strategy is to initiate trades that achieve their desired position net the orders from investors, hedgers and other dealers they expect to fill. In rounds III and IV dealers condition their trades on the orders from investors $\mathbb{O}^d_{t:j}$ and hedgers $\mathbb{H}^d_t$. In contrast, dealers cannot condition their trades on the orders from other dealers, $Z^d_{t:j}$, because inter-dealer trading decisions are made simultaneously. Instead, their BNE strategy is based on expected orders, $\mathrm{E}[Z^d_{t:j}|\mathcal{F}^d_{t:j}]$. As a consequence, dealers actual end-of-round positions are $A^d_{t:j} = \wp^d_{t:j} - \xi^d_{t:j}$, where $\xi^d_{t:j} = Z^d_{t:j} - \mathrm{E}[Z^d_{t:j}|\mathcal{F}^d_{t:j}]$ is the error in forecasting the incoming orders from other dealers.

Dealers choose their desire positions $\wp^d_{t:j}$ to maximize expected utility subject to the sequence of the budget constraints:

$$W^d_{t:\text{II}} = W^{\text{I}}_{d,t} + A^d_{t:\text{I}}\Delta S_{t:\text{II}}, \tag{9a}$$

$$W^d_{t:\text{III}} = W^d_{t:\text{II}} + (\wp^d_{t:\text{II}} - \xi^d_{t:\text{II}})\Delta S_{t:\text{III}}, \tag{9b}$$

$$W^d_{t:\text{IV}} = W^d_{t:\text{III}} + (\wp^d_{t:\text{III}} - \xi^d_{t:\text{III}})\Delta S_{t:\text{IV}}, \qquad \text{and} \tag{9c}$$

$$W^d_{t+1:\text{I}} = (1+r)W^d_{t:\text{IV}} + (\wp^d_{t:\text{IV}} - \xi^d_{t:\text{IV}})R_{t+1}, \tag{9d}$$

where $A^d_{t:\text{I}} = A^d_{t-1:\text{IV}} - \mathbb{O}^d_{t:\text{I}}$. In round II, dealer d chooses $\wp^d_{t:\text{II}}$ to maximize $\mathcal{U}^d_{t:\text{II}}$ subject to (9) with information $\mathcal{F}^d_{t:\text{II}}$. To implement this choice, they initiate inter-dealer trades $T^d_{t:\text{II}}$ identified in (8a). Similarly, dealer d chooses $\wp^d_{t:\text{III}}$ to maximize $\mathcal{U}^d_{t:\text{III}}$ with information $\mathcal{F}^d_{t:\text{III}}$, and $\wp^d_{t:\text{IV}}$ to maximize $\mathcal{U}^d_{t:\text{IV}}$ with information $\mathcal{F}^d_{t:\text{IV}}$ both subject to (9), and these choices are implemented by trades $T^d_{t:\text{III}}$ and $T^d_{t:\text{IV}}$ identified in (8b) and (8c). Notice that all of these decisions take account of the fact that the orders from other dealers are in general unpredictable, so the associated forecast errors $\xi^d_{t:j}$ represent an additional source of risk.

1.3 Market Clearing Conditions

There are three sets of market clearing conditions to consider: those for investors' orders, hedgers' orders, and inter-dealer trading. In round I, the aggregate imbalance in investors' orders, $\mathbb{O}_{t:\text{I}}$ defined by the integral on the left, must equal the sum of the orders received by the D dealers, on the right.

$$\int_0^1 (A^n_{t:\text{I}} - A^n_{t-1:\text{IV}} - Y^n_{t:\text{I}})dn = \mathbb{O}_{t:\text{I}} = \sum\nolimits_{d=1}^{\text{D}} \mathbb{O}^d_{t:\text{I}}. \tag{10}$$

Furthermore, in the BNE where all dealers quote the same prices, investors' orders are randomly assigned across dealers. I therefore assume that the order received by dealer d is $\mathbb{O}^d_{t:\text{I}} = \frac{1}{\text{D}}\mathbb{O}_{t:\text{I}} + \zeta^d_{t:\text{I}}$ with $\zeta^d_{t:\text{I}} \sim i.d.N(0, \sigma^2_\zeta)$. The allocation shocks $\zeta^d_{t:\text{I}}$ are negatively correlated across the dealers, with correlation $\rho = -(\text{D}-1)^{-1}$, so that $\sum_{d=1}^{\text{D}} \zeta^d_{t:\text{I}} = 0$, as required by market clearing.

In rounds III and IV, market clearing similarly requires that

$$\int_0^1 (A^n_{t:\text{III}} - A^n_{t:\text{I}})dn = \mathbb{O}_{t:\text{III}} = \sum\nolimits_{d=1}^{\text{D}} \mathbb{O}^d_{t:\text{III}}, \tag{11a}$$

$$\int_0^1 (A^n_{t:\text{IV}} - A^n_{t:\text{III}})dn = \mathbb{O}_{t:\text{IV}} = \sum\nolimits_{d=1}^{\text{D}} \mathbb{O}^d_{t:\text{IV}} \tag{11b}$$

The orders received by individual dealers in rounds III and IV are subject to mean-zero normally distributed allocation shocks, $\zeta^d_{t:\text{III}}$ and $\zeta^d_{t:\text{IV}}$, respectively. The market clearing condition for round-III hedge orders is given by

$$\int_0^1 \mathbb{H}^h_t dh = \mathbb{H}_t = \sum\nolimits_{d=1}^{\text{D}} \mathbb{H}^d_t \tag{12}$$

As above, the hedge orders received by dealer d are $\mathbb{H}^d_t = \mathbb{H}_t + \eta^d_t$ with $\eta^d_t \sim i.d.N(0, \sigma^2_\eta)$, where η^d_t is an allocation shock which is negatively correlated across dealers so that $\sum_{d=1}^{\text{D}} \eta^d_t = 0$.

The market clearing condition for inter-dealer trades is given by

$$\sum\nolimits_{d=1}^{\text{D}} Z^d_{t:j} = \sum\nolimits_{d=1}^{\text{D}} T^d_{t:j} = X_{t:j} \qquad \text{for} \quad j = \{\text{II,III,IV}\}. \tag{13}$$

Here, the sum of orders received by dealers on the left equals the total imbalance in orders initiated by dealers, which in turn defines aggregate inter-dealer order flow. Under the trading protocol, orders are equally split between dealers quoting the same price, so in the BNE each dealer receives a equal fraction of aggregate order flow: $Z^d_{t:j} = \frac{1}{\text{D}} X_{t:j}$.

In all the equilibria I study, investors hold the entire stock of Forex overnight. Daily changes in this stock, A_t, must therefore reflect differences between aggregate foreign income received by investors and the aggregate imbalance in Forex orders by hedgers: $A_t = A_{t-1} + Y_t - \mathbb{H}_t$. Combining this expression with equation (5) gives

$$A_t = \psi A_{t-1} + Y_t - H_t. \tag{14}$$

Since both Y_t and H_t are mean-zero random variables, the Forex stock follows a stationary AR(1) process. Forex prices are ultimately driven by this process and the random walk for the Forex payoff in (2).

2 The Benchmark Equilibrium

An equilibrium in the benchmark model comprises: (i) the BNE strategies that identify the prices dealers quote at the start of each trading round, (ii) the inter-dealer trades initiate by dealers in rounds II, III and IV, and (iii) the set of investors' orders in rounds I, III and IV. All of these trading decisions must be optimal in the sense that they maximize the expected utility of the respective agent based on the available information and they must be consistent with market clearing given the exogenously determined orders of hedgers. I further restrict my attention to efficient risk-sharing equilibria in which investors and hedgers hold the entire stock of Forex overnight. The focus on such equilibria is standard (see, e.g., Lyons, 1997, Evans and Lyons, 2002, and Evans, 2011), and is consistent with the empirical fact that dealers generally do not hold open positions overnight and the half-lives of the intraday positions are measured in minutes (see, e.g., Lyons, 1995, and Bjønnes and Rime, 2005).

The following theorem describes the benchmark equilibrium.[4]

Theorem 1. *In an efficient risk-sharing equilibrium, overnight returns and intraday price changes follow*

$$R_t \equiv S_{t:\text{I}} + \mathcal{P}_t - (1+r)S_{t-1:\text{IV}} = \Lambda_{\text{I}} A_{t-1} + \tfrac{1+r}{r} V_t, \tag{15a}$$

$$\Delta S_{t:\text{II}} \equiv S_{t:\text{II}} - S_{t:\text{I}} = 0, \tag{15b}$$

$$\Delta S_{t:\text{III}} \equiv S_{t:\text{III}} - S_{t:\text{II}} = \Lambda_{\text{III}} A_{t-1} - \tfrac{1}{1-\psi+r}\Lambda Y_t - \tfrac{1}{\psi}\Lambda_{\text{IV}} Y_t, \qquad \textit{and} \tag{15c}$$

$$\Delta S_{t:\text{IV}} \equiv S_{t:\text{IV}} - S_{t:\text{III}} = \Lambda_{\text{IV}} A_{t-1} + \tfrac{1}{\psi}\Lambda_{\text{IV}} Y_t + \tfrac{1}{1-\psi+r}\Lambda H_t, \tag{15d}$$

where $\Lambda = \Lambda_{\text{I}} + \Lambda_{\text{III}} + \Lambda_{\text{IV}}$ *for some coefficients* Λ_i. *Investors' orders in rounds* I *and* III *are given by*

$$\mathbb{O}^n_{t:\text{I}} = \beta^{\text{Y}}_{\text{I}} Y^n_t + \beta^{\text{A}}_{\text{I}} A_{t-1}, \qquad \textit{and} \tag{16a}$$

$$\mathbb{O}^n_{t:\text{III}} = \beta^{\text{Y}}_{\text{III}} Y_t + \beta^{\text{A}}_{\text{III}} A_{t-1}, \tag{16b}$$

[4] Mathematical derivations and proofs are in the Appendix.

for some coefficients β_j. Unexpected order flows from inter-dealer trading in rounds II *and* III *are*

$$X_{t:\text{II}} - \mathrm{E}[X_{t:\text{II}}|\mathcal{I}_{t:\text{II}}] = \Gamma^{\text{Y}} Y_t, \qquad \textit{and} \tag{17a}$$

$$X_{t:\text{III}} - \mathrm{E}[X_{t:\text{III}}|\mathcal{I}_{t:\text{III}}] = \Gamma^{\text{H}} H_t, \tag{17b}$$

for some coefficients Γ^{Y} and Γ^{H}, where $\mathcal{I}_{t:j} = \cap_d \mathcal{I}^d_{t:j}$ denote dealers' common information at the start of round j.

2.1 Qualitative Analysis

To explain the economic intuition behind this equilibrium, it is useful to start with the determination of round IV prices. Efficient risk-sharing requires that the outstanding stock of Forex is held by the continuum of investors rather than any of the D dealers at the end of the day. In equilibrium, dealers have sufficient information to find this price by inverting investors' aggregate demand for Forex to give

$$\begin{aligned} S_{t:\text{IV}} &= \sum_{i=1}^{\infty} \left(\tfrac{1}{1+r}\right)^i \mathrm{E}[\mathcal{P}_{t+i} - \Lambda A_{t+i-1}|\mathcal{I}_{t:\text{IV}}], \\ &= \tfrac{1}{r}\mathcal{P}_t - \tfrac{1}{1-\psi+r}\Lambda A_t, \end{aligned} \tag{18}$$

where $\Lambda = \Lambda_{\text{I}} + \Lambda_{\text{III}} + \Lambda_{\text{IV}}$. The second line in this expression follows from the processes for $\mathcal{P}_t$ and A_t in (2) and (14). The Λ_i coefficients identify equilibrium expected returns and price changes: $\mathrm{E}\left[R_t|\mathcal{I}_{t-1:\text{IV}}\right] = \Lambda_{\text{I}} A_{t-1}$, $\mathrm{E}\left[\Delta S_{t:\text{III}}|\mathcal{I}_{t:\text{I}}\right] = \Lambda_{\text{III}} A_{t-1}$, and $\mathrm{E}\left[\Delta S_{t:\text{IV}}|\mathcal{I}_{t:\text{I}}\right] = \Lambda_{\text{IV}} A_{t-1}$. Equation (18) shows that dealers quote a round-IV price equal to the present value of future payoffs adjusted for the risk premia necessary to ensure efficient current and future risk-sharing.

The evolution of prices, investors orders and aggregate order flow shown in Theorem 1 represent the process through which dealers acquire the information needed to quote $S_{t:\text{IV}}$. Dealers observe the value of $\mathcal{P}_t$ directly, and the information is immediately incorporated into their round I quote, as is reflected in (15a). In contrast, dealers become informed about the value of A_t $(= \psi A_{t-1} + Y_t - H_t)$ via trading. In particular, dealers learn about foreign income Y_t through a two-stage process. In the first stage dealers receive information via investors' round-I orders because they (optimally) depend on each investors' income, $Y^n_t = Y_t + \varepsilon^n_t$, as shown in (16a). Because dealers use the orders they received in round I as the basis for their inter-dealer trades in round II, in the second stage the information originally contained in investors' orders is revealed to dealers by their observation on aggregate order flow $X_{t:\text{II}}$. Equation (17a) shows that unexpected order flow is proportional to foreign income Y_t. Dealers incorporate this information into their round III quotes, so Y_t contributes to $\Delta S_{t:\text{III}}$, as shown by (15c). Dealers learn about the hedging shock H_t through an analogous process. The inter-dealer trades initiated in round III depend, in part, on the hedgers' orders each dealer receives at the start of the round and so carry information about H_t. This information is aggregated and revealed to all dealers by order flow $X_{t:\text{III}}$, as is shown by (17b). In sum, inter-dealer trading in rounds II and III aggregates the information about foreign income and the hedging shock that is initially conveyed to dealers in dispersed form by investors' round-I orders, and hedgers' round-III orders. Dealers then embed this information into the prices they quote in rounds III and IV, so as to achieve an efficient risk-sharing allocation by the end of the day.

In anticipation of the analysis below, it proves useful to examine two features of this process in greater detail. The first concerns the dependency of investors' round-I orders on their foreign income, Y_t^n. Investors choose these orders optimally, so it is worthwhile examining how and why these choices depend on Y_t^n. The second feature concerns the link between dealers' trading decisions in rounds II and III and aggregate order flows, $X_{t:\text{II}}$ and $X_{t:\text{III}}$. Again, inter-dealer trades are chosen optimally (as part of each dealer's BNE strategy), so it is useful to understand why the values of $X_{t:\text{II}}$ and $X_{t:\text{III}}$ induced by these trades convey information on Y_t and H_t.

To begin, consider the equations that describe investor n's optimal Forex holdings:

$$A^n_{t:\text{I}} = \Theta^n_{\text{I}|s_{\text{III}}}\text{E}[S_{t:\text{III}} - S_{t:\text{I}}|\Omega^n_{t:\text{I}}] + \Theta^n_{\text{I}|s_{\text{IV}}}\text{E}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{I}}] + \Theta^n_{\text{I}|\text{R}}\text{E}[R_{t+1}|\Omega^n_{t:\text{I}}], \tag{19a}$$

$$A^n_{t:\text{III}} = \Theta^n_{\text{III}|s_{\text{IV}}}\text{E}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{III}}] + \Theta^n_{\text{III}|\text{R}}\text{E}[R_{t+1}|\Omega^n_{t:\text{III}}], \qquad \text{and} \tag{19b}$$

$$A^n_{t:\text{IV}} = \Theta^n_{\text{IV}|\text{R}}\text{E}[R_{t+1}|\Omega^n_{t:\text{IV}}], \tag{19c}$$

where the coefficients, $\Theta^n_{\text{I}|j}$, $\Theta^n_{\text{III}|j}$ and $\Theta^n_{\text{IV}|j}$ depend on the conditional second moments of $\Delta S_{t:\text{III}}$, $\Delta S_{t:\text{IV}}$ and R_{t+1}. Here we see that the investor's optimal Forex holdings depend linearly on expected future intraday changes in prices and overnight returns. As in standard mean-variance portfolio problems, the round-IV choice is proportional to the expected overnight return $\text{E}[R_{t+1}|\Omega^n_{t:\text{IV}}]$. In this instance the coefficient $\Theta^n_{\text{IV}|\text{R}}$ equals $1/(\omega\mathbb{V}[R_{t+1}|\Omega^n_{t:\text{IV}}])$, were $\mathbb{V}[.|.]$ denotes the conditional variance. In the earlier rounds, the investor faces a more complex problem of exploiting expected price changes in the near term while hedging against future risks. These hedging motives make $A^n_{t:\text{I}}$ and $A^n_{t:\text{III}}$ dependent on the expected overnight return $\text{E}[R_{t+1}|\Omega^n_{t:}]$, and $A^n_{t:\text{I}}$ dependent on $\text{E}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{I}}]$.

By definition, the round-I order of investor n is $\mathbb{O}^n_{t:\text{I}} \equiv A^n_{t:\text{I}} - A^n_{t-1:\text{IV}} - Y^n_t$. Combing this expression with (19a) and the equilibrium price dynamics in (15) gives

$$\begin{aligned}\mathbb{O}^n_{t:\text{I}} \quad &= \quad \left\{\tfrac{1}{\psi}\Lambda_{\text{IV}}(\Theta^n_{\text{I}|s_{\text{IV}}} - \Theta^n_{\text{I}|s_{\text{III}}}) - \tfrac{1}{1-\psi+r}\Lambda\Theta^n_{\text{I}|s_{\text{III}}} + \Theta^n_{\text{I}|\text{R}}\Lambda_{\text{I}}\right\}\text{E}[Y_t|\Omega^n_{t:\text{I}}] \\ &\quad + \left\{\Theta^n_{\text{I}|s_{\text{III}}}\Lambda_{\text{III}} + \Theta^n_{\text{I}|s_{\text{IV}}}\Lambda_{\text{IV}} + \psi\Theta^n_{\text{I}|\text{R}}\Lambda_{\text{I}} - 1\right\}A_{t-1} - Y^n_t.\end{aligned} \tag{20}$$

The first term on the right-hand-side identifies the impact of private information on $A^n_{t:\text{I}}$. Because each investors' income comprises an aggregate and idiosyncratic component, their estimate of of Y_t is given by

$$\mathrm{E}[Y_t|\Omega^n_{t:\text{I}}] = \mathcal{G}^n_{\text{Y}} Y^n_t, \qquad \text{with} \qquad \mathcal{G}^n_{\text{Y}} = \frac{\sigma^2_{\text{Y}}}{\sigma^2_{\text{Y}} + \sigma^2_{\varepsilon}}.$$

Thus, foreign income Y^n_t has both an indirect speculated effect on the investor's order via the first term in (20), and a direct effect via the last term. The strength of the speculative effect depends on the precision of the information as measured by the gain coefficient $\mathcal{G}^n_{\text{Y}}$. It also depends on the investor's risk aversion and the "riskiness" of future returns via the $\Theta^n_{\text{I}|j}$ coefficients. So, taken together, the size of the speculative effect depends on the precision of the information in Y^n_t and the willingness of the investor to speculate on the information. This endogenous link between foreign income and investors' orders plays an important role in the analysis below.

Next, I consider the link between aggregate order flow and inter-dealer trades in round II. In equilibrium dealers do not hold overnight Forex positions, so their round II trades from (8a) become $T^d_{t:\text{II}} = \wp^d_{t:\text{II}} + \mathrm{E}\left[Z^d_{t:\text{II}}|\Omega^d_{t:\text{II}}\right] + \mathbb{O}^d_{t:\text{I}}$, where $\mathbb{O}^d_{t:\text{I}}$ denotes the investors' orders received by dealer d in round I. Furthermore, because dealers quote the same round-II price, they expect to receive an equal fraction of aggregate order flow from other dealers: $\mathrm{E}\left[Z^d_{t:\text{II}}|\Omega^d_{t:\text{II}}\right] = \frac{1}{\text{D}}\mathrm{E}\left[X_{t:\text{II}}|\Omega^d_{t:\text{II}}\right]$. Combining these expressions with the definition of aggregate order flow, gives

$$X_{t:\text{II}} = \sum\nolimits_{d=1}^{\text{D}} \wp^d_{t:\text{II}} + \tfrac{1}{\text{D}} \sum\nolimits_{d=1}^{\text{D}} \mathrm{E}[X_{t:\text{II}}|\Omega^d_{t:\text{II}}] + \mathbb{O}_{t:\text{I}}, \tag{21}$$

where $\mathbb{O}_{t:\text{I}} = \sum_{d=1}^{\text{D}} \mathbb{O}^d_{t:\text{I}}$ is the aggregate imbalance in the investors' round-I orders. This imbalance contains information concerning Y_t because it aggregates investors' orders in (16a): $\mathbb{O}_{t:\text{I}} = \int_0^1 \left\{\beta^{\text{Y}}_{\text{I}} Y^n_{t:\text{I}} + \beta^{\text{A}}_{\text{I}} A_{t-1}\right\} dn = \beta^{\text{Y}}_{\text{I}} Y_t + \beta^{\text{A}}_{\text{I}} A_{t-1}$. Thus, order flow aggregates information on foreign income because each dealer follows a BNE trading strategy that conditions on a subset of investors' round-I orders that individually contain dispersed information on Y_t.

Equation (21) identifies two further channels of information aggregation. The first operates through the aggregation of dealers' positions, $\sum_{d=1}^{\text{D}} \wp^d_{t:\text{II}}$. Each dealer's desired position can be written as

$$\wp^d_{t:\text{II}} = \Phi^d_{\text{II}|s_{\text{III}}} \mathrm{E}[\Delta S_{t:\text{III}}|\Omega^d_{t:\text{II}}] + \Phi^d_{\text{II}|s_{\text{IV}}} \mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^d_{t:\text{II}}], \qquad \text{and} \tag{22a}$$

$$\wp^d_{t:\text{III}} = \Phi^d_{\text{III}|s_{\text{IV}}} \mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^d_{t:\text{III}}], \tag{22b}$$

where the coefficients $\Phi^d_{\text{II}|j}$ and $\Phi^d_{\text{III}|j}$ are determined by the conditional second moments of $\Delta S_{t:\text{III}}$, $\Delta S_{t:\text{IV}}$ and unexpected inter-dealer orders. Notice that each dealers' round-II position, $\wp^d_{t:\text{II}}$ depends on their private forecasts for $\Delta S_{t:\text{III}}$ and $\Delta S_{t:\text{IV}}$. In equilibrium, these forecasts depend on the dealer's estimate of Y_t:

$$\begin{aligned}\mathrm{E}[\Delta S_{t:\text{III}}|\Omega^d_{t:\text{II}}] &= \Lambda_{\text{III}}A_{t-1} - (\tfrac{1}{1-\psi+r}\Lambda + \tfrac{1}{\psi}\Lambda_{\text{IV}})\mathrm{E}[Y_t|\Omega^d_{t:\text{II}}], \qquad \text{and} \\ \mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^d_{t:\text{II}}] &= \Lambda_{\text{IV}}A_{t-1} + \tfrac{1}{\psi}\Lambda_{\text{IV}}\mathrm{E}[Y_t|\Omega^d_{t:\text{II}}].\end{aligned}$$

Because dealers quote the same round-I price, they receive a random fraction of investors orders, $\mathbb{O}^d_{t:\text{I}} = \frac{1}{\text{D}}\mathbb{O}_{t:\text{I}} + \zeta^d_{t:\text{I}}$, which represent a noisy signal on Y_t. Each dealer can therefore estimate Y_t based on these orders as

$$\mathrm{E}[Y_t|\Omega^d_{t:\text{II}}] = \mathcal{G}^d_{\text{Y}}(\mathbb{O}^d_{t:\text{I}} - \mathrm{E}[\mathbb{O}^d_{t:\text{I}}|\mathcal{I}^d_{t:\text{I}}]) = \mathcal{G}^d_{\text{Y}}(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}Y_t + \zeta^d_{t:\text{I}}), \qquad \text{with} \qquad \mathcal{G}^d_{\text{Y}} = \frac{\frac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}\sigma^2_{\text{Y}}}{(\frac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}})^2\sigma^2_{\text{Y}} + \sigma^2_{\zeta}}.$$

Consequently, round-II order flow aggregates the private information dealers use to determine their desired position via the $\sum_{d=1}^{\text{D}} \wp^d_{t:\text{II}}$ term in (21). The remaining channel of information aggregate operates through dealers' expectations of incoming orders, $\sum_{d=1}^{\text{D}} \mathrm{E}\left[Z^d_{t:\text{II}}|\Omega^d_{t:\text{II}}\right] = \frac{1}{\text{D}}\sum_{d=1}^{\text{D}} \mathrm{E}[X_{t:\text{II}}|\Omega^d_{t:\text{II}}]$. In this case dealers use their private estimates of income to compute $\mathrm{E}\left[X_{t:\text{II}}|\Omega^d_{t:\text{II}}\right] = \Gamma^{\text{Y}}\mathrm{E}[Y_t|\Omega^d_{t:\text{II}}]$, which forms part of the BNE trading strategy.

Round-III order flow aggregates information in an analogous manner. In this case the counterpart to equation (21) is

$$X_{t:\text{III}} = \sum\nolimits_{d=1}^{\text{D}} \wp^d_{t:\text{III}} + \tfrac{1}{\text{D}}\sum\nolimits_{d=1}^{\text{D}} \mathrm{E}[X_{t:\text{III}}|\Omega^d_{t:\text{III}}] + \mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}}. \tag{23}$$

The last three terms equal the difference between the aggregate imbalance in round-III orders by investors and hedgers $\mathbb{H}_t + \mathbb{O}_{t:\text{III}} = \sum_{d=1}^{\text{D}}(\mathbb{H}^d_t + \mathbb{O}^d_{t:1ii})$, and existing dealer holdings $\sum_{d=1}^{\text{D}} A^d_{t:\text{II}}$, which equal $-\mathbb{O}_{t:\text{I}}$ by market clearing. The aggregate imbalance in hedge orders aggregates information on the H_t shock directly, because $\mathbb{H}_t = (1-\psi)A_{t-1} + H_t$. By contrast, the aggregate imbalance in investors' orders, $\mathbb{O}_{t:\text{III}}$, carries no new information because Y_t was revealed to dealers by $X_{t:\text{II}}$. As in round II, order flow also aggregates the private information dealers use to forecast returns and incoming orders. In this case, hedge orders are the source of dealers' private information, producing estimates of H_t:

$$\mathrm{E}[H_t|\Omega^d_{t:\text{III}}] = \mathcal{G}^d_{\text{H}}(\mathbb{H}^d_t - \mathrm{E}[\mathbb{H}^d_t|\mathcal{I}^d_{t:\text{III}}]) = \mathcal{G}^d_{\text{H}}(\tfrac{1}{\text{D}}H_t + \eta^d_t), \qquad \text{wih} \qquad \mathcal{G}^d_{\text{H}} = \frac{\frac{1}{\text{D}}\sigma^2_{\text{H}}}{(\frac{1}{\text{D}})^2\sigma^2_{\text{H}} + \sigma^2_{\eta}}.$$

These estimates provide dealers' forecasts of $\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^d_{t:\text{III}}]$ used to determine desired positions $\wp^d_{t:\text{III}}$, and their forecasts $\mathrm{E}\left[X^d_{t:\text{III}}|\Omega^d_{t:\text{III}}\right]$.

One further feature of the benchmark equilibrium deserves comment: the origin of the A_{t-1}

terms in (15). Under efficient risk sharing, investors hold the available stock of Forex overnight, so $A_t = \int_0^1 A_t^h dn = \int_0^1 \Theta^n_{\text{IV}|\text{R}} \mathrm{E}[R_{t+1}|\Omega^n_{t:\text{IV}}]dn$. In equilibrium, investors have common forecasts of returns (i.e., $\mathrm{E}[R_{t+1}|\Omega^n_{t:\text{IV}}] = \mathrm{E}[R_{t+1}|\mathcal{I}_{t:\text{IV}}]$ for all n), so this condition pins down the overnight risk premium as

$$\mathrm{E}[R_{t+1}|\mathcal{I}_{t:\text{IV}}] = \Lambda_{\text{I}} A_t, \qquad \text{with} \qquad \Lambda_{\text{I}} = \left\{ \int_0^1 \Theta^n_{\text{IV}|\text{R}} dn \right\}^{-1}. \tag{24}$$

The intraday risk premia are similarly determined by market clearing. In particular, (21) and (23) imply that $\sum_{d=1}^{\text{D}} \mathrm{E}[\wp^d_{t:\text{II}}|\mathcal{I}_{t:\text{I}}] = -\mathrm{E}[\mathbb{O}_{t:\text{I}}|\mathcal{I}_{t:\text{I}}]$ and $\sum_{d=1}^{\text{D}} \mathrm{E}\left[\wp^d_{t:\text{III}}|\mathcal{I}_{t:\text{I}}\right] = -\mathrm{E}[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}}|\mathcal{I}_{t:\text{I}}]$. The left-hand-side of these equations identify the expected dealer demand to hold Forex in rounds II and III, while the right-hand-side identifies the available supply from the external orders from investors and hedgers. When combined with (19) and (22), these expressions imply that

$$\mathrm{E}[\Delta S_{t:\text{III}}|\mathcal{I}_{t:\text{I}}] = \Lambda_{\text{III}} A_{t-1}, \qquad \Lambda_{\text{III}} = \frac{1 - \psi\Theta_{\text{I}|R}\Lambda_{\text{I}}}{\Phi_{\text{II}|s_{\text{III}}} + \Theta_{\text{I}|s_{\text{III}}}} - \frac{\Phi_{\text{II}|s_{\text{IV}}} + \Theta_{\text{I}|s_{\text{IV}}}}{\Phi_{\text{II}|s_{\text{III}}} + \Theta_{\text{I}|s_{\text{III}}}} \Lambda_{\text{IV}}, \tag{25a}$$

$$\mathrm{E}[\Delta S_{t:\text{IV}}|\mathcal{I}_{t:\text{I}}] = \Lambda_{\text{IV}} A_{t-1}, \qquad \Lambda_{\text{IV}} = \frac{\psi(1 - \Theta_{\text{III}|R}\Lambda_{\text{I}})}{\Phi_{\text{III}|s_{\text{IV}}} + \Theta_{\text{III}|s_{\text{IV}}}}, \tag{25b}$$

where $\Phi_{j|i} = \sum_{d=1}^{\text{D}} \Phi^d_{j|i}$ and $\Theta_{i|j} = \int_0^1 \Theta^n_{i|j} dn$. Notice that the intraday premia depend on both dealers' and investors' preferences via the $\Phi_{j|i}$ and $\Theta_{i|j}$ terms, whereas the overnight premia reflect investors' preferences alone. In general, the intraday risk premia are smaller when there are a larger number of dealers available to share the (ex ante) risks of open intraday positions.

