Social power as a solution to the Bertrand Paradox

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Abstract

We show that in a duopoly with homogeneous consumers, if these are negatively influenceable by each other behavior (e.g. congestion/ snob/ Veblen/ network effects), a pure price equilibrium with positive profits for both firms exists. Furthermore, even in the case products are undifferentiated, an equilibrium where firms charge different (positive) prices and have different profits exists. Thus, when firms engage in uniform price competition, heterogeneity, and in particular non-atomicity in the distribution of preferences, is neither a necessary condition to ensure existence, nor to achieve asymmetries. We further show that in the case products are differentiated, social differentiation overcomes the effect of standard differentiation in creating price asymmetries.

Keywords: Social influence; Bertrand duopoly; Bertrand competition; network effects; product differentiation; homogeneous products; pure price equilibrium; linear demand.

Introduction

Price dispersion and periods of price stability are fairly common phenomenons in many service markets. In fact they are embedded in consumption behavior: we compare prices (thus expect price dispersion), but not on a daily basis (we expect some price stability). Yet, pure (asymmetric) price equilibria, above marginal costs, persisted as problematic to achieve in the context of uniform price competition within oligopoly theory.[1] The discrepancy between what is a natural expectancy but a theoretical modelling difficulty is troubling and well reflected in the so called paradoxical and unconvincing nature of Bertrand's zero profit equilibrium.

The problem has in general been attributed to the simplistic and homogeneous nature of the original Bertrand framework ([6], [32]). Standard and most convincing solutions are essentially two-folded: introducing some form of (exogeneous) heterogeneity, either in the firm or consumer side, accompanyed by some further assumption which can ensure demand continuity and stabilize competition in pure price strategies.[2] If both product and consumers are homogeneous, and heterogeneity is introduced in the cost structure, '*there*

[1] Tirole [32] refered to the study of price competition as simultaneously *a fundamental part* and *one of* [the] *weakest links of oligopoly theory*, and Vives [33] even defined the oligopoly problem as '*centered around the potential indeterminateness of price equilibria with a few number of competitors*'.

[2] Alternative approaches would involve leaving the standard Bertrand framework and considering other strategic variables for firms, for example, by allowing firms to compete in quantities, as in Cournot, choosing/investing in quality, or others. Note that solutions based on the temporal dimension can in fact be seen as introducing timing as a strategic variable.

are serious existence problems' (Maskin [26]). Heterogeneity in costs introduces an asymmetric lower point in the undercutting descent dynamics, which prevents firms from stabilizing at a common zero profit stopping point, but requires a further assumption to be stabilized. With differentiable and increasing cost functions, equilibrium prices above marginal costs in general involve mixed strategies, and in the extreme cases of constant costs or capacity constrains, prices stay at marginal costs, for the former, and for the latter pure equilibria exist only for small capacities and are symmetric (see for example [32] or [13]). As such, since Edgeworth showed this price indeterminacy problem, the stabilization on pure price strategies has been based on product differentiation and heterogeneity of consumer preferences specified by a non-atomic distribution (Hotelling's approach). Caplin and Nalebuff [12] show that for multi-dimensional product differentiation, if preferences are linear in the weights assigned to product benefits, and the distribution of consumer types can be represented by a density function which satisfies a weak form of concavity, then pure price equilibria exist. Palma et al [27] show that if firms treat the utility of a particular consumer as a random variable due to a lack of information regarding the tastes of that consumer, sufficient heterogeneity ensures a pure price equilibrium.

The caveat in these approaches is that they provide sufficient, but not necessary conditions for existence (as emphasized by Caplin and Nalebuff [12]). In particular, it is not established the necessity of imposing some form of heterogeneity to achieve a pure price equilibrium with prices above marginal costs. Rather, the framework is changed from homogeneous to heterogeneous, and it is this new problem which is solved. As a result, the existence and asymmetry of an equilibrium with positive profits often becomes a byproduct of exogenously imposed heterogeneity. Whether this is always the case is seemingly a question still to be addressed.[3] Burdett and Judd [10] use consumer search behavior to show that, with infinite firms, ex ante heterogeneity is not a necessary condition to achieve an equilibrium with price dispersion. However, a search friction is not enough to ensure price dispersion for small markets (or a countable number of firms), as it drives all equilibrium prices from marginal cost to the monopoly price, as shown by Diamond [14]. As such, for small markets the open questions still remain: do positive asymmetric pure price solutions exist for a priori completely homogeneous markets? Do positive asymmetric pure price solutions exist for atomic distributions of consumer preferences?

