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Abstract

We study a Schumpeterian model of long-run growth with endogenous fertility and with *three* interacting dimensions of innovation. Scientific research is the fundamental dimension of innovation that creates new technological knowledge. This is allocated over new working prototypes in the horizontal dimension. New firms finance scientific research by obtaining the property rights of new working prototypes, and existing firms invest in developing the blueprint mode of working prototypes into the more productive modes of production in the vertical dimension. Balanced growth in the standards of living is fully endogenous without scale effects, and a new parameter, i.e., the elasticity of scientific knowledge with respect to existing collective scientific knowledge, nonlinearly accelerates long-run growth. With exogenous population growth, the model generates a semi-endogenous result due to the endogenously determined bound on technological opportunity.

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> [...] economists have not gotten into the "black box" of knowledge evolution in the past. [...] Models of endogenous growth have attempted to open these black boxes, but have just found another black box inside.
>
> — Joel Mokyr, *The Gifts of Athena* (2002, p.116)

1 Introduction

Any model of economic growth is a mapping from ideas to the standards of living. This is strictly true even for a toy economy where Robinson Crusoe hunts, cooks and eats fishes. First, he establishes a set of useful knowledge by using his relevant ideas. Applying useful knowledge in a certain way, he then hunts raw fishes. Finally, he applies useful knowledge possibly in some other way to cook them. Without ideas, naturally, he could not survive.

Robinson's tale suggests that the mapping from ideas to the standards of living is formed by at least three intermediate mappings. In sufficiently general terms, the Ideas-to-Knowledge mapping (I-K) is characterized by some number of processes along which *individuals* select some *ideas* to form a set of useful knowledge. Next, the Knowledge-to-Technology mapping (K-T) is characterized by some number of processes along which *individuals* apply useful knowledge to create new useful *ideas* (and knowledge) and to produce some useful *objects*. The Technology-to-Consumption mapping (T-C), finally, is characterized by the production of other useful *objects* to be consumed. Technology with capitalized "T", defined here as the complex system of all useful things including individuals with their numbers and skills, is utilized at this ultimate stage of production.

Neoclassical theory concludes that long-run growth in the standards of living is sustainable only through long-run growth in an elusive variable called technology or productivity level and denoted by A or X. The I-K and the K-T mappings are simplified away, and the so-called fate of human societies is bounded with this mysterious A or X.

Two main streams of theories offer solutions to this mystery. The first one is the growth theory of Marshallian externalities developed by Arrow (1962) and Uzawa (1965) and revolutionarily extended by Romer (1986) and Lucas (1988). This theory emphasizes the role of increasing returns with respect to physical and human capital under price-taking behavior as the source of sustainable growth in the standards of living. The second one developed by Romer (1990), Grossman and Helpman (1991) and Aghion and Howitt (1992) and extended by many others since then is the growth theory of Schumpeterian creative destruction that endogenizes the incentives for the accumulation of certain forms of knowledge under price-setting behavior.

Both the Marshallian and the Schumpeterian theories explain how the K-T mapping works, i.e., how economies convert knowledge into Technology. However, the I-K mapping within which individuals use ideas to form knowledge is simplified by the presumption that any form of knowledge, say K, can be treated as a variable with the domain $[K_0, \infty) \subset \mathbb{R}_{++}$ over the horizon $t \in [0, \infty)$. Thus, ideas and knowledge are "equivalent" "things"; there exists an implicit correspondence from the number of ideas to the level of knowledge associated with them.

Building on Weitzman's (1998) proposition that new ideas are successful hybridizations of existing ideas, Olsson (2000) challenges this last presumption with a set theory of knowledge in which any form of knowledge is a subspace of the Euclidean space of ideas. Thus, ideas have several aspects represented by the dimensions of this Euclidean space, and the accumulation of knowledge is convincingly defined as the expansion of such subspaces through the elimination of nonconvexities. Olsson (2005) applies this set theory of knowledge to technological knowledge where the nonconvexity of the space of technological knowledge determines technological opportunity. Therefore, technological opportunity to be exploited is limited by the feasibility of new successful hybridizations. Without technological paradigm shifts that create new nonconvexities, technological knowledge stagnates.

To the growth theorist, none of the three mappings is more important than the others *a priori*. However, the T-C mapping must be rich enough for the derivation of testable implications since the I-K and, to some extent, the K-T mappings include many unobservable components, e.g., the number of ideas and the level of knowledge. The K-T mapping, on the other hand, must incorporate some necessary game theory of underlying industrial organization and some necessary contract theory of knowledge accumulation. Besides, Dynamic (Stochastic) General Equilibrium notion is, in general, desired. The last but not least of such difficulties is the so-called unified growth theory that extends the stylized facts of economic growth and development over the entire history of mankind. Therefore, the growth theorists optimize their models with respect to the strength of three intermediate mappings, to the choice between pure theory and testable implications and to the emphasis on unified growth issues. An outsider's first comment would be *Mission*: *Impossible*!

Quite not; considering the advances of the theory in the last two decades and the contributions recently collected in Aghion and Durlauf (2005). One of the most important contributions is the development of new Schumpeterian models of horizontal and vertical R&D in which scale effects disappear and the emphasis on the nexus between policy and growth is maintained.[1] However, even in these

[1] For early contributions to this line of Schumpeterian theory, see Dinopoulos and Thompson (1998), Peretto (1998a), Young (1998) and Howitt (1999). Laincz and Peretto (2006) and Madsen (2008) present empirical evidence in favor of such Schumpeterian models against the semi-endogenous solution of Jones (1995), Kortum (1997) and Segerstrom (1998).

two-dimensional models of endogenous technology, some interpretation difficulties are present regarding the ways in which the society converts knowledge-based resources into Technology. What simply motivate us for the model studied in this paper are some of these difficulties as we discuss now.

The main points of distress, to our belief, are the existing answers to the questions of how one can classify different types of knowledge and how certain types of knowledge are accumulated under interaction. Existing models incorporate product innovation (horizontal R&D) and process innovation (vertical R&D) —*a surface of knowledge*. However, this does not match the entire collection of knowledge-creation activities that (modern) societies pursue. The evident example, recently remarked by Comin (2004, p.414), is the NSF's classification of R&D activities: *Basic Research* as the planned search for new knowledge, *Applied Research* as the application of existing knowledge to create new products and processes, and *Development* as the application of existing knowledge to improve existing products and processes. Thus, applied research in the horizontal dimension generates new products and new processes, and development in the vertical dimension improves existing ones. What is missing is the third, fundamental dimension of basic research.

An early description of such a three-dimensional structure of R&D is discussed by Lewis (1955, Ch.4), and the idea is best summarized by Nelson (1982, p.463) who describes the distinction and the connection between science and technology as follows:

> Research in the basic sciences is guided largely by the internal logic of the quest for understanding of a set of fundamental scientific questions. These questions are not generally defined in terms of knowledge needed for the advance of a particular technology. However, there are a number of so-called applied sciences where research priorities are directly tied to technological problems and opportunities.

The literature on the role of scientific knowledge in shaping the history of mankind through technological advancement in its broad sense actually dates back to the revolutionary essays of Bacon (1620[2004], p.85) who provides perhaps the earliest description of purposeful innovation by asserting that

> [...] many more things, better things, and at more frequent intervals, are to be hoped from human reason, hard work, direction and concentration than from chance, animal instinct and so on [...]

For our purposes, suffice it to admit that the increasing role of scientific knowledge as the fundamental source of innovation throughout the history is a major stylized fact of R&D.[2]

[2]See Gomulka (1990) for a classification of other stylized facts of R&D.

Three-dimensional structure of the knowledge-base leads us to a very large body of literature to which scholars from different fields contribute by providing some valuable insights on how, indeed, these different forms of knowledge are accumulated and converted into Technology. Among the most influential scholars of this particular subject are Paul A. David, Giovanni Dosi, Simon Kuznets, David S. Landes, Joel Mokyr, Richard R. Nelson, Nathan Rosenberg and Jacob Schmookler. Fortunately, we find a systematic understanding of this *space* of knowledge, with a strong emphasis on the timing and the location of the Industrial Revolution, in Mokyr's (2002) *Gifts of Athena* —an informal theory of what is called *useful knowledge*.[3]

But, *why* is it useful for our purposes? Mokyr (2002) builds on the distinctions between *propositional* (episteme) and *prescriptional* (techne) knowledge and between *aggregate* and *collective* knowledge. The former, as we demonstrate below, allows us to model knowledge spillovers more rigorously and less elusively than in some of the existing models, mostly in the tradition of Romer (1990) and Grossman and Helpman (1991), in which *blueprint knowledge is not explicitly incorporated*. The latter, in connection with the former, helps us to clarify which certain forms of knowledge are nonrival and/or nonexcludable and which are rival and/or excludable. In a strong sense, what Dasgupta and David (1994) call *the open Republic of Science* and *the proprietary Realm of Technology* become explicit.

We show that, in a private ownership economy with perfect protection of intellectual property rights, scientific research can be properly integrated with horizontal and vertical dimensions of innovation. Three balanced growth results follow:

- With endogenous population dynamics, a unique balanced growth equilibrium exists with the properties that (*i*) both horizontal and vertical R&D are active and (*ii*) scale effects are sterilized.
- In this fully endogenous growth equilibrium, the balanced growth rate of the standards of living crucially depends on the concentration of scientific research on existing scientific knowledge.
- Once the population growth becomes purely an exogenous process, semi-endogenous growth emerges due to the endogenously determined bound on technological opportunity.

Due to high degrees of dimensionality and nonlinearity, however, the model is not tractable enough, and this raises two technicalities. First, the analysis of transitional dynamics requires some simplifications and (perhaps) some numerical work which we leave for future research. Second, the (steady-state) balanced

[3]Evolutionary aspects of knowledge creation in general and micro(techno)economics of knowledge and innovation in particular are well-surveyed by Dosi (1988).

growth equilibrium can be solved only numerically. This is the major weakness of this model that trades off *simplicity* for *structure*.

The paper is organized as follows: In the next section, we develop the model economy and define the dynamic general equilibrium. In Section 3, we define balanced growth equilibrium and derive some of the conditions required for its existence and uniqueness. In Section 4, we draw some implications for economies with sustained growth. Section 5 concludes. Some derivations omitted in the main text are presented in the appendix.

2 A Model Economy

Model time is continuous with horizon $t \in [0, \infty)$.

Notation 1.

- *For any variable X_t, the initial value is denoted by X_0, the first order derivative with respect to time is denoted by $\dot{X}$, and the instantaneous growth rate is denoted by g_X.*
- *Time indicator t is omitted almost everywhere, and all* (*time-invariant*) *model parameters are denoted by lowercase Greek letters.* ■

Consider a closed economy with L intelligent individuals. With some limited neurobiological capacity, these individuals create, obtain and memorize *information*. In general, information can be stored in some special devices. Ancient tablets, encyclopedias and JSTOR are such devices that store information which then becomes available for retrieval. We assume that our economy is endowed with some number of these *external storage devices* (see below).