2.2 Quantitative Analysis

The equations describing the benchmark equilibrium in Theorem 1 contain coefficients that are themselves defined by a set of highly non-linear equations. It is therefore necessary to consider numerical solutions to these equations if we are to examine the benchmark equilibrium in any greater detail. I will also use these numerical solutions to study the effects of front-running and information-sharing below.

Table 1 shows the parameter values used in the numerical analysis of the benchmark equilibrium. I set the number of dealers in the market to 20. This choice is consistent with the fact that most Forex dealing activity is concentrated in ten or so global banks. The daily interest rate r is set equal to 0.0001, which implies an annualized rate of approximately two percent. Investors and dealers are assumed to have the same preferences, with a coefficient of absolute risk aversion ω equal to 2. The ψ coefficient governing the equilibrium A_t process is 0.999. This choice makes daily changes in Forex prices extremely hard to forecast.

Table 1: Parameter Values

Parameter		Value
Number of dealers	D	20
Risk free rate	r	0.0001
Risk aversion	ω	2
AR(1) coeff.	ψ	0.999
Payoff Shocks	σ_{V}	0.00001
Income Shocks	σ_{Y}	0.007
Hedge Shocks	σ_{H}	0.0035
Investor Gain	$\mathcal{G}^n_{\mathrm{Y}}$	[0.0001,.......0.01]
Dealer Gain	$\mathcal{G}^d_{\mathrm{H}}$	[0.02,.......0.04]

Notes: The table shows the parameter values used in the numerical analysis of the benchmark equilibrium.

Three shocks drive Forex prices in the model: the payoff shocks V_t, the shocks to foreign income Y_t, and the shocks to hedgers' Forex orders, H_t. I calibrate the standard deviations of these shocks so that payoff shocks account for approximately ten percent of the daily change in Forex prices. In equilibrium V_t shocks are directly incorporated into dealers' quotes whereas Y_t and H_t shocks affect quotes via their impact on order flows. So this calibration ensures that order flows are the proximate cause of approximately 90 percent of the variance in equilibrium Forex prices. My calibration also implies that the H_t shocks contribute 20 percent of the variance in A_t. This means that shocks to foreign income are the primary driver of external Forex orders.

The two remaining parameters have important quantitative implications for the benchmark equilibrium. The inferences investors draw about aggregate income Y_t depend on the relative variance of idiosyncratic income shocks σ^2_ε and aggregate income σ^2_{Y}: $\mathrm{E}[Y_t|\Omega^n_{t:\mathrm{I}}] = \mathcal{G}^n_{\mathrm{Y}} Y^n_t$, where $\mathcal{G}^n_{\mathrm{Y}} = \sigma^2_{\mathrm{Y}}/(\sigma^2_{\mathrm{Y}} + \sigma^2_\varepsilon)$. To examine how the precision of these inferences affect the equilibrium, I consider different values for σ^2_ε than imply gain coefficients $\mathcal{G}^n_{\mathrm{Y}}$ ranging from 0.0001 to 0.01. Larger values for $\mathcal{G}^n_{\mathrm{Y}}$ imply that individual investors' income Y^n_t conveys more precise information about aggregate income, which they take into account when determining their Forex orders.

The second parameter affects dealers' inferences concerning the H_t shock based on hedgers' orders: $\mathrm{E}[H_t|\Omega^d_{t:\mathrm{III}}] = \mathcal{G}^d_{\mathrm{H}}(\frac{1}{\mathrm{D}}H_t + \eta^d_t)$, where $\mathcal{G}^d_{\mathrm{H}} = (\frac{1}{\mathrm{D}}\sigma^2_{\mathrm{H}})/((\frac{1}{\mathrm{D}})^2\sigma^2_{\mathrm{H}} + \sigma^2_\eta)$. Here I consider different values for the variance of the distribution shock σ^2_η that imply gain coefficients ranging from 0.02 to 0.04. Again, larger values for the gain coefficient imply that dealers have more precise information about H_t based on the hedgers' orders they receive.[5] I also assume that the distribution shocks affecting hedgers' orders have the same variance as the distribution shocks affecting the investors' orders dealers receive in rounds I and III: i.e., $\sigma^2_\zeta = \sigma^2_\eta$.

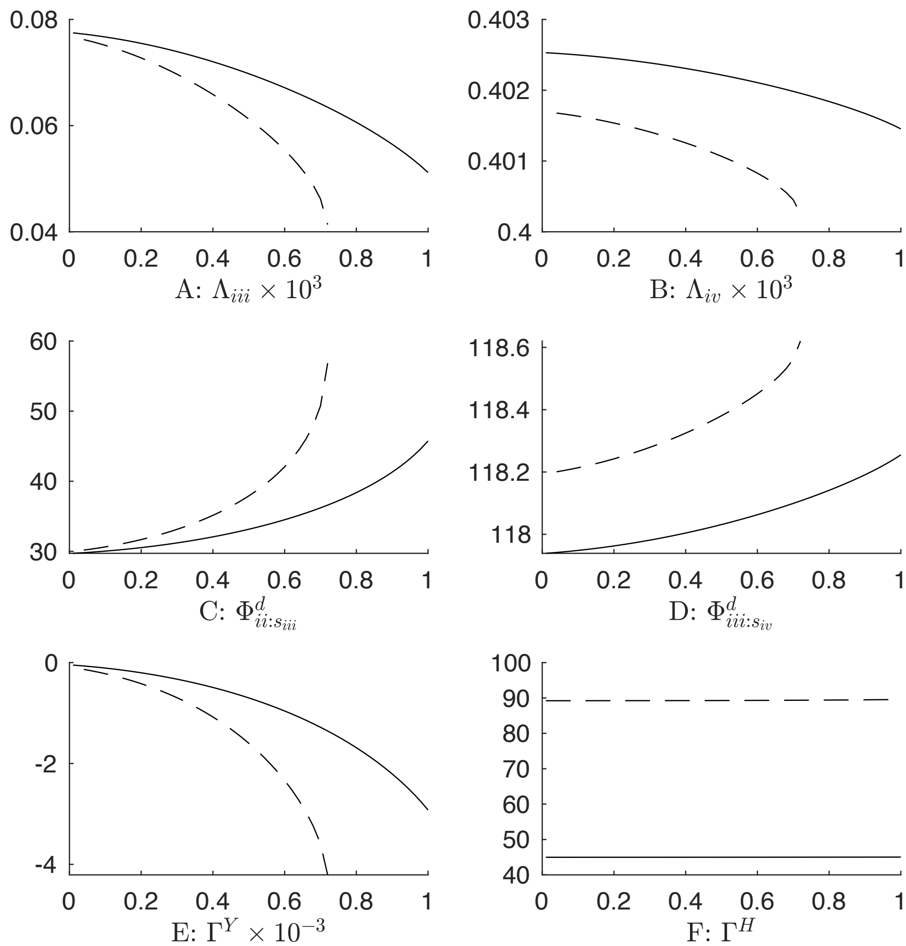
Figure 2 plots six key coefficients from the benchmark equilibrium. Panels A and B plot the coefficients governing the intraday risk premia, Λ_{III} and Λ_{IV}, against the investor gain coefficient $\mathcal{G}^n_{\text{Y}}$ for the case where the dealers' gain $\mathcal{G}^d_{\text{H}}$ equals 0.02 and 0.04. In this calibration of the model $\Lambda_{\text{I}} = 0.02$, so the intraday risk premia implied by the values of Λ_{III} and Λ_{IV} are an order of magnitude smaller than the overnight risk premia. These plots show that the intraday risk premia are smaller (in absolute value) in equilibria where investors have more precise information about aggregate income; i.e., where $\mathcal{G}^n_{\text{Y}}$ is larger. Intuitively, investors require less compensation for holding risky intraday Forex positions when they have more precise information about the shock to aggregate income that affects intraday changes in Forex prices. Panels A and B also show that Λ_{III} and Λ_{IV} are smaller in equilibria where hedgers' orders contain more precise information on the H_t shocks. Again, this feature arises because dealers face less risk from intraday price changes.

Panels C and D show how the sensitivity of dealers' desired positions (in rounds II and III) to expected price changes varies across the equilibria. As one would expect, dealers take more aggressive positions in equilibria where they have more precise information (i.e., the $\Phi^d_{\text{II}|s_{\text{III}}}$ and $\Phi^d_{\text{III}|s_{\text{IV}}}$ are bigger in equilibria where the dealers' gain $\mathcal{G}^d_{\text{H}}$ is larger), because the risk associated with H_t shocks is smaller. Notice, also, that dealers take on more aggressive positions in equilibria where investors have more precise information. This feature arises because dealers learn more about Y_t based on the investors' orders they receive in round I. When individual investors have more precise information about aggregate income, their round-Iorders contain a larger speculative component, so the orders received by each dealer are more informative about Y_t. In other words, dealers become more aggressive in their position-taking when they obtain more precise price-relevant information from external orders.

The lower panels of Figure 2 show how aggregate order flow in rounds II and III relate to the Y_t and H_t shocks across the equilibria. Recall that unexpected order flow in round II is given by $X_{t:\text{II}} - \text{E}[X_{t:\text{II}}|\mathcal{I}_{t:\text{II}}] = \Gamma^{\text{Y}} Y_t$. The plots in Panel E show that Γ^{Y} is more negative in equilibria where

[5] For more perspective on the size of the gain coefficients, it is useful to compare the conditional and unconditional variances of the underlying variables. The variance of foreign income conditioned on an investor's round-I information is $\text{V}[Y_t|\Omega^n_{t:\text{I}}] = (1 - \mathcal{G}^n_{\text{Y}})\sigma^2_{\text{Y}}$, so values for $\mathcal{G}^n_{\text{Y}}$ between 0.0001 to 0.01 mean that an observation on Y^n_t reduces the conditional variance for Y_t by between 0.01 and 1 percent. The conditional variance of H_t based on dealers' hedge orders is $\text{V}[H_t|\Omega^d_{t:\text{III}}] = (1 - \frac{1}{\text{D}}\mathcal{G}^d_{\text{H}})\sigma^2_{\text{H}}$, so the values for $\mathcal{G}^d_{\text{H}}$ imply that the conditional variance is reduced by between 0.1 and 0.2 percent.

Figure 2: Benchmark Equilibrium



Notes: The figure plots coefficients from the benchmark equilibrium against the investors gain, $\mathcal{G}^n_{\mathrm{Y}}$ ×100, for the case where the dealers' gain $\mathcal{G}^d_{\mathrm{H}}$ equals 0.02 (solid plots) and 0.04 (dashed plots).

investors and dealers have more precise information. As was noted above, investors' round-I orders contain a larger speculative component when they have more precise information, so that dealers' round-II trading strategies are more sensitive to aggregate income. In addition, dealers learn more about Y_t from the investors' orders they receive and so aim to take more aggressive round-II positions.

The net effect is that positive income shocks induced individual dealers to initiate greater sales of Forex in inter-dealer trading, which in aggregate produces a larger unexpected negative order flow. The incentives for dealers to take aggressive round-II positions are magnified when they have more precise information about H_t shocks, so income shocks have larger order flow effects under these circumstances. In round III, unexpected order flow is given by $X_{t:\text{III}} - \mathrm{E}[X_{t:\text{III}}|\mathcal{I}_{t:\text{III}}] = \Gamma^{\text{H}} H_t$. As panel F shows, the size of the Γ^{H} coefficient does not vary across equilibria with different investor gain coefficients. By round III the value of Y_t is common knowledge to dealers and investors, so the precision of investors' round-I information is no longer relevant. In contrast, the precision of dealers' information concerning H_t has a sizable impact on Γ^{H}. When dealers receive more precise information concerning a positive H_t shock from the hedgers' orders they receive, they aim to take more aggressive long positions and so initiate larger purchases in inter-dealer trading which in aggregate produce a larger positive order flow.

To summarize, in the benchmark equilibrium dispersed information about the Y_t and H_t shocks is transmitted to dealers via the external orders they receive from investors and hedgers, aggregated via inter-dealer trading, and then embedded into Forex prices.

3 Front-Running

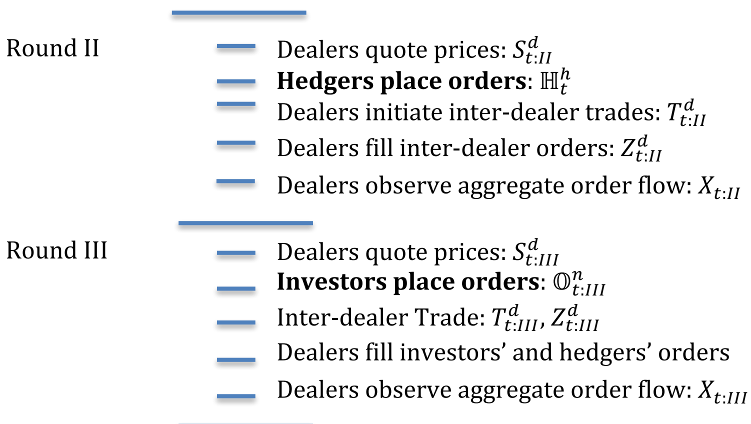
I make one small modification to the model in order to accommodate front-running. I now assume that hedgers place their Forex orders immediately after dealers quote prices at the start of round II rather than at the start of round III. This means that dealers know the hedgers' orders they will need to fill at the end of round III before they begin inter-dealer trading in round II. The new timing of events in rounds II and III are shown in Figure 3.

In this section, I analyze how dealers trade on hedger-order information in round II and how the trades affect round III prices and other aspects of the equilibrium. To facilitate this analysis, I proceed in two steps. First, I consider how an individual dealer would act if he alone had access to hedgers' orders at the start of round II. Then, I analyze the equilibrium in which all dealers receive their hedgers' orders in round II.

3.1 Unilateral Front-Running

To examine the incentives for front-running, consider how a single dealer ($d = 1$) would use the information contained in hedgers' orders received in round II if prices still followed the benchmark equilibrium process in (15). Because hedgers' orders are exogenous, the orders received by the dealer at the start of round II are still given by $\mathbb{H}_t^1 = \mathbb{H}_t + \eta_t^1$. The dealer can therefore estimate H_t before the start of round-II inter-dealer trading as $\mathrm{E}[H_t|\Omega^1_{t:\text{II}}] = \mathcal{G}^d_{\text{H}}(\frac{1}{\text{D}}H_t + \eta^1_t)$. Recall that in

Figure 3: Front-Running Model Timing



Notes: Modified timing of trades in Front-Running Model. Changes shown in bold type.

the benchmark equilibrium H_t shocks have no effect on $\Delta S_{t:\text{III}}$ because round-II order flow only reveals the value of Y_t. So the advanced information on H_t contained in the hedgers' orders has no forecasting power for the near-term change in prices, $\Delta S_{t:\text{III}}$. In contrast, the advanced information is useful in forecasting $\Delta S_{t:\text{IV}}$ because dealers incorporate the value of H_t revealed by round-III order flow when quoting prices at the start of round IV. In particular, the dealer's forecast is now given by

$$\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^1_{t:\text{II}}] = \Lambda_{\text{IV}} A_{t-1} + \tfrac{1}{\psi}\Lambda_{\text{IV}}\mathrm{E}[Y_t|\Omega^1_{t:\text{II}}] + \tfrac{1}{1-\psi+r}\Lambda\mathrm{E}[H_t|\Omega^1_{t:\text{II}}].$$

As above, the dealer 1 initiates inter-dealer trades in round II according to $T^1_{t:\text{II}} = \wp^1_{t:\text{II}} + \frac{1}{\text{D}}\mathrm{E}\left[X_{t:\text{II}}|\Omega^1_{t:\text{II}}\right] + \mathbb{O}^1_{t:\text{I}}$, so advanced information can affect his trade by changing either the desired position $\wp^1_{t:\text{II}}$ and/or expected order flow $\mathrm{E}\left[X_{t:\text{II}}|\Omega^1_{t:\text{II}}\right]$. Under the assumption that the dealer ignores the effects of his own trades on equilibrium order flow, the advanced information only affects the desired position. In particular, from (22a) the change in the desired position due to advanced information is $\frac{1}{1-\psi+r}\Phi^d_{\text{II}|s_{\text{IV}}}\Lambda\mathrm{E}[H_t|\Omega^1_{t:\text{II}}]$. In the benchmark equilibrium, dealers choose their round-II positions to hedge against future shocks with a positive value for the $\Phi^d_{\text{II}|s_{\text{IV}}}$ coefficient. Consequently, the effect of advanced information that produces a positive value for $\mathrm{E}[H_t|\Omega^1_{t:\text{II}}]$ is to increase their orders to purchase Forex from other dealers in round II. In other words, the dealer will infer from hedgers' orders to purchase Forex that they should to aim to take a larger round-II position as a hedge against the likelihood that dealers will quote higher round-IV prices to accommodate a positive H_t shock.

This analysis illuminates several important points. First, the dealer uses the advanced information about the hedgers' Forex orders to "trade ahead" of those orders. Ceteris paribus, the dealer initiates Forex purchases (sales) in inter-dealer trading when he knows that he must fill hedgers Forex purchase (sales) orders in the future. Second, the incentive for dealers to front-run hedgers' orders only arises here because dealers use to the information in those orders to better hedge against future shocks. If dealers chose their round-II positions without regard to hedging future shocks (i.e., if $\Phi^1_{\text{II}|s_{\text{IV}}} = 0$), advanced information on hedgers' orders would have no effect on their round-II trading decisions. Actual Forex dealers often refer to the front-running of external orders as "pre-hedging"; apparently to suggest that their trades are aimed at reducing their own risks. The analysis here provides some support for this view insofar as front-running only appears because the dealer aims to better hedge against the price-impact of future shocks.

Finally, it is important to recognize that this is a partial equilibrium analysis of front-running because I assumed that the change in the dealer's round-II trades had no effect on the benchmark equilibrium. This may be approximately true when there are a large number of dealers in the market and the unilateral front-running by one dealer occurs without the knowledge of the others. Under these special circumstances, front-running has no impact on other market participants and the benefit to the dealer comes solely from enhanced hedging. Note that there is no increase in dealers' expected trading profit because the difference in their round-II position is uncorrelated with $\Delta S_{t:\text{III}}$. So the increase in the dealer's expected utility from using advanced information arises from a reduction in the conditional variance of their future wealth.

3.2 Multilateral Front-Running

Since there is a clear incentive for any dealer to unilaterally front-run, it is unreasonable to assume that prices and order flows will continue to follow the benchmark equilibrium process when many dealers have advanced knowledge of their hedgers' orders. Rather, in equilibrium, round-II order flow will reflect the decisions of these dealers to front-run, which will, in turn, affect the determination of prices and order flows in subsequent trading rounds. I now examine the behavior of prices and order flows in an equilibrium where all dealers receive advanced knowledge of their hedgers' orders in round II.

Theorem 2. *In an efficient risk-sharing equilibrium with multilateral front-running, overnight returns and intraday price changes follow*

$$R_t = \Lambda_{\text{I}} A_{t-1} + \tfrac{1+r}{r} V_t, \tag{26a}$$

$$\Delta S_{t:\text{II}} = 0, \tag{26b}$$

$$\Delta S_{t:\text{III}} = \Lambda_{\text{III}} A_{t-1} + \lambda_{\text{III}}(X_{t:\text{II}} - \text{E}[X_{t:\text{II}}|\mathcal{I}_{t:\text{II}}]), \qquad \textit{and} \tag{26c}$$

$$\Delta S_{t:\text{IV}} = \Lambda_{\text{IV}} A_{t-1} + \lambda_{\text{IV}_1}(X_{t:\text{III}} - \text{E}[X_{t:\text{III}}|\mathcal{I}_{t:\text{III}}]) + \lambda_{\text{IV}_2}(X_{t:\text{II}} - \text{E}[X_{t:\text{II}}|\mathcal{I}_{t:\text{II}}]), \tag{26d}$$

for some coefficients Λ_i and λ_i. Investors' orders in rounds I *and* III *are given by*

$$\mathbb{O}^n_{t:\text{I}} = \beta^{\text{Y}}_{\text{I}} Y^n_t + \beta^{\text{A}}_{\text{I}} A_{t-1}, \qquad \textit{and} \tag{27a}$$

$$\mathbb{O}^n_{t:\text{III}} = \beta^{\text{Y}}_{\text{III}} Y^n_t + \beta^{\text{X}}_{\text{III}}(X_{t:\text{II}} - \text{E}[X_{t:\text{II}}|\mathcal{I}_{t:\text{II}}]) + \beta^{\text{A}}_{\text{III}} A_{t-1}, \tag{27b}$$

for some coefficients β_j. Unexpected order flows from inter-dealer trading in rounds II *and* III *are*

$$X_{t:\text{II}} - \text{E}[X_{t:\text{II}}|\mathcal{I}_{t:\text{II}}] = \Gamma^{\text{Y}}_{\text{II}} Y_t + \Gamma^{\text{H}}_{\text{II}} H_t, \qquad \textit{and} \tag{28a}$$

$$X_{t:\text{III}} - \text{E}[X_{t:\text{III}}|\mathcal{I}_{t:\text{III}}] = \Gamma^{\text{Y}}_{\text{III}} Y_t + \Gamma^{\text{H}}_{\text{III}} H_t, \tag{28b}$$

for some coefficients Γ^{Y}_i and Γ^{H}_i, where $\mathcal{I}_{t:j}$ denotes dealers' common information at the start of round j.

Theorem 2 shows that multilateral front-running has far-reaching effects on the behavior of equilibrium prices, order flows and investors' Forex orders. In this equilibrium, unexpected aggregate order flows in rounds II and III depend on both the Y_t and H_t shocks. As a consequence, while dealers continue to use aggregate order flows to determine the prices to quote in rounds III and IV, these flows contain different information about the underlying Y_t and H_t shocks. So while (26c) and (26d) link $\Delta S_{t:\text{III}}$ and $\Delta S_{t:\text{IV}}$ to order flows, actual price changes are different functions of the underlying shocks than in the benchmark equilibrium. Multilateral front-running also affects investors' orders. Equation (27a) shows that investors' round-I orders take the same form as in the benchmark equilibrium, but the coefficients are quantitatively different. In round III, investors' orders depend on individual income Y^n_t (rather than Y_t) and on unexpected order flow from round II.

To understand the economics behind these equilibrium effects, it is useful to again start with the determination of round IV prices. In the benchmark equilibrium, order flows in rounds II and III sequentially reveal the value of the Y_t and H_t shocks so that dealers can quote a round-IV price that achieves efficient risk-sharing. This remains true in the front-running equilibrium, but dealers learn the value of the Y_t and H_t shocks using the order flows from rounds II and III. Thus the price dealers quote in round IV still takes the form shown in equation (18).

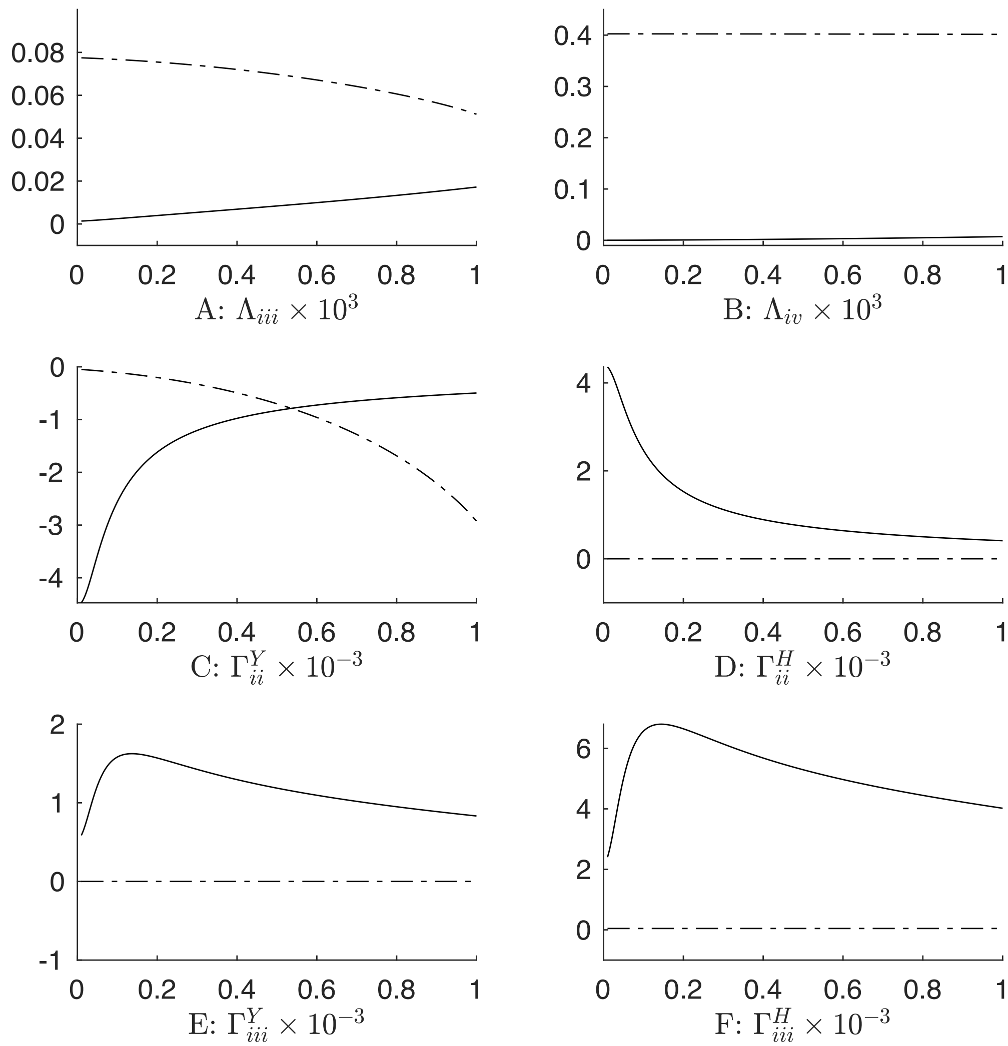
Front-running has more far-reaching effects on dealers' round-III quotes because order flow in round II no longer reveals the value of Y_t. The reason is that dealers' front-running trades create an information externality. Each dealer finds it optimal to initiate trades using their private estimates of Y_t and H_t, but these actions make aggregate order flow dependent on both shocks, so it is impossible for dealers to precisely estimate either Y_t or H_t from their observations on $X_{t:\text{II}}$. Thus, in comparison with the benchmark equilibrium, the information externality slows down the process by which inter-dealer trading aggregates the information on Y_t contained in investors' orders.

The slowing down of information aggregation has several effects on the equilibrium. First, the price dealers quote in round III depends on their common estimate of Y_t, $\mathrm{E}[Y_t|X_{t:\text{II}}]$, rather than its actual value. This means that uncertainty concerning Y_t, and its impact on prices, is not fully resolved until round IV, which is a period later in the benchmark equilibrium. Second, dealers also base their round-III quotes on their estimate of H_t, $\mathrm{E}[H_t|X_{t:\text{II}}]$, because this ensures efficient risk-sharing. As a result, H_t shocks affect round-III prices, via their impact on $\mathrm{E}[Y_t|X_{t:\text{II}}]$ and $\mathrm{E}[H_t|X_{t:\text{II}}]$. This represents a new source of risk that investors will factor into their round-I orders. Third, the impact of H_t shocks on round-III prices creates a stronger incentive for dealers to front-run their hedgers' orders. Recall that there was only a hedging incentive to front-run when prices follow the benchmark equilibrium process. In this equilibrium dealers also have a speculative incentive to front-run because they have private information concerning H_t that will affect $\Delta S_{t;\text{III}}$. This dependency is similar to the "free-riding" feature of dealer front-running in Osler and Turnbull (2017).

