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Abstract

This article considers an inter-temporal optimization problem in a fairly general form and give sufficient conditions ensuring the convergence to infinity of the economy. These conditions can be easily verified and applied for a large class of problems in literature. As examples, some applications for different economies are also given.

Keywords: Unbounded growth, sustained growth, non-convex dynamic programming

JEL classification: C61, O40, O41

1 Introduction

Initiated by Bellman [4], the dynamic programming literature becomes rapidly a workhorse of economic dynamic analysis. The tradition approach, culminated in Stokey & Lucas (with Prescott) in [24], gives a good explanation and prediction for many economic phenomena. The theory of dynamic programming described

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in Stokey & Lucas (with Prescott) [24] bases on a relatively strong structure of convexity. One of its implications is that generally, the economy converges to a steady state independently of the initial state.

Many works have been given in the configurations where this strong convexity structure is not satisfied. Clark [7], Majumdar and Mitra [19], Majumdar & Nermuth [18], Skiba [23] consider the economies where production functions exhibit an early phase of increasing returns, usually known as *convex-concave* functions. Dechert & Nishimura [10] extend their works to a general non-concave production function. These works prove the existence of a critical level of capital stock, usually named "*Dechert-Nishimura-Skiba*" point[1]. Beginning with a level capital stock under this level, the economy shrinks and collapses to zero, otherwise it increases to a steady state[2].

Kamihigashi & Roy [14] extend the analysis to a larger class of production function, by assuming only the upper-semi continuity. They characterize the critical point below it the economy collapses in long run and above which survival (bounded away from zero) is possible.

Another line of literature studies conditions allowing the convergence to infinity of the economy. Jones & Manuelli in [11], [12] work with concave production function which keeps sufficiently high productivity even with a large accumulation of capital. Under this condition, the economy always converges to infinity.

Kamihigashi & Roy [15] relax not only the concavity but also the continuity of production, and prove that under the condition that the productivity is sufficiently high for large accumulation of capital stock, if the economy begins with a initial state higher than a critical level, it will increasing to infinity[3].

As Majumdar & Nermuth [19], Dechert & Nishimura [10], Mitra & Ray [20], Kamihigashi & Roy [15] use the notion of *net gain function*, representing the discounted net returns on investment. They prove that the economy always evolves

[1]For a more detail survey, see Akao & al [1].

[2]For an analysis in continuous time, see Akao & al [2].

[3]Since the production function is not continuous, their condition must be stated under the form of upper and under derivatives of this function.

in order to increase the value of *net gain function.* It is interesting and surprising to see how the use of this notion provides such rich results, and gives us deep insights in economic dynamics.

Roy [22] studies an economy with *wealth effects*, where the utility depends not only on the consumption but also on the capital level. He prove that the high *capitalism spirit* (represented under a condition requiring that the marginal rate of substitution between capital-consumption in the preferences is sufficiently large) can compensate the low productivity. If the sum of this two quantities overcomes the discount rate, beginning with a level of capital accumulation, the economy continues accumulate and hence converges to infinity.

In this article, we consider the same question about conditions ensuring sustained growth, in the most generalized possible general case, *i.e.* where the dynamics of the economy can be characterized as a solution of

$$\max \left[\sum_{s=0}^{\infty} \delta^s V(x_s, x_{s+1}) \right],$$

where $\delta \in (0, 1)$ is the discount factor and V denotes the payoffs function.

We prove that with some mild conditions, the following inequality is sufficient for characterizing sustained growth:

$$V_2(x, x) + \delta V_1(x, x) > 0, \tag{1.1}$$

for any x large enough[4].

The intuition for (1.1) is given as follow. If between saving and remaining in *status quo*, choice saving always prevails, then the sustained growth is possible.

The results in this article allow us, in our subjective opinion, gather a large class of cases studied in the literature under a same viewpoint. It can be applied for the situations where the Kamihigashi & Roy's [15] techniques for one-sector economy

[4]For any (x, y) belonging to the definition domain of V, the notations $V_1(x, y)$ and $V_2(x, y)$ denote respectively the partial derivatives corresponding to the first and the second arguments.

can not be used. For example the two-sectors economies, the economy with *wealth effects* presented by Roy [22] and *capitalism spirit* of Kamihigashi [13], or the economy with accumulation of human capital, presented in this article.

The article is organized as follows. Section 2 presents the fundamentals of the model. Under the *tail-insensitivity* condition, optimal solution exists, and under the *super-modularity*, its monotonicity is ensured. Section 3 studies the conditions ensuring sustained growth, with the main one being (1.1). Section 4 concludes and Section 5 gives some applications in different configurations in literature. Proofs are gathered in Appendix.

2 Fundamentals

2.1 The model

Time is discrete: $s = 0, 1, 2, \ldots$. The discount factor is $0 < \delta < 1$. The technology of this economy is characterized by a correspondence $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$. For any $x_0 \geq 0$, denote by $\Pi(x_0)$ the set of feasible paths $\{x_s\}_{s=0}^{\infty}$ satisfying $x_{s+1} \in \Gamma(x_s)$ for any $s \geq 0$.

Given capital stocks at some consecutive dates x_s and x_{s+1}, the indirect utility level at date s is $V(x_s, x_{s+1})$, where V is a real function whose domain of definition is the graph of Γ: the set (x, y) such that $y \in \Gamma(x)$.

For given $x_0 \geq 0$, the economy solves the following inter-temporal optimization problem

$$\max \left[\sum_{t=0}^{\infty} \delta^s V(x_s, x_{s+1}) \right],$$

$$\text{s.c } x_{s+1} \in \Gamma(x_s), \forall\ s \geq 0.$$

Denote by v the value function of this problem:

$$v(x_0) = \sup_{\Pi(x_0)} \left[\sum_{s=0}^{\infty} \delta^s V(x_s, x_{s+1}) \right].$$

2.2 Existence of solution and the Bell-man functional equation

Assumption **A1** establishes standard conditions ensuring the existence of solution for the maximization problem. For the detailed comments about these conditions, curious readers can refer to Le Van & Morhaim [16].

Assumption A1. i) *The correspondence Γ is no-empty, convex compact values and ascending*[5].

ii) *The function V is continuous in graph of Γ, strictly increasing with respect to the first argument and decreasing with respect to the second one.*

iii) Non-triviality: *For any $x_0 > 0$, there exists $\{x_s\}_{s=0}^{\infty} \in \Pi(x_0)$ such that*

$$\sum_{s=0}^{\infty} \delta^s V(x_s, x_{s+1}) > -\infty.$$

iv) Tail-insensitivity: *Fixed $x_0 > 0$, for any $\epsilon > 0$, there exist T_0, a neighbourhood $\mathscr{V}$ of x_0 such that for any $x'_0 \in \mathscr{V}$, any $\{x'_s\}_{s=0}^{\infty} \in \Pi(x'_0)$, any $T \geq T_0$:*

$$\sum_{s=T}^{\infty} \delta^s V(x'_s, x'_{s+1}) < \epsilon.$$

The conditions (*i*), (*ii*) and (*iii*) are usual in literature, characterizing the main properties of the technology, the trade-off between consume today and invest tomorrow, and ensure that the problem is not trivial.