Any piece of information about anything is defined as an *idea* if at least one individual knows it or at least one storage device is loaded with it or both. *Knowledge*, in any form, is a collection of some ideas.

Assumption 1. *There exists a strictly increasing, scalar-valued function that maps the number of ideas to the level of knowledge.* ■

Assumption 1 significantly simplifies the I-K mapping since any idea has a unique trajectory contrary to Olsson's (2000) set theory. Thus, any form of knowledge can be treated as a variable with some domain $\mathbb{D} \subset \mathbb{R}_{++}$.

2.1 External Storage Devices

Let $X_A \in \mathbb{R}_{++}$ denote the aggregate level of some form of knowledge. By construction, only some fraction of it can be stored in external devices. Thus, aggregate knowledge X_A is *ex ante* useful, but only *collective knowledge* is *ex post* used. This leads us to reinterpret the well-known notions of rivalry and excludability since, in the present setup, a piece of information is either known or not known by an individual and is either stored or not stored in external devices. Thus, a set of ideas (and the particular body of knowledge associated with these ideas) becomes perfectly nonrival and nonexcludable if stored in external devices and remains perfectly rival and excludable otherwise.

External storage devices, broadly defined, are specific products and services that lower the access costs to information. All communication and transportation products and services, e.g., the printing press, the telegraph and the rail transport, lower these costs.[4] We assume that the *number* of such distinct products and services is given by

$$\gamma N$$

where $\gamma \in (0, 1)$ is a fraction parameter and N is the number of all products and services in the economy. However, since external storage devices are rival, an increasing number of individuals who use these devices lowers the aggregative efficiency. A convenient way to integrate these two effects is to impose

$$X := \eta \left(\frac{\gamma N}{H} \right)^{\delta} X_A \tag{1}$$

where X is the collective fraction of X_A and H is the number of individuals with aggregate knowledge X_A.[5] Then, $\delta > 0$ determines the marginal effect of the number of external storage devices *per individual* where $\eta > 0$ is a necessary scaling parameter that guarantees

$$X < X_A$$

for any t. We hereafter assume that the access to external storage devices is a free-of-charge (public) service.

[4]To clarify the distinction, we can compare two economies with respect to their ability to diffuse knowledge through external storage devices. In a 4000 B.C. economy, the only way of accessing knowledge is to visit the place where a collection of stone tablets located. In a 2000 A.D. economy, individuals have high-speed wireless Internet connection even in the bathrooms of their houses, and the Internet offers free access to an online source called Wikipedia.

[5]Cozzi and Spinesi (2004) explicitly model such access costs measured in time units and incorporate them directly to the knowledge production function. We offer a general form of diffusion process related to communication and transportation opportunities under rivalry.

2.2 Scientific Research

We now introduce scientific research —credited *always* and *everywhere* as the driving force behind Technology but not differentiated from Technology by a large number of growth theorists. Suppose that H_{S_u} number of individuals called *scientists* devote their time to pursue scientific research in places like *research universities* with the continuum $u \in [0,1]$. Consider the representative university with $H_{S_u} = H_S$. Scientists in this university create new scientific knowledge $\dot{S}_A$ in any infinitesimal epoch of time $\mathrm{d}t > 0$ by using (*i*) the existing level of *collective* scientific knowledge (S) and (*ii*) some useful products and services such as the microscope, the telescope and the personal computer $(\gamma_S N)$, again subject to rivalry. A simple multiplicative law of motion is

$$\dot{S}_A = \beta H_S S^{\sigma_S} \left(\frac{\gamma_S N}{H_S}\right)^{1-\sigma_S} \tag{2}$$

where $\beta \in \left(0, \overline{\beta}\right]$ denotes the neurobiological research productivity with $\overline{\beta} < \infty$ and $\gamma_S \in (0,1)$ is the fraction of products and services that are useful for scientific research activity. The parameter that turns out to be crucial in the long-run is σ_S. This is the degree of the *concentration* of scientific research on existing level of collective scientific knowledge. At this stage of our analysis, we assume that $\sigma_S \in (0,1)$ and $\gamma + \gamma_S \in (0,1)$. The former restriction implies that the scientific knowledge exhibits constant returns to knowledge-base resources.

Collective scientific knowledge $S < S_A$ is simply determined by (1):

$$S = \eta \left(\frac{\gamma N}{H_S}\right)^{\delta} S_A \tag{3}$$

Together with (2), this yields the growth rate of collective scientific knowledge as

$$g_S = \delta \left(g_N - g_{H_S}\right) + \frac{\beta \eta H_S}{S^{1-\sigma_S}} \left(\frac{\gamma_S N}{H_S}\right)^{1-\sigma_S} \left(\frac{\gamma N}{H_S}\right)^{\delta} \tag{4}$$

Why is scientific knowledge subject to diffusion inefficiency? Collectiveness of scientific knowledge is nothing but the main defining characteristic of what is called *open science*. In his leading study of the sociology of science, Merton (1973) remarks collectiveness as an ideal norm of scientific knowledge.[6] It is evident that knowledge is useful if it is available to the *right* people in the *right* place at the *right* time (Foray, 2004, p.18). The history of science and technology indeed records a large number of simultaneous discoveries and inventions such as calculus by Newton and Leibniz and telephone by Bell and Gray, as documented in

[6] See Hess and Ostrom (2006) for a collection of essays on open science.

detail by Merton (1961). Research in any field of science is subject to the same friction in 2000 A.D. economies. However, the frequency of simultaneous discoveries and inventions is much lower due to the highly efficient communication through online scholar archives.

2.3 Technology

In the treatment above, we have denoted all products and services in the economy by N and mentioned a couple of examples such as the personal computer and the rail transport. The entire collection of these products and services are *macroinventions*, i.e., the set of all consumption goods and services that firms produce and individuals consume in the T-C mapping. Leaving these production and consumption problems to later sections, we now answer how scientific knowledge determines Technology.

In any $\mathrm{d}t > 0$, new scientific knowledge denoted by $\dot{S}_A$ is created. Axioms, laws, theorems and all other *propositional* components of $\dot{S}_A$ constitute what we call nontechnological scientific knowledge. The remaining fraction of $\dot{S}_A$, i.e., technological knowledge, on the other hand, is formed by recipes, procedures, blueprints and all other *prescriptional* components. Compare, for example, Pontryagin's maximum principle and Schumpeterian growth theory to the recipe of cheesecake and the set of instructions that explain how to construct a space station. They are fundamentally different.

Formally let $\dot{T} < \dot{S}_A$ denote newly created technological scientific knowledge. Following Olsson (2000, p.270), the mapping from $\dot{S}_A$ to $\dot{T}$ can be formalized as

$$\dot{T} := \theta \dot{S}_A \tag{5}$$

where $\theta \in (0, 1)$ represents the economy's technological creativity to convert new scientific knowledge into new technological knowledge.

What determines the frontier of inventiveness of the economy in the horizontal dimension of macroinventions is $\dot{T}$. Then, there must exist a way in which the economy allocates new technological knowledge $\dot{T}$ over the horizontal continuum of new products $\dot{N}$. Challenging the conventional wisdom of treating N as a knowledge-based variable *per se*, we assert that the only type of knowledge embodied in the flow $\dot{N}$ of new goods is the prescriptional knowledge of the working prototypes. This body of prescriptional knowledge is the minimum amount of technological knowledge required to produce the working prototype of a product. Formally, the working prototype of each new macroinvention $j \in \left[0, \dot{N}\right]$ in the horizontal dimension is endowed with some blueprint knowledge denoted by $B_j > 0$. Thus, the technological allocation problem faced by (applied) scientists (or engineers) is to create a collection of working prototypes such that

$$\dot{T} \geq \int_{0}^{\dot{N}} B_j \mathrm{d}j \tag{6}$$

holds with strict equality. Only in that case, the efficient allocation of new technological knowledge $\dot{T}$ is guaranteed.[7]

At any point in time, then, the role of N as a proxy for the technological advancement of the economy in the horizontal dimension is extended with $\{B_i\}_{i=0}^{N}$ in the present setup. A higher N, across time and space, corresponds to some more advanced technology level if some sort of externality with respect to N exists, e.g., the spillover effect on the scientific research. However, the blueprint knowledge plays a central role in determining the actual number of working prototypes.

Notice that S_A and therefore T are *known* by scientists, and $\dot{T}$ is converted to $\dot{N}$ working prototypes. These working prototypes, in our analysis, is the only source of funding for scientific research. That is to say, there exists a large number of profit-seeking entrepreneurs (measured with mass 0 in individuals' space) who are willing to purchase the intellectual property rights of these working prototypes. Setting the labor supply as the numéraire with wage rate $W = 1$, the zero-profit condition for scientific research is given by

$$\int_{0}^{\dot{N}} \left(P_{N_j} \times 1\right) \mathrm{d}j = H_S \tag{7}$$

where P_{N_j} denote the competitive patent price of working prototype j.

Remark 1. *What is being traded is the blueprint knowledge B_j transformed into some physical form of a unique object, e.g., a book of instructions. Apparently, B_j is indivisible since it is the minimum amount of prescriptional knowledge required to produce the working prototype.* ■

Once a working prototype with its blueprint knowledge B_j is owned by a firm, the macroinvention process is completed. The firm then invests in developing the prototype mode of production into more productive (less costly) modes. The typical output of this development activity for macroinvention $i \in [0, N]$ is the process knowledge or *microinventions* denoted by $\dot{Z}_i$. As in Peretto and Connolly (2007), we assume that this development activity is pursued by independent groups of

[7] In Mokyr's (2002, p.17) Figure 1, this technological allocation problem is described as the selection of "manifest entities" from "feasible techniques". Kortum (1997) models such a technological allocation problem in the vertical dimension as a stochastic search process.

skilled individuals in places like *technology parks*. However, we extend their formulation with spillovers from relevant fraction of macroinventions $\gamma_Z N$. Formally, we impose the following multiplicative law of motion that characterizes the development activity for each macroinvention $i \in [0, N]$

$$\dot{Z}_i = \beta H_{Z_i} Z_i^{\zeta_Z} S^{\zeta_S} \left(\frac{\gamma_Z N}{H_{Z_i}} \right)^{1-\zeta_Z-\zeta_S} \tag{8}$$

where β is the neurobiological parameter defined above. For each macroinvention, H_{Z_i} denotes the number of product specialists (or *technologists*) that use relevant macroinventions $\gamma_Z N$ as well as the existing levels of process knowledge Z_i and scientific knowledge S. Again, we put no further restrictions other than $\gamma + \gamma_S + \gamma_Z \in (0, 1)$ and $\zeta_Z + \zeta_S \in (0, 1)$, all strictly positive.