Figure 4 provides more perspective on the front-running equilibrium. The figure plots the values for the risk premia coefficients, Λ_{III} and Λ_{IV}, and the order flow coefficients, $\Gamma_{\text{II}}^{\text{Y}}$, $\Gamma_{\text{II}}^{\text{H}}$, $\Gamma_{\text{III}}^{\text{Y}}$, and $\Gamma_{\text{III}}^{\text{H}}$ from the front-running and benchmark equilibria against investors' gain coefficient $\mathcal{G}_{\text{Y}}^{n}$ (as in Figure 2). I compare the equilibria in the case where $\mathcal{G}_{\text{H}}^{d} = 0.02$ so hedgers' orders provide dealers with relatively imprecise information about H_t. All the other parameters are set to the values shown in Table 1.

Panels A and B of Figure 4 compare the risk premia coefficients across the two equilibria. To interpret these plots, recall that $\mathrm{E}\left[\Delta S_{t:\text{III}}|\mathcal{I}_{t:\text{I}}\right] = \Lambda_{\text{III}} A_{t-1}$ and $\mathrm{E}\left[\Delta S_{t:\text{IV}}|\mathcal{I}_{t:\text{I}}\right] = \Lambda_{\text{IV}} A_{t-1}$, so lower values for the coefficients imply smaller positive premia when dealers and investors collectively hold long Forex positions. The plots in panels A and B show that font-running reduces the risk premium required to hold long positions during rounds II and III. Recall that these premia provide the compensation necessary to share risk efficiently across investors and dealers. In the front-running equilibrium, dealers have advanced information on their hedgers' orders, which reduces the risk they face in round II. At the same time, the information externality created by front-running makes

Figure 4: Front-Running Equilibrium I



Notes: The figure plots coefficients from the front-running equilibrium (solid) and benchmark equilibrium (dot-dashed) against the investors gain, $\mathcal{G}^n_{\text{Y}} \times 100$. All other parameters are equal to the values in Table 1 with dealers' gain $\mathcal{G}^d_{\text{H}}$ set equal to 0.02.

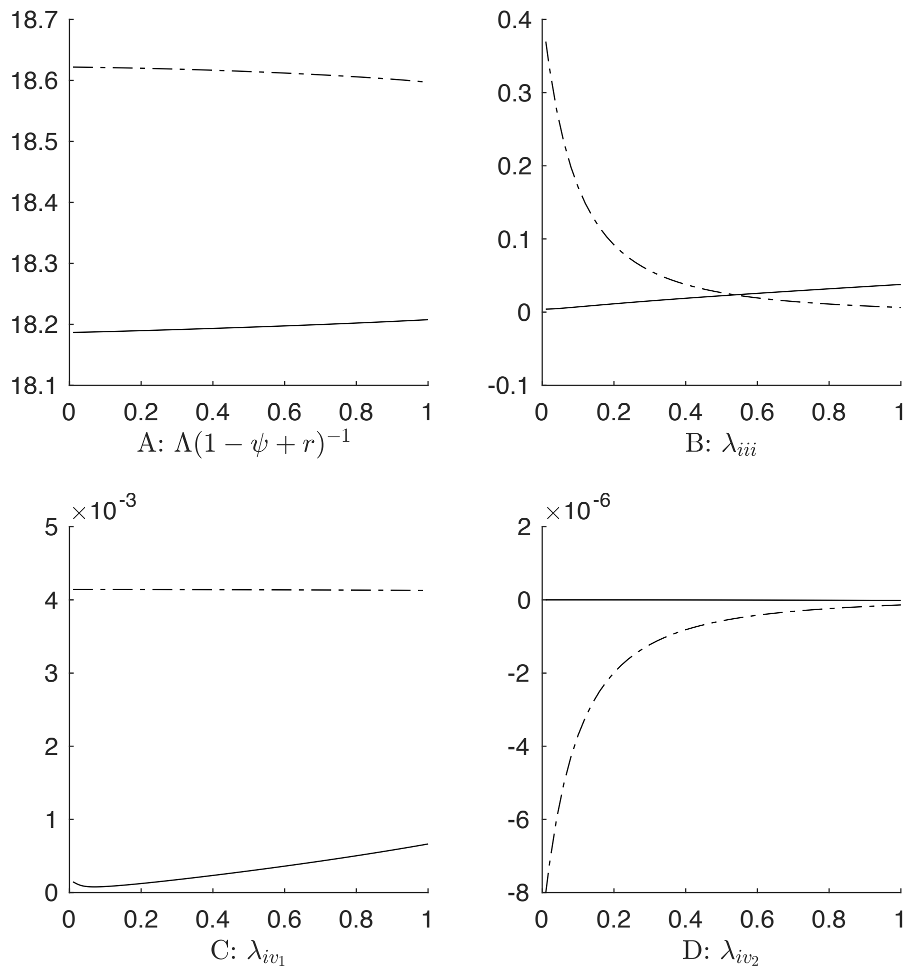
$\Delta S_{t:\text{III}}$ susceptible to H_t shocks, which increases the risk faced by investors. These forces push the round II premium in opposite directions, but on balance the effects on dealers' risk dominate. So, as panel A shows, the premium is smaller than in the benchmark equilibrium. Panel B shows that front-running significantly reduces the risk premium in round III. In the benchmark equilibrium, the risk premium primarily compensates investors for the risk associated with the impact of H_t shocks on $\Delta S_{t:\text{IV}}$. This risk is diminished in the front-running equilibrium because the change in prices $\Delta S_{t:\text{III}}$ provides investors with (imprecise) information on H_t.

The remaining panels in Figure 4 compare the order flow coefficients in the front-running and benchmark equilibria. The plots for $\Gamma_{\text{II}}^{\text{Y}}$ and $\Gamma_{\text{II}}^{\text{H}}$ in panels C and D show that Y_t and H_t shocks have roughly opposite impacts on round-II order flow in the front-running equilibrium. Efficient risk-sharing requires that prices in rounds III and IV incorporate information on $Y_t - H_t$ that dealers learn from order flows in rounds II and III. So dealers initiate trades in round II based on $\text{E}[Y_t - H_t|\Omega_{t:\text{II}}^d]$ using the information on Y_t from their round-I investors' orders and information on H_t from their pending hedgers' orders. As a result, dealers have both a speculative and hedging motive to trade in the same direction as their investor and hedgers' orders. Positive Y_t shocks that induce investors to sell Forex to dealers across the market in round I produce negative order flow in inter-dealer trading, as shown by the plot for $\Gamma_{\text{II}}^{\text{Y}}$ in panel C. Similarly, positive H_t shocks that produce pending hedgers' orders to purchase Forex from dealers across the market, produce positive order flow as dealers "trade ahead" of their pending orders, so the values for $\Gamma_{\text{II}}^{\text{H}}$ are positive as shown in panel D. Notice, also, that both shocks have smaller impacts on order flow in equilibria where investors have more precise information about income (i.e., in equilibria where $\mathcal{G}_{\text{Y}}^{n}$ is larger). In these cases, investors' round-I orders contain a larger speculate component so dealers are more likely to find themselves with large unwanted inventory positions that mitigate the impact of pending hedgers' orders on their round-II trades.

Finally, panels E and F show how front-running affects the composition of order flow in round III. Because the information externality makes it impossible to precisely infer Y_t or H_t from $X_{t:\text{II}}$, dealers still have an incentive to use their private estimates of Y_t and H_t in their round-III trading strategies. As in round II, this private information is aggregated into order flow via inter-dealer trading. We see in panel E that Y_t shocks have a positive impact on unexpected order flow via this aggregation process. The plots for $\Gamma_{\text{III}}^{\text{H}}$ in panel F show that H_t shocks have a much larger positive impact on order flow in the front-running equilibrium than in the benchmark equilibrium. The reason is that round-II order flow provides dealers with more precise information about H_t in the front-running equilibrium so that their round-III trades use this information more aggressively.

Figure 5 shows how front-running affects the behavior of equilibrium prices. Recall that dealers quote prices as part of their BNE strategies, so the price-impact of shocks varies with these strategies

Figure 5: Front-Running Equilibrium II



Notes: The figure plots coefficients from the front-running equilibrium (solid) and benchmark equilibrium (dot-dashed) against the investors gain, $\mathcal{G}^n_{\mathrm{Y}}$ $\times 100$. All other parameters are equal to the values in Table 1 with dealers' gain $\mathcal{G}^d_{\mathrm{Y}}$ set equal to 0.02.

across different equilibria.[6] Panel A plots the values for $\frac{1}{1-\psi+r}\Lambda$ in the front-running and benchmark equilibria against the investors' gain coefficient $\mathcal{G}^n_{\mathrm{Y}}$. These coefficients identify the cumulative effects

[6] This feature differentiates my analysis from earlier research on dual and strategic trading (cited above) in which the price-impact of trade is specified exogenously.

of $H_t - Y_t$ shocks on dealers' quotes by round IV. The cumulative price effects are approximately two percent smaller in the front-running equilibrium because dealers and investors require smaller risk premia to hold long intraday Forex positions, as discussed above.

The remaining panels in Figure 5 plot the price-impact coefficients that measure how dealers' revise their price quotes in response to unexpected order flow. The coefficients λ_{III} and λ_{IV_1} identify how dealers revise their quotes at the start of rounds III and IV in response to the unexpected aggregate order flows they observe at the end of the previous rounds. These coefficients are uniformly positive, indicating the dealers revise their quotes upwards (downwards) in response to unexpected positive (negative) flows. Panel B shows that the coefficient λ_{III} is much less sensitive to differences in the precision of investors information (measured by $\mathcal{G}^n_{\text{Y}}$) than in the benchmark equilibrium. Because Y_t and H_t shocks have approximately symmetric impacts on round-II order flows, dealers draw similar inferences from their observations on these flows across different values for $\mathcal{G}^n_{\text{Y}}$. By contrast, the impact of Y_t shocks on round-II order flows varies with $\mathcal{G}^n_{\text{Y}}$ in the benchmark equilibrium, so dealers adjust their inferences. Panel C shows that the price-impact of round-III order flow is much smaller in the front-running than the benchmark equilibrium. In general, the price-impact coefficients depend on the incremental information carried by the flow. The price-impact coefficient is large in the benchmark equilibrium because the round-III flow fully reveals the value of H_t, which is important in determining the round-IV price level that shares risk efficiently. By contrast, in the front-running equilibrium, the round-III flow provides incremental information on Y_t and H_t that have opposite risk-sharing implications for round-IV prices. The plots in panel D show that dealers also use the information in round-II flows when quoting round-IV prices. These flows contain information about the intraday risk premia embedded in round-III prices that must be adjusted so that round-IV prices share risk efficiently. These adjustments are very small in both the front-running and benchmark equilibria.

To summarize, multilateral front-running has far-reaching effects on the behavior of equilibrium prices and order flows. These effects originate with the information externality induced by front-running that alters the composition and price-impact of order flows, and the optimal trading decisions of dealers and investors. I next examine how these effects depend on the degree of collusion among dealers

4 Front-Running with Collusion

To this point, I have considered the effects of front-running in equilibria where the hedgers' orders received by each dealer contained relatively imprecise information about aggregate shocks. I now analyze how the effects of front-running change when dealers can make more precise inferences about aggregate shocks through the collusive sharing of information on hedgers' orders. This analysis is motivated by the many regulator reports (cited in the Introduction) that document how dealers at major banks collusively shared information about their pending external orders.

To study the effects of collusion, I assume that the D dealers in the market are split into equal groups of size G. Dealers within each group share information on the hedgers' orders they receive in round II before making their own trading decisions. To keep things simple, I assume that there is no netting of hedgers' orders among the dealers in each group, so collusion only involves the sharing of information.

Recall that the hedgers' orders received by dealer d are $\mathbb{H}_t^d = \frac{1}{\mathrm{D}}\mathbb{H}_t + \eta_t^d$, where $\mathbb{H}_t$ is the aggregate imbalance in hedgers' orders and $\eta_t^d \sim N(0, \sigma_\eta^2)$ is a distribution shock that allocates orders across dealers. These shocks are negatively correlated across dealers with correlation $\rho = -\frac{1}{\mathrm{D}-1}$ to ensure that $\sum_{d=1}^{\mathrm{D}} \eta_t^d = 0$. In the absence of collusion, each dealer's estimate of H_t is based on their own hedgers' orders. But if a dealer is part of a collusive group with G members, they can obtain a more precise estimate of H_t based on the net imbalance in the orders received by the group: $\mathbb{H}_t^{\mathrm{G}} = \sum_{d \in \mathrm{G}} \mathbb{H}_t^d$. Aggregating across dealers gives $\mathbb{H}_t^{\mathrm{G}} = \frac{\mathrm{G}}{\mathrm{D}}\mathbb{H}_t + \eta_t^{\mathrm{G}}$, where $\eta_t^{\mathrm{G}} = \sum_{d \in \mathrm{G}} \eta_t^d$ is the group distribution shock which has a mean of zero and variance equal to $\frac{\mathrm{D}-\mathrm{G}}{\mathrm{D}-1}\mathrm{G}\sigma_\eta^2$. The estimate of H_t for dealers in a group of size G is therefore

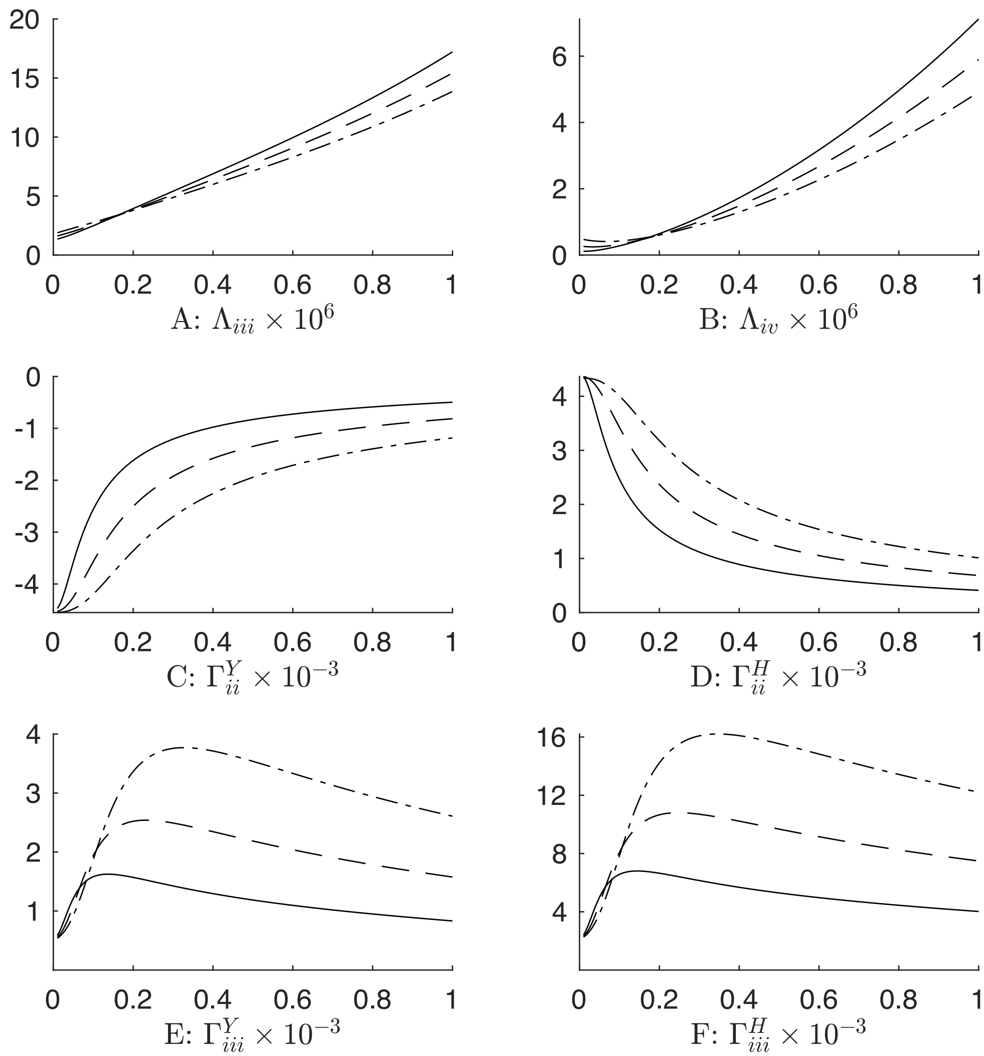
$$\mathrm{E}[H_t|\Omega_{t:\mathrm{II}}^{\mathrm{G}}] = \mathcal{G}_{\mathrm{H}}^{\mathrm{G}}(\mathbb{H}_t^{\mathrm{G}} - \mathrm{E}[\mathbb{H}_t^{\mathrm{G}}|\mathcal{I}_{t:\mathrm{II}}]) = \mathcal{G}_{\mathrm{H}}^{\mathrm{G}}(\frac{\mathrm{G}}{\mathrm{D}}H_t + \eta_t^{\mathrm{G}}), \qquad \text{where} \qquad \mathcal{G}_{\mathrm{H}}^{\mathrm{G}} = \frac{\frac{\mathrm{G}}{\mathrm{D}}\sigma_{\mathrm{H}}^2}{(\frac{\mathrm{G}}{\mathrm{D}})^2\sigma_{\mathrm{H}}^2 + \frac{\mathrm{D}-\mathrm{G}}{\mathrm{D}-1}\mathrm{G}\sigma_\eta^2}.$$

Dealers put more weight on the unexpected imbalance in the groups' hedge orders $\mathbb{H}_t^{\mathrm{G}}$ as the size of the group rises because the gain coefficient $\mathcal{G}_{\mathrm{H}}^{\mathrm{G}}$ is increasing in G. Since the distribution shocks η_t^d are negatively correlated across dealers, aggregating orders across dealers in the group produces a more precise signal on H_t, so the unexpected values for $\mathbb{H}_t^{\mathrm{G}}$ are given greater weight in dealers' estimates.

Figure 6 illustrates how different values for the group gain coefficient $\mathcal{G}_{\mathrm{H}}^{\mathrm{G}}$ affect the front-running equilibria. The figure plots the values for the risk premia coefficients, Λ_{III} and Λ_{IV}, and the order flow coefficients, $\Gamma_{\mathrm{II}}^{\mathrm{Y}}$, $\Gamma_{\mathrm{II}}^{\mathrm{H}}$, $\Gamma_{\mathrm{III}}^{\mathrm{Y}}$, and $\Gamma_{\mathrm{III}}^{\mathrm{H}}$ in equilibria where $\mathcal{G}_{\mathrm{H}}^{\mathrm{G}} = \{0.02, 0.03, 0.04\}$. If we assumed that the gain for an individual dealer is $\mathcal{G}_{\mathrm{H}}^d = 0.02$, these values correspond to groups ranging between one and 10 dealers. Since there are only a total of 20 dealers in model, these plots provide a good indication of how large differences in the degree of collusion affects the equilibria.

Figure 6 shows that collusion among dealers has a much larger impact on the composition of order flows than on the intraday risk premia. In panels A and B, there are only small differences between the risk premia coefficients, Λ_{III} and Λ_{IV}, across the three sets of equilibria. Collusion among dealers reduces the risk they face in holding intraday positions, but it generally increases the risks investors

Figure 6: Front-Running Equilibrium III



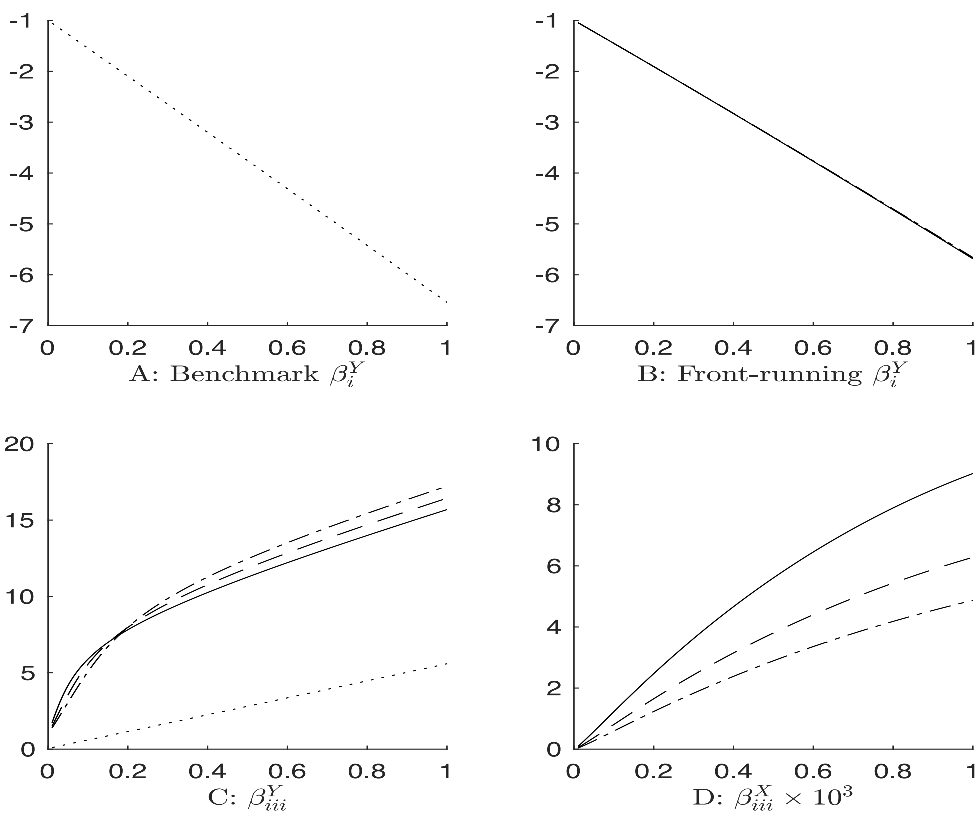
Notes: The figure plots coefficients from the front-running equilibrium against the investors gain, $\mathcal{G}^{n}_{\mathrm{Y}}$ $\times 100$. Solid, dashed and dot-dashed lines plot coefficients from equilibria with the dealers' group gain $\mathcal{G}^{\mathrm{G}}_{\mathrm{H}}$ equal to 0.02, 0.03 and 0.04, respectively. All other parameters are equal to the values in Table 1 face as front-running makes round-III prices susceptible to H_t shocks. The plots in panels A and B show how these different factors are reflected in the risk premia across equilibria with different degrees of dealer collusion and varying precision in investors' information. Notice that greater collusion either increases or reduces Λ_{III} and Λ_{IV} depending on precision of investors' information. This non-monotonicity appears in other features of the front-running equilibria examined below.

The effects of collusion on round-II order flow are shown in panels C and D. When information on hedgers' orders is shared among a larger group of dealers, individual dealers face less risk from H_t shocks. As a consequence, the dealers aim for round-II positions $\wp^d_{t:\text{II}}$ that make more aggressive use of their forecasts for $\Delta S_{t:\text{III}}$ and $\Delta S_{t:\text{IV}}$. Since these forecasts depend on the shared information in hedgers orders and information contained in round-I investors' orders, the trades initiated by each colluding dealer in round II become more dependent on external trades. In aggregate, this greater dependency means that Y_t shocks have a larger (negative) impact on (unexpected) order flow, as shown in panel C. Similarly, the plots in panel D show that H_t shocks have a larger positive impact on order flow when there is more collusion among dealers. Collusion also affects the composition of order flow in round III. As in round II, greater collusion leads dealers to make more aggressive use of their forecasts for $\Delta S_{t:\text{IV}}$. In this case, dealers use their shared information on hedgers orders, and the information in the investor orders they individually received in rounds I and III. In aggregate, this makes round-III order flow more susceptible to Y_t and H_t shocks, as can be seen in panels E and F.

In this model, hedgers' orders are strictly exogenous so the orders received by each dealer do not depend on whether they engage in unilateral or collusive front-running. In contrast, investors choose their orders in rounds I and III optimally, so the investor orders received by each dealer are potentially impacted by the equilibrium price effects of collusive front-running even though there is no opportunity for a dealer to front-run their investor orders. Figure 7 provides information on this market-wide effect.

Theorems 1 and 2 showed that each investor's round-I order follows $\mathbb{O}^n_{t:\text{I}} = \beta^{\text{Y}}_{\text{I}} Y^n_t + \beta^{\text{A}}_{\text{I}} A_{t-1}$. Panels A and B of Figure 7 show how the $\beta^{\text{Y}}_{\text{I}}$ coefficient varies with the investor's gain coefficient $\mathcal{G}^n_{\text{Y}}$ in the benchmark and front-running equilibria. Individual investors use the information in their own foreign income Y^n_t to establish speculative round-I positions. When this information is imprecise, the speculative positions are very small, so their orders simply hedge foreign income which makes $\beta^{\text{Y}}_{\text{I}}$ close to -1. In equilibria where investors have more precise information, they use their round-I orders to take larger speculative positions, which makes the $\beta^{\text{Y}}_{\text{I}}$ coefficient more negative. The plots in panel A and B show that investors' orders have a smaller speculative component in the front-running equilibria. In other words, investors establish smaller speculative positions in round I when they recognize that dealers are front-running hedgers' orders. Panel B also shows that there

Figure 7: Investors' Orders



Notes: Panels A and B plot the income coefficient for investors' round-I orders. Panel C plots the income coefficient for investors' round-III orders in the benchmark and front-running equilibria. Panel D plots the order flow coefficient on investors' round-III orders in the front-running equilibria. Solid, dashed and dot-dashed lines plot coefficients from the front-running equilibria with the dealers' gain $\mathcal{G}_{\mathrm{H}}^d$ equal to 0.02, 0.03 and 0.04, respectively. Dotted lines plot coefficients from the benchmark equilibria with $\mathcal{G}_{\mathrm{H}}^d = 0.02$. All other parameters are equal to the values in Table 1

are negligible differences between the $\beta_{\mathrm{I}}^{\mathrm{Y}}$ coefficients across front-running equilibria with different degrees of collusion. It appears that the presence of front-running rather than the degree of collusive front-running has the greatest impact on investors' willingness to take speculative round-I positions.

The lower panels of Figure 7 show how front-running and collusion affect the determinants of investors' round-III orders. In the benchmark equilibrium investors learn the value of Y_t by the start of round III, so Y_t supplants Y_t^n as the determinant of their individual orders: $\mathbb{O}^n_{t:\text{III}} = \beta^{\text{Y}}_{\text{III}} Y_t + \beta^{\text{A}}_{\text{III}} A_{t-1}$. Panel C shows that $\beta^{\text{Y}}_{\text{III}}$ increases with the precision of investor's information because under efficient risk sharing the intraday risk premium induces investors to hold longer Forex positions in round III. In the front-running equilibrium the information externality prevents investors from learning the value of Y_t by the start of round III, so their orders depend on Y_t^n and round-II order flow (which they infer from $\Delta S_{t:\text{III}}$ and A_{t-1}):

$$\mathbb{O}^n_{t:\text{III}} = \beta^{\text{Y}}_{\text{III}} Y_t^n + \beta^{\text{X}}_{\text{III}} (X_{t:\text{II}} - \text{E}[X_{t:\text{II}} | \mathcal{I}_{t:\text{II}}]) + \beta^{\text{A}}_{\text{III}} A_{t-1}.$$

The plots in Panel C show that the $\beta^{\text{Y}}_{\text{III}}$ coefficients are larger in the presence of front-running which indicates that investors are more willing to take speculative positions in round III based on the private information in Y_t^n. This arises because Y_t shocks contribute to unexpected order flow in round-III that dealers use to quote prices in round IV. Consequently, investors' income Y_t^n has forecasting power for $\Delta S_{t:\text{IV}}$. In a sense, the information externality created by front-running shifts investors' speculative activity from round I to round III. Panel D shows how the degree of collusion affects the dependency of investors' orders on prior unexpected order flow, $X_{t:\text{II}} - \text{E}[X_{t:\text{II}} | \mathcal{I}_{t:\text{II}}]$. Investors use $X_{t:\text{II}} - \text{E}[X_{t:\text{II}} | \mathcal{I}_{t:\text{II}}]$ to update their estimates of Y_t and H_t, which in turn affect their forecasts for $\Delta S_{t:\text{IV}}$ and R_{t+1}. The plots show that the net effect of the information conveyed by positive order flow is that investors increase their purchases of Forex in order to establish larger long speculative positions. This effect is larger when investors have more precise private information on Y_t and when the degree of collusion is lower because in both instances the incremental information conveyed by order flow is diminished.

These results show that the degree of collusive front-running primarily affects the composition of equilibrium order flows and the determinants of investors' orders. However, all in all, it appears that equilibrium trading patterns are affected more by the presence of front-running than by the degree to which dealers engage in collusive front-running.

5 Front-Running, Collusion and Trading Profits

To this point my analysis has focused on how front-running and collusion affect the behavior of equilibrium prices, orders flows, and external orders. In this section, I examine how front-running impacts dealers' and investors' welfare. Clearly, as part of a BNE strategy, individual dealers must benefit from front-running given the trading decisions of other dealers. However, since front-running creates an information externality, it is much less clear how dealers' welfare compares between the benchmark and the front-running equilibria. Furthermore, if dealers are better off in the front-running equilibrium, is this because they make more profitable trades with hedgers, investors, or both? In other words, do dealers benefit from front-running at the expense of just hedgers (i.e. the counter-parties in the front-run orders), or at the expense of both hedgers and investors?