From a game theory perspective, there is an inherent strategic asymmetry in the original Bertrand framework. The set of players is composed of firms and consumers, thus a strategy profile consists of prices set by firms and

[3] Hotelling himself [19] may arguably be asking this same question with his location game: may these type of solutions be achieved endogenously? Although d'Aspremont et al [4] later show that in fact they could not (with the original formulation), there is in Hotelling's model an entanglement between heterogeneity of product and consumers (a single space is used to represent both diversity of products and consumers). Once the disentanglement is made and solutions found for homogeneous products, for example through vertical differentiation [22] or different price sensitivities [2], one is ready to adress the issue of necessity of heterogeneity of consumers, independent of product heterogeneity. However, solutions which rely on non-atomic distributions of consumer preferences, to ensure continuity in aggregate demand, exclude the case of homogeneous consumers, by excluding the possibility of atomic distributions. Furthermore, these solutions for homogeneous products are often symmetric and/ or stay at marginal costs.

purchasing choices by consumers, but while firms best response depends on the whole strategy profile (its price, the prices set by other firms and demand based on consumption behavior), consumers best response is assumed to depend only on prices and not on the other consumers' choices, hence ignoring part of the game's strategic profile. Notwithstanding that in some markets this may still be an appropriate modeling assumption, in most markets today the role that consumers play in each other's choices, and in shaping markets, is of greater importance. The growth of Internet, the emergence of user-generated content digital platforms, the use of social networks and the increase of data availability, has not only emphasized this role, but also reduced the asymmetry between firms' and consumers' impact in determining outcomes.

Among the effects of considering social interdependence in consumption is the introduction of a new source of variability: with the (endogenous) dependence on each other, a consumer's choice is no longer determinate a priori whenever firms charge different prices, as in the original framework. When we introduce social influence and consider its strategic relevance, *indifferent* consumers are no longer confined to the one point domain of firms setting the same price (the unique indifference point in the original Bertrand setting). Let us emphasize that, in this context, indifference does not mean consumers do not care about the decision, but rather that they could *rationally* make different decisions when facing the same conditions. Under this interpretation it is natural to associate a probabilistic behavior to their consumption decisions, which we call *non-loyal* behavior, as opposed to *loyal* or non-probabilistic consumption behavior.[4] Observe, though, that non-loyal behavior is not a new, or modern phenomenon. In fact, there has always been intrinsic variability in human decision making. We do not always dress in our favorite color and choose our favorite mode of transportation to eat our favorite dish at our favorite restaurant. This does not appear to be an 'irrational' behavior, but a rather natural human characteristic, independently of whether it is conscious or not. The very meaning of the word *favorite* means we *favor* some choice in detriment of another, and not that it is our only (rational) choice. Social interdependence allows us to see non-loyal behavior, not as a minimal and sporadic phenomenon, but rather as something rooted in human (social) behavior.

The main idea behind the classic solutions to stabilize a pure price equilibria is turning the set of indifferent consumers into a negligible set, so to eliminate firms' incentive to capture a mass of indifferent consumers by undercutting. This means the main obstruction to the existence of pure price equilibria is eliminated by assuming indifferent consumers are a zero measure set, either because preferences are non-atomic or price quotations come from a non-degenerate mixed price strategy. Once we allow non-loyal behavior to be a locally persistent strategic phenomenon, it becomes clear that the reasoning allowing firms to capture a positive mass of indifferent consumers with a small price deviation, leading to non-existence of pure price equilibria, is anchored on two apparently innocuous, but quite strong assumptions

[4]We observe that loyalty differs from installed base, since being loyal is, in this context, a strategical behavior, thus endogenous, not imposed. Loyal consumers are just those who opt for pure strategies.

(which reflect the high asymmetry between firm and consumer power). First, firms capture this mass because it is assumed they are able to turn non-loyal consumers into loyal with a small price deviation. Second, when this mass is captured, hence, when the mass moves, it is assumed to have no impact on consumption choices.

Our approach is that non-loyal consumers are more sensitive and react to small price deviations by varying the probability of consuming (instead of becoming loyal), while loyal consumers are less sensitive and react only to higher price changes.[5] As such the partition of consumers into loyal and non-loyal is invariant for small price changes. The local stability of this partition creates a local coordination device for firms by having associated a unique continuous demand deviation, given by the non-loyal consumers' probability change. In sum, we propose that, with social interdependence, the set of indifferent consumers (non-loyal) is stable for small changes, and in general not a negligible phenomenon. Roughly saying, this justifies the existence of asymmetric pure price solutions, even in homogeneous frameworks, by using a positive mass of indifferent consumers to produce demand continuity, and loyal consumers to create asymmetries. Social influence being negative smoothly removes the incentive produced by a lower price. Hence, in an uniform price competition market with social interaction, rather than an intrinsic price indeterminacy problem due to determinate consumption behavior (for essentially all prices), we have in the consumers behavior variability a road to stabilize prices.

[5]By lower sensitivity of loyal consumers, we mean that the equilibrium condition inequality for these consumers is strict, hence their best response is constant for a neighborhood of the outcome. The idea that loyal consumers may have lower sensibility is rather natural, and intuitive to the very notion of brand loyalty.

Modeling approach, results and related literature.