Two points are worth to be emphasized. First, there is no diffusion inefficiency for Z_i among technologists H_{Z_i} since it is implicitly assumed that these small groups of skilled individuals pursue a very organized development work. Second, the process knowledge Z_j's of other macroinventions do not spill over Z_i since the process knowledge, by construction, is perfectly specific, e.g., the *recipe* of cheesecake is not relevant to the *set of instructions* that explain how to construct a space station. Instead, what spills over is the collective scientific knowledge S since what could be common in the process knowledge of an automobile and an aircraft is not the knowledge of how to produce an automobile or an aircraft in some certain ways but instead the knowledge of, for example, how engines work.

Notice that newly developed process knowledge $\dot{Z}_i$ is of economic value, and technologists H_{Z_i} for each macroinvention i exploit this value by selling $\dot{Z}_i$ to the firm i. Unlike the blueprint knowledge, however, $\dot{Z}_i$ is not necessarily indivisible. That is, for $\mathrm{d}t > 0$, $Z_i + \dot{Z}_i$ is the state-of-the-art mode of production but the firm, in principle, has the option of purchasing a fraction of $\dot{Z}_i$. Therefore, the zero-profit condition for any i is given by

$$\int_0^{\dot{Z}_i} \left(P_{Z_i}(z) \times 1 \right) \mathrm{d}z = H_{Z_i} \tag{9}$$

where $P_{Z_i}(z)$ denotes the patent price corresponding to the increment z of new process knowledge $\dot{Z}_i$. Thus, the true meaning of the term *micro*inventions becomes explicit; each micro development of original prototype indexed by z is a new product. For simplicity, though, we assume that the competitive patent prices $\{P_{Z_i}(z)\}_{z=0}^{\dot{Z}_i}$ are symmetric over z, i.e., $P_{Z_i}(z) = P_{Z_i}$ for all z. Then, the firm purchases the state-of-the-art mode,[8] and (9) can be rewritten as

$$P_{Z_i} \dot{Z}_i = H_{Z_i} \tag{9'}$$

2.4 From Macroinventions to Microinventions

In the model described so far, the value of the process knowledge and the blueprint knowledge *at the date of invention* must be identical for any macroinvention i invented at any point in time t, and what we formalize below is this link under some simplifying assumptions. Despite the fact that theorists in the tradition of Romer (1990) and Grossman and Helpman (1991) usually relax this connection by not addressing the fundamental difference between propositional and prescriptional forms of knowledge, this is crucial in the Schumpeterian models with two-dimensional R&D.[9]

Two technicalities are present. Throughout the horizon $t \in [0, \infty)$, the number of macroinventions N evolves as governed by the technological opportunity T. Since the blueprint knowledge embodied in new macroinventions at any $\mathrm{d}t > 0$ determines the actual number of macroinventions, the rigorous representation of the technological allocation problem should incorporate the date of innovation t^* explicitly as in

$$\dot{T} = \int_0^{\dot{N}} B_{jt^*} \mathrm{d}j \tag{6'}$$

with $\mathrm{d}t = t^* - t_0$ where t_0 is some reference initial point such that

$$N_{t_0} + \dot{N} = N_{t^*}$$

This implies that, given $\dot{T}$ and N_{t_0}, the number of products at date t^* is a function of B_{jt^*}'s through $\dot{N}$. Since innovation is a continuous process in this model, t^* is actually the generic time variable for the horizon $[0, \infty)$.

The second technicality is about the establishment of the new firms $\dot{N}$ and the initiation of development activity. Index, now, any one of these new firms by $i = j$ and let B_{it^*} denote its blueprint knowledge. Thus, at the point t^* of innovation

[8] This property is in line with Kortum's (1997) model in which, due to the imitation opportunity, only the most efficient techniques are actually used.

[9] A notable exception to this argument is the model of Peretto and Smulders (2002) in which the creation of new products alter the network externalities since new firms create new technological "problems" that other firms at some technological distance benefit. In the models that are based on creative destruction of Aghion and Howitt (1992), the creation of new products is complicated with some form of vertical process knowledge that represents the difficulty of R&D (e.g., Young, 1998; Howitt, 1999).

over the horizon $[0, \infty)$, this new firm i produces its product by using B_{it^*}. However, development activity instantly returns a more productive mode of production $Z_{i(t^*+\mathrm{d}t)}$ such that

$$B_{it^*} + \dot{Z}_i = Z_{i(t^*+\mathrm{d}t)}$$

Naturally, the initial value of the microinventions of macroinvention i is given by $Z_i^\circ = B_{it^*}$ where t^* in B_{it^*} denotes the generic time variable for the horizon $[0, \infty)$ *but "$\circ$" in Z_i° denotes the time variable of macroinvention i at the beginning of its lifetime.* The message is clearer in Figure 1.

[Insert Figure 1 around here.]

Note that the integral in (6′) complicates the matters without further structure imposed on B_{jt^*}'s. Thus, for simplicity, we state the following symmetry assumption:

Assumption 2. *For all new macroinventions $j \in \left[0, \dot{N}\right]$, the blueprint knowledge is identical, i.e., for any t^*, $B_{jt^*} = B_{t^*}$.* ■

Then, (6′) implies $\dot{T} = \dot{N} B_{t^*}$. Recalling that innovation is continuous and that t^* is the generic time variable for the horizon $[0, \infty)$, we further have $B_{t^*} = B$. Thus, the problem reduces into the question of how to specify B. We now extend the symmetry assumption to answer this question in a very simple way.[10]

Assumption 3. *Macroinventions are identical in all respects. Formally,*

$$Z_i = Z \qquad P_{Z_i} = P_Z \qquad H_{Z_i} = H_Z$$

■

Notice that, under Assumptions 2 and 3, $\mathrm{d}t \to 0$ immediately implies

$$B = Z$$

for any $t\ (= t^*)$ since $Z_i = Z$, $B_{jt^*} = B$ and $Z_i^\circ = B_{it^*}$ as argued above. Figure 2 pictures the impossibility of $B \neq Z$ under symmetry.

[Insert Figure 2 around here.]

It must be now clear that there is a crucial link between scientific and technological opportunity summarized as

$$\theta \dot{S}_A = \dot{T} = \dot{N} Z \tag{10}$$

[10] See below for the restriction that guarantees the existence of symmetric equilibria.

Eliminating $\dot{T}$ and $\dot{S}_A$and solving for g_N yield

$$g_N = \frac{\theta\beta\gamma_S^{1-\sigma_S} H_S^{\sigma_S} S^{\sigma_S}}{ZN^{\sigma_S}} \tag{11}$$

for any t. To complete the discussion of technology, we finally solve for the growth rate g_Z of microinventions under symmetry and obtain

$$g_Z = \frac{\beta\gamma_Z^{1-\zeta_Z-\zeta_S} H_Z^{\zeta_Z+\zeta_S} S^{\zeta_S} N^{1-\zeta_Z-\zeta_S}}{Z^{1-\zeta_Z}} \tag{12}$$

2.5 Technology-to-Consumption Mapping

In this subsection, we introduce markets, i.e., the T-C mapping in which production and consumption decisions are made. We should note, for integrity, that the "production" of individuals with their skills and numbers is a segment of the K-T mapping under our conceptualization since individuals are useful. However, the "production" of individuals is endogenous to consumption decisions through endogenous fertility.

2.5.1 The Household's Problem

We model the household's problem with endogenous fertility as in Connolly and Peretto (2003) who use Barro and Sala-i-Martin's (1998, Ch.9) formulation of Becker and Barro's (1988) model. Endogenous fertility is introduced through the positive marginal utility of the number of children.

Up to this point, we have proceeded with L individuals as the actors of the economy. Now, we make the necessary assumption that there exists a mass 1 continuum of identical households each with L identical members. Then, we can proceed further with a representative household such that L identical members of this representative household constitute the population of the economy with endogenous growth rate g_L.

We assume that each member of the household is endowed with a unique working force measured in time and normalized to unity. This working force, set as the numéraire, is supplied inelastically at the competitive wage rate $W = 1$ as already argued above.

Financial wealth of the household is accumulated through *firm ownership shares* in the form of a single asset. Denoting the aggregate asset holdings of the household by La, the law of motion for *per capita* asset holdings is given by

$$\dot{a} = (r - g_L)\, a + (1 - f)\, W - Pc \tag{13}$$

where $r > 0$ is the rate of return on asset holdings, $f \in [0, 1]$ is the variable that denotes the time endowment allocated to rearing children and Pc is per capita consumption expenditure over Dixit and Stiglitz's (1977) consumption aggregator c defined as

$$c := \left[\int_0^N c_i^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{14}$$

with (constant) elasticity of substitution $\varepsilon > 1$. Intertemporal utility function of the household is defined as

$$U_t := \int_t^\infty e^{-\rho(\tau-t)} \left[\log(c_\tau) + \nu \log(L_\tau) + \phi_1 \log\left(\dot{L}_\tau\right) \right] \mathrm{d}\tau \tag{15}$$

where $\nu > 0$ is the parameter that determines the marginal utility of the family size L and $\phi_1 > 0$ is a fertility parameter that determines the marginal utility of the children $\dot{L}$. $\rho > 0$ is the subjective discount rate.

Denoting the cost of reproduction for children measured in time units by $\phi_2 > 0$ and the exogenous mortality rate by $\mu > 0$, the endogenous population growth rate is defined as

$$g_L := \frac{f}{\phi_2} - \mu \tag{16}$$

where the first term represents the fertility rate for each household member.

Given price streams and relevant initial conditions, the household seeks to maximize (15), subject to (13), (14), (16) and the relevant no-Ponzi-game restriction, by choosing $\{c_t, f_t\}_{t=0}^\infty$ and $\{c_{it}\}_{i=0}^N$. As derived in the appendix, optimal allocation of resources yields

$$c_i = \left(\frac{P_i}{P}\right)^{-\varepsilon} c \qquad (\forall i) \tag{17}$$

$$r = g_L + g_P + g_c + \rho \tag{18}$$

$$g_L = \frac{\phi_1 \rho P c}{\rho(a + \phi_2) - (\nu + \phi_1) P c} \tag{19}$$

$$f = \phi_2 (\mu + g_L) \tag{20}$$

where P is the aggregate price index defined as

$$P := \left[\int_0^N P_j^{1-\varepsilon} \mathrm{d}j \right]^{\frac{1}{1-\varepsilon}} \tag{21}$$

2.5.2 Firms' Problem

We formulate firms' problem almost identical to Peretto and Connolly's (2007). The difference is that we neglect the fixed operating costs. In Peretto and Connolly

(2007), these costs affect entry decisions in horizontal R&D through the aggregate resource constraint on the labor force, and the number of firms that the economy can accommodate with nonnegative profits is finite. More importantly, a threshold level of the number of firms exists due to the option of shutting down the investment in vertical R&D at some proximity of the exit level. Thus, the global dynamics are characterized by two possible balanced growth equilibria: one in which both horizontal and vertical R&D are active and the other in which vertical R&D is shut down. Peretto and Connolly (2007) allegorize this as the Manhattan Metaphor since the number of buildings (i.e., the horizontal dimension) that can be constructed in Manhattan island is fixed due to the land constraint but the heights of such buildings (i.e., the vertical dimension) are potentially infinite.