To address these questions, it is useful to examine the equilibrium dynamics of investors' and dealers' wealth. In the case of investors, the budget constraints in (4) can be rewritten as

$$W^n_{t+1:\mathrm{I}} = A^n_{t:\mathrm{IV}}R_{t+1} + (1+r)\left\{W^n_{t:\mathrm{I}} + S_{t:\mathrm{I}}Y^n_t\right\} + (1+r)\left\{A^n_{t-1:\mathrm{IV}}(S_{t:\mathrm{IV}} - S_{t:\mathrm{I}})\right\} + \Pi^n_t, \quad (29a)$$

$$\text{where} \qquad \Pi^n_t = (1+r)\left\{(\mathbb{O}^n_{t:\mathrm{III}} + \mathbb{O}^n_{t:\mathrm{I}})\Delta S_{t:\mathrm{IV}} + \mathbb{O}^n_{t:\mathrm{I}}\Delta S_{t:\mathrm{III}}\right\}. \quad (29b)$$

Equation (29a) decomposes next day's wealth into four components: the excess return on the overnight Forex position, $A^n_{t:\mathrm{IV}}R_{t+1}$; the return on prior wealth and foreign income $(1+r)\times\{W^n_{t:\mathrm{I}} + S_{t:\mathrm{I}}Y^n_t\}$; the intraday capital gain on prior Forex holdings, $(1+r)\left\{A^n_{t-1:\mathrm{IV}}(S_{t:\mathrm{IV}} - S_{t:\mathrm{I}})\right\}$; and intraday trading profits, Π^n_t. Because investors hold identical overnight positions in the benchmark and front-running equilibria, in aggregate these positions equal the entire stock of Forex so $A_t = \int_0^1 A^n_{t:\mathrm{IV}}dn = A^n_{t:\mathrm{IV}}$ for all n. The overnight return and intraday capital gain components are therefore identical across investors. Equation (29b) shows how trading profits are determined by the intraday capital gains on positions the investor establishes through their orders in rounds I and III.

In the case of dealers, the budget constraints in (9) can be rewritten as

$$W^d_{t+1:\mathrm{I}} = (1+r)W^d_{t:\mathrm{I}} + \Pi^d_t, \quad (30a)$$

$$\text{where} \qquad \Pi^d_t = (1+r)\left\{(\wp^d_{t:\mathrm{III}} - \xi^d_{t:\mathrm{III}})\Delta S_{t:\mathrm{IV}} + (\wp^d_{t:\mathrm{II}} - \xi^d_{t:\mathrm{II}})\Delta S_{t:\mathrm{III}} - \mathbb{O}^d_{t:\mathrm{I}}\Delta S_{t:\mathrm{II}}\right\}. \quad (30b)$$

Since dealers do not hold Forex overnight in any equilibria, next day's wealth depends on just prior wealth and the dealer's trading profits, Π^d_t. Equation (30b) shows how these profits depend on the intraday capital gains dealers obtain on their end-of-round Forex positions. For example, dealers have a position of $-\mathbb{O}^d_{t:\mathrm{I}}$ at the end of round I after filling the investors' orders they receive of $\mathbb{O}^d_{t:\mathrm{I}}$. Each dealer's position at the end of rounds II and III depends on their desired position and the unexpected orders they must fill from other dealers, $\wp^d_{t:j} - \xi^d_{t:j}$. Notice that the hedgers' orders and investors' round-III orders received by dealer d affect profits via their impact on desired positions $\wp^d_{t:j}$, whereas the external orders received by all dealers affect positions via unexpected inter-dealer orders $\xi^d_{t:j}$, and price changes $\Delta S_{t:j}$.

The trading profits of investors and dealers are linked by the market clearing conditions. In particular, combining (29b), (30b) with (10) - (13), we find that the trading profits of all dealers are

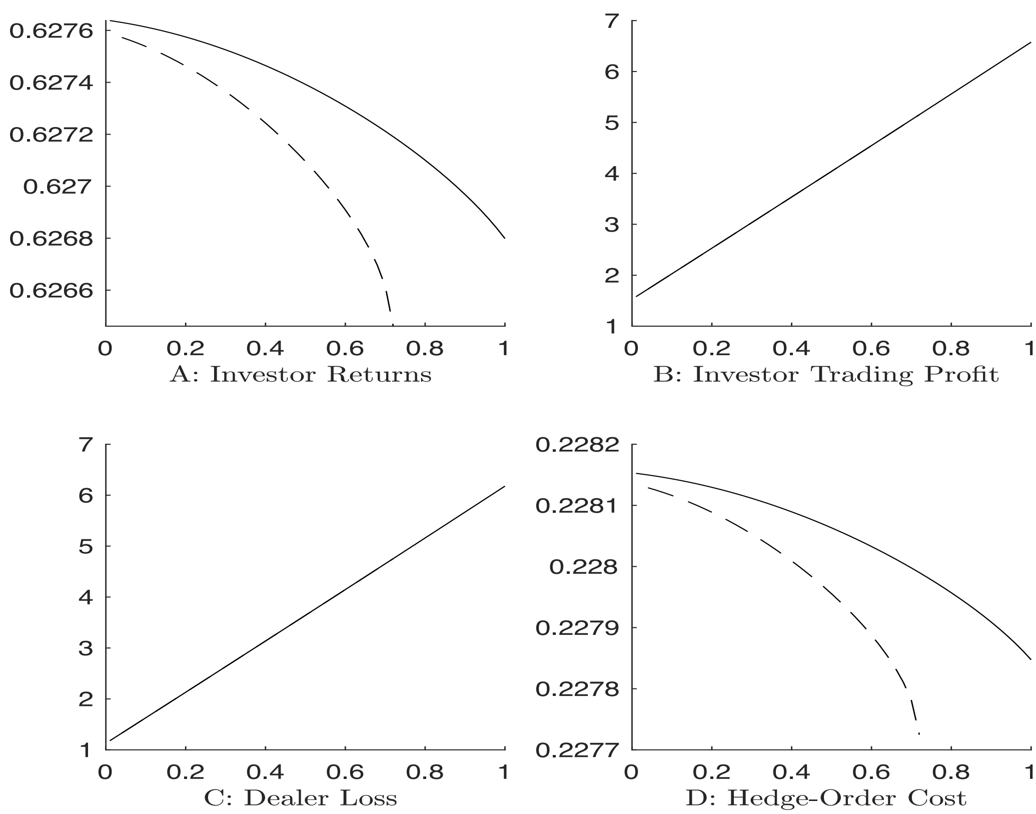
$$\Pi_t^{\text{D}} \equiv \sum_{d=1}^{\text{D}} \Pi_t^d = -(1+r)\left\{(\mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}})\Delta S_{t:\text{IV}} + \mathbb{O}_{t:\text{I}}\Delta S_{t:\text{III}}\right\} - (1+r)(\mathbb{H}_t \Delta S_{t:\text{IV}}), \tag{31}$$

where $\mathbb{O}_{t:j}$ and $\mathbb{H}_t$ are the aggregate imbalances in orders from investors and hedgers. The first term on the right-hand-side is equal to minus one times the aggregate trading profits of investors: $-\int_0^1 \Pi_t^n dn$, while the second term identifies the cost of filling hedgers' orders: $\mathbb{C}_t = (1+r)(\mathbb{H}_t \Delta S_{t:\text{IV}})$. These costs are equal to the capital gain the dealers would have captured if they had filled hedgers' orders at the round-IV price rather than the round-III price. Equation (31) shows that any increase in the aggregate trading profits of dealers caused by front-running must either come from a fall in investors' trading profits $\int_0^1 \Pi_t^n dn$, and/or a reduction in the hedge order costs $\mathbb{C}_t$.

Figure 8 shows how trading profits and costs are distributed across investors and dealers in the benchmark equilibrium. Panel A plots the return investors expect on their overnight and intraday positions $\mathrm{E}[A_{t:\text{IV}}^n R_{t+1} + (1+r)\left\{A_{t-1:\text{IV}}^n (S_{t:\text{IV}} - S_{t:\text{I}})\right\}]$ against the investor gain coefficient $\mathcal{G}_{\text{Y}}^n$ for two values of the dealers' gain: $\mathcal{G}_{\text{H}}^d = \{0.02, 0.04\}$. Expected overnight returns are unaffected by the precision of either investors' or dealers' information because under complete risk sharing the overnight risk premia only reflects investors' preferences. Thus the plots in Panel A show that the precision of investors' and dealers' information have some impact on the expected capital gains on intraday positions (i.e., $\mathrm{E}[(1+r)\left\{A_{t-1:\text{IV}}^n (S_{t:\text{IV}} - S_{t:\text{I}})\right\})$, but the effects are very small. Panel B shows that investors' information has a much larger impact on their expected trading profits, $\mathrm{E}[\Pi_t^n]$. Expected trading profits increase proportionately with the gain coefficient $\mathcal{G}_{\text{Y}}^n$ because investors willingness to take speculative positions with their round-I orders critically depends on the precision of their information concerning Y_t. When investors have more precise information, they are willing to take larger speculative positions which, in expectation, produce greater trading profits. Notice, also, that investors' expected trading profits are not materially affected by the precision of dealers' information. Panel B plots $\mathrm{E}[\Pi_t^n]$ against $\mathcal{G}_{\text{Y}}^n$ for $\mathcal{G}_{\text{H}}^d = \{0.02, 0.04\}$, but the two plotted lines are indistinguishable. The lower panels of Figure 8 show how the precision of information affects dealer welfare. Panel C plots expected trading losses across all dealers, $-\mathrm{E}[\Pi_t^{\text{D}}]$, which mirror investors' expected profits in panel B. Panel D shows the expected hedge cost $\mathrm{E}[\mathbb{C}_t]$ as a fraction of expected trading losses, $-\mathrm{E}[\Pi_t^{\text{D}}]$. Providing liquidity to hedgers adds approximately 23 percent to dealers' overall expected losses. The plots in panel D are similar in shape to those in panel A because $\mathrm{E}[\mathbb{C}_t]$ depends on the size of the intraday round-III risk premia.

Figure 8 shows that dealers' expected losses are minimally affected by the gain coefficient $\mathcal{G}_{\text{H}}^d$. One might have anticipated that dealers' losses would be smaller when they have more precise information on the aggregate imbalance in hedgers orders because they would have been willing to

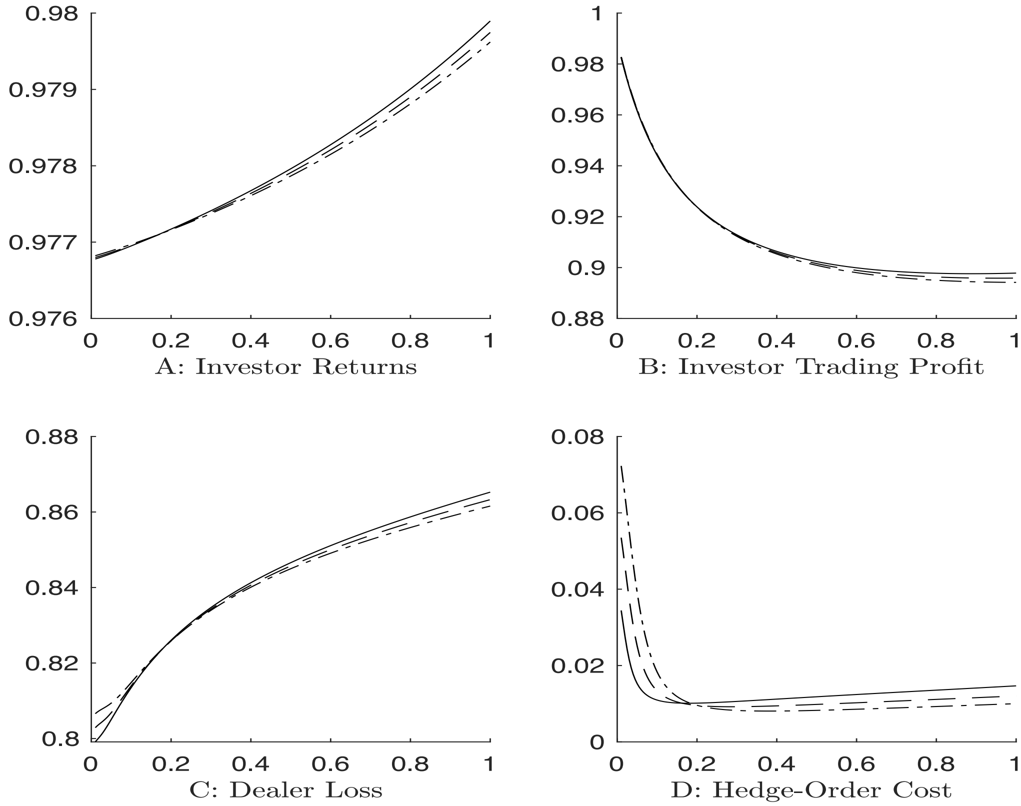
Figure 8: Benchmark Profits and Costs



Notes: Panel A plots the return investors expect on their overnight and intraday positions $\mathrm{E}[A^n_{t:\mathrm{IV}}R_{t+1}+(1+r)\left\{A^n_{t-1:\mathrm{IV}}(S_{t:\mathrm{IV}}-S_{t:\mathrm{I}})\right\}]$. Panel B plots investors' expected trading profits, $\mathrm{E}[\Pi^n_t]$. Panels C and D plot the expected dealer loss $-\mathrm{E}[\Pi^{\mathrm{D}}_t]$ and the ratio of the expected hedge order costs to the expect loss, $-\mathrm{E}[\mathbb{C}_t]/\mathrm{E}[\Pi^{\mathrm{D}}_t]$, respectively. All variables multiplied by 1000 and plotted against the investor gain coefficient, $\mathcal{G}^n_{\mathrm{Y}}\times 100$. Each panel contains solid and dashed lines from the benchmark equilibria with the dealers' gain $\mathcal{G}^d_{\mathrm{H}}$ equal to 0.02 and 0.04, respectively (but the lines are only distinguishable in panels A and D). All other parameters are equal to the values in Table 1

take more aggressive speculative positions. However, in the benchmark equilibrium greater precision only directly impacts inter-dealer trading in round III. Consequently, it affects the distribution of trading profits across dealers, but not the expected aggregate profit. One important implication of this result is that the collusive sharing of information about hedgers' orders does not materially

Figure 9: Front-Running Profits and Costs



Notes: Panel A plots the return investors expect on their overnight and intraday positions $\mathrm{E}[A^n_{t:\mathrm{IV}}R_{t+1}+(1+r)\left\{A^n_{t-1:\mathrm{IV}}(S_{t:\mathrm{IV}}-S_{t:\mathrm{I}})\right\}]$. Panel B plots investors' expected trading profits, $\mathrm{E}[\Pi^n_t]$. Panels C and D plot the expected dealer loss $-\mathrm{E}[\Pi^{\mathrm{D}}_t]$ and the expect cost of filling hedgers' orders $\mathrm{E}[\mathbb{C}_t]$. All variables are plotted as fractions of the corresponding value in the benchmark equilibrium (with $\mathcal{G}^d_{\mathrm{H}}=0.02$) against the investor gain coefficient, $\mathcal{G}^n_{\mathrm{Y}}\times 100$. Solid, dashed and dot-dashed lines plot coefficients from the front-running equilibria with $\mathcal{G}^{\mathrm{G}}_{\mathrm{H}}$ equal to 0.02, 0.03 and 0.04, respectively.

improve aggregate dealer profits when dealers are unable to front-run the orders. Information-sharing does have some influence on the size of intraday risk premia and expected hedge costs, but these effects are very small when compared to investors' expected trading profits. So, in sum, collusive information-sharing without front-running doesn't appear to materially lower investors' expected returns and trading profits.

The effects of front-running on returns, trading profits, and costs are illustrated in Figure 9. Panel A shows investors' expected returns in the front-running equilibria as a fraction of expected returns in the benchmark equilibria (with $\mathcal{G}^{d}_{\text{H}} = 0.02$) from collusive front-running equilibria where $\mathcal{G}^{\text{G}}_{\text{H}} = \{0.02, 0.03, 0.04\}$. The figure shows that front-running lowers expected returns by a little more than two percent (depending on the value for the investors' gain $\mathcal{G}^{n}_{\text{Y}}$) across equilibria with different degrees of collusion. These effects primarily reflect the impact of front-running on the intraday risk premia. Panel B plots investors' expected trading profits in the front-running equilibria as a fraction of profits in the benchmark equilibria. In this case, front-running reduces expected profits between approximately two and ten percent. Investors are most adversely affected by front-running in cases where they have more precise information about foreign income, measured by large values for $\mathcal{G}^{n}_{\text{Y}}$. By comparison, the impact of front-running on investors' profits is only minimally affected by the degree of collusion.

The lower panels of Figure 9 show how front-running impacts dealers' expected losses. Panel C plots expected losses for all dealers in the collusive font-running equilibria as a fraction of expected losses in the benchmark equilibrium. The plot shows that front-running produces sizable reductions in dealers' losses ranging from approximately 20 to 12 percent. Front-running benefits dealers most when investors have very imprecise information about foreign income. Under these circumstances, investors' round-I orders have a smaller speculative component, so dealers are less likely to find themselves with a large unwanted position that inhibits their willingness to front-run their pending hedgers' orders in round II. Front-running also significantly lowers the expected cost of providing liquidity to hedgers, $\mathrm{E}[\mathbb{C}]$. Panel D plots $\mathrm{E}[\mathbb{C}]$ in the three front-running equilibria as a fraction of the expected cost in the benchmark equilibria. The plots show that front-running almost eliminates these costs unless investors have very imprecise information, and even then the costs fall by over 90 percent.

Two aspects of these findings are particularly noteworthy. First, front-running by dealers has an adverse effect on *all* other market participants. As I noted in the introduction, the standard view of regulators and legal plaintiffs is that dealer front-running primarily harms the entity that submits the front-run order; i.e., the hedgers in this model. In contrast, the results above show that front-running can have significant adverse effects on investors' trading profits even through dealers had no opportunity to front-run investors' orders. Second, information-sharing among dealers has rather minor welfare effects in the front-running equilibria. Figures 6 and 7 showed that aggregate order flows and investors' trades varied significantly across equilibria with different values for $\mathcal{G}^{\text{G}}_{\text{H}}$, but the plots in Figure 9 are very similar. It appears that the presence or absence of front-running has much larger welfare implications than the degree to which dealers collusively front-run external orders.

To understand why front-running adversely affects investors' trading profits, it is useful to consider the source of these profits in the benchmark equilibrium. Figure 7 showed that investors' round-I trades had a larger speculative component when they had more precise private information on Y_t. On average these positions generate trading profits on investors' round-I trades as the information on Y_t is aggregated by inter-dealer trading and incorporated into round-III prices, and these profits are higher in expectation when investors take larger speculative positions (see Figure 8). By round III, investors' private information makes no contribution to their trading profits because the value of Y_t can be inferred from public information. So their round-III trades are simply used to establish a position that benefits from the intraday risk premium. Front-running reduces investors' expected trading profits through two channels. First, it makes investors less willing to take speculative positions based on their own private information in round I. Second, the dependency of investors' round III orders on round-II order flow creates a problem of "rational confusion" (see, e.g., Bacchetta and van Wincoop, 2008). Investors cannot differentiate between the effects of Y_t and H_t shocks on order flow, so inevitably their round-III orders contain "mistakes" that were missing in the benchmark equilibrium.

Front-running also effects the prices hedgers pay to fill their orders. In all equilibria, positive H_t shocks raise the price dealers quote in round IV to ensure efficient risk-sharing, but in the presence of front-running, positive H_t shocks also raise prices in round III via their impact on order flow. Consequently, positive H_t shocks not only increase hedgers' net orders to purchase Forex, but also raise the price hedgers pay in aggregate to have those orders filled. Similarly, negative H_t shocks reduce round-III prices in the front-running equilibrium and increase the net sales of Forex by hedgers, so front-running reduces the prices hedgers receive in aggregate for their net Forex sales. Clearly, these effects of front-running benefit dealers. In particular, they contribute to dealers' trading profits by reducing the capital gains that are missed when hedgers' orders are filled at the round-III price rather than the round-IV price.

Finally, let us turn to the effects of collusion. The plots in Figure 9 shows that collusive front-running harms investors marginally more than unilateral front-running, but it is the existence of front-running that does most of the harm. The reason is that front-running with or without collusion creates the information externality that fundamentally changes the behavior of Forex prices and order flows. In the absence of front-running, collusion between dealers doesn't change the process by which the information contained in order flow is embedded into prices, so its effects on investors and hedgers are very small.

6 Conclusion

The analysis in this paper shows that front-running affects the behavior of Forex prices and order flows in ways that impact all market-participants. These market-wide effects appear because front-running creates an externality that slows down the aggregation of information by inter-dealer trading. Hitherto, regulators and plaintiffs have focused on how front-running directly affects the counter-parties in front-run trades, but my analysis shows that this perspective is too narrow. Front-running directly benefits dealers by shifting the cost of providing liquidity to their counter-parties (i.e. hedgers), but it also harms investors that have no direct involvement in the front-running trades because they are less willing to take speculative positions based on their own private price-relevant information. This collateral harm to investors may be more empirically significant than the direct harm suffered by dealers' counter-parties because front-running trades likely represent a very small fraction of all trades in the market.

Many regulator reports highlight the fact that Forex dealers at several major banks used electronic chat rooms to share information on their inventory positions and customer orders. My analysis shows that while the sharing of information on customer orders has equilibrium effects that impact all participants, in and of itself, such information-sharing has relatively minor effects on welfare. As a result, the welfare implications of unilateral and collusive front-running are quite similar. However, this finding comes with a caveat. My analysis focuses on equilibria in which information-sharing occurs symmetrically across all dealers. Collusive front-running may have different equilibrium implications when some dealers share information on pending customer orders while others keep their customer order information confidential. I leave this as a topic for future research.

References

Admati, Anat R. and Paul Pfleiderer. 1991. "Sunshine Trading and Financial Market Equilibrium." 4 (3):443–481. 4

Allen, Franklin and Douglas Gale. 1992. "Stock-price manipulation." *Review of financial studies* 5 (3):503–529. 4

Bacchetta, Philippe and Eric van Wincoop. 2008. "Higher Order Expections in Asset Pricing." *Journal of Money, Credit and Banking* 40 (5):837–866. 41

Bernhardt, Dan and Bart Taub. 2008. "Front-running dynamics." *Journal of Economic*

Theory 138 (1):288 – 296. URL http://www.sciencedirect.com/science/article/pii/S0022053107000798. 4

Bessembinder, Hendrik, Allen Carrion, Laura Tuttle, and Kumar Venkataraman. 2016. "Liquidity, resiliency and market quality around predictable trades: Theory and evidence." *Journal of Financial Economics* 121 (1):142 – 166. URL http://www.sciencedirect.com/science/article/pii/S0304405X16300113. 4

Bjønnes, Geir H. and Dagfinn Rime. 2005. "Dealer Behavior and Trading Systems in Foreign Exchange Markets." *Journal of Financial Economics* 75 (3):571–605. 13

Brunnermeier, Markus K and Lasse Heje Pedersen. 2005. "Predatory trading." *The Journal of Finance* 60 (4):1825–1863. 4

Carlin, Brunce Ian, Miguel Sousa Lobo, and S. Viswanathan. 2007. "Episodic Liquidity Crises: Cooperative and Predatory Trading." *The Journal of Finance* 62 (5):2235–2274. 4

Comerton-Forde, Carole and Tālis J. PutniGnVs. 2011. "Measuring closing price manipulation." *Journal of Financial Intermediation* 20 (2):135 – 158. 4

Cushing, David and Ananth Madhavan. 2000. "Stock returns and trading at the close." *Journal of Financial Markets* 3 (1):45 – 67. URL http://www.sciencedirect.com/science/article/pii/S1386418199000129. 4

Evans, Martin D. D. 2011. *Exchange-Rate Dynamics*. Princeton Series in International Finance. 2, 3, 5, 13

Evans, Martin D. D. and Richard K. Lyons. 2002. "Order flow and exchange rate dynamics." *Journal of Political Economy* 110 (1):170–180. 2, 13

Evans, Martin DD. 2018. "Forex trading and the WMR fix." *Journal of Banking & Finance* 87:233–247. 4, 5

Hart, Oliver D. 1977. "On the profitability of speculation." *The Quarterly Journal of Economics* :579–597. 4

Hillion, Pierre and Matti Suominen. 2004. "The manipulation of closing prices." *Journal of Financial Markets* 7 (4):351 – 375. 4

Ito, Takatoshi and Masahiro Yamada. 2015. "Was the Forex Fixing Fixed?" Working Paper 21518, National Bureau of Economic Research. 5

Kumar, Praveen and Duane J. Seppi. 1992. “Futures Manipulation with ‘Cash Settlement’.” 47:1485–1502. 4

Kyle, Albert S and S Viswanathan. 2008. “How to define illegal price manipulation.” *American Economic Review* 98 (2):274–79. 3

Lyons, Richard K. 1995. “Tests of Microstructural Hypothesis in the Foreign Exchange Market.” *Journal of Financial Economics* 39 (2-3):321–351. 13

———. 1997. “A Simultaneous Trade Model of the Foreign Exchange Hot Potato.” *Journal of Financial Economics* 42 (3-4):275–298. 2, 3, 13

Melvin, Michael and John Prins. 2015. “Equity hedging and exchange rates at the London 4 p.m. fix.” *Journal of Financial Markets* 22:50–72. 4

Osler, Carol and Alasdair Turnbull. 2017. “Dealer Trading at the Fix.” Working Papers 101R, Brandeis University, Department of Economics and International Businesss School. URL `https://EconPapers.repec.org/RePEc:brd:wpaper:101r`. 5, 26

Rochet, Jean-Charles and Jean-Luc Vila. 1994. “Insider trading without normality.” *The review of economic studies* 61 (1):131–152. 4

Schied, Alexander and Torsten Schöneborn. 2009. “Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets.” *Finance and Stochastics* 13 (2):181–204. 4

Vila, Jean-Luc. 1989. “Simple games of market manipulation.” *Economics Letters* 29 (1):21 – 26. 4

Appendix

This appendix provides the derivations of key equations in the text and the proofs of the Theorems 1 and 2.

A.1 Derivations

The following lemma is used to derive the equations for investors' and dealers' optimal Forex holdings in (19) and (22):

Lemma. *Let $\mathcal{Z} = \varkappa(a) + \Upsilon'\mathcal{X} + \mathcal{X}'\varXi\mathcal{X}$, where $\mathcal{X}$ is a $k \times 1$ vector of normally distributed random variables with zero means and covariance Σ. $\varXi$ is a symmetric $k \times k$ matrix, Υ is a $k \times 1$ vector function of the scalar a and $\varkappa(a)$ is a function of a. Provided that $\varTheta = I - 2\varXi\Sigma$ is positive definite,*

$$\mathcal{U} = \mathrm{E}\exp(\mathcal{Z}) = |\Theta|^{-1/2}\exp\left(\delta + \Upsilon'\mu + \tfrac{1}{2}\Upsilon'\varTheta^{-1}\Sigma\Upsilon\right). \tag{A.32}$$

Differentiating $\mathcal{U}$ with respect to a gives the following first order condition

$$\frac{\partial\varkappa}{\partial a} + \frac{\partial\Upsilon}{\partial a}\left(\tfrac{1}{2}\left(\varTheta^{-1}\Sigma + \left[\varTheta^{-1}\Sigma\right]'\right)\Upsilon\right) = 0, \tag{A.33}$$

where $\frac{\partial\Upsilon}{\partial a} = [\frac{\partial\Upsilon_1}{\partial a}, \frac{\partial\Upsilon_2}{\partial a}, ...]$.

Derivation of Equation (19)

Consider investor n's choice for $A^n_{t:\text{IV}}$ in round IV. In both the benchmark and front-running equilibria, overnight returns are $R_{t+1} = \Lambda_{\text{I}}A_t + \frac{1+r}{r}V_{t+1}$. Since V_{t+1} is a normal random variable and A_t is know to investors by round IV, R_{t+1} is normally distributed conditional on information, $\Omega^n_{t:\text{IV}}$. Choosing $A^n_{t:\text{IV}}$ to maximize expected utility $\mathcal{U}^n_{t:\text{IV}}$ subject to (4b) is therefore equivalent to maximizing $\mathrm{E}[A^n_{t:\text{IV}}\mathcal{R}_{t+1}|\Omega^n_{t:\text{IV}}] - \frac{1}{2}\omega\mathrm{V}[A^n_{t:\text{IV}}\mathcal{R}_{t+1}|\Omega^n_{t:\text{IV}}]$. The resulting first-order condition implies that

$$A^n_{t:\text{IV}} = \Theta^n_{\text{IV}|\text{R}}\mathrm{E}[R_{t+1}|\Omega^n_{t:\text{IV}}], \qquad \text{with} \qquad \Theta^n_{\text{IV}|\text{R}} = \frac{1}{\omega\mathrm{V}[\mathcal{R}_{t+1}|\Omega^{\text{IV}}_{n,t}]} \tag{A.34}$$

as shown in (19c).