The study of a social component's impact in consumption behavior has a long tradition, in particular as an external effect (or nonfunctional aspect) of consumption/use of private or public goods/services. Notably, for example, in the seminal works of Rae, Veblen, Leibstein or Tiebout (see for example [25], [31] and references therein). Whether in a more intricate or simpler form, and while not necessarily equal in every market or context, technological and economic development, education, social awareness, financial autonomy and access to information, have definitely brought changes to the social interdependence level of our choices. In many cases, consumption itself is a social act, driven by social interactions or concerns. The study of such phenomena experiences particular growth and aroused interest, as the relevance and economic implications of social influence are pointed out as a crucial direction to follow (particularly the importance of network economics, see for example [21], [17], [8]), and have been persistent as a field of interest (for example [23], [20], [18], [5], and [11]). A survey including consumer demand under network effects and social influence can be found for example in [30].

In this work, we consider a duopoly with a finite set of consumers. The modeling approach stands on three more or less standard principles: (i) firms set prices to maximize profits; (ii) pricing and consumption have different time frames (prices only contribute to forming a behavioral intention, which can thus be represented by a probability distribution over options); (iii) consumers are influenced by each other behavior. In view of (ii) we consider a two stage game, where the first stage is a duopoly pricing game, and the second stage is a consumption game based on the observed prices. Each of the two subgames is a simultaneous move, complete information game. Outcomes are evaluated according to the notion of Nash equilibrium and subgame-perfection.

Following the theories of planned behavior and reasoned action, the consumers utility is constructed on two commensurable components, personal and social. The personal component reflects how a consumer values the characteristics of each product, and we make no restriction on this component. In particular the value could come from an atomic distribution of preferences, and may include costs. The social component reflects how a consumer is influenced by the decisions of others. Here, we follow the classic definitions of influence and power from social psychology, in which power and influence involve a dyadic relation between two consumers (e.g. French and Raven [16]). Social power is defined as a structural property of a particular social relationship, and reflects the consumer's relative capacity to modify others' payoffs, in some sense similar to Keltner et al [24] or Fiske and Berdahl [15]. We take social power as constant, and it is influence which varies according to the particular strategy profile in question. Influence refers to the effect on individual i produced by another individual j, and social power is the maximum potential ability of i to influence j. Individuals exert influence only through the action of choosing to consume from firm 1 or 2. When we talk about positive and negative influence, we refer to whether consumers 'like' to make the same decision or not (which may result from the passive presence of j). With this, the utility becomes of the von Neumann-Morgenstern type, and the behavioral intention a best response in the Nash sense.

The approach to the consumers game finds its inspiration in the socio-economic model of Brida et al. [9] that analyses how the choice of a service is influenced by the profile of its users. The mathematical modeling foundations for the consumers game and utility function are based on the two types dichotomic model by Soeiro et al. [29] and its more general version in [28], which, in turn, are grounded on the intersection of game theory and social psychology, through the theories of planned behavior and reasoned action (see for example [3]).

We show that, when social power is negative, a pure price subgame-perfect equilibrium with positive profits for both firms exists. Furthermore, even in the case products are not differentiated, asymmetric pure price equilibria always exist. There are some particular advantages in these solutions. A first advantage is being free from any limitation (or assumption) on the distribution of preferences that leads to product differentiation. In particular, the model could be seen as a discretization of the Hotelling line (the same for vertical product differentiation), but where the distribution is possibly discontinuous, atomic, etc... Namely, this is applicable to situations where the set of non-loyal (indifferent) consumers is not irrelevant. A second advantage is that one need not consider loyal consumers as exogeneously captive installed bases of non-strategic individuals, or possibly locked by a switch cost, but as strategic consumers using an integer probability behavior. We then observe that, when social power is negative, having loyal consumers reduces equilibrium prices. A fact which may at first seem counterintuitive, nonetheless, a natural consequence of consumers preferring uncertain to certain negative influence. A third advantage is that heterogeneity (whether it is product differentiation, capacity constraints, or other), need not be imposed exogenously to achieve asymmetric pure price equilibria. Furthermore, we show that the effect of *social product differentiation*, which is endogenous and possibly variable, as determined by social influence, overcomes the effect of standard product differentiation in creating price asymmetries. Note however that we are by no means mitigating the importance of product differentiation, or other type of heterogeneity introduced a priori. We are in fact empowering the study of heterogeneity, by allowing its study to be independent of concerns or restrictions guaranteeing existence. With social power, any form of heterogeneity may be studied on top of already existing solutions. Moreover, and relating to the work in [7], this approach can be used as yet another possible microeconomic foundation for linear demand.