The most important condition is the *tail-insensitivity* one. This condition not

[5] In the spirit of Amir [3]. For any $x \leq x'$, $y \in \Gamma(x)$, $y' \in \Gamma(x')$, we have $\min\{y, y'\} \in \Gamma(x)$ and $\max\{y, y'\} \in \Gamma(x')$.

only states that the value function should be finite, but moreover it allows the satisfaction of *upper semi-continuity* property, which is important for the existence of solution.

Under the conditions in **A1**, the value function is increasing and upper-semi continuous. This continuity ensures the existence of solution for optimization problem.

PROPOSITION 2.1. *Assume* ***A1***. *Then:*

i) *The value function v is strictly increasing and upper-semi continuous.*

ii) *Solution exists.*

iii) *The value function satisfies the* Bellman equation*:*

$$v(x_0) = \max_{x_1 \in \Gamma(x_0)} \big[V(x_0, x_1) + \delta v(x_1) \big].$$

iv) *A sequence $\{x_s\}_{s=0}^{\infty}$ is a optimal if and only if for any $s \geq 0$,*

$$v(x_s) = V(x_s, x_{s+1}) + \delta v(x_{s+1}).$$

From now on, for $x_0 \geq 0$, denote by ϕ the optimal policy correspondence:

$$\phi(x_0) = \underset{x_1 \in \Gamma(x_0)}{\operatorname{argmax}} \big[V(x_0, x_1) + \delta v(x_1) \big].$$

The Proposition 2.1 has a consequence that $\phi(x_0)$ is a no-empty, compact value correspondence[6].

2.3 Super-modularity and monotonicity

In this section, we will study the monotonicity of optimal path and optimal policy correspondence. It is intuitive to assume the super-modularity, a property stating

[6] As in Dechert & Nishimura [10], for $x_1 \in \phi(x_0)$, the set $\phi(x_1)$ is single-valued. Moreover, the value function v is differentiable at x_1. Almost everywhere, the correspondence ϕ is single-valued and the value function is differentiable. A generalization of this result for configurations with uncertainty is given in Nishimura & al [21].

the complementarity of capital accumulations.

Assumption A2. *The payoff function V is strictly super modular.*[7]

Under the super-modularity property, the optimal policy correspondence is "*increasing*", as stated in Proposition 2.1. This is an important result helping the understanding of optimal paths' behaviour. The super-modularity implies that every optimal path is monotonic. The result and proof of Lemma 2.1 are similar to the one-sector configuration studied in Dechert & Nishimura [10].

LEMMA **2.1.** *Assume **A1** and **A2**. Then*

i) *For all $x_0 < x'_0$, and $x_1 \in \phi(x_0)$, $x'_1 \in \phi(x'_0)$, we have $x_1 < x'_1$.*

ii) *Every optimal path is either monotonic or constant.*

A direct consequence of Lemma 2.1 is that every optimal path either converges to some real value, or to infinity. Moreover, Lemma 2.1 allows us to characterize a general feature of optimal paths, stated in Proposition 2.2. If for some initial state x_0, the optimal path converges to infinity, then the same property is also satisfied for any greater initial level of capital stock, thanks to the monotonicity of optimal policy correspondence. If such initial state x_0 does not exist, every optimal path is bounded from above.

PROPOSITION 2.2. *Assume **A1** and **A2**. Then one of the two following complementary statements is verified:*

i) *There exists $\overline{x} \geq 0$ such that for any $x_0 \geq \overline{x}$, any optimal path beginning from x_0 is increasing and converges to infinity.*

ii) *For any $x_0 \geq 0$, every optimal path beginning from x_0 is bounded from above.*

[7] The (strict) super-modularity is defined as: for every (x, x') and (y, y') that belong to Graph(Γ), $V(x, y) + V(x', y')(>) \geq V(x', y) + V(x, y')$ is verified whenever $(x', y')(>) \geq (x, y)$. When V is twice differentiable, (strict) super modularity sums up to positive cross derivatives: $V_{12}(x, y)(>) \geq 0$ for any x, y.

3 The sustained growth condition

With **A1**, solution exists. With **A2**, the monotonicity is satisfied. Each optimal path is hence either converges to a steady state, or converges to infinity. If for some x_0, there is an optimal beginning from x_0 converges to infinity, then this property is verified for any optimal path beginning from $x'_0 > x_0$.

In this section, we discuss condition ensuring the possibility of sustained growth *i.e.* the convergence to infinity of the economy.

3.1 The condition

The main idea runs as follows: for any capital accumulation level x, if between the choice of staying in *status quo* and saving a little, the economy always prefer the later one, then it can converges to infinity.

Assumption A3. *There exists $x' \geq 0$ such that for any $x > x'$, $x \in int\Gamma(x)$ and*

$$V_2(x, x) + \delta V_1(x, x) > 0.$$

Denote by x^* the smallest value satisfying this property[8]*.*

The Proposition 3.1 states our first main results, for the case $x^* = 0$. If the payoff function V is bounded from below, every optimal path beginning from a positive initial state converges to infinity. For the configuration where V is unbounded from below, there exist feasible paths converging to zero. For this case, we assume the *technical assumption.* The intuition of this condition is that the depreciate rate of capital is not *too high* comparing to the discount rate.

Assumption T1. Technical condition *Let the depreciate rate of capital stock be $0 < d < 1$: for any $x \geq 0$, $\min \Gamma(x) = (1-d)x$. For any $x > 0$:*

$$\lim_{T \to \infty} \delta^T V\left((1-d)^T x, (1-d)^T x\right) = 0.$$

[8]Denote by S the set of x' such that $V_2(x, x) + \delta V_1(x, x) > 0$ for any $x > x'$. Once S is no-empty, there exists $x^* = \inf S$. It is easy to verify that $x^* \in S$.

The idea of the main results runs as follows: under **A2** and **A3**, for any $x^* \leq y \leq x$, we have the following inequality[9]:

$$\frac{V(x,x)}{1-\delta} \geq V(x,y) + \frac{\delta V(y,y)}{1-\delta}. \tag{3.1}$$

Now suppose that for some $x_0 > x^*$, there is an optimal path $\{x_s\}_{s=0}^{\infty}$ beginning from x_0 which is decreasing. For each T such that $x_{T+1} \geq x^*$,

$$\begin{aligned}
v(x_0) &\geq \sum_{s=0}^{\infty} \delta^s V(x_0, x_0) \\
&= \frac{V(x_0,x_0)}{1-\delta} \\
&\geq V(x_0,x_1) + \frac{\delta V(x_1,x_1)}{1-\delta} \\
&\geq V(x_0,x_1) + \delta V(x_1,x_2) + \frac{\delta^2 V(x_2,x_2)}{1-\delta} \\
&\dots \\
&\geq \sum_{s=0}^{T} \delta^s V(x_s, x_{s+1}) + \frac{\delta^{T+1} V(x_{T+1}, x_{T+1})}{1-\delta}.
\end{aligned}$$

Consider the case $x^* = 0$. Let T converges to infinity, if V is bounded from below, the right-hand-side of the inequality converges to $v(x_0)$, which leads us to a contradiction. The case V is unbounded from below is more complicated, because we must avoid the possibility that the optimal path converges to zero with high speed. Assuming the *technical condition*, we can obtain the same conclusion as the case V is bounded from below.