In round I the investor chooses $A^n_{t:\text{I}}$ to maximize $\mathrm{E}\left[-\omega\exp(-\omega W^n_{t+1:\text{I}})|\Omega^n_{t:\text{I}}\right]$ subject to (4). In both the benchmark and front-running equilibria, intraday price changes and overnight returns are linear functions of normally distributed random variables, so applying the Lemma, the investor's problem is equivalent to minimizing $-\mathcal{U}$ in (A.32), with

$$\begin{aligned}
\mathcal{Z} &= -\omega W^n_{t+1:\mathrm{I}} = \varkappa + \Gamma'\mathcal{X} + \mathcal{X}'\varXi\mathcal{X}, \\
\varkappa &= -\omega(1+r)\left[(W^n_{t:\mathrm{I}} + S_{t:\mathrm{I}}Y^n_t) + A^{\mathrm{I}}_{n,t}\mathrm{E}[S_{t:\mathrm{III}} - S_{t:\mathrm{I}}|\Omega^n_{t:\mathrm{I}}] + \mathrm{E}[A^n_{t:\mathrm{III}}|\Omega^n_{t:\mathrm{I}}]\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^n_{t:\mathrm{I}}]\right] \\
&\qquad - \omega\mathrm{E}[A^n_{t:\mathrm{IV}}|\Omega^n_{t:\mathrm{I}}]\mathrm{E}[R_{t+1}|\Omega^n_{t:\mathrm{I}}], \\
\Upsilon &= \begin{bmatrix} -\omega(1+r)\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^n_{t:\mathrm{I}}] \\ -\omega\mathrm{E}[R_{t+1}|\Omega^n_{t:\mathrm{I}}] \\ -\omega\mathrm{E}[A^n_{t:\mathrm{IV}}|\Omega^n_{t:\mathrm{I}}] \\ -\omega(1+r)A^n_{t:\mathrm{I}} \\ -\omega(1+r)\mathrm{E}[A^n_{t:\mathrm{III}}|\Omega^n_{t:\mathrm{I}}] \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} \mathrm{U}[A^n_{t:\mathrm{III}}|\Omega^n_{t:\mathrm{I}}] \\ \mathrm{U}[A^n_{t:\mathrm{IV}}|\Omega^n_{t:\mathrm{I}}] \\ \mathrm{U}[R_{t+1}|\Omega^n_{t:\mathrm{I}}] \\ \mathrm{U}[\Delta S_{t:\mathrm{III}}|\Omega^n_{t:\mathrm{I}}] \\ \mathrm{U}[\Delta S_{t:\mathrm{IV}}|\Omega^n_{t:\mathrm{I}}] \end{bmatrix}, \\
\text{and} \qquad & \varXi = \begin{bmatrix} 0 & 0 & 0 & 0 & -\omega(1+r) \\ 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned} \tag{A.35}$$

where $\mathrm{U}[\chi|\Omega^n_{t:j}] = \chi - \mathrm{E}[\chi|\Omega^n_{t:j}]$ for any variable χ. The first-order condition for $A^n_{t:\mathrm{I}}$ is

$$0 = -\omega(1+r)\mathrm{E}^n_{t:\mathrm{I}}(S_{t:\mathrm{III}} - S_{t:\mathrm{I}}) + \begin{bmatrix} 0 & 0 & 0 & -\omega(1+r) & 0 \end{bmatrix} \left[\psi^{\mathrm{I}}_{ij}\right] \begin{bmatrix} -\omega(1+r)\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^n_{t:\mathrm{I}}] \\ -\omega\mathrm{E}[R_{t+1}|\Omega^n_{t:\mathrm{I}}] \\ -\omega\mathrm{E}[A^n_{t:\mathrm{IV}}|\Omega^n_{t:\mathrm{I}}] \\ -\omega(1+r)A^n_{t:\mathrm{I}} \\ -\omega(1+r)\mathrm{E}[A^n_{t:\mathrm{III}}|\Omega^n_{t:\mathrm{I}}] \end{bmatrix},$$

with $\left[\Psi^{\mathrm{I}}_{ij}\right] = \frac{1}{2}\left((I - 2\varXi\Sigma)^{-1}\Sigma + \left[(I - 2\varXi\Sigma)^{-1}\Sigma\right]'\right)$ and $\Sigma = \mathrm{V}[\mathcal{X}|\Omega^n_{t:\mathrm{I}}]$, where $\mathcal{X}$ and $\varXi$ are defined in (A.35). It proves convenient to rewrite this equation as

$$\begin{aligned}
&\begin{bmatrix} 1 & -\Psi^{\mathrm{I}}_{14}\omega(1+r) \end{bmatrix} \begin{bmatrix} \mathrm{E}[S_{t:\mathrm{III}} - S_{t:\mathrm{I}}|\Omega^n_{t:\mathrm{I}}] \\ \mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^n_{t:\mathrm{I}}] \end{bmatrix} = \\
&\qquad \omega(1+r)\begin{bmatrix} \Psi^{\mathrm{I}}_{44} & \Psi^{\mathrm{I}}_{45} \end{bmatrix} \begin{bmatrix} A^n_{t:\mathrm{I}} \\ \mathrm{E}[A^n_{t:\mathrm{III}}|\Omega^n_{t:\mathrm{I}}] \end{bmatrix} + \omega \begin{bmatrix} \Psi^{\mathrm{I}}_{24} & \Psi^{\mathrm{I}}_{34} \end{bmatrix} \begin{bmatrix} \mathrm{E}[R_{t+1}|\Omega^n_{t:\mathrm{I}}] \\ \mathrm{E}[A^n_{t:\mathrm{IV}}|\Omega^n_{t:\mathrm{I}}] \end{bmatrix}.
\end{aligned} \tag{A.36}$$

In round III the investor chooses $A^n_{t:\mathrm{III}}$ to minimizing $-\mathcal{U}$ in (A.32), with $\mathcal{Z} = -\omega W^n_{t+1:\mathrm{I}} =$

$\varkappa + \Gamma'\mathcal{X} + \mathcal{X}'\Xi\mathcal{X}$,

$$\varkappa = -\omega\left[(1+r)A^n_{t:\text{III}}\text{E}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{III}}] + \text{E}[A^n_{t:\text{IV}}|\Omega^n_{t:\text{III}}]\text{E}[R_{t+1}|\Omega^n_{t:\text{III}}]\right],$$

$$\Upsilon = \begin{bmatrix} -\omega\text{E}[R_{t+1}|\Omega^n_{t:\text{III}}] \\ -\omega\text{E}[A^n_{t:\text{IV}}|\Omega^n_{t:\text{III}}] \\ -\omega(1+r)A^n_{t:\text{III}} \end{bmatrix}, \qquad \mathcal{X} = \begin{bmatrix} \text{U}[A^n_{t:\text{IV}}|\Omega^n_{t:\text{III}}] \\ \text{U}[R_{t+1}|\Omega^n_{t:\text{III}}] \\ \text{U}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{III}}] \end{bmatrix}, \qquad \text{and} \qquad \Xi = \begin{bmatrix} 0 & -\omega & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{A.37}$$

The first order condition for $A^n_{t:\text{III}}$ is

$$0 = -\omega(1+r)\text{E}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{III}}] + \begin{bmatrix} 0 & 0 & -\omega(1+r) \end{bmatrix}\left[\Psi^{\text{III}}_{ij}\right]\begin{bmatrix} -\omega\text{E}[R_{t+1}|\Omega^n_{t:\text{III}}] \\ -\omega\text{E}[A^n_{t:\text{IV}}|\Omega^n_{t:\text{III}}] \\ -\omega(1+r)A^n_{t:\text{III}} \end{bmatrix},$$

with $\left[\Psi^{\text{III}}_{ij}\right] = \frac{1}{2}\left((I-2\Xi\Sigma)^{-1}\Sigma + \left[(I-2\Xi\Sigma)^{-1}\Sigma\right]'\right)$ and $\Sigma = \text{V}[\mathcal{X}|\Omega^n_{t:\text{III}}]$ where $\mathcal{X}$ and Ξ are defined in (A.37). Again, it is convenient to rewrite this equation as

$$\text{E}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{III}}] = \omega(1+r)\Psi^{\text{III}}_{33}A^n_{t:\text{III}} + \omega\begin{bmatrix} \Psi^{\text{III}}_{13} & \Psi^{\text{III}}_{23} \end{bmatrix}\begin{bmatrix} \text{E}[R_{t+1}|\Omega^n_{t:\text{III}}] \\ \text{E}[A^n_{t:\text{IV}}|\Omega^n_{t:\text{III}}] \end{bmatrix}. \tag{A.38}$$

To find investors round I position, we take expectations conditional on $\Omega^n_{t,\text{I}}$ on both sides of this equation and combine the result with (A.36) to give

$$\begin{bmatrix} 1 & -\omega(1+r)\Psi^{\text{I}}_{14} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \text{E}[S_{t:\text{III}} - S_{t:\text{I}}|\Omega^n_{t:\text{I}}] \\ \text{E}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{I}}] \end{bmatrix} =$$
$$\omega(1+r)\begin{bmatrix} \Psi^{\text{I}}_{44} & \Psi^{\text{I}}_{45} \\ 0 & \Psi^{\text{III}}_{33} \end{bmatrix}\begin{bmatrix} A^n_{t:\text{I}} \\ \text{E}[A^n_{t:\text{III}}|\Omega^n_{t:\text{I}}] \end{bmatrix} + \omega\begin{bmatrix} \Psi^{\text{I}}_{24} & \Psi^{\text{I}}_{34} \\ \Psi^{\text{III}}_{13} & \Psi^{\text{III}}_{23} \end{bmatrix}\begin{bmatrix} \text{E}[R_{t+1}|\Omega^n_{t:\text{I}}] \\ \text{E}[A^n_{t:\text{IV}}|\Omega^n_{t:\text{I}}] \end{bmatrix}.$$

Using the fact that $\text{E}[A^n_{t:\text{IV}}|\Omega^n_{t:\text{I}}] = \Theta^n_{\text{IV}|\text{R}}\text{E}[R_{t+1}|\Omega^n_{t:\text{I}}]$, from (A.34) we can rewrite this expression as

$$\begin{bmatrix} A^n_{t:\text{I}} \\ \text{E}[A^n_{t:\text{III}}|\Omega^n_{t:\text{I}}] \end{bmatrix} = \tfrac{1}{\omega(1+r)}\begin{bmatrix} \Psi^{\text{I}}_{44} & \Psi^{\text{I}}_{45} \\ 0 & \Psi^{\text{III}}_{33} \end{bmatrix}^{-1}$$
$$\times\left(\begin{bmatrix} 1 & -\omega(1+r)\Psi^{\text{I}}_{14} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \text{E}[S_{t:\text{III}} - S_{t:\text{I}}|\Omega^n_{t:\text{I}}] \\ \text{E}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{I}}] \end{bmatrix} - \omega\begin{bmatrix} \Psi^{\text{I}}_{24} + \Psi^{\text{I}}_{34}\Theta^n_{\text{IV}|\text{R}} \\ \Psi^{\text{III}}_{13} + \Psi^{\text{III}}_{23}\Theta^n_{\text{IV}|\text{R}} \end{bmatrix}\text{E}[R_{t+1}|\Omega^n_{t:\text{I}}]\right).$$

The first row of this matrix equation gives the equation for $A^n_{t:\text{I}}$ shown in (19a).

To find investors round III position, we combine (A.38) with the fact that $\mathrm{E}[A^n_{t:\text{IV}}|\Omega^n_{t:\text{III}}] = \Theta^n_{\text{IV}|\text{R}}\mathrm{E}[R_{t+1}|\Omega^n_{t:\text{III}}]$ from (A.34) to give

$$A^n_{t:\text{III}} = \frac{1}{\omega(1+r)\Psi^{\text{III}}_{33}}\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{III}}] - \frac{\Psi^{\text{III}}_{13} + \Psi^{\text{III}}_{23}\Theta^n_{\text{IV}|\text{R}}}{(1+r)\Psi^{\text{III}}_{33}}\mathrm{E}[R_{t+1}|\Omega^n_{t:\text{III}}],$$

as shown in equation (19b).

Derivation of Equation (22)

In round III, dealers choose $\wp^d_{t:\text{III}}$ to maximize $\mathcal{U}^d_{t:\text{III}}$ with information $\Omega^d_{t:\text{III}}$. In both the benchmark and front-running equilibria, the dealers' budget constraint becomes

$$W^d_{t+1:\text{I}} = (1+r)\left[W^{\text{I}}_{d,t} - Z^d_{t:\text{I}}\Delta S_{t:\text{II}} + (\wp^d_{t:\text{II}} - \xi^d_{t:\text{II}})\Delta S_{t:\text{III}} + (\wp^d_{t:\text{III}} - \xi^d_{t:\text{III}})\Delta S_{t:\text{IV}}\right].$$

By Lemma, choosing $\wp^d_{t:\text{III}}$ to maximize $-\mathrm{E}[\exp(-\theta W^d_{t+1:\text{I}})|\Omega^d_{t:\text{III}}]$ is equivalent to minimizing $-\mathcal{U}$ in (A.32), with $\mathcal{Z} = -\omega W^d_{t+1:\text{I}} = \varkappa + \Upsilon'\mathcal{X} + \mathcal{X}'\Xi\mathcal{X}$, where

$$\varkappa = -\omega(1+r)\left[W^d_{t:\text{III}} + \wp^d_{t:\text{III}}\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^d_{t:\text{III}}]\right]$$

$$\Upsilon = \begin{bmatrix} \omega(1+r)\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^d_{t:\text{III}}] \\ -\omega(1+r)\wp^d_{t:\text{III}} \end{bmatrix}, \qquad \mathcal{X} = \begin{bmatrix} \xi^d_{t:\text{III}} \\ \mathrm{U}[\Delta S_{t:\text{IV}}|\Omega^d_{t:\text{III}}] \end{bmatrix}, \qquad \text{and} \qquad \Xi = \begin{bmatrix} 0 & \omega(1+r) \\ 0 & 0 \end{bmatrix}. \tag{A.39}$$

The first order condition for $\wp^d_{t:\text{III}}$ is

$$-\omega(1+r)\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^d_{t:\text{III}}] + \begin{bmatrix} 0 & -\omega(1+r) \end{bmatrix}\begin{bmatrix} \Psi^{\text{III}}_{11} & \Psi^{\text{III}}_{21} \\ \Psi^{\text{III}}_{21} & \Psi^{\text{III}}_{22} \end{bmatrix}\begin{bmatrix} \omega(1+r)\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^d_{t:\text{III}}] \\ -\omega(1+r)\wp^d_{t:\text{III}} \end{bmatrix} = 0,$$

with $\left[\Psi^{\text{III}}_{ij}\right] = \frac{1}{2}\left((I-2\Xi\Sigma)^{-1}\Sigma + \left[(I-2\Xi\Sigma)^{-1}\Sigma\right]'\right)$ and $\Sigma = \mathrm{V}[\mathcal{X}|\Omega^d_{t:\text{III}}]$ where $\mathcal{X}$ and Ξ are defined in (A.39). This expression simplifies to

$$\wp^d_{t:\text{III}} = \Phi^d_{\text{III}|s_{\text{IV}}}\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^d_{t:\text{III}}] \qquad \text{where} \qquad \Phi^d_{\text{III}|s_{\text{IV}}} = \frac{1+\omega(1+r)\Psi^{\text{III}}_{21}}{\omega(1+r)\Psi^{\text{III}}_{22}},$$

as shown in (22b).

In round II, dealers choose $\wp^d_{t:\text{II}}$to minimizing $-\mathcal{U}$ in (A.32), with $\mathcal{Z} = -\omega W^d_{t+1:\text{I}} = \varkappa + \Upsilon'\mathcal{X} +$

$\mathcal{X}'\varXi\mathcal{X}$, where

$$\varkappa = -\omega(1+r)\left[W^d_{t:\mathrm{III}} + \wp^d_{t:\mathrm{II}}\mathrm{E}[\Delta S_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{II}}] + \mathrm{E}[\wp^d_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{II}}]\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^d_{t:\mathrm{II}}]\right],$$

$$\Upsilon = \begin{bmatrix} \omega(1+r)\mathrm{E}[\Delta S_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{II}}] \\ -\omega(1+r)\wp^d_{t:\mathrm{II}} \\ -\omega(1+r)\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^d_{t:\mathrm{II}}] \\ -\omega(1+r)\mathrm{E}[\wp^d_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{II}}] \end{bmatrix}, \qquad \mathcal{X} = \begin{bmatrix} \xi^d_{t:\mathrm{II}} \\ \mathrm{U}[\Delta S_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{II}}] \\ \mathrm{U}[\wp^d_{t:\mathrm{III}} - \xi^d_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{II}}] \\ \mathrm{U}[\Delta S_{t:\mathrm{IV}}|\Omega^d_{t:\mathrm{II}}] \end{bmatrix}, \qquad \text{and}$$

$$\varXi = \begin{bmatrix} 0 & \omega(1+r) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega(1+r) \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{A.40}$$

The first order condition for $\wp^d_{t:\mathrm{II}}$ is

$$0 = -\omega(1+r)\mathrm{E}[\Delta S_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{II}}] + \begin{bmatrix} 0 & -\omega(1+r) & 0 & 0 \end{bmatrix}\left[\Psi^{\mathrm{II}}_{i,j}\right]\begin{bmatrix} \omega(1+r)\mathrm{E}[\Delta S_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{II}}] \\ -\omega(1+r)\wp^d_{t:\mathrm{II}} \\ -\omega(1+r)\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^d_{t:\mathrm{II}}] \\ -\omega(1+r)\mathrm{E}[\wp^d_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{II}}] \end{bmatrix},$$

with $\left[\Psi^{\mathrm{II}}_{ij}\right] = \frac{1}{2}\left((I-2\varXi\Sigma)^{-1}\Sigma + \left[(I-2\varXi\Sigma)^{-1}\Sigma\right]'\right)$ and $\Sigma = \mathrm{V}[\mathcal{X}|\Omega^d_{t:\mathrm{III}}]$ where $\mathcal{X}$ and $\varXi$ are defined in (A.40). Simplifying this expression using the fact that $\mathrm{E}[\wp^d_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{II}}] = \Phi^d_{\mathrm{III}|s_{\mathrm{IV}}}\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^d_{t:\mathrm{II}}]$ gives

$$\wp^d_{t:\mathrm{II}} = \left(\frac{1}{\omega(1+r)\Psi^{\mathrm{II}}_{22}} + \frac{\Psi^{\mathrm{II}}_{21}}{\Psi^{\mathrm{II}}_{22}}\right)\mathrm{E}[\Delta S_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{II}}] + \left[\left(1 - \frac{\Psi^{\mathrm{II}}_{24}}{\Psi^{\mathrm{II}}_{22}}\right)\Phi^d_{\mathrm{III}|s_{\mathrm{IV}}} - \frac{\Psi^{\mathrm{II}}_{23}}{\Psi^{\mathrm{II}}_{22}}\right]\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^d_{t:\mathrm{II}}],$$

as shown in (22a).

Dealer Price Quotes

Since dealers simultaneously quote prices at the start of each round, and prices are good for orders of any size, all dealers must quote the same price to avoid the expected utility loss associated with arbitrage. This means that equilibrium prices can only be a function of dealers common information at the start of each trading round, $\mathcal{I}_{t,j} = \bigcap_d \mathcal{I}^d_{t,j}$. This restriction allows us to derived the following equations for prices from the market clearing and efficient risk-sharing that apply in both the benchmark and front-running equilibria:

$$S_{t:\text{I}} = S_{t:\text{II}} \tag{A.41a}$$
$$S_{t:\text{II}} = \mathrm{E}\left[S_{t:\text{III}}|\mathcal{I}_{t:\text{II}}\right] + \tfrac{\Phi_{\text{II}|s_{\text{IV}}}}{\Phi_{\text{II}|s_{\text{III}}}}\mathrm{E}\left[\Delta S_{t:\text{IV}}|\mathcal{I}_{t:\text{II}}\right] + \tfrac{1}{\Phi_{\text{II}|s_{\text{III}}}}\mathrm{E}\left[Z_{t:\text{I}}|\,\mathcal{I}_{t:\text{II}}\right] \tag{A.41b}$$
$$S_{t:\text{III}} = \mathrm{E}\left[S_{t:\text{IV}}\ |\mathcal{I}_{t:\text{III}}\right] + \tfrac{1}{\Phi_{\text{III}|s_{\text{IV}}}}\mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}}|\,\mathcal{I}_{t:\text{III}}\right] \tag{A.41c}$$
$$S_{t:\text{IV}} = \tfrac{1}{1+r}\mathrm{E}[S_{t+1:\text{I}} + \mathcal{P}_{t+1}|\mathcal{I}_{t:\text{IV}}] - \tfrac{1}{(1+r)\Theta_{\text{IV}|\text{R}}}\mathrm{E}\left[A_t|\mathcal{I}_{t:\text{IV}}\right] \tag{A.41d}$$

where $\Phi_{j|i} = \sum_{d=1}^{\text{D}} \Phi_{j|i}^d$ and $\Theta_{i|j} = \int_0^1 \Theta_{i|j}^n dn$.

To derive these equations, first note that in an efficient risk sharing equilibrium, investors hold the entire stock of Forex in round IV, so

$$A_t = \int_0^1 A_{t:\text{IV}}^n dn = \int_0^1 \Theta_{\text{IV}|\text{R}}^n \mathrm{E}[R_{t+1}|\Omega_{t:\text{IV}}^n]dn.$$

Dealers' common information $\mathcal{I}_{t:\text{IV}}$ is a subset of investors information $\Omega_{t:\text{IV}}^n$, so applying the Law of Iterated expectations

$$\mathrm{E}\left[A_t|\mathcal{I}_{t:\text{IV}}\right] = \left(\int_0^1 \Theta_{\text{IV}|\text{R}}^n dn\right)\mathrm{E}[R_{t+1}|\mathcal{I}_{t:\text{IV}}]. \tag{A.42}$$

Combining this equation with the definition of overnight returns, gives (A.41d).

In round III, market clearing implies

$$X_{t:\text{III}} = \sum\nolimits_{d=1}^{\text{D}} \wp_{t:\text{III}}^d + \tfrac{1}{\text{D}}\sum\nolimits_{d=1}^{\text{D}} \mathrm{E}\left[X_{t:\text{III}}|\Omega_{t:\text{III}}^d\right] + \mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}}. \tag{A.43}$$

Substituting for $\wp_{t:\text{III}}^d$ from (22b), taking expectations conditioned on $\mathcal{I}_{t:\text{III}}$, and simplifying gives

$$0 = \Phi_{\text{III}|s_{\text{IV}}}\mathrm{E}\left[\Delta S_{t:\text{IV}}\ |\mathcal{I}_{t:\text{III}}\right] + \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}}|\,\mathcal{I}_{t:\text{III}}\right], \tag{A.44}$$

where $\Phi_{\text{III}|s_{\text{IV}}} = \sum_{d=1}^{\text{D}} \Phi_{\text{III}|s_{\text{IV}}}^d$. Rearranging this equation using the fact that $\Delta S_{t:\text{IV}} = S_{t:\text{IV}} - S_{t:\text{III}}$, produces (A.41c).

In round II, market clearing implies

$$X_{t:\text{II}} = \sum\nolimits_{d=1}^{\text{D}} \wp_{t:\text{II}}^d + \tfrac{1}{\text{D}}\sum\nolimits_{d=1}^{\text{D}} \mathrm{E}\left[X_{t:\text{II}}|\Omega_{t:\text{II}}^d\right] + \mathbb{O}_{t:\text{I}}. \tag{A.45}$$

As above, we substitute for $\wp_{t:\text{II}}^d$ from (22a), take expectations conditioned on $\mathcal{I}_{t:\text{II}}$ and simplify to obtain

$$0 = \Phi_{\text{II}|s_{\text{III}}}\text{E}[\Delta S_{t:\text{III}}|\mathcal{I}_{t:\text{II}}] + \Phi_{\text{II}|s_{\text{IV}}}\text{E}[\Delta S_{t:\text{IV}}|\mathcal{I}_{t:\text{II}}] + \text{E}\left[Z_{t:\text{I}}|\mathcal{I}_{t:\text{II}}\right], \tag{A.46}$$

where $\Phi_{\text{II}|s_{\text{III}}} = \sum_{d=1}^{\text{D}} \Phi^{d}_{\text{II}|s_{\text{III}}}$ and $\Phi_{\text{II}|s_{\text{IV}}} = \sum_{d=1}^{\text{D}} \Phi^{d}_{\text{II}|s_{\text{IV}}}$. Rearranging this equation produces (A.41b).

Finally, dealers quote the same price in rounds I and II to eliminate the risk of capital losses from filling investors' round-I orders. These losses arise from the $(A^{d}_{t-1:\text{IV}} - \mathbb{O}^{d}_{t:\text{I}})\Delta S_{t:\text{II}}$ term in (9a). This is feasible because there is no change in dealers' common information between the start of rounds Iand II.

Risk Premia

In both the benchmark and front-running equilibria, dealers' observations on $X_{t:\text{II}}$ and $X_{t:\text{III}}$ are sufficient to reveal the values of Y_t and H_t by the start of round IV, so dealers' estimates of A_t are $\text{E}[A_t|\mathcal{I}_{t:\text{IV}}] = \sum_{j=0}^{\infty}(Y_{t-j} - H_{t-j}) = A_t$. Combining this result with (A.42) gives the following equation for the overnight risk premium

$$\text{E}[R_{t+1}|\mathcal{I}_{t:\text{IV}}] = \Theta^{-1}_{\text{IV}|\text{R}} A_t = \Lambda_{\text{I}} A_t,$$

as shown in (24).

The first step in determining the intraday risk premia is to identify aggregate investors' orders. Combining the definitions $\mathbb{O}^{n}_{t:\text{I}} \equiv A^{n}_{t:\text{I}} - A^{n}_{t-1:\text{IV}} - Y^{n}_{t}$ and $\mathbb{O}^{n}_{t:\text{III}} \equiv A^{n}_{t:\text{III}} - A^{n}_{t:\text{I}}$ with (19) and aggregating, gives

$$\mathbb{O}_{t:\text{I}} = \int_0^1 \left\{ \Theta^{n}_{\text{I}|s_{\text{III}}}\text{E}[\Delta S_{t:\text{III}}|\Omega^{n}_{t:\text{I}}] + \Theta^{n}_{\text{I}|s_{\text{IV}}}\text{E}[\Delta S_{t:\text{IV}}|\Omega^{n}_{t:\text{I}}] + \Theta^{n}_{\text{I}|\text{R}}\text{E}[R_{t+1}|\Omega^{n}_{t:\text{I}}] \right\} dn - A_{t-1} - Y_t, \tag{A.47a}$$

$$\mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} = \int_0^1 \left\{ \Theta^{n}_{\text{III}|s_{\text{IV}}}\text{E}[\Delta S_{t:\text{IV}}|\Omega^{n}_{t:\text{III}}] + \Theta^{n}_{\text{III}|R}\text{E}[\mathcal{R}_{t+1}|\Omega^{n}_{t:\text{III}}] \right\} dn - A_{t-1} - Y_t. \tag{A.47b}$$

To find the round III premium, $\text{E}\left[\Delta S_{t:\text{IV}}|\mathcal{I}_{t:\text{I}}\right]$, we take expectations on both sides of (A.47b)

$$\begin{aligned}\text{E}\left[\mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}}|\,\mathcal{I}_{t:\text{I}}\right] &= \Theta_{\text{III}|s_{\text{IV}}}\text{E}[\Delta S_{t:\text{IV}}|\mathcal{I}_{t:\text{I}}] + \Theta_{\text{III}|R}\text{E}[R_{t+1}|\mathcal{I}_{t:\text{I}}] - A_{t-1} - \text{E}\left[Y_t|\,\mathcal{I}_{t:\text{I}}\right], \\ &= \left(\Theta_{\text{III}|s_{\text{IV}}}\Lambda_{\text{IV}} + \Theta_{\text{III}|R}\Lambda_{\text{I}} - 1\right) A_{t-1},\end{aligned} \tag{A.48}$$

where $\Theta_{\text{III}|s_{\text{IV}}} = \int_0^1 \Theta^{n}_{\text{III}|s_{\text{IV}}} dn$ and $\Theta_{\text{III}|R} = \int_0^1 \Theta^{n}_{\text{III}|R} dn$. Notice that $\text{E}\left[Y_t|\,\mathcal{I}_{t:\text{I}}\right] = 0$ because Y_t is an i.i.d. mean-zero variable and dealers have no information about the realization of Y_t at the start of round I. Next, we take expectations of (A.44) conditioned $\mathcal{I}_{t:\mathrm{I}}$ to give

$$0 = \Phi_{\mathrm{III}|s_{\mathrm{IV}}} \mathrm{E}\left[\Delta S_{t:\mathrm{IV}} \; |\mathcal{I}_{t:\mathrm{I}}\right] + \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\mathrm{III}} + \mathbb{O}_{t:\mathrm{I}} | \, \mathcal{I}_{t:\mathrm{I}}\right]. \tag{A.49}$$

Since dealers have no common information at the start of round I about the H_t shock, $\mathrm{E}\left[\mathbb{H}_t | \, \mathcal{I}_{t:\mathrm{I}}\right] = (1-\psi)A_{t-1}$. Combining this condition with (A.48), (A.49) and the definition, $\mathrm{E}\left[\Delta S_{t:\mathrm{IV}} | \mathcal{I}_{t:\mathrm{I}}\right] = \Lambda_{\mathrm{IV}} A_{t-1}$, gives

$$0 = \left[\Phi_{\mathrm{III}|s_{\mathrm{IV}}} \Lambda_{\mathrm{IV}} + \left(\Theta_{\mathrm{III}|s_{\mathrm{IV}}} \Lambda_{\mathrm{IV}} + \Theta_{\mathrm{III}|R} \Lambda_{\mathrm{I}} \psi - 1\right) + 1 - \psi\right] A_{t-1}.$$

This condition must hold for all values of A_{t-1}, so

$$\Lambda_{\mathrm{IV}} = \frac{\psi(1 - \Lambda_{\mathrm{I}} \Theta_{\mathrm{III}|R})}{\Phi_{\mathrm{III}|s_{\mathrm{IV}}} + \Theta_{\mathrm{III}|s_{\mathrm{IV}}}},$$

as shown in (25b).