1 Model and main results

There are two firms and n consumers. The game begins with a *pricing stage*, where firms act simultaneous and independently, followed by a *consumption stage*, where consumers act simultaneous and independently. In the first stage, each firm ($j = 1, 2$) sets a price p_j for the service it provides. For each pair of prices $\mathbf{p} \equiv (p_1, p_2) \in (\mathbb{R}_0^+)^2$, the second stage is a standard simultaneous move game, with the set of players (consumers) denoted by $\mathcal{I}$ and two possible actions representing choosing one of the two firms. Consumption is thus mandatory and the behavior strategy of a consumer i is represented by a point $(\sigma_1^i(\mathbf{p}), \sigma_2^i(\mathbf{p}))$ in the standard probability simplex Δ^1, where $\sigma_1^i(\mathbf{p}) \equiv \sigma^i(\mathbf{p})$ and $\sigma_2^i(\mathbf{p}) \equiv 1 - \sigma^i(\mathbf{p})$ represent, respectively, the probability of consumer i using the service provided by firm 1 and 2 at prices $\mathbf{p}$. A strategy profile for the game is thus a pair denoted $(\mathbf{p}^*, \boldsymbol{\sigma}(\mathbf{p}))$ formed by a pair of prices $\mathbf{p}^*$ and a consumption behavior for every possible $\mathbf{p}$ summarized by the profile $\boldsymbol{\sigma}(\mathbf{p}) \equiv (\sigma^{i_1}(\mathbf{p}), \ldots, \sigma^{i_n}(\mathbf{p})) \in (\Delta^1)^n$. We will sometimes omitt the dependence on $\mathbf{p}$ when there is no ambiguity and simplifies notation. Nevertheless, we will make use of the distinction between a strategy profile $(\mathbf{p}^*, \boldsymbol{\sigma}(\mathbf{p}))$ and the outcome $(\mathbf{p}^*, \boldsymbol{\sigma}(\mathbf{p}^*))$. For example, payoffs for firms and consumers depend only on outcomes, whereas behavior strategies are used to evaluate the profitability of firm's price deviations. We will study and characterize subgame-perfect equilibria and the corresponding outcomes of the two stage game.

Firms. For simplicity, we assume firms have no costs, neither in producing nor in providing the service. In a given outcome $(\mathbf{p}, \boldsymbol{\sigma})$, the demand and profit for each firm $j = 1, 2$, are determined, respectively, by functions

$D_j : (\mathbb{R}_0^+)^2 \times (\Delta^1)^n \to [0, n]$, and $\Pi_j : (\mathbb{R}_0^+)^2 \times (\Delta^1)^n \to \mathbb{R}_0^+$, given by[6]

$$D_j(\mathbf{p}, \boldsymbol{\sigma}) \equiv \sum_{i \in \mathcal{I}} \sigma_j^i(\mathbf{p}), \text{ and } \Pi_j(\mathbf{p}, \boldsymbol{\sigma}) = p_j D_j(\mathbf{p}, \boldsymbol{\sigma}).$$

Consumers. For the construction of utility, one may think of the following process: suppose there are two distributions characterizing, respectively, personal and social preferences. These determine two components: (i) the benefit $b_1^i, b_2^i \in \mathbb{R}$ a consumer i derives from each product/service[7]; and (ii) two matrices of dyadic interactions (one for each firm) whose entries are social weights $\alpha_1^{ii'}, \alpha_2^{ii'} \in \mathbb{R}$, which represent how much a consumer i is influenced by i' when both choose to use the service from firm 1 or 2, respectively. The aim of this work is to show that, even if the distributions of consumer types are represented by density functions f_b and f_α which are concentrated at a single point (leading to homogeneous consumers), outcomes with positive profits for both firms exist, and may be asymmetrical.

We will thus consider an a priori homogeneous set of consumers. Therefore, the *personal benefit* derived from the use of each service is independent of the consumer, i.e. for every $i \in \mathcal{I}$ we have $b_1^i = b_1, b_2^i = b_2 \in \mathbb{R}$. The *social influence* exerted at each service by the choice of other consumers is also determined by only two *social weights* $\alpha_1, \alpha_2 \in \mathbb{R}$. These may be interpreted as representing how much a consumer *likes/dislikes* to share the service from

[6]The probabilistic nature of demand can be interpreted either from a frequentist point of view (i.e. consumption behavior over a period), or from the point of view that pricing and consumption have a different time frame, so prices in fact only determine a behavioral intention.

[7]This benefit may be a result of aggregating different characteristics, in particular it may also involve costs. Hence, we impose no restriction to positive benefits.

firm 1 or 2, respectively. These are the same for all consumers, nevertheless, social influence, and thus payoffs in a given outcome, may differ, as they depend on the strategy profile. Personal and social parameters determine the following pure strategy payoffs for a consumer i,

$$u_j(\mathbf{p}, \boldsymbol{\sigma}_{-i}) \equiv -p_j + b_j + \alpha_j \sum_{i' \neq i} \sigma_j^{i'}(\mathbf{p}), \quad j = 1, 2.$$

The payoffs for consumers in a given outcome $(\mathbf{p}, \boldsymbol{\sigma})$ are determined by the expected utility function $u : \mathcal{I} \times (\mathbb{R}_0^+)^2 \times (\Delta^1)^n \to \mathbb{R}$ given by

$$u^i(\mathbf{p}, \boldsymbol{\sigma}) \equiv \sigma^i(\mathbf{p}) u_1(\mathbf{p}, \boldsymbol{\sigma}_{-i}) + (1 - \sigma^i(\mathbf{p})) u_2(\mathbf{p}, \boldsymbol{\sigma}_{-i}).$$

Social Power and product differentiation.