Simply put, the Manhattan Metaphor builds on the resource constraint with respect to *individuals*. Our paper, as we demonstrate in Section 3, does focus on the resource constraint with respect to *ideas* since integrated technological opportunity for horizontal and vertical dimensions of innovation is bounded with the scientific opportunity through blueprint knowledge. Thus, even if the firms do not face fixed operating costs, some bound on the growth potential exists solely due to technological boundedness.

Now suppose that each firm uses the production technology

$$Y_i = Z_i^{\alpha} H_{Y_i} \tag{22}$$

which is embodied in the process knowledge Z_i. H_{Y_i} denotes the demand for labor and $\alpha \in (0,1)$ denotes the elasticity of output with respect to Z_i. Physical capital is neglected to keep the analysis as simple as possible.

Notice that firm i's total demand is given by $Y_i = Lc_i$. Then, (17) and (22) respectively imply the following conditional demand schedules:

$$Y_i\left(P_i\right) = \left(\frac{P_i}{P}\right)^{-\varepsilon} Lc \qquad H_{Y_i}\left(P_i, Z_i\right) = \left(\frac{P_i}{P}\right)^{-\varepsilon} LcZ_i^{-\alpha} \tag{23}$$

The instantaneous profit level of firm i is therefore defined as

$$\Pi_i := P_i Y_i\left(P_i\right) - W H_{Y_i}\left(P_i, Z_i\right) - P_{Z_i}\dot{Z}_i \tag{24}$$

where $P_{Z_i}\dot{Z}_i$ is the cost associated with current patent purchases. Taking L, c, P, W, r and P_{Z_i} as given, firm i maximizes the present discounted sum of profits over the horizon $\tau \in [0, \infty)$, i.e.,

$$V_i := \int_0^{\infty} e^{-\int_0^{\tau} r \mathrm{d}t} \Pi_i \mathrm{d}\tau, \tag{25}$$

by choosing P_i and $\dot{Z}_i$, subject to (23), (24) and (8). Invoking $W = 1$, the solution to this problem, as derived in the appendix, is characterized by

$$P_i = \left(\frac{\varepsilon}{\varepsilon - 1}\right) Z_i^{-\alpha} \tag{26}$$

$$r = \left(\frac{\alpha L c_i}{Z_i^{1+\alpha}}\right) \frac{1}{P_{Z_i}} + \frac{\dot{P}_{Z_i}}{P_{Z_i}} \tag{27}$$

where (26) is the intratemporal pricing rule and (27) is the no-arbitrage equation in which the right-hand side is the rate of return on microinventions Z_i.

Remark 2. *In the horizontal dimension, the users of new knowledge are new firms, and, therefore, Arrow (replacement) effect is absent. In the vertical dimension, the users of new knowledge are existing firms, and Arrow (replacement) effect is internalized via profit maximization.* ■

In the light of Remark 2, we now derive the rate of return on macroinventions N_j. Since each newly established firm $j \in \left[0, \dot{N}\right]$ operates in the same way as an existing firm $i \in [0, N]$ after establishment, the standard asset-pricing equation for a new firm j is simply given by

$$r = \frac{\Pi_j}{V_j} + \frac{\dot{V}_j}{V_j} \tag{28}$$

with the restriction $V_j = P_{N_j}$ which is implied by the fact that universities have the opportunity to establish new firms by themselves. Invoking $W = 1$ and using the solution of firms' problem, one can rewrite this asset-pricing equation as the no-arbitrage equation

$$r = \left(\frac{L c_j}{(\varepsilon - 1) Z_j^{\alpha}} - P_{Z_j} \dot{Z}_j\right) \frac{1}{P_{N_j}} + \frac{\dot{P}_{N_j}}{P_{N_j}} \tag{29}$$

where the right-hand side is the rate of return on macroinventions N_j.

2.6 Closing the Model under Symmetry

Recall that (*i*) firms in the horizontal dimension are Bertrand competitors, (*ii*) the process knowledge Z_i is the state variable with diminishing returns to output much like the physical capital stock and (*iii*) the intratemporal pricing rule incorporates the cost-reducing effect of $Z_i^{-\alpha}$. The existence and the stability of Nash equilibrium in strategies $\left(P_i, \dot{Z}_i\right)$, therefore, critically depends on some sort of Inada condition in Z_i, as explained in Peretto (1998a, 1998b). The idea is that the patent price (and, equivalently, the shadow value) P_{Z_i} of process knowledge Z_i must be increasing when Z_i converges to zero. We emphasize this important notion in the following lemma.

Lemma 1. *A symmetric Bertrand-Nash equilibrium in strategies* $\left(P_i, \dot{Z}_i\right)$ *exists and satisfies stability if*

$$\frac{1}{\varepsilon - 1} > \alpha$$

$\blacksquare$

Proof— See Peretto (1998b, p.76). □

Now note that, under symmetry, (21) and (26) imply

$$P = \left(\frac{\varepsilon}{\varepsilon - 1}\right) Z^{-\alpha} N^{-\frac{1}{\varepsilon - 1}}$$

Then, no-arbitrage equations (27) and (29) can be rewritten as

$$r = \left(\frac{(\varepsilon - 1)\,\alpha LPc}{\varepsilon ZN}\right) \frac{1}{P_Z} + \frac{\dot{P}_Z}{P_Z} \qquad (27')$$

$$r = \left(\frac{LPc}{\varepsilon N} - H_Z\right) \frac{1}{P_N} + \frac{\dot{P}_N}{P_N} \qquad (29')$$

where P_N and P_Z are symmetric patent prices that satisfy

$$P_N = \frac{Z H_S^{1-\sigma_S}}{\theta \beta \gamma_S^{1-\sigma_S} S^{\sigma_S} N^{1-\sigma_S}} \qquad P_Z = \frac{H_Z^{1-\zeta_Z-\zeta_S}}{\beta \gamma_Z^{1-\zeta_Z-\zeta_S} Z^{\zeta_Z} S^{\zeta_S} N^{1-\zeta_Z-\zeta_S}} \qquad (30)$$

under zero-profit conditions of scientific research and technological development activities and other results developed earlier.

We now close the model with two market clearing conditions. Labor markets clear if

$$(1 - f)\, L = N\left(H_Z + H_Y\right) + H_S \qquad (31)$$

where (23) under symmetry implies

$$H_Y = \frac{(\varepsilon - 1)\, LPc}{\varepsilon N} \qquad (32)$$

Then, eliminating P_N and P_Z using (30) and H_Z using (31) and (32) yields

$$\begin{aligned} r &= \frac{\alpha(\varepsilon-1)LPcg_Z}{\varepsilon H_Z N} + (1-\zeta_Z-\zeta_S)(g_{H_Z}-g_N) - \zeta_Z g_Z - \zeta_S g_S && (27'') \\ r &= \frac{L[Pc-(1-f)]g_N}{H_S} + g_{H_S} - \frac{\dot{g}_N}{g_N} && (29'') \end{aligned}$$

Finally, in the stock markets, the total value of assets is equal to the total market value of firms with $V = P_N$. Thus, stock markets clear if

$$La = NP_N \tag{33}$$

Definition 1. *The (symmetric) dynamic general equilibrium of the economy over the horizon $t \in [0, \infty)$, if exists, is defined by the collections*

$$\{S_A, S, N, Z\}_{t=0}^{\infty} \qquad \{r, W, P\}_{t=0}^{\infty} \qquad \{L, H_Y, H_Z, H_S\}_{t=0}^{\infty} \qquad \{a, c, f\}_{t=0}^{\infty}$$

that satisfy the equilibrium equations

$$\begin{aligned} g_{S_A} &= \frac{\beta\eta H_S}{S^{1-\sigma_S}}\left(\frac{\gamma_S N}{H_S}\right)^{1-\sigma_S}\left(\frac{\gamma N}{H_S}\right)^{\delta} && (E1) \\ g_S &= \delta\left(g_N - g_{H_S}\right) + g_{S_A} && (E2) \\ g_N &= \frac{\theta\beta\gamma_S^{1-\sigma_S} H_S^{\sigma_S} S^{\sigma_S}}{ZN^{\sigma_S}} && (E3) \\ g_Z &= \frac{\beta\gamma_Z^{1-\zeta_Z-\zeta_S} H_Z^{\zeta_Z+\zeta_S} S^{\zeta_S} N^{1-\zeta_Z-\zeta_S}}{Z^{1-\zeta_Z}} && (E4) \\ r &= \frac{\alpha(\varepsilon-1)LPcg_Z}{\varepsilon H_Z N} + (1-\zeta_Z-\zeta_S)(g_{H_Z}-g_N) - \zeta_Z g_Z - \zeta_S g_S && (E5) \\ r &= \frac{L[Pc-(1-f)]g_N}{H_S} + g_{H_S} - \frac{\dot{g}_N}{g_N} && (E6) \\ W &= 1 && (E7) \\ P &= \left(\frac{\varepsilon}{\varepsilon-1}\right) Z^{-\alpha} N^{-\frac{1}{\varepsilon-1}} && (E8) \\ g_L &= \frac{\phi_1\rho Pc}{\rho(a+\phi_2) - (\nu+\phi_1)Pc} && (E9) \\ H_Y &= \frac{(\varepsilon-1)LPc}{\varepsilon N} && (E10) \\ H_S &= (1-f)L - N(H_Z + H_Y) && (E11) \\ a &= \left(\frac{H_S}{L}\right)\frac{1}{g_N} && (E12) \\ g_c &= r - g_L - g_P - \rho && (E13) \\ f &= \phi_2(\mu + g_L) && (E14) \end{aligned}$$

given initial values $S_{A0}, S_0, N_0, Z_0, L_0, a_0 > 0$ *and subject to the relevant transversality conditions* (see appendix), *to the feasibility constraints*

$$r > 0 \qquad \Pi > 0 \qquad f \in [0,1]$$

and to the nonnegativity constraints on the growth rates of $\{S_A, S, N, Z\}_{t=0}^{\infty}$. ■

Remark the loss of tractability due to the dimensionality and the nonlinearity of the dynamical system summarized in Definition 1. This problem restricts the analysis with (steady-state) balanced growth equilibria if this general structure of the model is preserved. As we see below, the loss of tractability is so strict that the fully parametric (analytical) solution of the steady-state equilibria is not obtained.