The case $x^* > 0$ challenges us in another way. If x^* is a steady state of the economy, we can use the same arguments as the case $x^* = 0$ to prove that there is no decreasing optimal path beginning from $x_0 > x^*$. The reason is that such optimal path must be bounded from below by x^*.

In the situation where there is no argument assuring that x^* is steady states, difficulties arise, since from some date T, we may have $x_{T+1} < x^*$, and the funda-

[9] The idea is inspired by similar consideration in Cao & Werming [6].

mental inequality (3.1) presented above can not be verified for any $s \geq T$. In order to circumvent this difficulty, we assume that not only $V_2(x,x) + \delta V_1(x,x) > 0$ for x big, but these values are sufficiently high such that

$$\int_{x^*}^{\infty} \big(V_2(x,x) + \delta V_1(x,x)\big)dx = \infty.$$

Under this condition, we obtain the similar results as the case $x^* = 0$. We can even relax the *technical condition* and do not need make distinction between bounded and unbounded from below functions.

3.2 Super-modularity and sustained growth

In this subsection, we prove that if the condition $V_2(x,x)+\delta V_1(x,x) > 0$ is satisfied for any $x > 0$, then the sustained growth is ensured.

Proposition 3.1. *Assume **A1**, **A2** and **A3** with* $x^* = 0$.

i) *Consider the case V is bounded from below. Then for any $x_0 > 0$, any optimal path beginning from x_0 is increasing and converges to infinity.*

ii) *Consider the case V is unbounded from below. Under the* technical condition*, any optimal path beginning from x_0 is increasing and converges to infinity.*

In many configurations, for example the *convex-concave* production function, the condition in **A3** is satisfied only for sufficiently large level of initial capital stock. In these situations, $x^* > 0$. With an additional condition, we can ensure the sustained growth for the economies beginning from a sufficiently high value of capital stock. Moreover, we can relax the *technical condition* for the case V is unbounded from below.

Proposition 3.2. *Assume **A1**, **A2** and **A3** with $x^* > 0$. Suppose that*

$$\int_{x^*}^{\infty} \big(V_2(x,x) + \delta V_1(x,x)\big)dx = \infty.$$

Then there exists $\overline{x}$ such that for any $x_0 > \overline{x}$, every optimal path beginning from x_0 is increasing and converges to infinity.

Though Corollary 3.1 is direct consequence of Proposition 3.2.

Corollary 3.1. *Assume **A1**, **A2**. Suppose that*

$$\lim_{x\to\infty} V(x,x) = \infty,$$

and for some $\epsilon > 0$, there exists $x^* > 0$ such that for any $x > x^*$,

$$\frac{-V_2(x,x)}{\delta V_1(x,x)} \leq 1 - \epsilon.$$

Then there exists $\overline{x}$ such that for any $x_0 > \overline{x}$, every optimal path beginning from x_0 is increasing and converges to infinity.

In opposition to the condition for sustained growth, we can also characterize the one under which the economy is always bounded.

Proposition 3.3. *Assume **A1** and **A2**. Suppose that there exists some $\tilde{x}$ such that for any $x > \tilde{x}$, either $x \leq \min \Gamma(x)$, or $x \in int\Gamma(x)$ and*

$$V_2(x,x) + \delta V_1(x,x) < 0.$$

Then for any x_0, every optimal path beginning from x_0 is bounded.

3.3 Convexity and sustained growth

Under the strict concavity of payoff function, interestingly, we can obtain the same results as Propositions 3.1 and 3.2 without the super-modularity and the condition stated in Proposition 3.2.

With the convexity structure, it is well known in the dynamic programming literature[10] that the value function v is concave. The optimal policy correspondence ϕ becomes function.

[10]See Stokey, Lucas (with Prescott) [24].

The critical level of capital stock x^* is either equal to zero, or strictly positive and, satisfying the simultaneously Ramsey-Euler equation and transversality condition, becomes the biggest steady state of the economy. Moreover, since x^* is the biggest steady state, either $\phi(x) > x$ for any $x > x^*$, or $\phi(x) < x$ for any $x > x^*$. This property has an important consequence is that for any initial state $x_0 > x^*$, the optimal path beginning from x_0 is monotonic.

Using the same arguments in the proofs of Propositions 3.1 and 3.2, we obtain Proposition 3.4.

Proposition 3.4. *Suppose that V is strictly concave. Assume **A1** and **A3**.*

i) *Consider the case V is bounded from below. For any $x_0 > x^*$, the optimal path beginning from x_0 is increasing and converges to infinity.*

ii) *Consider the case V is unbounded from below and $x^* > 0$. For any $x_0 > x^*$, the optimal path beginning from x_0 is increasing and converges to infinity.*

iii) *Consider the case V is unbounded from below and $x^* = 0$. Under* technical condition*, for any $x_0 > 0$, the optimal path beginning from x_0 is increasing and converges to infinity.*

Similarly to the condition in Proposition 3.3, we have Proposition 3.5. The concavity of V allows us to relax the super-modularity property.

Proposition 3.5. *Suppose that V is strictly concave. Assume **A1**. Suppose that there exists $\tilde{x}$ such that for any $x > \tilde{x}$, either $x \geq \max \Gamma(x)$, or $x \in int\Gamma(x)$ and*

$$V_2(x, x) + \delta V_1(x, x) < 0.$$

Then for any x_0, the optimal path beginning from x_0 is bounded.

3.4 Statistical comparative

We establish the conditions under which for sufficiently initial of capital stock, the economy can growth and converges to infinity. In our analysis, the critical level x^* plays an important role. That naturally raises the question: how this level depends on small changes of fundamentals of the economy, for example the discount factor?

For each value δ, denote by $x^*(\delta)$ the critical threshold. Since $V_1(x, y) \geq 0$ for any $(x, y) \in \text{Graph}(\Gamma)$, for any $\delta' \geq \delta$, we have $x^*(\delta') \leq x^*(\delta)$. The critical level x^* is hence a non increasing function with respect to the discount rate. By adding the hypothesis that in a neighbourhood of x^*, the function $V_2(x, x) + \delta V_1(x, x)$ is injective, the level $x^*(\delta')$ becomes a strictly decreasing function with respect to δ' belonging to a neighbourhood of δ.

PROPOSITION 3.6. *Suppose that V is differentiable. Given the discount rate δ and assume that $x^*(\delta) > 0$. Suppose that in a neighbourhood of $(x^*(\delta) - \epsilon, x^*(\delta) + \epsilon)$, the function $V_2(x, x) + \delta V_1(x, x)$ is injective. Then there exist a neighbourhood of δ such that in this interval, $x^*(\delta')$ is strictly decreasing.*

3.5 Remarks

In this article, the most important condition is $V_2(x, x) + \delta V_1(x, x) > 0$. Naturally, that raises the question about what happens if the differentiability of V is not satisfied.

Notice that the condition in **A3** can be replaced by the following one, which is weaker and does not require neither differentiability nor continuity: for $x \geq y \geq x^*$,

$$\frac{V(x, x)}{1 - \delta} \geq V(x, y) + \frac{\delta V(y, y)}{1 - \delta}. \tag{3.2}$$

We can hence extend the result in Proposition (3.2) with $x^* = 0$ to the case where V is not differentiable or continuous, for example the one-sector economy case presented in Kamihigashi & Roy [15]. In their set up, the inequality (3.2) is satisfied.