To find the round II premium, $\mathrm{E}\left[\Delta S_{t:\mathrm{III}} | \mathcal{I}_{t:\mathrm{I}}\right]$, we take expectations on both sides of (A.47a)

$$\begin{aligned}
\mathrm{E}\left[\mathbb{O}_{t:\mathrm{I}} | \, \mathcal{I}_{t:\mathrm{I}}\right] &= \Theta_{\mathrm{I}|s_{\mathrm{III}}} \mathrm{E}[\Delta S_{t:\mathrm{III}} | \mathcal{I}_{t:\mathrm{I}}] + \Theta_{\mathrm{I}|s_{\mathrm{IV}}} \mathrm{E}[\Delta S_{t:\mathrm{IV}} | \mathcal{I}_{t:\mathrm{I}}] + \Theta_{\mathrm{I}|\mathrm{R}} \mathrm{E}[R_{t+1} | \mathcal{I}_{t:\mathrm{I}}] - A_{t-1} - \mathrm{E}\left[Y_t | \, \mathcal{I}_{t:\mathrm{I}}\right], \\
&= \Theta_{\mathrm{I}|s_{\mathrm{III}}} \Lambda_{\mathrm{III}} A_{t-1} + \Theta_{\mathrm{I}|s_{\mathrm{IV}}} \Lambda_{\mathrm{IV}} A_{t-1} + \Theta_{\mathrm{I}|\mathrm{R}} \Lambda_{\mathrm{I}} A_{t-1} - A_{t-1},
\end{aligned}$$

where $\Theta_{\mathrm{I}|s_{\mathrm{III}}} = \int_0^1 \Theta^n_{\mathrm{I}|s_{\mathrm{III}}} dn$, $\Theta_{\mathrm{I}|s_{\mathrm{IV}}} = \int_0^1 \Theta^n_{\mathrm{I}|s_{\mathrm{IV}}} dn$ and $\Theta_{\mathrm{I}|\mathrm{R}} = \int_0^1 \Theta^n_{\mathrm{I}|\mathrm{R}} dn$. We then combine this equation with the expectation of (A.46)

$$0 = \Phi_{\mathrm{II}|s_{\mathrm{III}}} \mathrm{E}[\Delta S_{t:\mathrm{III}} | \mathcal{I}_{t:\mathrm{I}}] + \Phi_{\mathrm{II}|s_{\mathrm{IV}}} \mathrm{E}[\Delta S_{t:\mathrm{IV}} | \mathcal{I}_{t:\mathrm{I}}] + \mathrm{E}\left[Z_{t:\mathrm{I}} | \mathcal{I}_{t:\mathrm{I}}\right],$$

to give

$$0 = \left[\Phi_{\mathrm{II}|s_{\mathrm{III}}} \Lambda_{\mathrm{III}} + \Phi_{\mathrm{II}|s_{\mathrm{IV}}} \Lambda_{\mathrm{IV}} + \left(\Theta_{\mathrm{I}|s_{\mathrm{III}}} \Lambda_{\mathrm{III}} + \Theta_{\mathrm{I}|s_{\mathrm{IV}}} \Lambda_{\mathrm{IV}} + \Theta_{\mathrm{I}|\mathrm{R}} \Lambda_{\mathrm{I}} \psi - 1\right)\right] A_{t-1}.$$

As above, the term in [.] must equal zero, which implies that

$$\Lambda_{\mathrm{III}} = \frac{1 - \Theta_{\mathrm{I}|\mathrm{R}} \Lambda_{\mathrm{I}} \psi}{\Phi_{\mathrm{II}|s_{\mathrm{III}}} + \Theta_{\mathrm{I}|s_{\mathrm{III}}}} - \left(\frac{\Phi_{\mathrm{II}|s_{\mathrm{IV}}} + \Theta_{\mathrm{I}|s_{\mathrm{IV}}}}{\Phi_{\mathrm{II}|s_{\mathrm{III}}} + \Theta_{\mathrm{I}|s_{\mathrm{III}}}}\right) \Lambda_{\mathrm{IV}},$$

as shown in (25a).

Returns and Intraday Price Changes

In both the benchmark and front-running equilibria, intraday price changes and overnight returns implied by dealers' quotes are given by

$$
\begin{aligned}
R_t &= \Lambda_{\text{I}} A_{t-1} + \tfrac{1+r}{r} V_t, && \text{(A.50a)}\\
\Delta S_{t:\text{III}} &= \Lambda_{\text{III}} A_{t-1} - \tfrac{1}{r}\Lambda(\mathrm{E}[A_t|\mathcal{I}_{t:\text{III}}] - \mathrm{E}[A_t|\mathcal{I}_{t:\text{II}}]), && \text{(A.50b)}\\
&\quad + \Phi^{-1}_{\text{III}|s_{\text{IV}}} \left\{ \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} | \, \mathcal{I}_{t:\text{III}}\right] - \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} | \, \mathcal{I}_{t:\text{II}}\right] \right\}, \\
\Delta S_{t:\text{IV}} &= \Lambda_{\text{IV}} A_{t-1} - \tfrac{1}{r}\Lambda(A_t - \mathrm{E}[A_t|\mathcal{I}_{t:\text{III}}]) \\
&= -\Phi^{-1}_{\text{III}|s_{\text{IV}}} \left\{ \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} | \, \mathcal{I}_{t:\text{III}}\right] - \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} | \, \mathcal{I}_{t:\text{II}}\right] \right\}. && \text{(A.50c)}
\end{aligned}
$$

To derive these equations, we first identify the prices dealers quote in round IV. (25a) and (A.41d) imply that $\mathrm{E}\left[S_{t+1:\text{IV}} - S_{t+1:\text{I}}|\mathcal{I}_{t:\text{IV}}\right] = (\Lambda_{\text{IV}} + \Lambda_{\text{III}})\mathrm{E}\left[A_t|\mathcal{I}_{t:\text{IV}}\right]$. Combining this expression with (25a) and iterating forward gives

$$
S_{t:\text{IV}} = \sum_{i=1}^{\infty} \left(\tfrac{1}{1+r}\right)^i \mathrm{E}[\mathcal{P}_{t+1} - \Lambda A_{t+i-1}|\mathcal{I}_{t:\text{IV}}], \tag{A.51}
$$

where $\Lambda = \Lambda_{\text{I}} + \Lambda_{\text{III}} + \Lambda_{\text{IV}}$. Substituting for the $\mathcal{P}_t$ process from (2) and the A_t process from (14), (A.51) simplifies to

$$
S_{t:\text{IV}} = \tfrac{1}{r} F_t - \tfrac{\Lambda}{1-\psi+r} A_t. \tag{A.52}
$$

By definition, $S_{t:\text{I}} = \mathrm{E}\left[S_{t:\text{IV}} \; |\mathcal{I}_{t:\text{I}}\right] - (\Lambda_{\text{IV}} + \Lambda_{\text{III}})A_{t-1}$, so substituting for $S_{t:\text{IV}}$ gives

$$
S_{t:\text{I}} = \frac{1}{r}\mathcal{P}_t - \left[\tfrac{\psi}{1-\psi+r}\Lambda + \Lambda_{\text{III}} + \Lambda_{\text{II}}\right] A_{t-1}.
$$

Leading this expression forward by one day, and substituting in the definition for overnight returns produces

$$
R_{t+1} = S_{t+1:\text{I}} + \mathcal{P}_{t+1} - (1+r)S_{t:\text{IV}} = \frac{1+r}{r} V_{t+1} + \Lambda_{\text{I}} A_t,
$$

as shown in (A.50a).

To find expressions for the intraday price changes, we first use (A.41c) to substitute for $S_{t:\text{III}}$ in the definition $\Delta S_{t:\text{III}} = \Lambda_{\text{III}} A_{t-1} + S_{t:\text{III}} - \mathrm{E}[S_{t:\text{III}}|\mathcal{I}_{t:\text{II}}]$. This produces

$$\begin{aligned}
\Delta S_{t:\text{III}} &= \Lambda_{\text{III}} A_{t-1} + \mathrm{E}\left[S_{t:\text{IV}}\ |\mathcal{I}_{t:\text{III}}\right] - \mathrm{E}\left[S_{t:\text{IV}}\ |\mathcal{I}_{t:\text{II}}\right] \\
&\qquad + \Phi^{-1}_{\text{III}|s_{\text{IV}}} \left\{ \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{III}}\right] - \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{II}}\right] \right\}, \\
&= \Lambda_{\text{III}} A_{t-1} - \tfrac{1}{1-\psi+r} \Lambda (\mathrm{E}[A_t|\mathcal{I}_{t:\text{III}}] - \mathrm{E}[A_t|\mathcal{I}_{t:\text{II}}]) \\
&\qquad + \Phi^{-1}_{\text{III}|s_{\text{IV}}} \left\{ \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{III}}\right] - \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{II}}\right] \right\},
\end{aligned}$$

as shown in (A.50b). Equation (A.41c) also implies that

$$\begin{aligned}
\Delta S_{t:\text{IV}} &= -\Phi^{-1}_{\text{III}|s_{\text{IV}}} \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{III}}\right] + S_{t:\text{IV}} - \left[S_{t:\text{IV}}\ |\mathcal{I}_{t:\text{III}}\right], \\
&= -\Phi^{-1}_{\text{III}|s_{\text{IV}}} \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{II}}\right] + S_{t:\text{IV}} - \left[S_{t:\text{IV}}\ |\mathcal{I}_{t:\text{III}}\right] \\
&\qquad - \Phi^{-1}_{\text{III}|s_{\text{IV}}} \left(\mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{III}}\right] - \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{II}}\right] \right), \\
&= \Lambda_{\text{IV}} A_{t-1} - \tfrac{1}{1-\psi+r} \Lambda (A_t - \mathrm{E}[A_t|\mathcal{I}_{t:\text{III}}]) \\
&\qquad - \Phi^{-1}_{\text{III}|s_{\text{IV}}} \left(\mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{III}}\right] - \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{II}}\right] \right),
\end{aligned}$$

as shown in (A.50c).

A.2 Proof of Theorem 1

There are two steps in the Proof: In step 1, I show that intraday price changes and overnight returns follow (15) under the conjectured equilibrium process for aggregate inter-dealer order flows in (17). In step 2 I verify that the optimal trading decisions of dealers and investors produce these order flows.

Step One

Since aggregate order flows augment dealers' common information at the start of rounds III and IV under the conjectured process for order flows in (17), dealers have common knowledge concerning Y_t when quoting round III prices, and common knowledge concerning H_t when quoting round IV prices. Consequently, $A_t - \mathrm{E}[A_t|\mathcal{I}_{t:\text{III}}] = -H_t$ and $\mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{III}}\right] - \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{II}}\right] = \mathrm{E}\left[\mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{III}}\right] - \mathrm{E}\left[\mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} |\, \mathcal{I}_{t:\text{II}}\right] = -\varsigma Y_t$, for some unknown parameter ς. Hence

$$\Delta S_{t:\text{IV}} = \Lambda_{\text{IV}} A_{t-1} + \tfrac{1}{1-\psi+r} \Lambda H_t + \Phi^{-1}_{\text{III}|s_{\text{IV}}} \varsigma Y_t,$$

and so

$$\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{III}}] = \Lambda_{\text{IV}} A_{t-1} + \Phi^{-1}_{\text{III}|s_{\text{IV}}} \varsigma Y_t.$$

Overnight returns are $R_{t+1} = \frac{1+r}{r} V_{t+1} + \Lambda_{\text{I}} A_t$, so

$$\mathrm{E}[R_{t+1}|\Omega^n_{t:\text{III}}] = \Lambda_{\text{I}} \left(Y_t + \psi A_{t-1} \right).$$

Substituting these equations into (A.47b) gives

$$\begin{aligned} \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} &= \int_0^N \left\{ \Theta^n_{\text{III}|s_{\text{IV}}} \left(\Lambda_{\text{IV}} A_{t-1} + \Phi^{-1}_{\text{III}|s_{\text{IV}}} \varsigma Y_t \right) + \Theta^n_{\text{III}|R} \Lambda_{\text{I}} \left(Y_t + \psi A_{t-1} \right) \right\} dn - A_{t-1} - Y_t, \\ &= \left(\Theta_{\text{III}|s_{\text{IV}}} \Phi^{-1}_{\text{III}|s_{\text{IV}}} \varsigma + \Theta_{\text{III}|R} \Lambda_{\text{I}} - 1 \right) Y_t + \left(\Theta_{\text{III}|s_{\text{IV}}} \Lambda_{\text{IV}} + \Theta_{\text{III}|R} \Lambda_{\text{I}} \psi - 1 \right) A_{t-1}. \end{aligned}$$

So $-\varsigma = \Theta_{\text{III}|s_{\text{IV}}} \Phi^{-1}_{\text{III}|s_{\text{IV}}} \varsigma + \Theta_{\text{III}|R} \Lambda_{\text{I}} - 1$. Solving for ς gives $\varsigma = \frac{1-\Theta_{\text{III}|R}\Lambda_{\text{I}}}{1+\Theta_{\text{III}|s_{\text{IV}}}\Phi^{-1}_{\text{III}|s_{\text{IV}}}}$ and hence

$$\mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} = -\frac{\Phi_{\text{III}|s_{\text{IV}}} \Lambda_{\text{IV}}}{\psi} Y_t + \left(\Theta_{\text{III}|s_{\text{IV}}} \Lambda_{\text{IV}} + \Theta_{\text{III}|R} \Lambda_{\text{I}} \psi - 1 \right) A_{t-1}.$$

Now from (A.50b)

$$\begin{aligned} \Delta S_{t:\text{III}} &= \Lambda_{\text{III}} A_{t-1} - \tfrac{1}{1-\psi+r} \Lambda (\mathrm{E}[A_t|\mathcal{I}_{t:\text{III}}] - \mathrm{E}[A_t|\mathcal{I}_{t:\text{II}}]) \\ &\qquad + \Phi^{-1}_{\text{III}|s_{\text{IV}}} \left\{ \mathrm{E}\left[H_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} | \, \mathcal{I}_{t:\text{III}} \right] - \mathrm{E}\left[H_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} | \, \mathcal{I}_{t:\text{II}} \right] \right\}, \\ &= \Lambda_{\text{III}} A_{t-1} - \left(\tfrac{1}{1-\psi+r} \Lambda + \tfrac{1}{\psi} \Lambda_{\text{IV}} \right) Y_t \end{aligned}$$

as shown in (15c). Similarly, $A_t - \mathrm{E}[A_t|\mathcal{I}_{t:\text{III}}] = -H_t$, so (A.50c) becomes

$$\begin{aligned} \Delta S_{t:\text{IV}} &= \Lambda_{\text{IV}} A_{t-1} - \tfrac{1}{1-\psi+r} \Lambda (A_t - \mathrm{E}[A_t|\mathcal{I}_{t:\text{III}}]) \\ &\qquad - \Phi^{-1}_{\text{III}|s_{\text{IV}}} \left(\mathrm{E}\left[H_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} | \, \mathcal{I}_{t:\text{III}} \right] - \mathrm{E}\left[H_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} | \, \mathcal{I}_{t:\text{II}} \right] \right), \\ &= \Lambda_{\text{IV}} A_{t-1} + \tfrac{1}{\psi} \Lambda_{\text{IV}} Y_t + \tfrac{1}{1-\psi+r} \Lambda H_t \end{aligned}$$

as shown in (15d). This completes our verification of (15).

Step Two

Now we verify that equilibrium order flows follow (17). To begin, consider the implications of (15) for investors' individual forecasts. Recall that individual investors estimate aggregate foreign income in round I based on their own income is $Y^n_t = Y_t + \varepsilon^n_t$, where Y_t and ε^n_t are i.i.d. mean zero random variables with variances, σ^2_{Y} and σ^2_{ε}, respectively. By the Projection Theorem, $\mathrm{E}[Y_t|\Omega^n_{t:\mathrm{I}}] = \mathcal{G}^n_{\mathrm{I}} Y^n_t$, with $\mathcal{G}^n_{\mathrm{Y}} = \frac{\sigma^2_{\mathrm{Y}}}{\sigma^2_{\mathrm{Y}}+\sigma^2_{\varepsilon}}$. In the benchmark equilibrium, investors hold the entire stock of Forex overnight, so $A_{t-1} = \mathrm{E}[A_{t-1}|\Omega^n_{t:\mathrm{I}}]$. Combining these results with (15c) and (15d), gives

$$\begin{aligned}
\mathrm{E}[\Delta S_{t:\mathrm{III}}|\Omega^n_{t:\mathrm{I}}] &= \Lambda_{\mathrm{III}} A_{t-1} - \left(\tfrac{1}{1-\psi+r}\Lambda + \tfrac{1}{\psi}\Lambda_{\mathrm{IV}}\right)\mathcal{G}^n_{\mathrm{Y}} Y^n_t, \\
\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^n_{t:\mathrm{I}}] &= \Lambda_{\mathrm{IV}} A_{t-1} + \tfrac{1}{\psi}\Lambda_{\mathrm{IV}}\mathcal{G}^n_{\mathrm{Y}} Y^n_t, \qquad \text{and} \\
\mathrm{E}[R_{t+1}|\Omega^n_{t:\mathrm{I}}] &= \Lambda_{\mathrm{I}}\psi A_{t-1} + \Lambda_{\mathrm{I}}\mathcal{G}^n_{\mathrm{Y}} Y^n_t.
\end{aligned}$$

Substituting these forecasts in into (A.47a) gives the following expression for investors' round-Iorders:

$$\begin{aligned}
\mathbb{O}_{t:\mathrm{I}} &= \int_0^1 \left\{-\Theta^n_{\mathrm{I}|s_{\mathrm{III}}}\left(\tfrac{1}{1-\psi+r}\Lambda + \tfrac{1}{\psi}\Lambda_{\mathrm{IV}}\right)\mathcal{G}^n_{\mathrm{Y}} Y^n_t + \Theta^n_{\mathrm{I}|s_{\mathrm{IV}}}\tfrac{1}{\psi}\Lambda_{\mathrm{IV}}\mathcal{G}^n_{\mathrm{Y}} Y^n_t + \Theta^n_{\mathrm{I}|\mathrm{R}}\Lambda_{\mathrm{I}}\mathcal{G}^n_{\mathrm{Y}} Y^n_t\right\} dn - Y_t \\
&\qquad + \int_0^1 \left\{\Theta^n_{\mathrm{I}|s_{\mathrm{III}}}\Lambda_{\mathrm{III}} A_{t-1} + \Theta^n_{\mathrm{I}|s_{\mathrm{IV}}}\Lambda_{\mathrm{IV}} A_{t-1} + \Theta^n_{\mathrm{I}|\mathrm{R}}\Lambda_{\mathrm{I}}\psi A_{t-1} - A_{t-1}\right\} dn, \\
&= \left[\left\{\tfrac{1}{\psi}\Lambda_{\mathrm{IV}}(\Theta_{\mathrm{I}|s_{\mathrm{IV}}} - \Theta_{\mathrm{I}|s_{\mathrm{III}}}) - \tfrac{1}{1-\psi+r}\Lambda\Theta_{\mathrm{I}|s_{\mathrm{III}}} + \Theta_{\mathrm{I}|\mathrm{R}}\Lambda_{\mathrm{I}}\right\}\mathcal{G}^n_{\mathrm{Y}} - 1\right] Y_t \\
&\qquad + \left\{\Theta_{\mathrm{I}|s_{\mathrm{III}}}\Lambda_{\mathrm{III}} + \Theta_{\mathrm{I}|s_{\mathrm{IV}}}\Lambda_{\mathrm{IV}} + \Theta_{\mathrm{I}|\mathrm{R}}\Lambda_{\mathrm{I}}\psi - 1\right\} A_{t-1}, \\
&= \beta^{\mathrm{Y}}_{\mathrm{I}} Y_t + \beta^{\mathrm{A}}_{\mathrm{I}} A_{t-1},
\end{aligned}$$

Next, consider the implications of (15) for dealers' forecasts. In round I, each dealer receives a random allocation of investors orders (because dealers quote the same round-I price), so $\mathbb{O}^d_{t:\mathrm{I}} = \frac{1}{\mathrm{D}}\mathbb{O}_{t:\mathrm{I}} + \zeta^d_{t:\mathrm{I}}$, with $\sum_{d=1}^{\mathrm{D}} \zeta^d_{t:\mathrm{I}} = 0$. Their best estimate of foreign income based on this order is

$$\mathrm{E}[Y_t|\Omega^d_{t:\mathrm{II}}] = \mathcal{G}^d_{\mathrm{Y}}(\mathbb{O}^d_{t:\mathrm{I}} - \mathrm{E}[\mathbb{O}^d_{t:\mathrm{I}}|\mathcal{I}^d_{t:\mathrm{I}}]) = \mathcal{G}^d_{\mathrm{Y}}(\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}} Y_t + \zeta^d_{t:\mathrm{I}}), \qquad \text{where} \qquad \mathcal{G}^d_{\mathrm{Y}} = \frac{\frac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}\sigma^2_{\mathrm{Y}}}{(\frac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}})^2\sigma^2_{\mathrm{Y}} + \sigma^2_{\zeta}}.$$

Dealers' individual forecasts in round II are therefore

$$\begin{aligned}
\mathrm{E}[\Delta S_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{II}}] &= \mathrm{E}[\Delta S_{t:\mathrm{III}}|\mathcal{I}_{t:\mathrm{II}}] - \left(\tfrac{1}{1-\psi+r}\Lambda + \tfrac{1}{\psi}\Lambda_{\mathrm{IV}}\right)\mathcal{G}^d_{\mathrm{Y}}\left(\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}} Y_t + \zeta^d_{t:\mathrm{I}}\right) \qquad \text{and} \\
\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^d_{t:\mathrm{II}}] &= \mathrm{E}[\Delta S_{t:\mathrm{IV}}|\mathcal{I}_{t:\mathrm{II}}] + \tfrac{1}{\psi}\Lambda_{\mathrm{IV}}\mathcal{G}^d_{\mathrm{Y}}\left(\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}} Y_t + \zeta^d_{t:\mathrm{I}}\right).
\end{aligned}$$

Combining equations (22) and (21)gives

$$\begin{aligned}\mathrm{U}\left[X_{t:\mathrm{II}}|\,\mathcal{I}_{t:\mathrm{II}}\right] = \sum\nolimits_{d=1}^{\mathrm{D}} \Phi^{d}_{\mathrm{II}|s_{\mathrm{III}}} \left\{\mathrm{E}[\Delta S_{t:\mathrm{III}}|\Omega^{d}_{t:\mathrm{II}}] - \mathrm{E}[\Delta S_{t:\mathrm{III}}|\mathcal{I}_{t:\mathrm{II}}]\right\} \\ + \sum\nolimits_{d=1}^{\mathrm{D}} \Phi^{d}_{\mathrm{II}|s_{\mathrm{IV}}} \left\{\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^{d}_{t:\mathrm{II}}] - \mathrm{E}[\Delta S_{t:\mathrm{IV}}|\mathcal{I}_{t:\mathrm{II}}]\right\} \\ + \tfrac{1}{\mathrm{D}} \sum\nolimits_{d=1}^{\mathrm{D}} \left\{\mathrm{E}[X_{t:\mathrm{II}}|\Omega^{d}_{t:\mathrm{II}}] - \mathrm{E}[X_{t:\mathrm{II}}|\mathcal{I}_{t:\mathrm{II}}]\right\} + \mathbb{O}_{t:\mathrm{I}} - \mathrm{E}[\mathbb{O}_{t:\mathrm{I}}|\mathcal{I}_{t:\mathrm{II}}].\end{aligned}$$

So substituting in dealers' forecasts

$$\begin{aligned}\mathrm{U}\left[X_{t:\mathrm{II}}|\,\mathcal{I}_{t:\mathrm{II}}\right] &= -\sum\nolimits_{d=1}^{\mathrm{D}} \Phi^{d}_{\mathrm{II}|s_{\mathrm{III}}} \left\{\left(\tfrac{1}{1-\psi+r}\Lambda + \tfrac{1}{\psi}\Lambda_{\mathrm{IV}}\right)\mathcal{G}^{d}_{\mathrm{Y}}\left(\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}Y_t + \zeta^{d}_{t:\mathrm{I}}\right)\right\} \\ &\quad + \sum\nolimits_{d=1}^{\mathrm{D}} \Phi^{d}_{\mathrm{II}|s_{\mathrm{IV}}} \left\{\tfrac{1}{\psi}\Lambda_{\mathrm{IV}}\mathcal{G}^{d}_{\mathrm{Y}}\left(\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}Y_t + \zeta^{d}_{t:\mathrm{I}}\right)\right\} \\ &\quad + \sum\nolimits_{d=1}^{\mathrm{D}} \tfrac{1}{\mathrm{D}}\Gamma^{\mathrm{Y}}_{\mathrm{II}}\mathcal{G}^{d}_{\mathrm{Y}}\left(\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}Y_t + \zeta^{d}_{t:\mathrm{I}}\right) + \beta^{\mathrm{Y}}_{\mathrm{I}}Y_t, \\ &= \left\{\left[\Phi_{\mathrm{II}|s_{\mathrm{IV}}}\tfrac{1}{\psi}\Lambda_{\mathrm{IV}} - \Phi_{\mathrm{II}|s_{\mathrm{III}}}\left(\tfrac{1}{1-\psi+r}\Lambda + \tfrac{1}{\psi}\Lambda_{\mathrm{IV}}\right) + \Gamma^{\mathrm{Y}}_{\mathrm{II}}\right]\tfrac{1}{\mathrm{D}}\mathcal{G}^{d}_{\mathrm{Y}} + 1\right\}\beta^{\mathrm{Y}}_{\mathrm{I}}Y_t, \\ &= \Gamma^{\mathrm{Y}}_{\mathrm{II}}Y_t.\end{aligned}$$

This verifies that round-II order flow follows (17a) with

$$\Gamma^{\mathrm{Y}}_{\mathrm{II}} = \frac{\beta^{\mathrm{Y}}_{\mathrm{I}}}{1-\frac{1}{\mathrm{D}}\mathcal{G}^{d}_{\mathrm{Y}}\beta^{\mathrm{Y}}_{\mathrm{I}}}\left\{1 + \left[\Phi_{\mathrm{II}|s_{\mathrm{IV}}}\tfrac{1}{\psi}\Lambda_{\mathrm{IV}} - \Phi_{\mathrm{II}|s_{\mathrm{III}}}\left(\tfrac{1}{1-\psi+r}\Lambda + \tfrac{1}{\psi}\Lambda_{\mathrm{IV}}\right)\right]\tfrac{1}{\mathrm{D}}\mathcal{G}^{d}_{\mathrm{Y}}\right\}.$$

Finally, to verify that round III order flow follows (17b), we note that (15) implies

$$\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^{d}_{t:\mathrm{III}}] = \mathrm{E}[\Delta S_{t:\mathrm{IV}}|\mathcal{I}_{t:\mathrm{III}}] + \tfrac{1}{1-\psi+r}\Lambda\mathrm{E}[H_t|\Omega^{d}_{t:\mathrm{III}}].$$

Each dealer receives a hedging order of $\mathbb{H}^{d}_{t} = \frac{1}{\mathrm{D}}\mathbb{H}_t + \eta^{d}_{t}$, with $\sum_{d=1}^{\mathrm{D}}\eta^{d}_{t} = 0$, so their estimate of the H_t shock is

$$\mathrm{E}[H_t|\Omega^{d}_{t:\mathrm{III}}] = \mathcal{G}^{d}_{\mathrm{H}}(\mathbb{H}^{d}_{t} - \mathrm{E}[\mathbb{H}^{d}_{t}|\mathcal{I}^{d}_{t:\mathrm{III}}]) = \mathcal{G}^{d}_{\mathrm{H}}(\tfrac{1}{\mathrm{D}}H_t + \eta^{d}_{t}) \qquad \text{where} \qquad \mathcal{G}^{d}_{\mathrm{H}} = \frac{\frac{1}{\mathrm{D}}\sigma^2_{\mathrm{H}}}{(\frac{1}{\mathrm{D}})^2\sigma^2_{\mathrm{H}} + \sigma^2_{\eta}}.$$

We now combine this equation with (22) and (23) to give

$$
\begin{aligned}
\mathrm{U}\left[X_{t:\text{III}}|\mathcal{I}_{t:\text{III}}\right] &= \sum\nolimits_{d=1}^{\mathrm{D}} \Phi^{d}_{\text{III}|s_{\text{IV}}} \Big(\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^{d}_{t:\text{III}}] - \mathrm{E}[\Delta S_{t:\text{IV}}|\mathcal{I}_{t:\text{III}}]\Big) \\
&\qquad + \tfrac{1}{\mathrm{D}} \sum\nolimits_{d=1}^{\mathrm{D}} \Big(\mathrm{E}\left[X_{t:\text{III}}|\Omega^{d}_{t:\text{III}}\right] - \mathrm{E}\left[X_{t:\text{III}}|\mathcal{I}_{t:\text{III}}\right]\Big) \\
&\qquad\qquad + (\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}}) - \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}}|\,\mathcal{I}_{t:\text{III}}\right], \\
&= \sum\nolimits_{d=1}^{\mathrm{D}} \Phi^{d}_{\text{III}|s_{\text{IV}}} \tfrac{1}{1-\psi+r} \Lambda \mathrm{E}[H_t|\Omega^{d}_{t:\text{III}}] + \sum\nolimits_{d=1}^{\mathrm{D}} \tfrac{1}{\mathrm{D}} \Gamma^{\text{H}}_{\text{III}} \mathrm{E}[H_t|\Omega^{d}_{t:\text{III}}] + H_t, \\
&= \sum\nolimits_{d=1}^{\mathrm{D}} \Phi^{d}_{\text{III}|s_{\text{IV}}} \tfrac{1}{1-\psi+r} \Lambda \mathcal{G}^{d}_{\text{H}} (\tfrac{1}{\mathrm{D}} H_t + \eta^{d}_{t}) + \sum\nolimits_{d=1}^{\mathrm{D}} \tfrac{1}{\mathrm{D}} \Gamma^{\text{H}}_{\text{III}} \mathcal{G}^{d}_{\text{H}} (\tfrac{1}{\mathrm{D}} H_t + \eta^{d}_{t}) + H_t, \\
&= \Phi_{\text{III}|s_{\text{IV}}} \tfrac{1}{1-\psi+r} \Lambda \tfrac{1}{\mathrm{D}} \mathcal{G}^{d}_{\text{H}} H_t + \Gamma^{\text{H}}_{\text{III}} \mathcal{G}^{d}_{\text{H}} \tfrac{1}{\mathrm{D}} H_t + H_t, \\
&= \Big\{ \Phi_{\text{III}|s_{\text{IV}}} \tfrac{1}{1-\psi+r} \Lambda \tfrac{1}{\mathrm{D}} \mathcal{G}^{d}_{\text{H}} + \Gamma^{\text{H}}_{\text{III}} \tfrac{1}{\mathrm{D}} \mathcal{G}^{d}_{\text{H}} + 1 \Big\} H_t, \\
&= \Gamma^{\text{H}}_{\text{III}} H_t.
\end{aligned}
$$

This verifies that round-III order flow follows (23) with

$$
\Gamma^{\text{H}}_{\text{III}} = \frac{1 + \frac{\Lambda}{1-\psi+r} \frac{1}{\mathrm{D}} \mathcal{G}^{d}_{\text{H}} \Phi_{\text{III}|s_{\text{IV}}}}{1 - \frac{1}{\mathrm{D}} \mathcal{G}^{d}_{\text{H}}}.
$$

■

A.3 Proof of Theorem 2

The Proof proceeds in four steps. In step one, I compute the conditional expectations and second moments for individual investors given the conjectured behavior of equilibrium prices in (26) and order flows in (28). Step two repeats these calculations for individual dealers. In step three, I verify that the equilibrium order flows follow (28). In the last step, I verify that equilibrium prices follow (26).