3 Balanced Growth Equilibrium

Definition 2. *A balanced growth path is a steady-state equilibrium trajectory along which all variables grow at constant* (*but not necessarily identical*) *rates.* ■

Notation 2. *Any variable along any steady-state equilibrium trajectory is denoted by an asterisk* ($*$). ■

An *educated guess* for the steady-state is the one in which the number of products *per capita*, defined as

$$n^* := \left(\frac{N}{L}\right)^*,$$

is constant. This in turn yields balanced growth equilibria that exhibit strong empirical validity as summarized by, for example, Laincz and Peretto (2006). To match other stylized facts, we further assume that, along any steady-state equilibrium trajectory, r^* and $(Pc)^*$ are constant. Therefore, a^* and H_Y^* are also constant as implied respectively by (E9) and (E10). By (E12), then, the number of scientists *per capita*, defined as

$$h_S^* := \left(\frac{H_S}{L}\right)^*,$$

is constant as well. The resource constraint finally implies the constancy of H_Z^*.

Incorporating these properties into (E1)-(E4) yields the balanced growth rates of three dimensions of innovation as

$$\begin{aligned}
g_S^* &= \beta\eta\gamma^{\delta}\gamma_S^{1-\sigma_S} h_S^{*\sigma_S-\delta} n^{*1-\sigma_S+\delta}\left(\frac{L}{S^{1-\sigma_S}}\right) = g_{S_A}^* \\
g_N^* &= \theta\beta\gamma_S^{1-\sigma_S} h_S^{*\sigma_S} n^{*-\sigma_S}\left(\frac{S^{\sigma_S}}{Z}\right) = g_L^* = g_{H_S}^* \\
g_Z^* &= \beta\gamma_Z^{1-\zeta_Z-\zeta_S} H_Z^{*\zeta_Z+\zeta_S}\left(\frac{S^{\zeta_S} N^{1-\zeta_Z-\zeta_S}}{Z^{1-\zeta_Z}}\right)
\end{aligned}$$

where the constancy of these growth rates implies that the three ratios in the parentheses are constant as well. Accordingly, we define

$$\kappa_S^* := \left(\frac{L}{S^{1-\sigma_S}}\right)^* > 0 \qquad \kappa_N^* := \left(\frac{S^{\sigma_S}}{Z}\right)^* > 0 \qquad \kappa_Z^* := \left(\frac{S^{\zeta_S} N^{1-\zeta_Z-\zeta_S}}{Z^{1-\zeta_Z}}\right)^* > 0$$

Thus, in any steady-state equilibrium, the following must be satisfied:

$$g_N^* = (1-\sigma_S)\, g_S^* \tag{34}$$

$$g_Z^* = \sigma_S g_S^* \tag{35}$$

$$(\zeta_Z + \zeta_S - 1)\, g_N^* + (1-\zeta_Z)\, g_Z^* = \zeta_S g_S^* \tag{36}$$

This has two important implications: First, the sum of the *endogenous* balanced growth rates of the vertical and the horizontal dimensions of innovation is bounded by the *endogenous* balanced growth rate of collective scientific knowledge, and the share of each is determined by the concentration parameter $\sigma_S \in (0,1)$.[11] This is quite intuitive, at least in the framework of this paper with constant returns in knowledge creation, since the very construction of the innovation process in the horizontal dimension is extended with blueprint knowledge as argued above. More clearly, we have

$$g_N^* + g_Z^* = g_S^*$$

which stems from the fact that applied scientists in the horizontal dimension always achieve an efficient allocation of technological resources over two dimensions of innovation; see (6′).

The second implication is a knife-edge restriction on the parameters that immediately follows from (34)-(36):

Restriction 1. *Model parameters σ_S, ζ_Z and ζ_S satisfy*

$$\sigma_S = \frac{1-\zeta_Z}{2(1-\zeta_Z)-\zeta_S}, \quad 2(1-\zeta_Z) > \zeta_S \quad \text{and} \quad \zeta_Z + \zeta_S < 1$$

where the latter two are implied by $\sigma_S \in (0,1)$ and the last one holds by construction. ■

Many could see such a strong knife-edge restriction as a caveat. However, as rigorously argued by Growiec (2007, Theorem 1), it is impossible to construct a growth model that supports strictly positive balanced growth without forcing such knife-edge assumptions. In our model, though, a simplifying alternative is possible at a limited cost of losing some intuition:

[11] The role of σ_S in determining the shares of each component is not explicit in this formulation. To uncover this, consider the case that $\sigma_S \to 1$ where the spillover from n to S_A vanishes. In this case, however, g_S and g_Z explode. Thus, the constant returns assumption, i.e., one of the fundamental notions of Schumpeterian paradigm, is necessary for the existence of balanced growth equilibria in this model.

Assumption 4.

$$\zeta_Z = 1 \qquad \zeta_S = 0$$

■

Under Assumption 4, the creation of process knowledge in the vertical dimension under symmetry reduces into the simplest form of

$$\dot{Z} = \beta H_Z Z$$

with $\kappa_Z = \kappa_Z^* = 1$ for any t, and $\sigma_S \in (0,1)$ turns out to be a free parameter such that the existence of balanced growth equilibria does no longer require the knife-edge property stated in Restriction 1. For the rest of the analysis, we utilize Assumption 4.

Invoking the balanced growth properties discussed above and eliminating some variables such as r^*, a^* and f^*, we can write the steady-state version of equilibrium equations as follow:

$$\begin{aligned}
g_L^* + g_Z^* &= \beta\eta\gamma^{\delta}\gamma_S^{1-\sigma_S} h_S^{*\sigma_S-\delta} n^{*1-\sigma_S+\delta}\kappa_S^* && (B1)\\
g_L^* &= \theta\beta\gamma_S^{1-\sigma_S} h_S^{*\sigma_S} n^{*-\sigma_S}\kappa_N^* && (B2)\\
g_Z^* &= \left(\frac{\sigma_S}{1-\sigma_S}\right) g_L^* && (B3)\\
g_L^* + \rho &= \frac{\alpha\beta\left(\varepsilon-1\right)\left(Pc\right)^*}{\varepsilon n^*} - g_Z^* && (B4)\\
\rho &= \frac{\left[\left(Pc\right)^* - \left(1-\phi_2\left(\mu+g_L^*\right)\right)\right] g_L^*}{h_S^*} && (B5)\\
g_L^* &= \frac{\phi_1\rho\left(Pc\right)^*}{\rho\left(\frac{h_S^*}{g_L^*}+\phi_2\right) - \left(\nu+\phi_1\right)\left(Pc\right)^*} && (B6)\\
h_S^* &= \left(1-\phi_2\left(\mu+g_L^*\right)\right) - n^*\left(\frac{g_Z^*}{\beta} + \frac{\left(\varepsilon-1\right)\left(Pc\right)^*}{\varepsilon n^*}\right) && (B7)
\end{aligned}$$

Notice that (*i*) κ_S^* and κ_N^* are determined respectively by $(B1)$ and $(B2)$ and (*ii*) g_Z^* is determined by $(B3)$. Thus, eliminating g_Z^* by $(B3)$ and omitting $(B1)$ and $(B2)$ yield a system of nonlinear equations, i.e., $(B4)$ to $(B6)$, in four endogenous variables $\left(\left(Pc\right)^*, n^*, h_S^*, g_L^*\right)$ and in nine parameters $(\alpha, \beta, \varepsilon, \rho, \phi_1, \phi_2, \nu, \mu, \sigma_S)$. The question is whether a unique solution to this system exists. If so, the rest of the equations completely determine the balanced growth rates and steady-state values of all variables including κ_S^* and κ_N^*.

The difficulty is the nonlinearity of this four-dimensional system, and an analytical solution is not available. Fortunately, *given a set of parameter values*, a numerical solution exists; the Jacobian of the system is invertible, and the implicit function theorem holds. Our numerical experiments further prove that, once the equilibrium restrictions and nonnegativity constraints are imposed, the solution is unique. It is now convenient to discuss these restrictions and constraints.[12]

First note that the household never chooses a value for f that makes the population growth rate zero. This is not surprising since the elasticity of instantaneous utility with respect to the number of children denoted by ϕ_1 is strictly positive; see Connolly and Peretto (2003, p.124) on this. Then, the condition of

$$g_L^* > 0$$

extends the nonnegativity restrictions on growth rates such that

$$g_S^*, g_Z^*, h_S^*, H_Z^* > 0$$

Thus, if a balanced growth equilibrium exists, all three dimensions of innovation exhibit positive growth in this equilibrium with the steady-state level of fertility choice restricted within the interval

$$f^* \in (\phi_2 \mu, 1)$$

This provides new insight on Peretto and Connolly's (2007) Manhattan Metaphor; indeed, converting the dimension of causality. As argued above, the relevant version of the Manhattan Metaphor of the model economy developed so far builds on a knowledge-based resource constraint. Formally, we have the following result:

Proposition 1. (*Steady-State Manhattan Metaphor*) *If a unique steady-state equilibrium trajectory* (*with its defining properties*) *exists* (*given some set of parameters*), *then*

(*i*) *all three dimensions of innovation exhibit long-run growth, i.e.,* $g_S^*, g_N^*, g_Z^* > 0$,

and, as implied by this,

(*ii*) *the steady-state number of firms per capita is bounded above such that*

$$n^* < n_{\max}^* := \frac{\alpha\beta\,(\varepsilon - 1)\,(Pc)^*}{\varepsilon\rho}$$

■

[12]Since the restriction on the determinant of the Jacobian is an equality-to-zero restriction, a solution exists for infinitely many sets of model parameters. If one selects a set of model parameters that makes the Jacobian invertible, then four distinct solutions exist where two of them are not real. Finally, nonnegativity constraints eliminate a third one, and the remaining one constitutes the unique steady-state equilibrium.

Proof— (*i*) has already been established. For (*ii*), rewrite ($B4$) by collecting the growth rates in the left-hand side. Invoking strict positivity of growth rates completes the proof. □

Corollary of the strict positivity of g_L^* is the fact that the constraint $r^* > 0$ is satisfied given $\rho > 0$. An insightful representation of this corollary is obtained if we rewrite ($B4$) in terms of H_Y^* and H_Z^* after some arrangements:

$$r^* = \alpha\beta H_Y^* - \beta H_Z^* > 0$$

This implies that

$$\frac{H_Z^*}{H_Y^*} < \alpha \tag{37}$$

which characterizes the upper bound of the ratio of development sector technologists to manufacturing sector workers. On the other hand, the strict positivity of symmetric profits, i.e.,

$$\Pi^* = \frac{(Pc)^*}{\varepsilon n^*} - H_Z^* > 0,$$

implies that

$$\frac{H_Z^*}{H_Y^*} < \frac{1}{\varepsilon - 1} \tag{38}$$

Inequalities (37) and (38) in terms of the ratio of H_Z^* to H_Y^* govern the steady-state decisions of firms, respectively, in vertical and horizontal dimensions of innovation. Recalling Remark 2, the elasticity of output with respect to process knowledge (α) determines the upper bound for vertical R&D since it is, *ceteris paribus*, more profitable to invest in increasing process knowledge given higher α. In the horizontal dimension, the inverse of this ratio has a lower bound since decreases in the elasticity of substitution (ε) *ceteris paribus* yield higher profit opportunities. Together with the assumption of $1 > \alpha(\varepsilon - 1)$, (37) implies (38), i.e., provided that the steady-state rate of return is strictly positive, then entry in the horizontal dimension is maintained.