Let us set up notation for the price difference,

$\Delta p \equiv p_1 - p_2$, (*price difference*),

and the differences of personal benefit and social weights, which characterize product differentiation, and are denoted by

$\Delta b \equiv b_1 - b_2$ (*standard product differentiation*);

$\Delta \alpha \equiv \alpha_1 - \alpha_2$ (*social product differentiation*).

Let us also denote the difference in pure strategy payoffs for consumer i by $\Delta u(\mathbf{p}, \boldsymbol{\sigma}_{-i}) \equiv u_1(\mathbf{p}, \boldsymbol{\sigma}_{-i}) - u_2(\mathbf{p}, \boldsymbol{\sigma}_{-i})$. The payoff of consumer i can be rewritten as $u^i(\mathbf{p}, \boldsymbol{\sigma}) = \sigma^i(\mathbf{p}) \Delta u(\mathbf{p}, \boldsymbol{\sigma}_{-i}) + u_2(\mathbf{p}, \boldsymbol{\sigma}_{-i})$. The choice of a behavior strategy by a given consumer i is determined by $\Delta u(\mathbf{p}, \boldsymbol{\sigma}_{-i})$. That is, if $\Delta u^i(\mathbf{p}, \boldsymbol{\sigma}_{-i}) > 0$ (resp. <0) consumer i's best response is firm 1 (resp.

firm 2), and consumer i is indifferent when $\Delta u^i(\mathbf{p}, \boldsymbol{\sigma}_{-i}) = 0$. This difference in pure strategy payoffs is

$$\Delta u(\mathbf{p}, \boldsymbol{\sigma}_{-i}) = -\Delta p + \Delta b + \underbrace{\alpha_1 \sum_{i' \neq i} \sigma^{i'} - \alpha_2 \sum_{i' \neq i} (1 - \sigma^{i'})}_{\text{social influence}}.$$

In a given outcome $(\mathbf{p}, \boldsymbol{\sigma})$, the *social influence* in the decision of consumer i is a weighted difference of the consumption behavior of others. This reveals the contextual nature of social product differentiation as opposed to the intrinsic nature of standard differentiation. On the other hand, social power is the potential influence a consumer has over another, i.e. the potential to change the difference in pure strategy payoffs (to alter the aforementioned relation). That is, social power of a consumer j over i, is the maximum effect j can produce on $\Delta u^i(\mathbf{p}, \boldsymbol{\sigma})$.[8] The social influence term can be rewritten as $(\alpha_1 + \alpha_2) \sum_{i' \neq i} \sigma^{i'} + \alpha_2(n-1)$. If a consumer $j \neq i$, changes from σ^j to $\sigma^j + \varepsilon$, leading the profile $\boldsymbol{\sigma}$ to become $\boldsymbol{\sigma'}$, then the influence change on i is given by $\Delta u(\mathbf{p}, \boldsymbol{\sigma}'_{-i}) = \Delta u(\mathbf{p}, \boldsymbol{\sigma}_{-i}) + \varepsilon(\alpha_1 + \alpha_2)$, for some $\varepsilon \in [-\sigma^j, 1 - \sigma^j]$. How much power is exerted by j with this change depends on the choice of ε. As such, in our approach, the ***social power*** of an individual (consumer) is

$$A \equiv \alpha_1 + \alpha_2.$$

The maximum influence that can be exerted upon a consumer is $A(n-1)$, which happens only when all other consumers change from one pure strategy to the other. As we are in an homogeneous case, with indistinguishable consumers, not only is product differentiation the same, but all consumers

[8]The issue of intention is beyond the scope of this work.

have the same social power. Influence, depends on the behavior strategy used by each consumer. The characterization of a consumers preference profile depends on standard product differentiation Δb, but on both α_1 and α_2, not only on their difference. Hence, the consumers preference profile is determined by the pair $(\Delta b, \boldsymbol{\alpha})$, where $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2)$.

Main results.

The equilibrium behavior drastically differs whether social power is positive or negative. Positive social power, $A > 0$, is associated with a type of conformity, bandwagon, or herd behavior, leading to type symmetries. Negative social power, $A < 0$, is usually associated to some type of congestion/snob/Veblen effect. The case $A = 0$ contains the original Bertrand framework.

We will show that, given a consumers preference profile $(\Delta b, \boldsymbol{\alpha})$, equilibria may be characterized in terms of social power A as follows:

(i) when $A \geq 0$, in a subgame-perfect equilibrium at least one firm has zero profit, i.e. an equilibrium is either a monopoly or the Bertrand zero profit equilibrium;

(ii) when $A < 0$, if consumers are *influenceable*, then non-monopolistic subgame-perfect equilibria exist, with positive profits for both firms.