Before getting into the details, it is useful to rewrite equilibrium equations for $\Delta S_{t:\text{III}}$ and $\Delta S_{t:\text{IV}}$ in terms of the exogenous shocks:

$$
\begin{aligned}
\Delta S_{t:\text{III}} &= \Lambda_{\text{III}} A_{t-1} + \lambda_{\text{III}} \Gamma^{\text{Y}}_{\text{II}} Y_t + \lambda_{\text{III}} \Gamma^{\text{H}}_{\text{II}} H_t, \\
&= \Lambda_{\text{III}} A_{t-1} + \mathcal{B}^{\text{Y}}_{\text{II}} Y_t + \mathcal{B}^{\text{H}}_{\text{II}} H_t,
\end{aligned}
$$

$$
\begin{aligned}
\Delta S_{t:\text{IV}} &= \Lambda_{\text{IV}} A_{t-1} + \lambda_{\text{IV}_1} (\Gamma^{\text{Y}}_{\text{III}} Y_t + \Gamma^{\text{H}}_{\text{III}} H_t) + \lambda_{\text{IV}_2} (\Gamma^{\text{Y}}_{\text{II}} Y_t + \Gamma^{\text{H}}_{\text{II}} H_t), \\
&= \Lambda_{\text{IV}} A_{t-1} + (\lambda_{\text{IV}_1} \Gamma^{\text{Y}}_{\text{III}} + \lambda_{\text{IV}_2} \Gamma^{\text{Y}}_{\text{II}}) Y_t + (\Gamma^{\text{H}}_{\text{III}} + \lambda_{\text{IV}_2} \Gamma^{\text{H}}_{\text{II}}) H_t, \\
&= \Lambda_{\text{IV}} A_{t-1} + \mathcal{B}^{\text{Y}}_{\text{III}} Y_t + \mathcal{B}^{\text{Y}}_{\text{III}} H_t.
\end{aligned}
$$

Step One

Individual investors choose their round-I orders based on their conditional expectations for Y_t and H_t and their associated variances. As above, the conditional moments for Y_t are

$$
\mathrm{E}[Y_t|\Omega^n_{t:\text{I}}] = \frac{\sigma^2_{\text{Y}}}{\sigma^2_{\text{Y}} + \sigma^2_{\varepsilon}} (Y_t + \varepsilon^n_t) = \mathcal{G}^n_{\text{Y}} Y^n_t, \qquad \text{and} \qquad \mathbb{V}[Y_t|\Omega^n_{t:\text{I}}] = \sigma^2_{\text{Y}} - \frac{\sigma^4_{\text{Y}}}{\sigma^2_{\text{Y}} + \sigma^2_{\varepsilon}} = (1 - \mathcal{G}^n_{\text{Y}}) \sigma^2_{\text{Y}},
$$

and those for H_t are $\mathrm{E}[H_t|\Omega^n_{t:\text{I}}] = 0$ and $\mathrm{E}[H_t|\Omega^n_{t:\text{I}}] = \sigma^2_{\text{H}}$. On the basis of these estimates, investor n's private forecasts are

$$
\begin{aligned}
\mathrm{E}[\Delta S_{t:\text{III}}|\Omega^n_{t:\text{I}}] &= \mathcal{B}^{\text{Y}}_{\text{II}} \mathcal{G}^n_{\text{Y}} Y^n_t + \Lambda_{\text{III}} A_{t-1}, \\
\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{I}}] &= \mathcal{B}^{\text{Y}}_{\text{III}} \mathcal{G}^n_{\text{Y}} Y^n_t + \Lambda_{\text{IV}} A_{t-1}, \\
\mathrm{E}[R_{t+1}|\Omega^n_{t:\text{I}}] &= \Lambda_{\text{I}} \left(\psi A_{t-1} + \mathcal{G}^n_{\text{Y}} Y^n_t \right).
\end{aligned}
$$

In round III, investors estimate Y_t and H_t based on Y^n_t and round-II order flow – which is derived from their observation of prices and knowledge of A_{t-1} as: $\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}] = \lambda^{-1}_{\text{III}} (\Delta S_{t:\text{III}} - \Lambda_{\text{III}} A_{t-1})$. Applying the Projection Theorem, we find that

$$
\begin{aligned}
\mathrm{E}[Y_t|\Omega^n_{t:\text{III}}] &= \begin{bmatrix} \sigma^2_{\text{Y}} & \Gamma^{\text{Y}}_{\text{II}} \sigma^2_{\text{Y}} \end{bmatrix} \begin{bmatrix} \sigma^2_{\text{Y}} + \sigma^2_{\varepsilon} & \Gamma^{\text{Y}}_{\text{II}} \sigma^2_{\text{Y}} \\ \Gamma^{\text{Y}}_{\text{II}} \sigma^2_{\text{Y}} & (\Gamma^{\text{Y}}_{\text{II}})^2 \sigma^2_{\text{Y}} + (\Gamma^{\text{H}}_{\text{II}})^2 \sigma^2_{\text{H}} \end{bmatrix}^{-1} \begin{bmatrix} Y_t + \varepsilon^n_t \\ \mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}] \end{bmatrix} \\
&= \mathcal{K}^n_{\text{Y}|Y} Y^n_t + \mathcal{K}^n_{\text{Y}|X} \mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}],
\end{aligned}
$$

$$
\mathbb{V}[Y_t|\Omega^n_{t:\text{III}}] = \sigma^2_{\text{Y}} - \begin{bmatrix} \sigma^2_{\text{Y}} & \Gamma^{\text{Y}}_{\text{II}} \sigma^2_{\text{Y}} \end{bmatrix} \begin{bmatrix} \sigma^2_{\text{Y}} + \sigma^2_{\varepsilon} & \Gamma^{\text{Y}}_{\text{II}} \sigma^2_{\text{Y}} \\ \Gamma^{\text{Y}}_{\text{II}} \sigma^2_{\text{Y}} & (\Gamma^{\text{Y}}_{\text{II}})^2 \sigma^2_{\text{Y}} + (\Gamma^{\text{H}}_{\text{II}})^2 \sigma^2_{\text{H}} \end{bmatrix}^{-1} \begin{bmatrix} \sigma^2_{\text{Y}} \\ \Gamma^{\text{Y}}_{\text{II}} \sigma^2_{\text{Y}} \end{bmatrix},
$$

and

$$\mathrm{E}[H_t|\Omega^n_{t:\text{III}}] = \left[\Gamma^{\text{H}}_{\text{II}}\sigma^2_{\text{Y}}\right]\left[(\Gamma^{\text{Y}}_{\text{II}})^2\sigma^2_{\text{Y}} + (\Gamma^{\text{H}}_{\text{II}})^2\sigma^2_{\text{H}}\right]^{-1}\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}] = \mathcal{K}^n_{\text{H}|X}\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}],$$

$$\mathbb{V}[H_t|\Omega^n_{t:\text{III}}] = \sigma^2_{\text{H}} - \left[\Gamma^{\text{H}}_{\text{II}}\sigma^2_{\text{H}}\right]\left[(\Gamma^{\text{Y}}_{\text{II}})^2\sigma^2_{\text{Y}} + (\Gamma^{\text{H}}_{\text{II}})^2\sigma^2_{\text{H}}\right]^{-1}\left[\Gamma^{\text{H}}_{\text{II}}\sigma^2_{\text{H}}\right] = \left(1 - \Gamma^{\text{H}}_{\text{II}}\mathcal{K}^n_{\text{H}|X}\right)\sigma^2_{\text{H}}.$$

The investor's private round-III forecasts are therefore

$$\begin{aligned}\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{III}}] &= \mathcal{B}^{\text{Y}}_{\text{III}}\left(\mathcal{K}^n_{\text{Y}|Y}Y^n_t + \mathcal{K}^n_{\text{Y}|X}\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}]\right) + \mathcal{B}^{\text{H}}_{\text{III}}\mathcal{K}^n_{\text{H}|X}\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}] + \Lambda_{\text{IV}}A_{t-1},\\ \mathrm{E}[R_{t+1}|\Omega^n_{t:\text{III}}] &= \Lambda_{\text{I}}\left(\mathcal{K}^n_{\text{Y}|Y}Y^n_t + \mathcal{K}^n_{\text{Y}|X}\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}]\right) - \Lambda_{\text{I}}\mathcal{K}^n_{\text{H}|X}\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}] + \Lambda_{\text{I}}\psi A_{t-1}.\end{aligned}$$

These forecasts are used to compute investors' round-III orders in Step Three.

Step Two

Individual dealers initiate their inter-dealer trades in round II base on their private estimates of Y_t and H_t that are conditioned on the orders they receive from investors in round I and the hedgers in round II. These estimates are given by

$$\mathrm{E}[Y_t|\Omega^d_{t:\text{II}}] = \frac{\frac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}\sigma^2_{\text{Y}}}{(\frac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}})^2\sigma^2_{\text{Y}} + \sigma^2_{\zeta}}\left(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}Y_t + \zeta^d_{t:\text{I}}\right) = \mathcal{G}^d_{\text{Y}}\left(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}Y_t + \zeta^d_{t:\text{I}}\right), \qquad \text{and}$$

$$\mathrm{E}[H_t|\Omega^d_{t:\text{II}}] = \frac{\frac{1}{\text{D}}\sigma^2_{\text{H}}}{(\frac{1}{\text{D}})^2\sigma^2_{\text{H}} + \sigma^2_{\eta}}\left(\tfrac{1}{\text{D}}H_t + \eta^d_t\right) = \mathcal{G}^d_{\text{H}}\left(\tfrac{1}{\text{D}}H_t + \eta^d_t\right).$$

The dealer's forecasts for $\Delta S_{t:\text{III}}$ and $\Delta S_{t:\text{IV}}$ are therefore

$$\mathrm{E}[\Delta S_{t:\text{III}}|\Omega^d_{t:\text{II}}] - \mathrm{E}[\Delta S_{t:\text{III}}|I_{t:\text{II}}] = \mathcal{B}^{\text{Y}}_{\text{II}}\mathcal{G}^d_{\text{Y}}\left(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}Y_t + \zeta^d_{t:\text{I}}\right) + \mathcal{B}^{\text{H}}_{\text{II}}\mathcal{G}^d_{\text{H}}\left(\tfrac{1}{\text{D}}H_t + \eta^d_t\right), \qquad \text{and}$$

$$\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^d_{t:\text{II}}] - \mathrm{E}[\Delta S_{t:\text{IV}}|I_{t:\text{II}}] = \mathcal{B}^{\text{Y}}_{\text{III}}\mathcal{G}^d_{\text{Y}}\left(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}Y_t + \zeta^d_{t:\text{I}}\right) + \mathcal{B}^{\text{Y}}_{\text{III}}\mathcal{G}^d_{\text{H}}\left(\tfrac{1}{\text{D}}H_t + \eta^d_t\right).$$

In round III, each dealer revises their estimates of Y_t and H_t to incorporate information in round-II order flow and the round-III orders they receive from investors. The revised estimates are

$$\mathrm{E}[Y_t|\Omega^d_{t:\text{III}}] = \begin{bmatrix}\mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{I}}} & \mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{III}}} & \mathcal{K}^d_{\text{Y}|X}\end{bmatrix}\begin{bmatrix}\frac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}Y_t + \zeta^d_{t:\text{I}}\\ \frac{1}{\text{D}}\beta^{\text{Y}}_{\text{III}}Y_t + \frac{1}{\text{D}}\beta^{\text{X}}_{\text{III}}\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}] + \zeta^d_{t:\text{III}}\\ \mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}]\end{bmatrix},$$

where

$$\left[\begin{array}{ccc}\mathcal{K}^d_{\mathrm{Y}|\mathbb{O}_{\mathrm{I}}} & \mathcal{K}^d_{\mathrm{Y}|\mathbb{O}_{\mathrm{III}}} & \mathcal{K}^d_{\mathrm{Y}|X}\end{array}\right] = \left[\begin{array}{ccc}\frac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}\sigma^2_{\mathrm{Y}} & \beta_{\mathrm{III}}\sigma^2_{\mathrm{Y}} & \Gamma^{\mathrm{Y}}_{\mathrm{II}}\sigma^2_{\mathrm{Y}}\end{array}\right]\times$$
$$\left[\begin{array}{ccc}\left(\frac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}\right)^2\sigma^2_{\mathrm{Y}}+\sigma^2_{\zeta} & \frac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}\beta_{\mathrm{III}}\sigma^2_{\mathrm{Y}} & \frac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}\Gamma^{\mathrm{Y}}_{\mathrm{II}}\sigma^2_{\mathrm{Y}} \\ \frac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}\beta_{\mathrm{III}}\sigma^2_{\mathrm{Y}} & \beta^2_{\mathrm{III}}\sigma^2_{\mathrm{Y}}+\left(\frac{1}{\mathrm{D}}\beta^{\mathrm{X}}_{\mathrm{III}}\Gamma^{\mathrm{H}}_{\mathrm{II}}\right)^2\sigma^2_{\mathrm{H}}+\sigma^2_{\zeta} & \beta_{\mathrm{III}}\Gamma^{\mathrm{Y}}_{\mathrm{II}}\sigma^2_{\mathrm{Y}}+\frac{1}{\mathrm{D}}\beta^{\mathrm{X}}_{\mathrm{III}}\left(\Gamma^{\mathrm{H}}_{\mathrm{II}}\right)^2\sigma^2_{\mathrm{H}} \\ \frac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}\Gamma^{\mathrm{Y}}_{\mathrm{II}}\sigma^2_{\mathrm{Y}} & \beta_{\mathrm{III}}\Gamma^{\mathrm{Y}}_{\mathrm{II}}\sigma^2_{\mathrm{Y}}+\frac{1}{\mathrm{D}}\beta^{\mathrm{X}}_{\mathrm{III}}\left(\Gamma^{\mathrm{H}}_{\mathrm{II}}\right)^2\sigma^2_{\mathrm{H}} & \left(\Gamma^{\mathrm{Y}}_{\mathrm{II}}\right)^2\sigma^2_{\mathrm{Y}}+\left(\Gamma^{\mathrm{H}}_{\mathrm{II}}\right)^2\sigma^2_{\mathrm{H}}\end{array}\right]^{-1}$$

with $\beta_{\mathrm{III}} = (\frac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{III}} + \frac{1}{\mathrm{D}}\beta^{\mathrm{X}}_{\mathrm{III}}\Gamma^{\mathrm{Y}}_{\mathrm{II}})$ and

$$\mathrm{E}[H_t|\Omega^d_{t:\mathrm{III}}] = \left[\begin{array}{cc}\mathcal{K}^d_{\mathrm{H}|H} & \mathcal{K}^d_{\mathrm{H}|X}\end{array}\right]\left[\begin{array}{c}\frac{1}{\mathrm{D}}H_t+\eta^d_t \\ \mathrm{U}[X_t|\mathcal{I}_{t:\mathrm{II}}]\end{array}\right],$$

where

$$\left[\begin{array}{cc}\mathcal{K}^d_{\mathrm{H}|H} & \mathcal{K}^d_{\mathrm{H}|X}\end{array}\right] = \left[\begin{array}{cc}\frac{1}{\mathrm{D}}\sigma^2_H & \Gamma^{\mathrm{H}}_{\mathrm{II}}\sigma^2_{\mathrm{H}}\end{array}\right]\left[\begin{array}{cc}\left(\frac{1}{\mathrm{D}}\right)^2\sigma^2_H+\sigma^2_\eta & \frac{1}{\mathrm{D}}\Gamma^{\mathrm{H}}_{\mathrm{II}}\sigma^2_{\mathrm{H}} \\ \frac{1}{\mathrm{D}}\Gamma^{\mathrm{H}}_{\mathrm{II}}\sigma^2_{\mathrm{H}} & \left(\Gamma^{\mathrm{Y}}_{\mathrm{II}}\right)^2\sigma^2_{\mathrm{Y}}+\left(\Gamma^{\mathrm{H}}_{\mathrm{II}}\right)^2\sigma^2_{\mathrm{H}}\end{array}\right]^{-1}.$$

The individual dealer's forecast for $\Delta S_{t:\mathrm{IV}}$ is computed from these estimates as

$$\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^d_{t:\mathrm{III}}] = \mathcal{B}^{\mathrm{Y}}_{\mathrm{III}}\mathrm{E}[Y_t|\Omega^d_{t:\mathrm{III}}] + \mathcal{B}^{\mathrm{H}}_{\mathrm{III}}\mathrm{E}[H_t|\Omega^d_{t:\mathrm{III}}] + \Lambda_{\mathrm{IV}}A_{t-1}.$$

We also require forecasts based on dealers' common information. The order flow equations in (28) imply that

$$\left[\begin{array}{c}Y_t \\ H_t\end{array}\right] = \left[\begin{array}{cc}\Gamma^{\mathrm{Y}}_{\mathrm{II}} & \Gamma^{\mathrm{H}}_{\mathrm{II}} \\ \Gamma^{\mathrm{Y}}_{\mathrm{III}} & \Gamma^{\mathrm{H}}_{\mathrm{III}}\end{array}\right]^{-1}\left[\begin{array}{c}\mathrm{U}[X_t|\mathcal{I}_{t:\mathrm{II}}] \\ \mathrm{U}[X_t|\mathcal{I}_{t:\mathrm{III}}]\end{array}\right] = \frac{1}{\Gamma^{\mathrm{Y}}_{\mathrm{II}}\Gamma^{\mathrm{H}}_{\mathrm{III}}-\Gamma^{\mathrm{H}}_{\mathrm{II}}\Gamma^{\mathrm{Y}}_{\mathrm{III}}}\left[\begin{array}{cc}\Gamma^{\mathrm{H}}_{\mathrm{III}} & -\Gamma^{\mathrm{H}}_{\mathrm{II}} \\ -\Gamma^{\mathrm{Y}}_{\mathrm{III}} & \Gamma^{\mathrm{Y}}_{\mathrm{II}}\end{array}\right]\left[\begin{array}{c}\mathrm{U}[X_t|\mathcal{I}_{t:\mathrm{II}}] \\ \mathrm{U}[X_t|\mathcal{I}_{t:\mathrm{III}}]\end{array}\right]. \tag{A.53}$$

Thus, values of Y_t and H_t are common knowledge among dealers by the start of round IV: i.e., $\mathrm{E}[Y_t|\mathcal{I}_{t:\mathrm{IV}}] = Y_t$ and $\mathrm{E}[H_t|\mathcal{I}_{t:\mathrm{IV}}] = H_t$. In round II, the estimates of Y_t and H_t are conditioned on order flow from round II:

$$\mathrm{E}\left[Y_t|\,I_{t:\mathrm{III}}\right] = \mathcal{K}_{\mathrm{Y}|X}\mathrm{U}[X_t|\mathcal{I}_{t:\mathrm{II}}] \qquad \text{and} \qquad \mathrm{E}\left[H_t|\,I_{t:\mathrm{III}}\right] = \mathcal{K}_{\mathrm{H}|X}\mathrm{U}[X_t|\mathcal{I}_{t:\mathrm{II}}]$$

where

$$\mathcal{K}_{\mathrm{Y}|X} = \frac{\Gamma^{\mathrm{Y}}_{\mathrm{II}}\sigma^2_{\mathrm{Y}}}{\left(\Gamma^{\mathrm{Y}}_{\mathrm{II}}\right)^2\sigma^2_{\mathrm{Y}}+\left(\Gamma^{\mathrm{H}}_{\mathrm{II}}\right)^2\sigma^2_{\mathrm{H}}} \qquad \text{and} \qquad \mathcal{K}_{\mathrm{H}|X} = \frac{\Gamma^{\mathrm{H}}_{\mathrm{II}}\sigma^2_{\mathrm{H}}}{\left(\Gamma^{\mathrm{Y}}_{\mathrm{II}}\right)^2\sigma^2_{\mathrm{Y}}+\left(\Gamma^{\mathrm{H}}_{\mathrm{II}}\right)^2\sigma^2_{\mathrm{H}}}.$$

In round II the estimates of Y_t and H_t are equal to the unconditional expectations: i.e., $\mathrm{E}[Y_t|\mathcal{I}_{t:\text{II}}] = 0$ and $\mathrm{E}[H_t|\mathcal{I}_{t:\text{II}}] = 0$. Below we use the differences between dealers' individual and common estimates, which are given by

$$\begin{aligned}
\mathrm{E}[Y_t|\Omega^d_{t:\text{III}}] - \mathrm{E}\left[Y_t|\, I_{t:\text{III}}\right] &= \mathcal{K}^d_{\text{Y}|\mathbb{O}_\text{I}}(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}Y_t + \zeta^d_{t:\text{I}}) + \mathcal{K}^d_{\text{Y}|\mathbb{O}_\text{III}}\left(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{III}}Y_t + \zeta^d_{t:\text{III}}\right) \\
&\quad + \left(\mathcal{K}^d_{\text{Y}|X} + \mathcal{K}^d_{\text{Y}|\mathbb{O}_\text{III}}(\tfrac{1}{\text{D}}\beta^{\text{X}}_{\text{III}}) - \mathcal{K}_{\text{Y}|X}\right)\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}], \qquad \text{and} \\
\mathrm{E}[H_t|\Omega^d_{t:\text{III}}] - \mathrm{E}\left[H_t|\, I_{t:\text{III}}\right] &= \mathcal{K}^d_{\text{H}|H}(\tfrac{1}{\text{D}}H_t + \eta^d_t) + \left(\mathcal{K}^d_{\text{H}|X} - \mathcal{K}_{\text{H}\;|X}\right)\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}].
\end{aligned}$$

From these results, we can compute the difference between dealers' individual and common forecasts as

$$\begin{aligned}
&\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^d_{t:\text{III}}] - \mathrm{E}[\Delta S_{t:\text{IV}}|I_{t:\text{III}}] = \\
&\quad \mathcal{B}^{\text{Y}}_{\text{III}}\left(\mathcal{K}^d_{\text{Y}|\mathbb{O}_\text{I}}(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}Y_t + \zeta^d_{t:\text{I}}) + \mathcal{K}^d_{\text{Y}|\mathbb{O}_\text{III}}\left(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{III}}Y_t + \zeta^d_{t:\text{III}}\right)\right) + \mathcal{B}^{\text{H}}_{\text{III}}\mathcal{K}^d_{\text{H}|H}(\tfrac{1}{\text{D}}H_t + \eta^d_t) \\
&\quad + \left[\mathcal{B}^{\text{Y}}_{\text{III}}\left(\mathcal{K}^d_{\text{Y}|X} + \mathcal{K}^d_{\text{Y}|\mathbb{O}_\text{III}}(\tfrac{1}{\text{D}}\beta^{\text{X}}_{\text{III}}) - \mathcal{K}_{\text{Y}|X}\right) + \mathcal{B}^{\text{H}}_{\text{III}}\left(\mathcal{K}^d_{\text{H}|X} - \mathcal{K}_{\text{H}\;|X}\right)\right]\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}].
\end{aligned} \tag{A.54}$$

Step Three

In this step, we compute unexpected aggregate order flows in rounds I and III. For round I, we first combine equations (22) and (21) to give

$$\begin{aligned}
\mathrm{U}\left[X_{t:\text{II}}|\,\mathcal{I}_{t:\text{II}}\right] &= \sum\nolimits_{d=1}^{\text{D}} \Phi^d_{\text{II}|s_\text{III}}\left\{\mathrm{E}[\Delta S_{t:\text{III}}|\Omega^d_{t:\text{II}}] - \mathrm{E}[\Delta S_{t:\text{III}}|\mathcal{I}_{t:\text{II}}]\right\} \\
&\quad + \sum\nolimits_{d=1}^{\text{D}} \Phi^d_{\text{II}|s_\text{IV}}\left\{\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^d_{t:\text{II}}] - \mathrm{E}[\Delta S_{t:\text{IV}}|\mathcal{I}_{t:\text{II}}]\right\} \\
&\quad + \tfrac{1}{\text{D}}\sum\nolimits_{d=1}^{\text{D}}\left\{\mathrm{E}[X_{t:\text{II}}|\Omega^d_{t:\text{II}}] - \mathrm{E}[X_{t:\text{II}}|\mathcal{I}_{t:\text{II}}]\right\} + \mathrm{U}[\mathbb{O}_{t:\text{I}}|\mathcal{I}_{t:\text{II}}].
\end{aligned} \tag{A.55}$$

The last term in this expression is the unexpected aggregate imbalance in investors' round-I orders. To find this term, we compute the round-I order from investor n:

$$\begin{aligned}
\mathbb{O}^n_{t:\text{I}} &= \Theta^n_{\text{I}|s_\text{III}}\mathrm{E}[\Delta S_{t:\text{III}}|\Omega^n_{t:\text{I}}] + \Theta^n_{\text{I}|s_\text{IV}}\mathrm{E}[\Delta S_{t:\text{IV}}|\Omega^n_{t:\text{I}}] + \Theta^n_{\text{I}|\text{R}}\mathrm{E}[R_{t+1}|\Omega^n_{t:\text{I}}] - A_{t-1} - Y^n_t, \\
&= \left\{\left[\Theta^n_{\text{I}|s_\text{III}}\mathcal{B}^{\text{Y}}_{\text{II}} + \Theta^n_{\text{I}|s_\text{IV}}\mathcal{B}^{\text{Y}}_{\text{III}} + \Theta^n_{\text{I}|\text{R}}\Lambda_\text{I}\right]\mathcal{G}^n_{\text{Y}} - 1\right\}Y^n_t + \left\{\Theta^n_{\text{I}|s_\text{III}}\Lambda_\text{III} + \Theta^n_{\text{I}|s_\text{IV}}\Lambda_\text{IV} + \Theta^n_{\text{I}|\text{R}}\Lambda_\text{I}\psi - 1\right\}A_{t-1}, \\
&= \beta^{\text{Y}}_{\text{I}}Y^n_t + \beta^{\text{A}}_{\text{I}}A_{t-1},
\end{aligned}$$

as shown in (27a). Aggregating across investors gives

$$\mathbb{O}_{t:\text{I}} = \beta^{\text{Y}}_{\text{I}}Y_t + \beta^{\text{A}}_{\text{I}}A_{t-1}, \tag{A.56}$$

so the last term in (A.55), $\mathrm{U}[\mathbb{O}_{t:\mathrm{I}}|\mathcal{I}_{t:\mathrm{II}}] = \beta^{\mathrm{Y}}_{\mathrm{I}} Y_t$.