Summarizing, we expect that a unique steady-state equilibrium exists with restrictions

$$\begin{aligned}
g_N^* &= g_L^* = g^* := g(\alpha, \beta, \varepsilon, \rho, \phi_1, \phi_2, \nu, \mu, \sigma_S) \\
g_Z^* &= \left(\frac{\sigma_S}{1-\sigma_S}\right) g^* \\
g_S^* &= \left(\frac{1}{1-\sigma_S}\right) g^* \\
\frac{H_Z^*}{H_Y^*} &< \alpha \qquad n^* < n_{\max}^* \qquad f^* \in (\phi_2\mu, 1)
\end{aligned}$$

given a set of parameter values that satisfy the symmetry condition $1 > \alpha(\varepsilon - 1)$.

4 Some Implications

In this section, some implications of the model's balanced growth equilibrium are being emphasized. To reduce the difficulties associated with the absence of analytical solution, we first provide a numerical comparative statics analysis of balanced growth rates and of some key steady-state ratios. Next, we introduce exogenous population growth to the otherwise identical model and briefly discuss the emerging semi-endogenous growth property.

4.1 Comparative Statics

Economic growth in the standards of living is defined now. Recalling that per capita consumption expenditure Pc is constant in the steady-state, the balanced growth rate of per capita consumption (in real terms) is given by

$$g_c^* = \alpha g_Z^* + \left(\frac{1}{\varepsilon - 1}\right) g_N^*$$

where we simply neglect external social returns to product variety. Using the results from Section 3, we can rewrite this as

$$g_c^* = \left(\frac{\alpha \sigma_S}{1 - \sigma_S} + \frac{1}{\varepsilon - 1}\right) g^*$$

where $g^* = g\left(\alpha, \beta, \varepsilon, \rho, \phi_1, \phi_2, \nu, \mu, \sigma_S\right)$ is uniquely solvable by numerical techniques. As clearly seen, economic growth in the standards of living is fully endogenous without scale effects.

The parameter of central interest is σ_S —*the only gift of Athena* that affects long-run growth in the standards of living. It does so in two ways. A higher level of σ_S *ceteris paribus* decreases the balanced growth rate of N, and this puts a downward pressure on economic growth. The upward pressure is due to the increase in the balanced growth rate of Z, and the latter dominates the former since g_Z^*'s weight increases faster than that of g_N^*'s. In all the experiments, we impose discrete increases in σ_S with

$$\sigma_S \in \{0.05, 0.1, 0.2, ..., 0.8, 0.9, 0.95\}$$

In Table 1, we collect the numerical values of other parameters used in the following comparative statics analysis. The values for parameters except β are borrowed from Connolly and Peretto (2003, Table 1). The neurobiological research productivity β is absent in their model where resources devoted to research is measured in terms of labor units in a simpler way. Regardless, this parameter which represents the contribution of a researcher to knowledge creation does not alter the growth rate of the economy in this model and set to 5.0, i.e., a researcher in any

R&D sector creates 5 units of knowledge per unit of time. In addition, our numerical work confirms that the elasticity of the household utility with respect to the family size, denoted by ν, has no effect on balanced population growth since aggregate consumption expenditure of the household increases with the family size, and endogenous population growth internalizes this. Hence, we actually have

$$g^* = g(\alpha, \varepsilon, \rho, \phi_1, \phi_2, \mu, \sigma_S)$$

[Insert Table 1 around here.]

In the first experiment, we analyze the effect of σ_S on g_c^* for different cases regarding the fundamentals of population growth. Higher mortality rate is set such that the life expectancy is 48 years (near to 2005 average of African countries) instead of its baseline value of 76 years. Higher marginal utility for children is associated with the exogenous increase in the elasticity parameter ϕ_1 from its baseline value 0.7 to the plausible maximum 1.2 which does not alter the qualitative results as in Connolly and Peretto (2003). Similarly, the parameter of reproduction cost (ϕ_2) is set to 45.0 that yields plausible decreases in the population growth. Results from this experiment are summarized in Figure 3.

[Insert Figures 3 and 4 around here.]

As clearly seen, the increases in the concentration parameter σ_S unambiguously increase the balanced growth rate of per capita consumption and unambiguously decrease the balanced growth rate of population, and these effects occur in a nonlinear fashion. Thus, the long-run effect of σ_S is robust to different demographic structures.

In the second experiment, we check the robustness with respect to α, ε and ρ. As seen in Figure 4, the nonlinear and expansionary effect of σ_S on g_c^* is robust, and changes in α, ε and ρ are of expected signs. Higher productivity of knowledge stimulates increased investment in vertical R&D, and higher elasticity of substitution among products increases the relative market size, defined as

$$\text{r. m. s.} = \left(\frac{n^*}{n^*_{\max}}\right) \in (0, 1),$$

and reduces the growth potential in the vertical dimension through (Steady-State) Manhattan Metaphor. The effect of increased discount rate is negative but nearly zero.

In the last three experiments, we simply answer how three key steady-state ratios respond to changes in the concentration parameter. These are

- the relative market size defined above,
- the Scientists-to-Population ratio (h_S^*), and

- the Technologists-to-Workers ratio $\left(H_Z^* \div H_Y^*\right)$.

Figures 5 to 7 respectively demonstrate the results of these experiments.

[Insert Figures 5, 6 and 7 around here.]

As seen in Figure 5, the nonlinear expansion in balanced growth rate of consumption is associated with decreases in the relative market size. This is, again, due to (Steady-State) Manhattan Metaphor such that economies that expand horizontal dimension n^* toward to its *endogenous* "frontier" $n^*_{\max}$ exhibit lower long-run growth in the vertical dimension. The striking, though highly intuitive, result is pictured in Figure 6 that shows the trade-off between scientific knowledge and scientists through increased concentration of knowledge. This deepens the new Schumpeterian view against the semi-endogenous solution such that the steady-state ratio of Scientists-to-Population may not be the correct variable to analyze for the evaluation of the scale effects since an economy with a higher concentration of scientific research achieves higher long-run rates of economic growth with a lower density of scientists. Connected with this, Figure 7 pictures that the *appropriate* choice of variable in understanding increased growth potential is the ratio of Technologists-to-Workers. That is, in a cross-section of economies, the increased relative weight of technologists at the firm (or industry) level corresponds to increased growth rates, yet the steady-state ratio of Technologists-to-Workers as well as the numerator and the denominator of this ratio are constant in any economy of this cross-section.

In short, permanent increases in the concentration parameter of scientific research unambiguously increase balanced growth rate of the standards of living, and main lessons of new Schumpeterian models without scale effects is maintained.

4.2 Semi-Endogenous Growth

Consider an alternative formulation of the household's problem with *exogenous population growth*. Formally, suppose that the representative household seeks to maximize

$$\int_0^\infty e^{-(\rho-\lambda)t} \log(c)\, \mathrm{d}t$$

where $\lambda \in (0, \rho)$ denotes the exogenous growth rate of population, and c is per capita consumption aggregator defined above. We further simplify away the necessity of resources devoted to rearing of children and write the budget constraint as

$$\dot{a} = (r - \lambda)\, a + W - Pc$$

The solution to this setup alters the Euler equation into the form of

$$g_P + g_c = r - \rho$$

Since the intratemporal allocation of consumption expenditure over varieties and firms' problem remain same as before, the only other change occurs in the resource constraint of labor services:

$$L = H_S + N\left(H_Z + H_Y\right)$$

The balanced growth in the standards of living along the isomorphic steady-state equilibrium path, characterized by the constancy of $\left(n^*, (Pc)^*, r^*, h_S^*, H_Z^*, H_Y^*\right)$, satisfies

$$g_c^* = \left(\frac{\alpha\sigma_S}{1-\sigma_S} + \frac{1}{\varepsilon - 1}\right)\lambda$$

This is a very strong semi-endogenous growth result, if the concentration parameter σ_S is not policy-based, and semi-endogenous growth in this model emerges due to the fact that the technological opportunity is (endogenously) bounded with blueprint knowledge similarly to Kortum's (1997) and Segerstrom's (1998) models.

Thus, we end up with an important result that provides some new insight on the controversy between Schumpeterian (fully endogenous) growth paradigm and the semi-endogenous solution against it. Provided that the model studied in this paper is a fairly accurate description of the K-T mapping, we could state the following. If blueprint knowledge puts a pressure on the frontier of inventiveness in the horizontal dimension, then what determines the full- or semi-endogeneity of long-run growth is the way in which the population growth is incorporated —a *thin red line*. Whether the semi-endogenous result emerges is crucially connected with the exogeneity of population growth, and if one introduces endogenous fertility, the nexus between policy and growth is settled again.

5 Conclusion

Simplicity is a virtue of any theory, and the most brilliant growth theorists have all exploited this. However, some interpretation difficulties always exist since the necessarily simplified *structure* is usually weak in some respects.

Recent advances in R&D-based growth theory build on the notion that product innovation is a fundamentally distinct type of innovative activity than the process innovation. In this paper, we have attempted to take one step further and studied a model with a third dimension that creates *useful knowledge*. Exactly as in Mokyr's (2002) informal theory of useful knowledge, we distinguish (*i*) the propositional and the prescriptional and (*ii*) the aggregate and the collective forms of knowledge. Taking these aspects of knowledge seriously allows us to challenge the interpretation difficulty associated with the convention that the scale of the horizontal dimension is a knowledge-based variable *per se*. It is certainly knowledge-based in the sense that "the economy" *knows* how to produce these different goods. However, the only form of knowledge embodied in these goods can be the process knowledge of how to actually produce these goods. This suggests that the creation of this special prescriptional knowledge could be somewhat complicated than the one suggested by a Romer-type (1990) knowledge production function of the form $\dot{N} = f(L_N, N)$. Instead, as Mokyr (2002) and many others in the same tradition suggest, there exists a mapping from some useful form of propositional knowledge into the selection of finalized working prototypes. Thus, $\dot{N}$ is a certain by-product of some more fundamental research activity.

What we offer is the purposeful search for new scientific knowledge which is subject to some diffusion inefficiency such that only *collective* scientific knowledge can be used. New technological knowledge, i.e., the excludable fraction of scientific knowledge, is then allocated over new working prototypes in the horizontal dimension by some subset of scientists. New firms obtain the property rights of new working prototypes by providing the only source funding for scientific activity, and existing firms invest in developing the blueprint mode of working prototypes into the more productive modes of production in the vertical dimension where technologists for each product is employed. Thus, the open Republic of Science creates the technological opportunity that the proprietary Realm of Technology is subject to.