The case $x^* > 0$ is more complicated, since in the proof we need the a technical result that

$$V(x,y) - V(y,y) = \int_y^x V_1(z,y)dz.$$

This condition can be assured for the case V is *absolutely continuous* on compact set, a condition which is weaker than *differentiability*.

If by some reason (for example V is concave), the critical threshold x^* is also a steady state, we do not need the condition $\int_{x^*}^{\infty}\big(V_2(x,x) + \delta V_1(x,x)\big)dx = \infty$. The argument is that for these cases, any optimal path beginning from $x_0 > x^*$ is bounded from below by x^*. Then we use the proof of Proposition 3.1.

The *technical condition* can be replaced by other conditions ensuring that for any optimal path $\{x_s\}_{s=0}^{\infty}$ beginning from $x_0 > 0$,

$$\lim_{T\to\infty} \delta^T V(x_T, x_T) = 0.$$

Obviously, this property is always satisfied for the bounded from below functions.

And, last but not least, the *strict* super-modularity in condition **A2** is not only for technical convenient. If the utility function satisfies only the super-modularity (but not strict), the optimal paths exhibit complicated behaviours. For example, in Kamihigashi & Roy [14], the instantaneous utility function is linear, the optimal path reaches one steady state in a finite time and can jump among different steady states afterwards[11]. A careful consideration for this case is interesting, but that must be the subject for another work.

4 Conclusions

We established conditions ensuring sustained growth. The threshold beyond which the economy converges to infinity is characterized. The conditions, in our subjective opinion, are simple and easy to verify. Moreover, we can apply them in a

[11] The monotonicity is not verified.

large class of inter-temporal optimization problems.

The critical threshold, intuitively, is a non-increasing function in respect to the discount rate, and in general, it is a decreasing function. This result echoes the one in Akao & al [1], which studies the dependency in discount rate of the thresholds for the collapse or convergence to steady state of the economy.

It is well known that if V is concave and $V_{12}(x, y) \leq 0$, there exist stable periodic cycles[12]. Using Proposition 3.4, with the satisfaction of condition **A3**, we prove that for high level of capital accumulation, the cycles disappear, and the economy follows an increasing path to infinity.

5 Applications

5.1 One sector economy

Consider the one sector economy where for given x_0, the agent solves:

$$\max \left[\sum_{s=0}^{\infty} \delta^s u(c_s) \right],$$
$$\text{s.c } c_t + x_{s+1} \leq f(x_s).$$

The utility function u is supposed to be strictly increasing and concave. The production function satisfies $f(x) > x$ for x sufficiently big.

The payoff function is $V(x, y) = u\big(f(x) - y\big)$.

We have

$$\begin{aligned} V_2(x, x) + \delta V_1(x, x) &= -u\left(f(x) - x\right) + \delta u\left(f(x) - x\right) f'(x) \\ &= u'\left(f(x) - x\right)\left(\delta f(x) - 1\right). \end{aligned}$$

The condition in **A3** is then equivalent to $f'(x) > \frac{1}{\delta}$. This is the same condition

[12] See Benhabib & Nishimura [5].

in Kamihigashi & Roy [15]. In order to simplify the exposition, assume that u is bounded from below.

Proposition 5.1. *Assume that u is concave, bounded from below, f is concave. Assume that for any $x > x^*$, $f'(x) > \frac{1}{\delta}$. Then for any $x_0 > x^*$, every optimal path beginning from x_0 is increasing and converges to infinity.*

Now consider the case f is not concave.

Proposition 5.2. *Assume that*

i) $\liminf_{x\to\infty} f'(x) > \frac{1}{\delta}$.

ii) $\limsup_{x\to\infty} f'(x) < \infty$.

iii) *The utility function is unbounded from above.*

Then there exists $\overline{x} \geq 0$ such that for any $x_0 \geq \overline{x}$, every optimal path beginning from x_0 is increasing and converges to infinity.

The Proposition 5.2 is direct consequence of Proposition 3.2. It adds a complementary feature to the result of Kamihigashi & Roy [15], which requires that $\lim_{c\to\infty} u'(c)c < \infty$ and hence rules out the constant elasticity and constant elasticity of marginal utility functions.

5.2 A two-sectors economy

Consider the two-sectors economy in Dana and Le Van[13] [9]. One sector produces consumption good, and the other one produces capital good.

At date s, the agent consumes c_s, produced by $f(x_s^1)$, the consumption production function. The capital good x_s^1 is produced by the sector 2. The capital used in the next date x_{s+1} is produced by the sector 2, which uses x_s^2 to produce a quantity $g(x_s^2)$ of capital good.

[13] Chapter 4, page 92.

The social planner solves the problem for given x_0:

$$\max \left[\sum_{s=0}^{\infty} \delta^s u(c_s)\right],$$
$$\text{s. c } 0 \le c_s \le f(x_s^c),$$
$$0 \le x_{s+1} \le g(x_s^2).$$

We assume that the functions u and f and g are strictly increasing and concave, satisfying Inada condition. The capital production function g satisfies

$$\lim_{x\to\infty} g'(x) = 1 + \lambda,$$

with λ is a strictly positive constant.

Define $\zeta(x) = g^{-1}(x)$, the inverse function of g. The function ζ is strictly increasing, differentiable and

$$\lim_{x\to\infty} \zeta'(x) = \frac{1}{1+\lambda}.$$

For each capital stock level x, the set of possible capital investment for the next day if $\Gamma(x) = [(1-d)x, g(x)]$. For each chosen level of capital stock of next day $0 \le y \le g(x)$, a level equal to y for capital sector of tomorrow, we must invest $\zeta(y)$ for the capital production sector.

The consumption level is $c = f(x - \zeta(y))$. The payoff function is hence:

$$V(x, y) = u\big[f\big(x - \zeta(y)\big)\big].$$

We have

$$\begin{aligned} V_2(x, x) + \delta V_1(x, x) &= -u'\big[f\big(x - \zeta(x)\big)\big] f'\big(x - \zeta(x)\big)\zeta'(x) \\ &\quad + \delta u'\big[f\big(x - \zeta(x)\big)\big] f'\big(x - \zeta(y)\big) \\ &= u'\big[f\big(x - \zeta(x)\big)\big] f'\big(x - \zeta(x)\big)\left(-\zeta'(x) + \delta\right). \end{aligned}$$

The condition $V_2(x,x) + \delta V_1(x,x) > 0$ is equivalent to $\zeta'(x) < \delta$. This can be satisfied if $\delta(1+\lambda) > 1$. This calculus allows the statement of Proposition 5.3, which is a consequence of Proposition 3.2.

PROPOSITION 5.3. *Assume that $\delta(1+\lambda) > 1$. Then for any $x_0 > 0$, the optimal path beginning from x_0 is increasing and converges to infinity.*

5.3 The economy with wealth effects

Consider the model of economic growth with wealth effects, presented in Kamihigashi [13] and Roy [22]. In this set up, the utility function depends on consumption level and capital stock level.

The maximization problem for some given x_0 is

$$\max \left[\sum_{s=0}^{\infty} \delta^s u(c_s, x_s)\right],$$
$$\text{s.c. } c_s + x_{s+1} \leq f(x_s),$$

where u is utility function and f is production function, being both concave, increasing and differentiable. Comme Roy [22], the function u is bounded from below.