The case where consumers have negative social power, case (ii), is the focus of this work. Let us discuss what we mean by influenceable in the case of negative social power. For a non-monopolistic equilibrium to exist we must guarantee that consumers *do* care about each other decisions, i.e. $|\Delta b|$ is not too high relative to social power. When products are standardly differentiated, $\Delta b \neq 0$, existence of a positive prices equilibrium does not follow without a limitation on the extent of asymmetry created by standard product differentiation (the magnitude of $|\Delta b|$). Namely, on how much one product or service is preferred over the other. We must thus impose consumers be *influenceable*, otherwise, they would just choose the firm with highest personal benefit, regardless of the choice of other consumers.

Let us make this precise. Define an upper and lower bound for standard product differentiation, respectively, $\underline{B} \equiv (A+\alpha_2)(n-1)$ and $\overline{B} \equiv -(\alpha_1 + A)(n-1)$. This defines an interval of size $-3A(n-1)$. Given a preference profile $(\Delta b, \boldsymbol{\alpha})$ we say that consumers are *negatively influenceable* if $A < 0$ and $\Delta b \in (\underline{B}, \overline{B})$.[9]

Theorem 1. *Every duopoly with homogeneous and negatively influenceable consumers has a pure price subgame-perfect equilibrium with positive profits for both firms. Furthermore, the unique symmetric credible consumers strategy is a focal equilibrium for consumers.*

The first part of the theorem establishes existence and has a straightforward important corollary: when $b_1 = b_2$ and $\alpha_1 = \alpha_2$, i.e. when there is no product differentiation, as $\overline{B} < 0$, $\overline{B} > 0$, and $\Delta b = 0$, negative social power $A < 0$ is sufficient to guarantee consumers are negatively influenceable.

[9]Note that this is a wide interval, which in particular will allow monopolies to be credible from the consumers game point of view. Nevertheless, firms will prefer to compete as that will produce higher profits. In terms of parameter interpretation from the consumers point of view, disregarding prices, it could be naturally reduced to $|A|(n-1)$. Here the choice of influenceability is made taking into account the game outcome.

Corollary 1. *Every undifferentiated duopoly with homogeneous consumers and negative social power has a pure price subgame-perfect equilibrium with positive profits for both firms.*

When both consumers and products (hence firms) are homogeneous, there is no a priori asymmetry, and, as such, it is not necessary to impose influenceability, because consumers are influenceable as long as $A \neq 0$ (there is no intrinsic preference for one of the products).

Once existence has been established, the question is whether prices and profit are the same. Note that equilibria with positive profits for both firms are not unique. The second part of the theorem refers to a comparison between several possible equilibria, where by symmetric we mean a strategy which is the same for all consumers. Let us develop this. Observe that a consumption profile induces a partition of the consumers set according to whether they use a pure or non-degenerate mixed strategy. We call the former *loyal* and the latter *non-loyal* consumers. Let us denote: the number of non-loyal consumers by $m(\mathbf{p}, \boldsymbol{\sigma}) \equiv \#\{i \in \mathcal{I} : 0 < \sigma^i(\mathbf{p}) < 1\}$; the number of consumers loyal to firm 1 by $l_1(\mathbf{p}, \boldsymbol{\sigma}) \equiv \#\{i \in \mathcal{I} : \sigma^i(\mathbf{p}) = 1\}$; and define $l_2(\mathbf{p}, \boldsymbol{\sigma})$ analogously. From here on we will omit the outcome dependence and use l_1, m and l_2. We call (l_1, m, l_2) a *loyalty characterization* of a given outcome. Note that, as the total number of consumers, n, is fixed and known, the knowledge of three coordinates is seemingly redundant. However, we will use it as a triplet because for equilibrium characterization the relevant information is the asymmetry of these three coordinates, rather than the number of consumers (which would in fact introduce a third coordinate anyway). Let us denote the difference in loyalty by $\Delta l \equiv l_1 - l_2$. We will show that in an equilibrium the price difference is determined by some loyalty characterization and given by

$$\Delta p^*(l_1, m, l_2; \Delta b, \boldsymbol{\alpha}) = \frac{1}{3}\left(2\Delta b + (n-1)\Delta\alpha + A\frac{\Delta l}{m}\right). \tag{1}$$

We observe that, according to Equation (1), the main driver of price asymmetries is social product differentiation. Furthermore, the effect of standard product differentiation is overcome even for small markets. Note also that, as social power is negative, increasing the number of loyal consumers for firm 1 ($\Delta l > 0$), decreases the equilibrium price difference towards firm 2.

The asymmetry in prices is dependent on three terms. This means that, when products are differentiated, an equilibrium may involve different prices even if the consumers strategy or the loyalty characterization is symmetric ($\Delta l = 0$). If there is no product differentiation equilibrium outcomes with positive profits may exist with a price difference proportional to a *relative loyalty ratio* ($\Delta l/m$) acording to social power A. The question remaining is whether asymmetric equilibria exist (i.e. whether asymmetric loyal consumption, $\Delta l \neq 0$, is credible) when $\Delta b = \Delta\alpha = 0$.