We construct the spillovers among these three dimensions of innovation as suggested by (*i*) the usefulness of collective scientific knowledge, (*ii*) the excludability of blueprint and process knowledge and (*iii*) the externalities associated with some specific products that reduce diffusion inefficiency and that increase research productivity. However, in this overly complicated general structure of the model, we fail to analyze the transitional dynamics. Even the solution of the unique steady-state equilibrium requires some numerical work, and, perhaps more importantly, the long-run growth rates in three dimensions of innovation is independent of the most of the new parameters that we introduce in this paper. In a sense, this is "discouraging" since most of the intellectual origins of economic growth create only some less interesting (steady-state) level effects.

The loss of tractability is the major weakness of the model studied in this paper, and the model is *almost useless* for the analysis of further issues without an appropriate simplification. Implications are available only for the study of long-run issues, and, as argued by Temple (2003), one should not seriously focus on such long-run implications due to the inherent elusiveness of the concept of the long-run itself. Nevertheless, we believe that the long-run implications of the model studied in this paper could be illuminating in two regards. First, the endogenous bound on technological opportunity is crucially linked to the role of blueprint knowledge in the creation of new products, and endogenous fertility is a necessary feature to obtain a fully endogenous long-run growth rate. Since every model has some specific knife-edge assumptions and some key steady-state ratios, small variations in the defining properties of balanced growth equilibria and in the functional forms of knowledge production functions could crucially alter the conclusions regarding semi-endogenous vs. Schumpeterian growth. Second, once the endogenous fertility is preserved, the model economy attains fully endogenous, strictly positive balanced growth rates in both product and process innovation, the main lessons of new Schumpeterian models without scale effects hold, and a specific parameter, i.e., the concentration of scientific research on collective scientific knowledge, accelerates long-run growth.

Finally, a piece of usefulness of the work presented in this paper is its immediate warning for the *mystery* of the Knowledge-to-Technology mapping even if one simplifies away the Ideas-to-Knowledge mapping by assuming that ideas and knowledge are identical things. In this respect, closing the gap between the scholars of knowledge and technology from different fields of social sciences and the growth theorists who always simplify stands as a noble task.

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Appendix

In what follows, $\mathcal{H}$ and $\mathfrak{h}_{\bullet}$ respectively denote the current-value Hamiltonian function and the associated co-state variable(s).

A1. The Household's Problem

For any t, the intratemporal allocation of resources over consumption varieties is determined simply as the solution of the problem to min $\int_0^N P_i c_i \mathrm{d}i$ subject to $c^{\frac{\varepsilon-1}{\varepsilon}} = \int_0^N c_i^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i$. As well-known, the interior minimum uniquely exists given strictly positive prices $\{P_i\}_{i=0}^N$ and satisfies $c_i = (P_i/P)^{-\varepsilon} c$ for any i where P is the aggregate price index defined in (21).

For the intertemporal choice, we formulate the current-value Hamiltonian as

$$\mathcal{H} = \log(c) + (\nu + \phi_1)\log(L) + \phi_1 \log\left(\frac{f}{\phi_2} - \mu\right) + \mathfrak{h}_a\left[\left(r - \frac{f}{\phi_2} + \mu\right)a + (1-f) - Pc\right] + \mathfrak{h}_L\left[\left(\frac{f}{\phi_2} - \mu\right)L\right]$$

The household chooses the paths of c and f given $a_0, L_0 > 0$ and subject to

$$\begin{aligned} \dot{a} &= \left(r - \frac{f}{\phi_2} + \mu\right)a + (1-f) - Pc \\ \dot{L} &= \left(\frac{f}{\phi_2} - \mu\right)L \end{aligned}$$

FOCs and TVCs are given by

$$\begin{aligned}
\frac{1}{c} &= \mathfrak{h}_a P && (c)\\
\frac{\phi_1}{\phi_2 g_L} + \frac{\mathfrak{h}_L L}{\phi_2} &= \frac{\mathfrak{h}_a a}{\phi_2} + \mathfrak{h}_a && (f)\\
\dot{\mathfrak{h}}_a - \rho \mathfrak{h}_a &= -\mathfrak{h}_a \left(r - g_L\right) && (\mathfrak{h}_a)\\
\dot{\mathfrak{h}}_L - \rho \mathfrak{h}_L &= -\left[\frac{\nu + \phi_1}{L} + \mathfrak{h}_L g_L\right] && (\mathfrak{h}_L)\\
\lim_{t\to\infty} e^{-\rho t} \mathfrak{h}_{at} a_t &= 0 && (\mathrm{TVC}_a)\\
\lim_{t\to\infty} e^{-\rho t} \mathfrak{h}_{Lt} L_t &= 0 && (\mathrm{TVC}_L)
\end{aligned}$$

The first and the third FOCs yield the Euler equation

$$r = g_c + g_P + g_L + \rho$$

The fourth equation can be written as

$$\left(g_{\mathfrak{h}_L} + g_L\right) = \rho - \frac{\nu + \phi_1}{\mathfrak{h}_L L}$$

This unstable differential equation of $\mathfrak{h}_L L$ implies that $\mathfrak{h}_L L$ instantaneously adjust to

$$\mathfrak{h}_L L = \frac{\nu + \phi_1}{\rho}$$

Combining this result with the second FOCs by eliminating $\mathfrak{h}_a$ yields

$$g_L = \frac{\phi_1 \rho P c}{\rho \left(a + \phi_2\right) - \left(\nu + \phi_1\right) P c}$$

A2. Firms' Problem

The current-value Hamiltonian of firm i's problem is defined as

$$\mathcal{H} = P_i \left(\frac{P_i}{P}\right)^{-\varepsilon} Lc - \left(\frac{P_i}{P}\right)^{-\varepsilon} Lc Z_i^{-\alpha} - P_{Z_i} \dot{Z}_i + \mathfrak{h}_i \dot{Z}_i$$

and the firm chooses P_i and $\dot{Z}_i$ subject to the law of motion of Z_i and the initial value $Z_{i0} > 0$. Thanks to the fundamental theorem of calculus, the variable discount factor does not complicate the problem of determining the optimal evolution of co-state variable. That is, FOCs can be written *as if* the discount factor is equal to the rate of return $r > 0$. Thus, we have

$$\varepsilon\left(\frac{P_i}{P}\right)^{-\varepsilon}\frac{LcZ_i^{-\alpha}}{P_i} = (\varepsilon-1)\left(\frac{P_i}{P}\right)^{-\varepsilon}Lc \tag{P_i}$$

$$\mathfrak{h}_i \geq P_{Z_i} \qquad \text{if } \dot{Z}_i > 0 \tag{$\dot{Z}_i$}$$

$$\dot{\mathfrak{h}}_i - r\mathfrak{h}_i = -\alpha\left(\frac{P_i}{P}\right)^{-\varepsilon}LcZ_i^{-\alpha-1} + (\mathfrak{h}_i - P_{Z_i})\left(\frac{\partial \dot{Z}_i}{\partial Z_i}\right) \tag{$\mathfrak{h}_i$}$$

However, the TVC is given by

$$\lim_{t\to\infty} e^{-\int_0^\tau r\mathrm{d}t}\mathfrak{h}_{it}Z_{it} = 0 \tag{TVC}$$

Note that the first FOC simply implies the intratemporal pricing rule in the text. Also note that the aggregate resource constraint on labor rules out the case in which $\mathfrak{h}_i > P_{Z_i}$ and $H_{Z_i} \to \infty$, and the general equilibrium with vertical R&D is characterized by

$$\mathfrak{h}_i = P_{Z_i}$$

as in Peretto (1998a). Then, incorporating $\mathfrak{h}_i = P_{Z_i}$ and $c_i = (P_i/P)^{-\varepsilon}c$ into the third FOC yields

$$r = \left(\frac{\alpha L c_i}{Z_i^{1+\alpha}}\right)\frac{1}{P_{Z_i}} + \frac{\dot{P}_{Z_i}}{P_{Z_i}}$$

Table 1. Parameter Sets for Comparative Statics Analysis

Parameter	Description	Baseline	Higher
α	Productivity of Knowledge	0.30000	0.55000
β	Research Productivity	5.00000	—
ε	Elasticity of Substitution	2.50000	4.00000
μ	Mortality Rate	0.01315	0.02083
ν	Utility Elasticity of the Household Size	0.70000	—
ρ	Subjective Discount Rate	0.04000	0.12000
ϕ_1	Utility Elasticity of the Children	0.70000	1.20000
ϕ_2	Reproduction Cost	40.00000	45.00000