Denote by u_c and u_x the corresponding partial derivatives of u in respect correspondingly to the first argument and the second one.

The indirect function is $V(x,y) = u\left(f(x) - y, x\right)$. It is easy to verify that under the concavity of utility function u and production function f, V is concave.

We have

$$V_2(x,x) + \delta V_1(x,x) = -u_c\left(f(x) - x, x\right) + \delta\left(u_c\left(f(x) - x, x\right) f'(x) + u_x\left(f(x) - x, x\right)\right).$$

The condition $V_2(x,x) + \delta V_1(x,x) > 0$ is equivalent to

$$f'(x) + \frac{u_x\left(f(x) - x, x\right)}{u_c\left(f(x) - x, x\right)} > \frac{1}{\delta}.$$

This is the same condition as Roy [22]. Define S the set of steady states, the set of solutions to

$$f'(x) + \frac{u_x\left(f(x) - x, x\right)}{u_c\left(f(x) - x, x\right)} = \frac{1}{\delta},$$

and $x^* = \sup S$.

By Proposition 3.4, we obtain the same result in Roy [22], without using his condition **U4** which assumes that consumption and capital are weakly complementary: $u_{cx}(c, x) \geq 0$.

PROPOSITION 5.4. *Denote by x^* the biggest steady state (if steady state does not exist, let $x^* = 0$). Suppose that for any $x > x^*$ we have*

$$f'(x) + \frac{u_x\left(f(x) - x, x\right)}{u_c\left(f(x) - x, x\right)} > \frac{1}{\delta}.$$

Then any optimal path beginning from $x_0 > x^*$ is increasing and converges to infinity.

5.4 Human capital accumulation

In this section, we consider a model in which investing in human capital may yield a sustainable economic growth. For the sake of simplicity, we suppose that there is no physical capital. The production is realised using the effective labor (human capital) through a production function which is supposed to be strictly increasing and strictly concave. The agent, or the social planer divides the production in consumption and investment in human capital, in order to maximize the intertemporal sum of utilities for each given human capital level h_0:

$$\max \left[\sum_{t=0}^{\infty} \delta^t u(c_t) \right],$$

$$\text{s.c } c_t + s_{t+1} \leq f(h_t),$$

$$\frac{h_{t+1}}{h_t} = \varphi(s_{t+1}).$$

The quantities c_t, s_t are respectively the consumption and the saving at period t and h_t is the human capital at the same period.

The output is obtained by using only the effective labor through a production function f which is concave, increasing, continuous. The utility function is strictly increasing, strictly concave. For the sake of simplicity, let $u(0) = 0$.

The rate of growth of the human capital depends on the investment s_{t+1} is defined similar to the spirit of Lucas [17].

$$h_{t+1} = h_t\varphi(s_{t+1}),$$

where φ is strictly increasing, differentiable, satisfying

$$\lim_{h\to\infty} \varphi(h) = 1 + \lambda,$$

with some $\lambda > 0$, representing the upper bound of the formation.

Define $\psi(s) = \varphi^{-1}(s)$, the inverse function of ϕ. This function is increasing, satisfying $\psi(1 + \lambda) = +\infty$.

At the optimum, $c_t = f(h_t) - \psi\left(\frac{h_{t+1}}{h_t}\right)$. We can re-write the optimization problem as:

$$v(h_0) = \max \sum_{t=0}^{\infty} \delta^t u\left(f(h_t) - \psi\left(\frac{h_{t+1}}{h_t}\right)\right),$$

$$0 \le h_{t+1} \le h_t\varphi(f(h_t)).$$

We have $\Gamma(x) = \{(x, y) \in \mathbb{R}^2_+ : (1 - d)x \le y \le x\varphi(x)\}$. The payoff function is defined as

$$V(x, y) = u\left(f(x) - \psi\left(\frac{y}{x}\right)\right).$$

Calculus give

$$V_2(x,y) = -u'\left(f(x) - \psi\left(\frac{y}{x}\right)\right)\psi'\left(\frac{y}{x}\right) \times \frac{1}{x},$$
$$V_1(x,y) = u'\left(f(x) - \psi\left(\frac{y}{x}\right)\right)\left(f'(x) + \psi'\left(\frac{y}{x}\right) \times \frac{y}{x^2}\right).$$

Fixing y, if x increases,

i) $u'\left(f(x) - \psi\left(\frac{y}{x}\right)\right)\psi'\left(\frac{y}{x}\right) \times \frac{1}{x}$ decreases, since $f(x) - \psi\left(\frac{y}{x}\right)$ increases,

ii) Since φ is concave, the inverse function ψ is convex, and $\psi'\left(\frac{y}{x}\right)$ decreases,

iii) Obviously, $\frac{1}{x}$ decreases.

This implies if we increase x, the value of $-u'\left(f(x) - \psi\left(\frac{y}{x}\right)\right)\psi'\left(\frac{y}{x}\right) \times \frac{1}{x}$ increases. Hence $V_{12}(x,y) > 0$ and the super-modularity condition is satisfied.

The condition in **A3** is equivalent to

$$V_2(x,x) + \delta V_1(x,x) = u'\left(f(x) - \psi(1)\right)\left(\delta f'(x) - (1-\delta)\frac{\psi'(1)}{x}\right) > 0.$$

PROPOSITION 5.5. i) *Suppose that* $f'(x) > \frac{1-\delta}{\delta}\frac{\psi'(1)}{x}$, *for any* $x > 0$. *For any initial level of human capital, the economy converges to infinity.*

ii) *Suppose that the utility function* u *is unbounded from above. If*

$$\liminf_{x\to\infty} f'(x) > 0,$$

then there exists $\overline{h} \geq 0$ *such that for any* $h_0 > \overline{h}$, *every optimal path beginning from* h_0 *is increasing and converges to infinity.*

Though $\liminf_{x\to\infty} f'(x) > 0$ is sufficient for sustained growth in Proposition 5.5, this condition is not necessary.

Consider for example the production function $f(x) = Ax^\alpha$, utility function $u(x) = x^\beta$ with $0 < \alpha, \beta < 1$. For x sufficiently big we have $\delta f'(x) > (1-\delta)\frac{\psi'(1)}{x}$. More over,

$$\begin{aligned}\delta f'(x) - (1-\delta)\frac{\psi'(1)}{x} &= O\left(\frac{1}{x^{1-\alpha}}\right),\\ u'\left(f(x) - \psi(1)\right) &= O\left(\frac{1}{x^{\alpha(1-\beta)}}\right).\end{aligned}$$

Hence

$$u'\left(f(x) - \psi(1)\right)\left(\delta f'(x) - (1-\delta)\frac{\psi'(1)}{x}\right) = O\left(\frac{1}{x^{1-\alpha\beta}}\right),$$

which implies for any h,

$$\int_h^\infty \left(V_2(x,x) + \delta V_1(x,x)\right) dx = \infty.$$

Applying Proposition 3.2, for initial human capital h_0 big enough, every optimal path beginning from h_0 is increasing and converges to infinity.