The other terms in (A.55) come from the dealers' forecasts computed in Step Two:

$$\begin{aligned}\mathrm{U}\left[X_{t:\mathrm{II}}|\,\mathcal{I}_{t:\mathrm{II}}\right] = &\sum\nolimits_{d=1}^{\mathrm{D}} \Phi^d_{\mathrm{II}|s_{\mathrm{III}}} \mathcal{B}^{\mathrm{Y}}_{\mathrm{II}} \mathcal{G}^d_{\mathrm{Y}} \left((\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}) Y_t + \zeta^d_{t:\mathrm{I}} \right) + \sum\nolimits_{d=1}^{\mathrm{D}} \Phi^d_{\mathrm{II}|s_{\mathrm{III}}} \mathcal{B}^{\mathrm{H}}_{\mathrm{II}} \mathcal{G}^d_{\mathrm{H}} \left(\tfrac{1}{\mathrm{D}} H_t + \eta^d_t \right) \\ &+ \sum\nolimits_{d=1}^{\mathrm{D}} \Phi^d_{\mathrm{II}|s_{\mathrm{IV}}} \mathcal{B}^{\mathrm{Y}}_{\mathrm{III}} \mathcal{G}^d_{\mathrm{Y}} \left((\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}) Y_t + \zeta^d_{t:\mathrm{I}} \right) + \sum\nolimits_{d=1}^{\mathrm{D}} \Phi^d_{\mathrm{II}|s_{\mathrm{IV}}} \mathcal{B}^{\mathrm{H}}_{\mathrm{III}} \mathcal{G}^d_{\mathrm{H}} \left(\tfrac{1}{\mathrm{D}} H_t + \eta^d_t \right) \\ &+ \tfrac{1}{\mathrm{D}} \sum\nolimits_{d=1}^{\mathrm{D}} \Gamma^{\mathrm{Y}}_{\mathrm{II}} \mathcal{G}^d_{\mathrm{Y}} \left((\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}) Y_t + \zeta^d_{t:\mathrm{I}} \right) + \tfrac{1}{\mathrm{D}} \sum\nolimits_{d=1}^{\mathrm{D}} \Gamma^{\mathrm{H}}_{\mathrm{II}} \mathcal{G}^d_{\mathrm{H}} \left(\tfrac{1}{\mathrm{D}} H_t + \eta^d_t \right) + \beta^{\mathrm{Y}}_{\mathrm{I}} Y_t,\end{aligned}$$

which simplifies to

$$\begin{aligned}\mathrm{U}\left[X_{t:\mathrm{II}}|\,\mathcal{I}_{t:\mathrm{II}}\right] = &\left[\Phi_{\mathrm{II}|s_{\mathrm{III}}} \mathcal{B}^{\mathrm{Y}}_{\mathrm{II}} \mathcal{G}^d_{\mathrm{Y}} (\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}) + \Phi_{\mathrm{II}|s_{\mathrm{IV}}} \mathcal{B}^{\mathrm{Y}}_{\mathrm{III}} \mathcal{G}^d_{\mathrm{Y}} (\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}) + \Gamma^{\mathrm{Y}}_{\mathrm{II}} \mathcal{G}^d_{\mathrm{Y}} (\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}) + \beta^{\mathrm{Y}}_{\mathrm{I}}\right] Y_t \\ &+ \left[\Phi_{\mathrm{II}|s_{\mathrm{III}}} \mathcal{B}^{\mathrm{H}}_{\mathrm{II}} (\tfrac{1}{\mathrm{D}} \mathcal{G}^d_{\mathrm{H}}) + \Phi_{\mathrm{II}|s_{\mathrm{IV}}} \mathcal{B}^{\mathrm{H}}_{\mathrm{III}} (\tfrac{1}{\mathrm{D}} \mathcal{G}^d_{\mathrm{H}}) + \Gamma^{\mathrm{H}}_{\mathrm{II}} (\tfrac{1}{\mathrm{D}} \mathcal{G}^d_{\mathrm{H}})\right] H_t.\end{aligned}$$

This is the form of the round-II order flow equation (28a). Equating coefficients on Y_t and H_t gives

$$\Gamma^{\mathrm{Y}}_{\mathrm{II}} = \Phi_{\mathrm{II}|s_{\mathrm{III}}} \mathcal{B}^{\mathrm{Y}}_{\mathrm{II}} \mathcal{G}^d_{\mathrm{Y}} (\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}) + \Phi_{\mathrm{II}|s_{\mathrm{IV}}} \mathcal{B}^{\mathrm{Y}}_{\mathrm{III}} \mathcal{G}^d_{\mathrm{Y}} (\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}) + \Gamma^{\mathrm{Y}}_{\mathrm{II}} \mathcal{G}^d_{\mathrm{Y}} (\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}) + \beta^{\mathrm{Y}}_{\mathrm{I}}, \qquad \text{and} \tag{A.57a}$$

$$\Gamma^{\mathrm{H}}_{\mathrm{II}} = \Phi_{\mathrm{II}|s_{\mathrm{III}}} \mathcal{B}^{\mathrm{H}}_{\mathrm{II}} (\tfrac{1}{\mathrm{D}} \mathcal{G}^d_{\mathrm{H}}) + \Phi_{\mathrm{II}|s_{\mathrm{IV}}} \mathcal{B}^{\mathrm{H}}_{\mathrm{III}} (\tfrac{1}{\mathrm{D}} \mathcal{G}^d_{\mathrm{H}}) + \Gamma^{\mathrm{H}}_{\mathrm{II}} (\tfrac{1}{\mathrm{D}} \mathcal{G}^d_{\mathrm{H}}). \tag{A.57b}$$

The equilibrium values of $\Gamma^{\mathrm{Y}}_{\mathrm{II}}$ and $\Gamma^{\mathrm{H}}_{\mathrm{II}}$ must satisfy these equations.

Next, we turn to the round-III flow. As above, (22) and (23) imply that

$$\begin{aligned}\mathrm{U}\left[X_{t:\mathrm{III}}|\,\mathcal{I}_{t:\mathrm{III}}\right] \;=\; &\sum\nolimits_{d=1}^{\mathrm{D}} \Phi^d_{\mathrm{III}|s_{\mathrm{IV}}} \left(\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^d_{t:\mathrm{III}}] - \mathrm{E}[\Delta S_{t:\mathrm{IV}}|\mathcal{I}_{t:\mathrm{III}}] \right) \\ &+ \tfrac{1}{\mathrm{D}} \sum\nolimits_{d=1}^{\mathrm{D}} \left(\mathrm{E}\left[X_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{III}}\right] - \mathrm{E}\left[X_{t:\mathrm{III}}|\mathcal{I}_{t:\mathrm{III}}\right] \right) \\ &+ \mathrm{U}\left[\mathbb{H}_t|\,\mathcal{I}_{t:\mathrm{III}}\right] + \mathrm{U}\left[\mathbb{O}_{t:\mathrm{III}} + \mathbb{O}_{t:\mathrm{I}}|\,\mathcal{I}_{t:\mathrm{III}}\right].\end{aligned} \tag{A.58}$$

The first term on the right-hand-side uses the difference between dealers' private forecasts and their common forecast for $\Delta S_{t:\mathrm{IV}}$, which was compute in (A.54). The next term is the analogous difference in forecasts for order flow:

$$\mathrm{E}[X_{t:\mathrm{III}}|\Omega^d_{t:\mathrm{III}}] - \mathrm{E}\left[X_{t:\mathrm{III}}|\mathcal{I}_{t:\mathrm{III}}\right] = \Gamma^{\mathrm{Y}}_{\mathrm{III}} \left(\mathrm{E}[Y_t|\Omega^d_{t:\mathrm{III}}] - \mathrm{E}\left[Y_t|\mathcal{I}_{t:\mathrm{III}}\right] \right) + \Gamma^{\mathrm{H}}_{\mathrm{III}} \left(\mathrm{E}[H_t|\Omega^d_{t:\mathrm{III}}] - \mathrm{E}\left[H_t|\mathcal{I}_{t:\mathrm{III}}\right] \right)$$

$$
\begin{aligned}
&= \Gamma^{\mathrm{Y}}_{\mathrm{III}}\mathcal{K}^{d}_{\mathrm{Y}|\mathbb{O}_{\mathrm{I}}}(\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}Y_t+\zeta^{d}_{t:\mathrm{I}})+\Gamma^{\mathrm{Y}}_{\mathrm{III}}\mathcal{K}^{d}_{\mathrm{Y}|\mathbb{O}_{\mathrm{III}}}\left(\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{III}}Y_t+\zeta^{d}_{t:\mathrm{III}}\right)\\
&\qquad +\Gamma^{\mathrm{Y}}_{\mathrm{III}}\left(\mathcal{K}^{d}_{\mathrm{Y}|X}+\mathcal{K}^{d}_{\mathrm{Y}|\mathbb{O}_{\mathrm{III}}}(\tfrac{1}{\mathrm{D}}\beta^{\mathrm{X}}_{\mathrm{III}})-\mathcal{K}_{\mathrm{Y}|X}\right)\mathrm{U}[X_t|\mathcal{I}_{t:\mathrm{II}}]\\
&\qquad +\Gamma^{\mathrm{H}}_{\mathrm{III}}\mathcal{K}^{d}_{\mathrm{H}|H}(\tfrac{1}{\mathrm{D}}H_t+\eta^{d}_t)+\Gamma^{\mathrm{H}}_{\mathrm{III}}\left(\mathcal{K}^{d}_{\mathrm{H}|X}-\mathcal{K}_{\mathrm{H}\ |X}\right)\mathrm{U}[X_t|\mathcal{I}_{t:\mathrm{II}}],\\
&= \left[\Gamma^{\mathrm{Y}}_{\mathrm{III}}\left(\mathcal{K}^{d}_{\mathrm{Y}|X}+\mathcal{K}^{d}_{\mathrm{Y}|\mathbb{O}_{\mathrm{III}}}(\tfrac{1}{\mathrm{D}}\beta^{\mathrm{X}}_{\mathrm{III}})-\mathcal{K}_{\mathrm{Y}|X}\right)+\Gamma^{\mathrm{H}}_{\mathrm{III}}\left(\mathcal{K}^{d}_{\mathrm{H}|X}-\mathcal{K}_{\mathrm{H}|X}\right)\right]\mathrm{U}[X_t|\mathcal{I}_{t:\mathrm{II}}]\\
&\quad +\Gamma^{\mathrm{Y}}_{\mathrm{III}}\mathcal{K}^{d}_{\mathrm{Y}|\mathbb{O}_{\mathrm{I}}}(\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{I}}Y_t+\zeta^{d}_{t:\mathrm{I}})+\Gamma^{\mathrm{Y}}_{\mathrm{III}}\mathcal{K}^{d}_{\mathrm{Y}|\mathbb{O}_{\mathrm{III}}}\left(\tfrac{1}{\mathrm{D}}\beta^{\mathrm{Y}}_{\mathrm{III}}Y_t+\zeta^{d}_{t:\mathrm{III}}\right)+\Gamma^{\mathrm{H}}_{\mathrm{III}}\mathcal{K}^{d}_{\mathrm{H}|H}(\tfrac{1}{\mathrm{D}}H_t+\eta^{d}_t).
\end{aligned}
$$

The aggregate imbalance in hedgers' orders is given by $\mathbb{H}_t=(1-\psi)A_{t-1}+H_t$, so the third term in (A.55) is

$$
\mathrm{U}\left[\mathbb{H}_t|\,\mathcal{I}_{t:\mathrm{III}}\right]=\mathrm{U}\left[H_t|\,\mathcal{I}_{t:\mathrm{III}}\right]=H_t-\mathcal{K}_{\mathrm{H}|X}\mathrm{U}[X_t|\mathcal{I}_{t:\mathrm{II}}].
$$

To find the last term, we consider the round-III order from investor n:

$$
\begin{aligned}
\mathbb{O}^{n}_{t:\mathrm{III}} &= \Theta^{n}_{\mathrm{III}|s_{\mathrm{IV}}}\mathrm{E}[\Delta S_{t:\mathrm{IV}}|\Omega^{n}_{t:\mathrm{III}}]+\Theta^{n}_{\mathrm{III}|R}\mathrm{E}[\mathcal{R}_{t+1}|\Omega^{n}_{t:\mathrm{III}}]-\mathbb{O}^{n}_{t:\mathrm{I}}-A_{t-1}-Y^{n}_t,\\
&= \left\{\left[\Theta^{n}_{\mathrm{III}|s_{\mathrm{IV}}}\mathcal{B}^{\mathrm{Y}}_{\mathrm{III}}+\Theta^{n}_{\mathrm{III}|R}\Lambda_{\mathrm{I}}\right]\mathcal{K}^{n}_{\mathrm{Y}|Y}-\beta^{\mathrm{Y}}_{\mathrm{I}}-1\right\}Y^{n}_t\\
&\qquad +\left[\Theta^{n}_{\mathrm{III}|s_{\mathrm{IV}}}\mathcal{B}^{\mathrm{Y}}_{\mathrm{III}}\mathcal{K}^{n}_{\mathrm{Y}|X}+\Theta^{n}_{\mathrm{III}|s_{\mathrm{IV}}}\mathcal{B}^{\mathrm{H}}_{\mathrm{III}}\mathcal{K}^{n}_{\mathrm{H}|X}+\Theta^{n}_{\mathrm{III}|R}\Lambda_{\mathrm{I}}\left(\mathcal{K}^{n}_{\mathrm{Y}|X}-\mathcal{K}^{n}_{\mathrm{H}|X}\right)\right]\mathrm{U}[X_t|\mathcal{I}_{t:\mathrm{II}}]\\
&\qquad +\left[\Theta^{n}_{\mathrm{III}|s_{\mathrm{IV}}}\Lambda_{\mathrm{IV}}+\Theta^{n}_{\mathrm{III}|R}\Lambda_{\mathrm{I}}\psi-\beta^{\mathrm{A}}_{\mathrm{I}}-1\right]A_{t-1},
\end{aligned}
$$

which is the same form as (27b). The last term in (A.58) is computed by aggregating investors orders from rounds I and III:

$$
\begin{aligned}
\mathbb{O}_{t:\mathrm{III}}+\mathbb{O}_{t:\mathrm{I}} &= \int_0^1(\mathbb{O}^{n}_{t:\mathrm{III}}+\mathbb{O}^{n}_{t:\mathrm{I}})dn,\\
&= \left\{\left[\Theta_{\mathrm{III}|s_{\mathrm{IV}}}\mathcal{B}^{\mathrm{Y}}_{\mathrm{III}}+\Theta_{\mathrm{III}|R}\Lambda_{\mathrm{I}}\right]\mathcal{K}^{n}_{\mathrm{Y}|Y}-1\right\}Y_t\\
&\qquad +\left[\Theta_{\mathrm{III}|s_{\mathrm{IV}}}\mathcal{B}^{\mathrm{Y}}_{\mathrm{III}}\mathcal{K}^{n}_{\mathrm{Y}|X}+\Theta_{\mathrm{III}|s_{\mathrm{IV}}}\mathcal{B}^{\mathrm{H}}_{\mathrm{III}}\mathcal{K}^{n}_{\mathrm{H}|X}+\Theta_{\mathrm{III}|R}\Lambda_{\mathrm{I}}\left(\mathcal{K}^{n}_{\mathrm{Y}|X}-\mathcal{K}^{n}_{\mathrm{H}|X}\right)\right]\mathrm{U}_{t:\mathrm{II}}X_t\\
&\qquad +\left[\Theta_{\mathrm{III}|s_{\mathrm{IV}}}\Lambda_{\mathrm{IV}}+\Theta_{\mathrm{III}|R}\Lambda_{\mathrm{I}}\psi-1\right]A_{t-1},\\
&= (\beta^{\mathrm{Y}}_{\mathrm{I}}+\beta^{\mathrm{Y}}_{\mathrm{III}})Y_t+\beta^{\mathrm{X}}_{\mathrm{III}}\mathrm{U}_{t:\mathrm{II}}X_t+(\beta^{\mathrm{A}}_{\mathrm{I}}+\beta^{\mathrm{A}}_{\mathrm{III}})A_{t-1}.
\end{aligned}
$$

The unexpected portion of $\mathbb{O}_{t:\mathrm{III}}+\mathbb{O}_{t:\mathrm{I}}$, based on dealer common round-III information is therefore

$$
\mathrm{U}\left[\mathbb{O}_{t:\mathrm{III}}+\mathbb{O}_{t:\mathrm{I}}|\,\mathcal{I}_{t:\mathrm{III}}\right] = (\beta^{\mathrm{Y}}_{\mathrm{I}}+\beta^{\mathrm{Y}}_{\mathrm{III}})\mathrm{U}\left[Y_t|\,\mathcal{I}_{t:\mathrm{III}}\right]=(\beta^{\mathrm{Y}}_{\mathrm{I}}+\beta^{\mathrm{Y}}_{\mathrm{III}})\left(Y_t-\mathcal{K}_{\mathrm{Y}|X}\mathrm{U}[X_t|\mathcal{I}_{t:\mathrm{II}}]\right).
$$

Substituting all of these results into the order flow equation gives

$$
\begin{aligned}
\mathrm{U}\left[X_{t:\text{III}}|\,\mathcal{I}_{t:\text{III}}\right] = \Phi_{\text{III}|s_{\text{IV}}} & \left[\mathcal{B}^{\text{Y}}_{\text{III}}\left(\mathcal{K}^d_{\text{Y}|X} + \mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{III}}}(\tfrac{1}{\text{D}}\beta^{\text{X}}_{\text{III}}) - \mathcal{K}_{\text{Y}|X}\right) + \mathcal{B}^{\text{H}}_{\text{III}}\left(\mathcal{K}^d_{\text{H}|X} - \mathcal{K}_{\text{H}\ |X}\right)\right]\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}] \\
& + \Phi_{\text{III}|s_{\text{IV}}}\mathcal{B}^{\text{Y}}_{\text{III}}\left(\mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{I}}}(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}) + \mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{III}}}\left(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{III}}\right)\right)Y_t + \Phi_{\text{III}|s_{\text{IV}}}\mathcal{B}^{\text{H}}_{\text{III}}\mathcal{K}^d_{\text{H}|H}\tfrac{1}{\text{D}}H_t \\
& + \left[\Gamma^{\text{Y}}_{\text{III}}\left(\mathcal{K}^d_{\text{Y}|X} + \mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{III}}}(\tfrac{1}{\text{D}}\beta^{\text{X}}_{\text{III}}) - \mathcal{K}_{\text{Y}|X}\right) + \Gamma^{\text{H}}_{\text{III}}\left(\mathcal{K}^d_{\text{H}|X} - \mathcal{K}_{\text{H}|X}\right)\right]\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}] \\
& + \Gamma^{\text{Y}}_{\text{III}}\left[\mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{I}}}(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}) + \Gamma^{\text{Y}}_{\text{III}}\mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{III}}}\left(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{III}}\right)\right]Y_t + \Gamma^{\text{H}}_{\text{III}}\mathcal{K}^d_{\text{H}|H}\tfrac{1}{\text{D}}H_t. \\
& + H_t - \mathcal{K}_{\text{H}|X}\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}] + (\beta^{\text{Y}}_{\text{I}} + \beta^{\text{Y}}_{\text{III}})\left(Y_t - \mathcal{K}_{\text{Y}|X}\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}]\right).
\end{aligned}
$$

So collecting terms

$$
\begin{aligned}
\mathrm{U}\left[X_{t:\text{III}}|\,\mathcal{I}_{t:\text{III}}\right] &= \pi_x\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}] + \pi_y Y_t + \pi_H H_t, \\
&= (\pi_x\Gamma^{\text{Y}}_{\text{II}} + \pi_y)Y_t + (\pi_x\Gamma^{\text{H}}_{\text{II}} + \pi_H)H_t.
\end{aligned}
$$

This is the form of equation (28b) with

$$
\begin{aligned}
\Gamma^{\text{Y}}_{\text{III}} &= \pi_{\text{X}}\Gamma^{\text{Y}}_{\text{II}} + \pi_{\text{Y}}, && \text{(A.59a)} \\
\Gamma^{\text{H}}_{\text{III}} &= \pi_{\text{X}}\Gamma^{\text{H}}_{\text{II}} + \pi_{\text{H}}, && \text{(A.59b)} \\
\pi_{\text{X}} &= \Phi_{\text{III}|s_{\text{IV}}}\left[\mathcal{B}^{\text{Y}}_{\text{III}}\left(\mathcal{K}^d_{\text{Y}|X} + \mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{III}}}(\tfrac{1}{\text{D}}\beta^{\text{X}}_{\text{III}}) - \mathcal{K}_{\text{Y}|X}\right) + \mathcal{B}^{\text{H}}_{\text{III}}\left(\mathcal{K}^d_{\text{H}|X} - \mathcal{K}_{\text{H}\ |X}\right)\right] \\
&\quad + \left[\Gamma^{\text{Y}}_{\text{III}}\left(\mathcal{K}^d_{\text{Y}|X} + \mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{III}}}(\tfrac{1}{\text{D}}\beta^{\text{X}}_{\text{III}}) - \mathcal{K}_{\text{Y}|X}\right) + \Gamma^{\text{H}}_{\text{III}}\left(\mathcal{K}^d_{\text{H}|X} - \mathcal{K}_{\text{H}|X}\right)\right] \\
&\quad - \mathcal{K}_{\text{H}|H} - (\beta^{\text{Y}}_{\text{I}} + \beta^{\text{Y}}_{\text{III}})\mathcal{K}_{\text{Y}|X}, && \text{(A.59c)} \\
\pi_{\text{H}} &= \Phi_{\text{III}|s_{\text{IV}}}\mathcal{B}^{\text{H}}_{\text{III}}\mathcal{K}^d_{\text{H}|H}\tfrac{1}{\text{D}} + \Gamma^{\text{H}}_{\text{III}}\tfrac{1}{\text{D}}\mathcal{K}^d_{\text{H}|H} + 1, \qquad \text{and} && \text{(A.59d)} \\
\pi_{\text{Y}} &= \Phi_{\text{III}|s_{\text{IV}}}\mathcal{B}^{\text{Y}}_{\text{III}}\left(\mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{I}}}(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}) + \mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{III}}}\left(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{III}}\right)\right) \\
&\quad + \Gamma^{\text{Y}}_{\text{III}}\left[\mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{I}}}(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{I}}) + \Gamma^{\text{Y}}_{\text{III}}\mathcal{K}^d_{\text{Y}|\mathbb{O}_{\text{III}}}\left(\tfrac{1}{\text{D}}\beta^{\text{Y}}_{\text{III}}\right)\right] + (\beta^{\text{Y}}_{\text{I}} + \beta^{\text{Y}}_{\text{III}}). && \text{(A.59e)}
\end{aligned}
$$

As above, the equilibrium values of $\Gamma^{\text{Y}}_{\text{III}}$ and $\Gamma^{\text{H}}_{\text{III}}$ must satisfy these equations.

Step Four

The final step in the Proof is to verify that $\Delta S_{t:\text{III}}$ and $\Delta S_{t:\text{IV}}$ follow the processes in (26c) and (26d). Equation (A.50b) gives the general equation for $\Delta S_{t:\text{III}}$. To use this equation, we compute

$$
\mathrm{E}[A_t|\mathcal{I}_{t:\text{III}}] - \mathrm{E}[A_t|\mathcal{I}_{t:\text{II}}] = \mathrm{E}[Y_t - H_t|\mathcal{I}_{t:\text{III}}] - \mathrm{E}[Y_t - H_t|\mathcal{I}_{t:\text{II}}] = \left(\mathcal{K}_{\text{Y}|X} - \mathcal{K}_{\text{H}|X}\right)\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}], \quad \text{(A.60)}
$$

and

$$\begin{aligned}
&\mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} \middle| \mathcal{I}_{t:\text{III}}\right] - \mathrm{E}\left[\mathbb{H}_t + \mathbb{O}_{t:\text{III}} + \mathbb{O}_{t:\text{I}} \middle| \mathcal{I}_{t:\text{II}}\right] \\
&\qquad = \mathrm{E}\left[H_t + (\beta^{\text{Y}}_{\text{I}} + \beta^{\text{Y}}_{\text{III}})Y_t \middle| \mathcal{I}_{t:\text{III}}\right] + \beta^{x}_{\text{III}}\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}] - \mathrm{E}\left[H_t + (\beta^{\text{Y}}_{\text{I}} + \beta^{\text{Y}}_{\text{III}})Y_t \middle| \mathcal{I}_{t:\text{II}}\right], \\
&\qquad = \left(\mathcal{K}_{\text{H}|X} + (\beta^{\text{Y}}_{\text{I}} + \beta^{\text{Y}}_{\text{III}})\mathcal{K}_{\text{Y}|X} + \beta^{x}_{\text{III}}\right)\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}]. \qquad \text{(A.61)}
\end{aligned}$$

Substituting these terms into (A.50b) gives

$$\Delta S_{t:\text{III}} = \Lambda_{\text{III}} A_{t-1} - \tfrac{1}{1-\psi+r}\Lambda\left(\mathcal{K}_{\text{Y}|X} - \mathcal{K}_{\text{H}|X}\right)\mathrm{U}_{t:\text{II}}X_t + \Phi^{-1}_{\text{III}|s_{\text{IV}}}\left(\mathcal{K}_{\text{H}|X} + (\beta^{\text{Y}}_{\text{I}} + \beta^{\text{Y}}_{\text{III}})\mathcal{K}_{\text{Y}|X} + \beta^{\text{X}}_{\text{III}}\right)\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}].$$

This is the form of equation (26c) with

$$\lambda_{\text{III}} = \Phi^{-1}_{\text{III}|s_{\text{IV}}}\left(\mathcal{K}_{\text{H}|X} + (\beta^{\text{Y}}_{\text{I}} + \beta^{\text{Y}}_{\text{III}})\mathcal{K}_{\text{Y}|X} + \beta^{\text{X}}_{\text{III}}\right) - \tfrac{1}{1-\psi+r}\Lambda\left(\mathcal{K}_{\text{Y}|X} - \mathcal{K}_{\text{H}|X}\right).$$

Equation (A.50c) gives the general equation for $\Delta S_{t:\text{IV}}$. To use this equation, I compute

$$\begin{aligned}
\mathrm{U}[A_t|\mathcal{I}_{t:\text{III}}] &= \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} \mathrm{U}[Y_t|\mathcal{I}_{t:\text{III}}] \\ \mathrm{U}[H_t|\mathcal{I}_{t:\text{III}}] \end{bmatrix}, \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} \Gamma^{\text{Y}}_{\text{II}} & \Gamma^{\text{H}}_{\text{II}} \\ \Gamma^{\text{Y}}_{\text{III}} & \Gamma^{\text{H}}_{\text{III}} \end{bmatrix}^{-1}\left(\begin{bmatrix} \mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}] \\ \mathrm{U}[X_t|\mathcal{I}_{t:\text{III}}] \end{bmatrix} - \begin{bmatrix} \mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}] \\ 0 \end{bmatrix}\right), \\
&= -\frac{\Gamma^{\text{H}}_{\text{II}} + \Gamma^{\text{Y}}_{\text{II}}}{\Gamma^{\text{Y}}_{\text{II}}\Gamma^{\text{H}}_{\text{III}} - \Gamma^{\text{H}}_{\text{II}}\Gamma^{\text{Y}}_{\text{III}}}\mathrm{U}[X_t|\mathcal{I}_{t:\text{III}}]. \qquad \text{(A.62)}
\end{aligned}$$

Substituting (A.62) and (A.61) into (A.50c) produces

$$\Delta S_{t:\text{IV}} = \Lambda_{\text{IV}} A_{t-1} + \tfrac{1}{1-\psi+r}\Lambda\frac{\Gamma^{\text{H}}_{\text{II}} + \Gamma^{\text{Y}}_{\text{II}}}{\Gamma^{\text{Y}}_{\text{II}}\Gamma^{\text{H}}_{\text{III}} - \Gamma^{\text{H}}_{\text{II}}\Gamma^{\text{Y}}_{\text{III}}}\mathrm{U}[X_t|\mathcal{I}_{t:\text{III}}] - \Phi^{-1}_{\text{III}|s_{\text{IV}}}\left(\mathcal{K}_{\text{H}|X} + (\beta^{\text{Y}}_{\text{I}} + \beta^{\text{Y}}_{\text{III}})\mathcal{K}_{\text{Y}|X} + \beta^{x}_{\text{III}}\right)\mathrm{U}[X_t|\mathcal{I}_{t:\text{II}}].$$

This is the form of equation (26d) with

$$\lambda_{\text{IV}_1} = \tfrac{1}{1-\psi+r}\Lambda\frac{\Gamma^{\text{H}}_{\text{II}} + \Gamma^{\text{Y}}_{\text{II}}}{\Gamma^{\text{Y}}_{\text{II}}\Gamma^{\text{H}}_{\text{III}} - \Gamma^{\text{H}}_{\text{II}}\Gamma^{\text{Y}}_{\text{III}}} \qquad \text{and} \qquad \lambda_{\text{IV}_2} = -\Phi^{-1}_{\text{III}|s_{\text{IV}}}\left(\mathcal{K}_{\text{H}|X} + (\beta^{\text{Y}}_{\text{I}} + \beta^{\text{Y}}_{\text{III}})\mathcal{K}_{\text{Y}|X} + \beta^{x}_{\text{III}}\right).$$

■