Theorem 2. *If $A < 0$, every undifferentiated duopoly has an asymmetric pure price subgame-perfect equilibrium with positive profits for both firms. Furthermore, in such an equilibrium firms have different prices and profits.*

These results show that neither heterogeneity nor product differentiation are necessary to resolve the classical Bertrand paradox, nor to achieve asymmetric pure price equilibria. In particular, no assumption on the distribution of consumer types is necessary.

2 Equilibrium demand and prices

2.1 Consumption stage equilibria

A consumption behavior profile $\boldsymbol{\sigma}(\mathbf{p})$ is *credible* for prices $\mathbf{p}^*$ if it is a second stage Nash equilibrium given the price pair $\mathbf{p}^*$. We say that a loyalty characterization (l_1^*, m^*, l_2^*) is credible for a price difference Δp^* if there is a consumption profile which is credible for $\mathbf{p}^*$ and has loyalty characterization $l_j(\mathbf{p}^*, \boldsymbol{\sigma}(\mathbf{p}^*)) = l_j^*$, for $j = 1, 2$ and $m(\mathbf{p}^*, \boldsymbol{\sigma}(\mathbf{p}^*)) = m^*$. The *price (difference) domain* $\mathcal{PD}(l_1, m, l_2; (\Delta b, \boldsymbol{\alpha}))$ of each loyalty characterization is the set of price diferences Δp for which (l_1, m, l_2) is credible when the consumers profile is $(\Delta b, \boldsymbol{\alpha})$. This will be abbreviated to $\mathcal{PD}(l_1, m, l_2)$.

The characterization of credible demand reduces to that of consumption stage Nash equilibria. The following threshold characterization is adapted from [29], where the full characterization of Nash equilibria is done for a one stage game with two types of players and two possible actions. The results there apply here as a reduction to the one-dimensional case of the second stage subgame (one type of players: homogeneous consumers). Here we include prices in the parameter space of the consumers subgame, and we will characterize the price differences for which a given behavior strategy is a Nash equilibrium.

Let us define the following *decision threshold* function,

$$T(l_1) \equiv T(l_1; (\Delta b, \boldsymbol{\alpha})) \equiv \Delta b + \alpha_1(l_1 - 1) - \alpha_2(n - l_1). \tag{2}$$

Note that the term $\alpha_1(l_1 - 1) - \alpha_2(n - l_1) = A(l_1 - 1) - \alpha_2(n - 1)$ reflects the aforementioned contextual nature of social product differentiation, which is then added to standard product differentiation to produce a price threshold for a consumption decision. The interpretation is a limit for the price difference, under which a set of strategies is still credible.

Strictly loyal consumption. Let $A \neq 0$, and let us start with the cases where there are only loyal consumers (i.e. $m = 0$), which form the base tiles of price domains. For non-monopolistic loyalty characterizations $(l_1, l_2 \neq n)$ there are right and left thresholds for a price difference increase/ decrease before a consumer is lost/ gain. Naturally, for monopolistic demand there is only an upper limit in price before a consumer is lost.

(i) $\mathcal{PD}(n, 0, 0) = (-\infty, T(n)]$;

(ii) $\mathcal{PD}(l_1, 0, n - l_1) = [T(l_1 + 1), T(l_1)]$, for $l_1 \in \{1, \ldots, n - 1\}$.

(iii) $\mathcal{PD}(0, 0, n) = [T(1), +\infty)$;[10]

Observe that $T(l_1 + k) = T(l_1) + kA$. When $A < 0$, the price domains of non-monopolistic characterizations are contained in the interval $(T(n), T(1))$ which is partitioned in $n - 1$ intervals of size $|A|$ by the set of thresholds $\{T(n), \ldots, T(1)\}$, and, as $T(l_1 + 1) < T(l_1)$, all these price domains are non-empty (or non-degenerate). When $A > 0$, as $T(n) > T(1)$ (and in fact $T(l_1 + 1) > T(l_1)$ for all l_1), the domains of non-monopolistic loyalty characterizations with $m = 0$ are empty. However, there is an interval where both monopolies coexist as credible outcomes, in this case $[T(1), T(n)]$.

[10]Note that the threshold is constructed in terms of firm 1 for simplicity. After loosing all consumers, there's no limit to price, thus $T(0)$ has no meaning. We observe also that there is no problem with infinite prices, since firms are not allowed to collude, and will have the incentive to deviate from a high price of the other firm.

Non-loyal consumption. Loyalty characterizations with non-loyal consumers connect two purely loyal equilibria. A loyalty characterization (l_1, m, l_2) with $m \geq 1$ connects $(l_1 + m, 0, l_2)$ to $(l_1, 0, l_2 + m)$, and the conection exists in the case the latter two are credible.

With positive social power, $A > 0$, consumers are either all loyal or all non-loyal, that is, all credible consumption outcomes are type-symmetric, a conformity effect (see [28, 29]). In the unique credible case, $m = n$, we have $\mathcal{PD}(0, n, 0) = (T(1), T(n))$.