5.5 Optimal growth model with investment enhancing labor

We consider the optimization problem presented by Crettez & al [8]. They consider an economy with investment enhancing labor. The labor force (normalized to 1) is divided in two parts: part $1-z$ is devoted to the production sector, transforming saving to capital for the next period, the other part z is used investment enhancing labor, for example the labor allocated for financial sector.

At date t, giving saving level s_t and investment enhancing labor z_t, the capital for the next day is $x_{t+1} = \phi(z_t)s_t$, where ϕ represents the efficiency of financial sector.

The economy solves

$$\max\left[\sum_{t=0}^{\infty}\delta^t u(c_t)\right]$$
$$\text{s.c } x_{t+1}=\phi(z_t)\left(f(x_t,1-z_t)-c_t\right),$$
$$0\leq z_t\leq 1.$$

Under standard conditions[14] for functions u, f and ϕ (concave, strictly increasing, satisfying Inada conditions), this problem has solution and can be re-written as

$$\max\left[\sum_{t=0}^{\infty}\delta^t V(x_t,x_{t+1})\right]$$
$$\text{s.c } 0\leq x_{t+1}\leq \max_{0\leq z_t\leq 1}\phi(z_t)f(x_t,1-z_t).$$

The payoff function is defined as

$$V(x,y)=\max u\left(F(x,y)\right),$$

where

$$F(x,y)=\max_{0\leq z\leq 1}\left(f(x,1-z)-\frac{y}{\phi(z)}\right).$$

The problem defining F is strictly concave in respect to z, so for each (x,y), there exists unique z maximizing the problem

$$F(x,y)=f\big(x,1-z(x,y)\big)-\frac{y}{\phi\big(z(x,y)\big)}.$$

The Inada conditions ensure that $z(x,y)$ belongs to the open interval $(0,1)$. Hence it is solution to

$$-f_2(x,1-z)+\frac{y\phi'(z)}{\big(\phi(z)\big)^2}=0.$$

[14] For details, see Crettez & al [8].

Under the assumption **H8**[15] in Crettez & al [8], the super-modularity of V is satisfied. By the envelope theorem,

$$\begin{aligned}
V_1(x,y) &= u'\big(F(x,y)\big)F_1(x,y) \\
&= u'\left(F(x,y)\right) f_1\big(x, 1 - z(x,x)\big), \\
V_2(x,y) &= u'\big(F(x,y)\big)F_2(x,y) \\
&= -\frac{u'\big(F(x,y)\big)}{\phi\big(z(x,y)\big)}.
\end{aligned}$$

Under the condition **H9**[16] in Crettez & al [8], there exists unique steady state, defined as solution to

$$\phi\left(z(x,x)\right) f_1\big(x, 1 - z(x,x)\big) = \frac{1}{\delta}.$$

The condition in **A3**, $V_2(x,x) + \delta V_1(x,x) > 0$ is equivalent to

$$\phi\left(z(x,x)\right) f_1(x, 1 - z(x,x)\big) > \frac{1}{\delta}.$$

The following proposition is a consequence of Propositions 3.1, 3.2.

PROPOSITION 5.6. i) *Assume that for any $x > 0$,*

$$\phi\big(z(x,x)\big) f_1\big(x, 1 - z(x,x)\big) > \frac{1}{\delta}.$$

Then for any $x_0 > 0$, every optimal path beginning from x_0 is increasing and converges to infinity.

ii) *If the steady state x^* is unique, and the condition in part (i) is satisfied for any $x > x^*$, then for any $x_0 > x^*$, every optimal path beginning from x_0 is increasing and converges to infinity.*

[15] This condition has a long and complicated statement, but can verified easily in the case of logarithmic utility function and Cobb-Douglass production function.

[16] Which is satisfied for the case of C.E.S production function.

iii) *Assume that u is unbounded from above and*

$$\liminf_{x\to\infty}\left(-\frac{1}{\phi\big(z(x,x)\big)}+\delta f_1(x,1-z(x,x))\right)>0.$$

Then for x large enough, every optimal path beginning from x is increasing and converges to infinity.

If the financial sector is sufficiently efficient, sustained growth may occur even in the case the marginal productivity is less than the discount rate.

Appendix

A Proof of Lemma 2.1

This proof goes in the same line of the one in Dechert and Nishimura [10].

Suppose that there exist $x_0 < x'_0$, $x_1 \in \phi(x_0)$, $x'_1 \in \phi(x'_1)$ and $x_1 \geq x'_1$. Hence $x'_1 \in \Gamma(x_0)$ and in the same definition as Amir [3]:

$$(x'_0,x_1)=(x_0,x_1)\vee(x'_0,x'_1),$$
$$(x_0,x'_1)=(x_0,x_1)\wedge(x'_0,x'_1).$$

We have

$$V(x_0,x_1)+\delta v(x_1)\geq V(x_0,x'_1)+\delta v(x'_1),$$
$$V(x'_0,x'_1)+\delta v(x'_1)\geq V(x'_0,x_1)+\delta v(x_1).$$

Combining these two equations we obtain

$$V(x_0,x_1)+V(x'_0,x'_1)\geq V(x_0,x'_1)+V(x'_0,x_1),$$

which is contradictory to the super-modularity assumption **A2**.

The monotonicity of optimal paths is direct consequence of the monotonicity of the optimal policy correspondence.

B Proof of Proposition 2.2

Assume that for some $\overline{x}$, there exists an optimal path $\{\overline{x}_s\}_{s=0}^{\infty}$ beginning from $\overline{x}$ converges to infinity. Then by induction, using Lemma 2.1, for any $x_0 > \overline{x}$, any optimal path $\{x_s\}_{s=0}^{\infty}$ beginning from x_0 satisfies $x_s > \overline{x}_s$ for any $s \geq 0$. Hence the sequence $\{x_s\}_{s=0}^{\infty}$ is increasing and $\lim_{s\to\infty} x_s = \infty$.

C Proof of Proposition 3.1

We begin the proof by Lemma C.1.

Lemma **C.1.** *Assume **A1**, **A2**. Assume also that for any $x > 0$ and:*

$$\frac{V(x,x)}{1-\delta} \geq V(x,y) + \frac{\delta}{1-\delta} V(y,y),$$

for any $0 \leq y \leq x$ with the strict inequality for $y < x$.

Then for any $x_0 > 0$, any optimal path $\{x_s\}_{s=0}^{\infty} \in \Pi(x_0)$ beginning from x_0 is increasing.

Proof. Consider $x_0 > 0$ and an optimal path $\{x_s\}_{s=0}^{\infty}$ beginning from x_0. By the Proposition 2.1, the sequence $\{x_s\}_{s=0}^{\infty}$ is either increasing or decreasing.

Suppose that this sequence is strictly decreasing: $x_s > x_{s+1}$ for any $t \geq 0$. Since the constant sequence $(x_0, x_0, \dots)$ belongs to $\Pi(x_0)$, we have:

$$\begin{aligned}
v(x_0) &\geq \frac{V(x_0, x_0)}{1-\delta} \\
&> V(x_0, x_1) + \frac{\delta}{1-\delta} V(x_1, x_1) \\
&\geq V(x_0, x_1) + \delta V(x_1, x_2) + \frac{\delta^2}{1-\delta} V(x_2, x_2) \\
&\dots \\
&> \sum_{s=0}^{T} \delta^s V(x_s, x_{s+1}) + \frac{\delta^{T+1}}{1-\delta} V(x_{T+1}, x_{T+1}).
\end{aligned}$$

Since either V is bounded from below, or V is unbounded from below and *technical condition* is satisfied, by letting T converges to infinity, the right-hand-side of the inequality converges to $v(x_0)$: a contradiction.