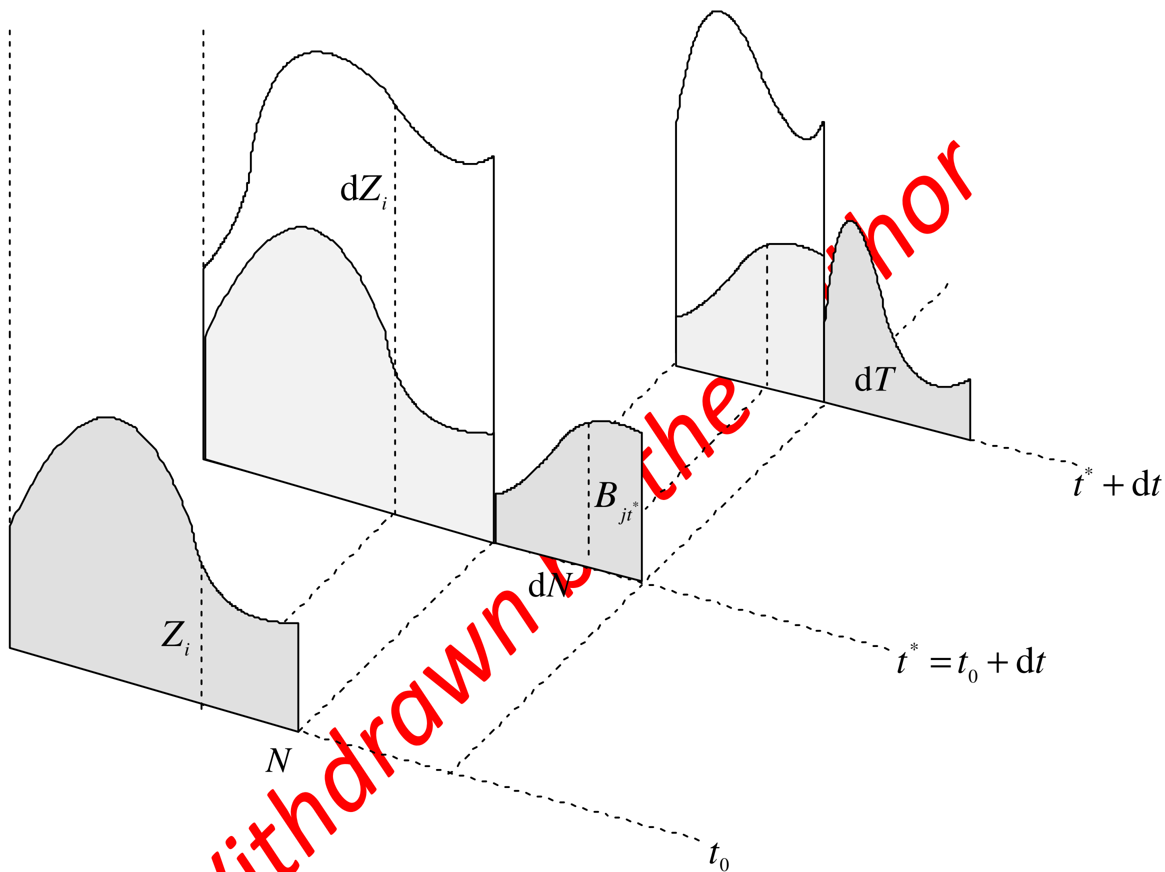

Figure 1: From Macroinventions to Microinventions

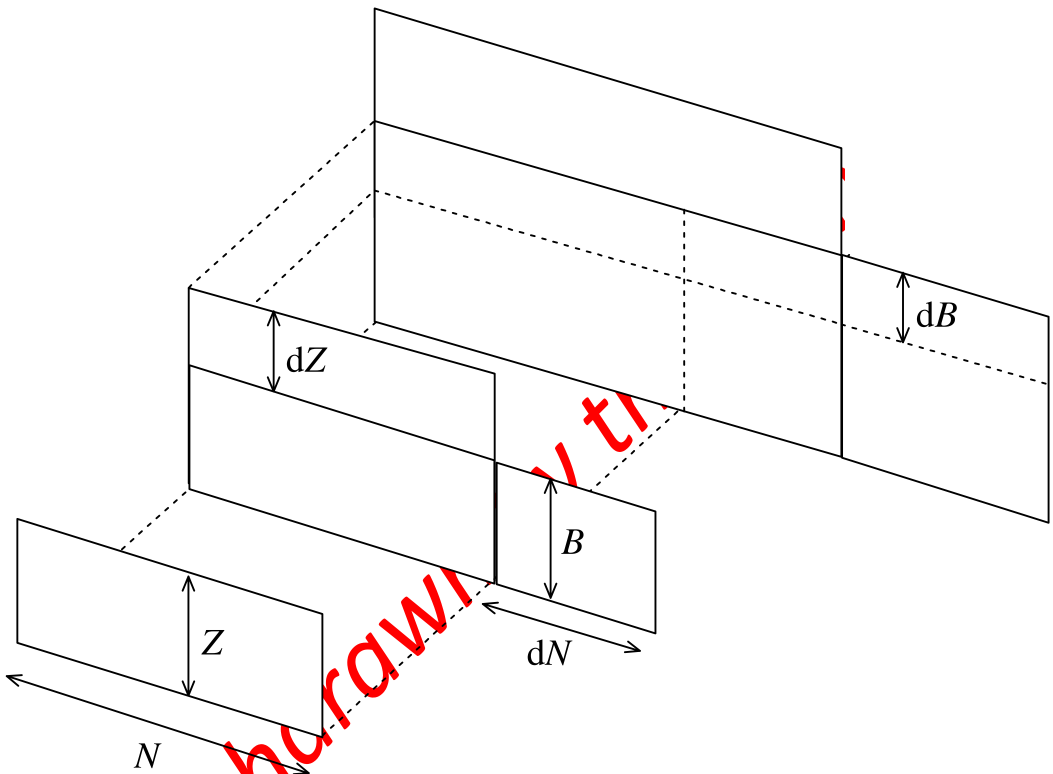

Figure 2: $B = Z$ under Symmetry

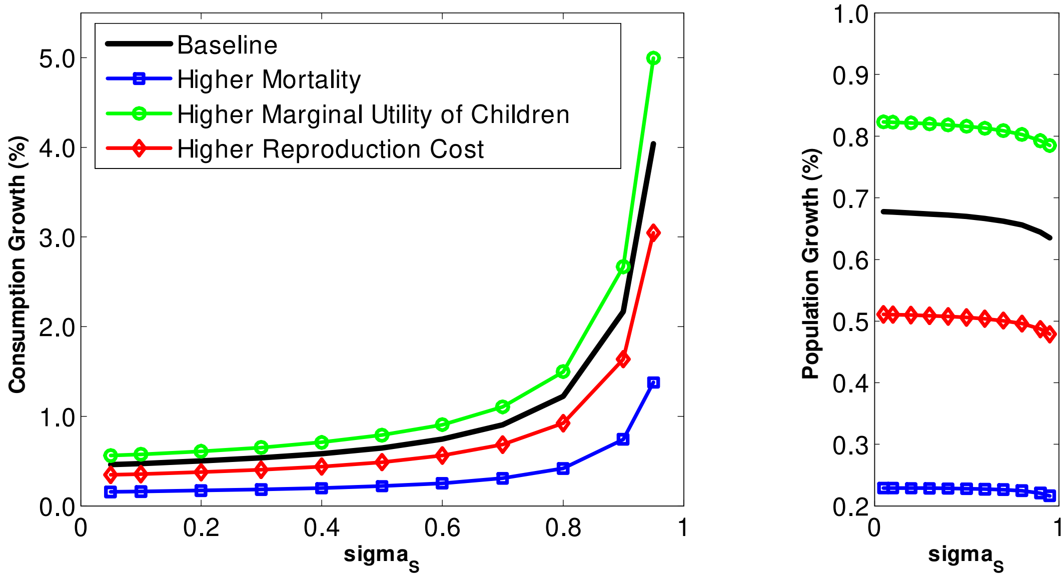

Figure 3: Balanced Growth: Comparative Statics I

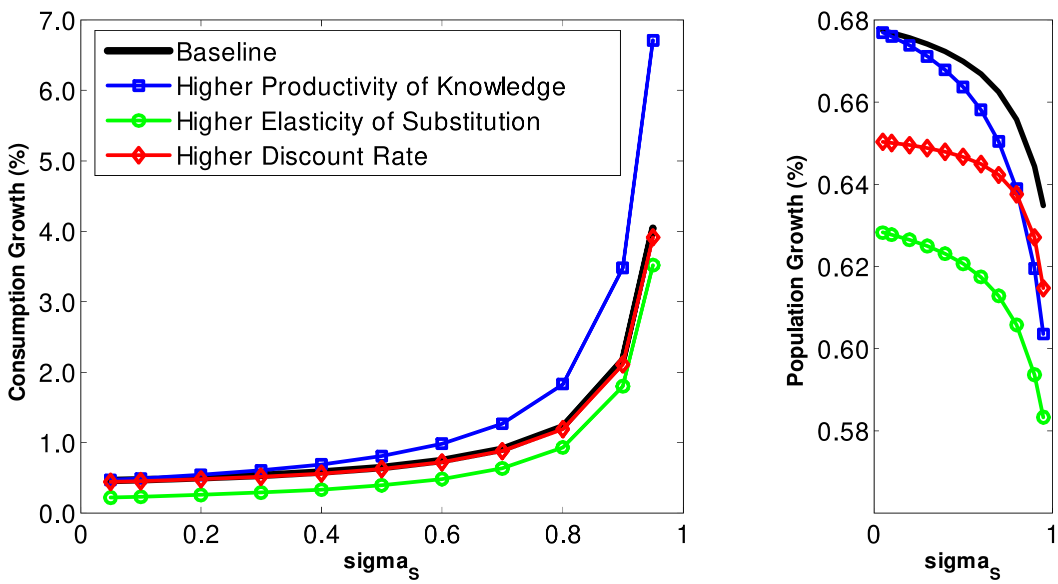

Figure 4: Balanced Growth: Comparative Statics II

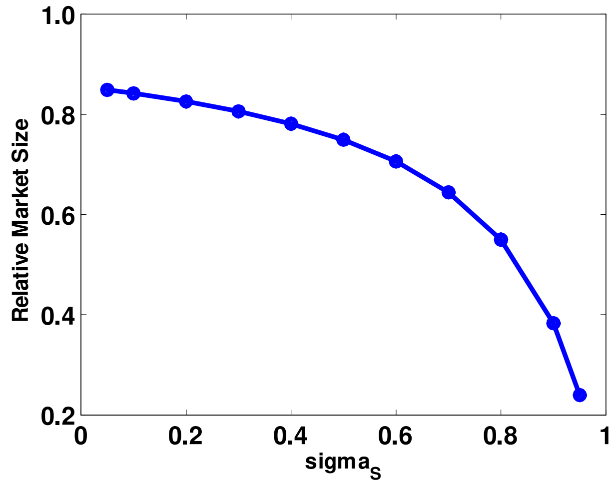

Figure 5: Relative Market Size in the Steady-State

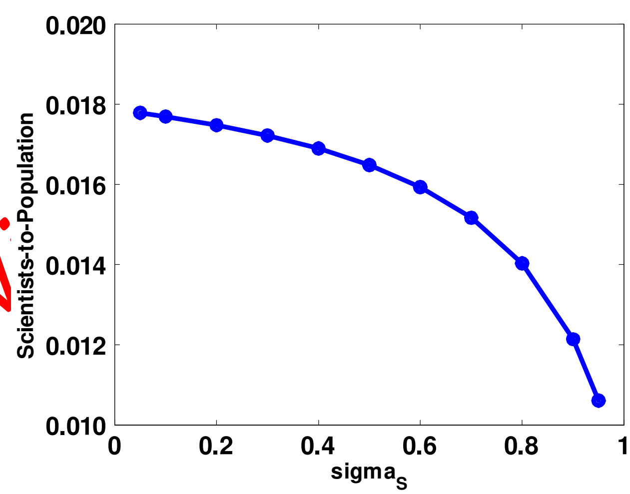

Figure 6: Scientists-to-Population in the Steady-State

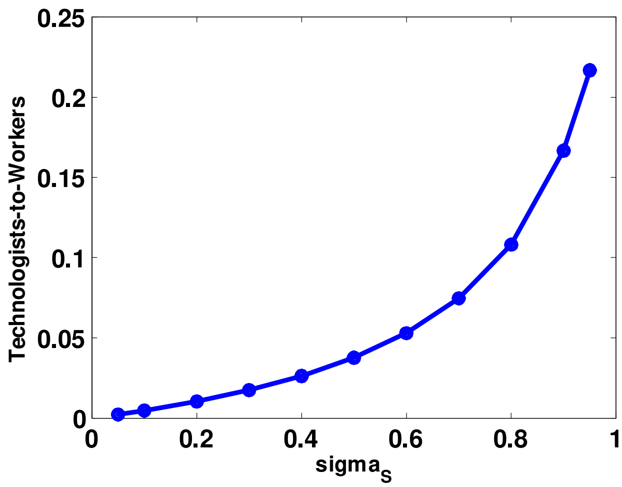

Figure 7: Technologists-to-Workers in the Steady-State