Hence the sequence $\{x_s\}_{s=0}^{\infty}$ is increasing. Suppose that this sequence does not converge to infinity, then $\lim_{s\to\infty} x_s = \tilde{x}$. By the upper semi-continuity of the value function v, we have $\tilde{x} \in \phi(\tilde{x})$: the limit value $\tilde{x}$ is a steady state. By Euler equation, we have

$$V_2(\tilde{x}, \tilde{x}) + \delta V_1(\tilde{x}, \tilde{x}) = 0,$$

a contradiction. The proof of Lemma C.1 is completed.

QED

First, we prove that for any $x > 0$, the function with variable y

$$h(y) = V(x, y) + \frac{\delta}{1-\delta} V(y, y)$$

is strictly increasing in $[0, x]$. Indeed,

$$\begin{aligned}
h'(y) &= V_2(x,y) + \frac{\delta}{1-\delta}\big(V_1(y,y) + V_2(y,y)\big) \\
&\geq V_2(y,y) + \frac{\delta}{1-\delta}\big(V_1(y,y) + V_2(y,y)\big) \\
&= \frac{1}{1-\delta}\big(V_2(y,y) + \delta V_1(y,y)\big) \\
&\geq 0.
\end{aligned}$$

Using the same arguments as in the proof of Lemma C.1, the optimal sequence $\{x_t\}_{t=0}^{\infty}$ is increasing and converse to infinity. The proof of Proposition 3.1 is completed.

D Proof of Proposition 3.2

Assume that for any $x_0 > 0$, every optimal path beginning from x_0 is strictly decreasing.

First, consider the case that for some $x_0 > x^*$, there is an optimal path $\{x_s\}_{s=0}^{\infty}$ beginning from x_0 satisfying $x_s \geq x^*$ for any s. Using the same argument as in the proof of Lemma C.1, we have

$$\begin{aligned}
v(x_0) &\geq \frac{V(x_0, x_0)}{1-\delta} \\
&\geq \sum_{s=0}^{T} \delta^s V(x_s, x_{s+1}) + \frac{\delta^{T+1} V(x_{T+1}, x_{T+1})}{1-\delta},
\end{aligned}$$

which converges to $v(x_0)$ when T converges to infinity: a contradiction.

Now consider the case for any $x_0 > 0$, every optimal path $\{x_s\}_{s=0}^{\infty}$ beginning from x_0, there exists T such that $x_{T+s} < x^*$ for any $s \geq 1$.

We will prove the following claim: for any $x \geq x^*$,

$$v(x) - v(x^*) \leq \int_{x^*}^{x} V_1(y,y)dy.$$

Indeed, take any optimal path $\{x_s\}_{s=0}^{\infty}$ beginning from x. There exists T such that $x_T > x^* \geq x_{T+1}$.

For any $0 \leq s \leq T-1$ we have

$$\begin{aligned}
v(x_s) - v(x_{s+1}) &= V(x_s, x_{s+1}) + \delta v(x_{s+1}) - v(x_{s+1}) \\
&\leq V(x_s, x_{s+1}) + \delta v(x_{s+1}) - V(x_{s+1}, x_{s+1}) - \delta v(x_{s+1}) \\
&= V(x_s, x_{s+1}) - V(x_{s+1}, x_{s+1}) \\
&= \int_{x_{s+1}}^{x_s} V_1(y, x_{s+1}) dy \\
&\leq \int_{x_{s+1}}^{x_s} V_1(y, y) dy.
\end{aligned}$$

The last inequality comes from the super-modularity: $V_{12}(x, y) \geq 0$ for any $y \in \Gamma(x)$ and hence $V_1(y, y) \geq V_1(y, x_{s+1})$ for $y \geq x_{s+1}$.

For $s = T$, observe that by the *ascending* property and the continuity of Γ, $x_{T+1} \in \Gamma(x^*)$. We then have

$$\begin{aligned}
v(x_T) - v(x^*) &\leq V(x_T, x_{T+1}) + \delta v(x_{T+1}) - V(x^*, x_{T+1}) - \delta v(x_{T+1}) \\
&= V(x_T, x_{T+1}) - V(x^*, x_{T+1}) \\
&= \int_{x^*}^{x_T} V_1(y, x_{T+1}) dy \\
&\leq \int_{x^*}^{x_T} V_1(y, y) dy.
\end{aligned}$$

This implies

$$\begin{aligned}
v(x) - v(x^*) &= \sum_{s=0}^{T-1} (v(x_s) - v(x_{s+1})) + (v(x_T) - v(x^*)) \\
&\leq \sum_{s=0}^{T-1} \int_{x_{s+1}}^{x_s} V_1(y, y) dy + \int_{x^*}^{x_T} V_1(y, y) dy \\
&= \int_{x^*}^{x} V_1(y, y) dy.
\end{aligned}$$

Fix $\epsilon = v(x^*) - \frac{V(x^*, x^*)}{1-\delta}$.

Take $\overline{x} \geq x^*$ such that for any $x > \overline{x}$:

$$\int_{x^*}^{x} (V_2(y,y) + \delta V_1(y,y))\, dy > (1-\delta)\epsilon.$$

Since $x \in \Gamma(x)$, the sequence $(x, x, x, \dots)$ belongs to $\Pi(x)$. Hence

$$\begin{aligned} v(x) &\geq \sum_{s=0}^{\infty} \delta^s V(x,x) \\ &= \frac{V(x,x)}{1-\delta}. \end{aligned}$$

We have

$$\begin{aligned} V(x,x) - V(x^*,x^*) &\leq (1-\delta)v(x) - (1-\delta)v(x^*) + (1-\delta)\epsilon \\ &\leq (1-\delta)\big(v(x) - v(x^*)\big) + (1-\delta)\epsilon \\ &\leq (1-\delta)\int_{x^*}^{x} V_1(y,y)dy + (1-\delta)\epsilon. \end{aligned}$$

This implies

$$\int_{x^*}^{x} (V_1(y,y) + V_2(y,y))\, dy \leq (1-\delta)\int_{x^*}^{x} V_1(y,y)dy + (1-\delta)\epsilon,$$

which is equivalent to

$$\int_{x^*}^{x} (V_2(y,y) + \delta V_1(y,y))\, dy \leq (1-\delta)\epsilon,$$

a contradiction.

Hence for any $x_0 \geq \overline{x}$, every optimal path beginning from x_0 is increasing and converges to infinity.

E Proof of Corollary 3.1

Obviously, the condition in the statement implies that $V_2(x,x) + \delta V_1(x,x) > 0$ for $x > x^*$. Since $V_2(y,y) \leq 0$ for any y and

$$V(x,x) - V(x^*,x^*) = \int_{x^*}^{x} \big(V_1(y,y) + V_2(y,y),\big)$$

we have

$$\int_{x^*}^{x} V_1(y,y)dy = \infty.$$

The condition in the statement also implies that for $x > x^*$,

$$V_2(x,x) + \delta V_1(x,x) \geq \epsilon V_1(x,x).$$

Hence

$$\begin{aligned}\int_{x^*}^{x} \left(V_2(y,y) + \delta V_1(y,y)\right) dy &\geq \int_{x^*}^{x} V_1(y,y)dy \\ &= \infty.\end{aligned}$$

Applying the same arguments as in proof of Proposition 3.2, this proof is completed.

F Proof of Proposition 3.3

For any $y \geq x > 0$,

$$h(y) = V(x,y) + \frac{\delta V(y,y)}{1-\delta}.$$

We have

$$
\begin{aligned}
h'(y) &= V_2(x,y) + \frac{\delta(V_1(y,y)+V_2(y,y))}{1-\delta} \\
&\leq V_2(y,y) + \frac{\delta(V_1(y,y)+V_2(y,y))}{1-\delta} \\
&= \frac{V_2(y,y)+\delta V_1(y,y)}{1-\delta} \\
&\leq 0.
\end{aligned}
$$

The function h is then decreasing in $[x,\infty)$. This implies

$$
\frac{V(x,x)}{1-\delta} \geq V(x,y) + \frac{\delta V(y,y)}{1-\delta},
$$

for $x^* \leq x \leq y$.

Assume that the statement in Proposition 3.3 is not true. For some $x_0 > \tilde{x}$, the optimal sequence beginning from $\{x_s\}_{s=0}^{\infty}$ is increasing and converges to infinity. Using the same argument as in the proof of Proposition 3.1,

$$
\begin{aligned}
v(x_0) &\geq \frac{V(x_0,x_0)}{1-\delta} \\
&> V(x_0,x_1) + \frac{\delta V(x_1,x_1)}{1-\delta} \\
&\ldots \\
&> \sum_{s=0}^{T} \delta^s V(x_s,x_{s+1}) + \frac{\delta^{T+1} V(x_{T+1},x_{T+1})}{1-\delta},
\end{aligned}
$$

which converges to $v(x_0)$: a contradiction.

G Proof of Proposition 3.4

Consider the case $x^* > 0$, then by the definition of x^*,

$$
V_2(x^*,x^*) + \delta V_1(x^*,x^*) = 0.
$$

By the convexity structure, x^* is the biggest steady state. Hence by the continuity op the optima policy function, either for any $x > x^*$, we have $\phi(x) > x$, or for any $x > x^*$, we have $\phi(x) < x$.

We will prove the claim: for any $x^* \leq y \leq x$,

$$\frac{V(x,x)}{1-\delta} \geq V(x,y) + \frac{\delta V(y,y)}{1-\delta}.$$

Indeed, observe that since $V(y,y)$ is a concave function in respect to y, the function $V_1(y,y) + V_2(y,y)$ is decreasing.

Let $h(y) = V(x,y) + \frac{\delta V(y,y)}{1-\delta}$. Observe that $V_2(x,y) \geq V_2(x,x)$, hence

$$\begin{aligned}
h'(y) &= V_2(x,y) + \frac{\delta\big(V_1(y,y) + V_2(y,y)\big)}{1-\delta} \\
&\geq V_2(x,x) + \frac{\delta\big(V_1(x,x) + V_2(x,x)\big)}{1-\delta} \\
&= \frac{V_2(x,x) + \delta V_1(x,x)}{1-\delta} \\
&> 0.
\end{aligned}$$

This implies for any $x^* \leq y \leq x$, we have

$$\frac{V(x,x)}{1-\delta} \geq V(x,y) + \frac{\delta V(y,y)}{1-\delta}.$$

The claim is proved. For the case $x^* = 0$, follows the same arguments. Using the same arguments as in the proof of Proposition 3.1, the Proposition 3.4 is proved.

H Proof of Proposition 3.5

Let ϕ be the optimal policy function. Remark that either $\phi(x) > x$ for any $x > \tilde{x}$, or $\phi(x) < x$ for any $x > \tilde{x}$. This property implies that for any $x_0 > \tilde{x}$, either the optimal sequence beginning from x_0 is increasing and converges to infinity, or it is bounded. Then we use the same arguments as in the proof of Proposition 3.3.

I Proof of Proposition 5.1

The Proposition 5.1 is direct consequence of Proposition 3.2.

J Proof of Proposition 5.2

Indeed, for x high enough, $f'(x) > \frac{1+\epsilon}{\delta}$ for some $\epsilon > 0$. The condition (i) in **??** is hence satisfied.

For the condition (ii), first fix $1 + a > \limsup_{x\to\infty} f'(x)$. We have

$$\begin{aligned}
\int_{x'}^{\infty} \big(V_2(y,y) + \delta V_1(y,y)\big) &= \int_{x'}^{\infty} u'\left(f(y) - y\right)\left(\delta f'(y) - 1\right) dy \\
&\geq \epsilon \int_{x'}^{\infty} u'\left(ay\right) dy \\
&= \frac{\epsilon\delta}{a} \lim_{x\to\infty} \left(u\left(ax\right) - u\left(ax'\right)\right) \\
&= \infty.
\end{aligned}$$

K Proof of Proposition 5.3

Since g is concave, the inverse function ζ is convex, and hence the payoff function V is concave.

The condition $\delta(1+\lambda) > 1$ implies that for any $x > 0$, $\zeta'(x) < \delta$. The assumption **A3** is satisfied with $x^* = 0$. Proposition 5.3 is then direct consequence of Proposition 3.4.

L Proof of Proposition 5.4

This is a consequence of Proposition 3.4.

M Proof of Proposition 5.5

The result in part (i) is consequence of Proposition 3.1.

Now consider the part (ii). Since $\liminf_{x\to\infty} f'(x) > 0$, for x big enough we have $\delta f'(x) - \frac{(1-\delta)\psi'(1)}{x} > \epsilon$, for some $\epsilon > 0$. We then have

$$\begin{aligned} V_2(x,x) + \delta V_1(x,x) &= u'\left(f(x) - \psi(1)\right)\left(\delta f'(x) - \frac{(1-\delta)\psi'(1)}{x}\right) \\ &\geq \epsilon u'\left(f(x) - \psi(1)\right). \end{aligned}$$

Since f is concave, there exists some $a > 0$ such that $f(x) < ax$ for x sufficiently big. Hence for any h sufficiently large,

$$\begin{aligned} \int_h^\infty \left(V_2(x,x) + \delta V_1(x,x)\right) &\geq \epsilon \int_h^\infty u'\left(f(x) - \psi(1)\right) dx \\ &\geq \epsilon \int_h^\infty u'\left(ax - \psi(1)\right) dx \\ &= \infty. \end{aligned}$$

Applying Proposition 3.2, the proof is completed.

N Proof of Proposition 5.6

The part (i) is direct consequence of Proposition 3.1.

The proof of the second part is similar to the one of Proposition 3.4.

Consider the third part. For x sufficiently large, we have $V_2(x,x) + \delta V_1(x,x) > 0$. Moreover, there exists $\epsilon > 0$ such that

$$-\frac{1}{\phi\big(z(x,x)\big)} + \delta f_1\big(x, 1 - z(x,x)\big) > \epsilon,$$

with x sufficiently big. The next of the proof follows the same arguments as in the proofs of Propositions 5.2 and 5.5.

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