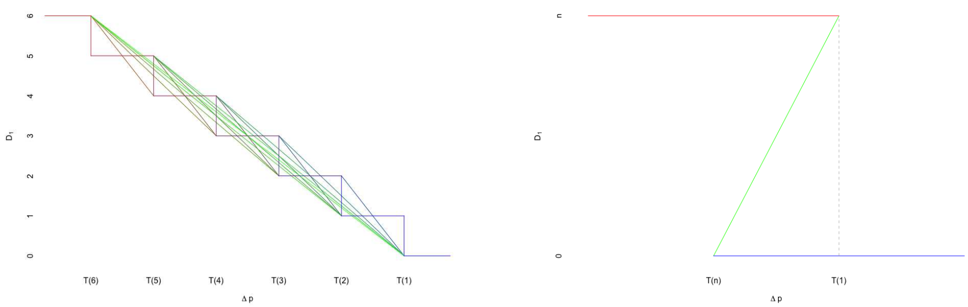
When $A < 0$, the price domain of characterizations with $m \geq 1$ is $\mathcal{PD}(l_1, m, l_2) = (T(l_1 + m), T(l_1 + 1))$. When $m = 1$ the domain reduces to a point, $\Delta p = T(l_1 + 1)$, and any value of $\sigma^i \in (0, 1)$ is credible for that consumer.

Following [29], in a Nash equilibrium, if $A \neq 0$, then for all $i, j \in \mathcal{I}$ with $0 < \sigma^i, \sigma^j < 1$ we must have $\sigma^i = \sigma^j$ (non-degenerate behavior strategies are type-symmetric when $A \neq 0$). This does not mean that a priori there is a unique credible demand for each pair of prices. Nevertheless, given a (fixed) loyalty characterization, if $m > 1$, demand is either credible and uniquely determined by the price difference, or non-credible for price differences outside its domain. With this, each strategy class (l_1, m, l_2) with $m > 1$ has associated to each price difference in its domain $\mathcal{PD}(l_1, m, l_2)$ a unique credible non-integer probability[11], given by

$$q(\Delta p; l_1, m, l_2) \equiv q(\Delta p; l_1, m, l_2; (\Delta b, \boldsymbol{\alpha})) = \frac{T(l_1 + 1) - \Delta p}{-A(m - 1)}. \qquad (3)$$

Therefore, there is a unique credible demand (for firm 1) which *preserves*

[11]If two consumers use different strategies, it leads to $\Delta u^i \neq \Delta u^j$ and they could not both be 0, hence one of them must use a pure strategy (see [29]).



(a) $A < 0$.

(b) $A > 0$.

Figure 1: A depiction of the fiber space of credible demand. On the left we consider $n = 6$ for ease of visualization. We have used RGB colors in the range $[0, 1]$ in the following way $(l_1/n, m/n, l_2/n)$.

the loyalty characterization in its credible price domain, given by the function $l_1 + mq(\Delta p; l_1, m, l_2)$. We observe that, in the respective price domain $0 < q(\Delta p; l_1, m, l_2) < 1$. This is the credible behavior strategy of non-loyal consumers; the probability of choosing firm 1.

Summing up, the characterization of consumption equilibria can be done completely using the partition into strategy classes according to loyalty. The set of second stage Nash equilibria is a fiber space representing the possibilities of credible demand for each price difference Δp. This fiber space is formed by the union of horizontal ($m = 0$), vertical ($m = 1$) and oblique ($m > 1$) line segments, each with a particular price domain (see Figure 1). Each segment (fiber) is identified by (l_1, m, l_2). Given Δp, a loyalty characterization (l_1, m, l_2) determines a unique point in this fiber space, i.e. it determines a unique value for credible demand at Δp.

2.2 Non-monopolistic outcomes

With the characterization of credible demand, we are now able to characterize the candidates to subgame-perfect outcomes positive profits for both firms.

Lemma 1. *Given a consumers preference profile* $(\Delta b, \boldsymbol{\alpha})$*, a strategy profile* $(\mathbf{p}^*, \boldsymbol{\sigma}^*(\mathbf{p}))$ *is a subgame-perfect equilibrium with positive profits for both firms only if there is a loyalty characterization* (l_1, m, l_2) *with* $m > 1$ *and a neighbourhood* $V(\Delta p^*) \subset \mathcal{PD}(l_1, m, l_2)$ *where for all* $\mathbf{p}$ *such that* $p_1 - p_2^* \in V(\Delta p^*)$ *and* $p_1^* - p_2 \in V(\Delta p^*)$*, we have*

$$D_1(p_1, p_2^*, \boldsymbol{\sigma}^*(\mathbf{p})) = l_1 + mq(p_1 - p_2^*; l_1, m, l_2);$$

$$D_2(p_1^*, p_2, \boldsymbol{\sigma}^*(\mathbf{p})) = l_2 + m(1 - q(p_1^* - p_2; l_1, m, l_2)).$$

Observe that: (i) the result does not apply to equilibria where at least one firm is with price equal to zero, i.e. these equilibria (both the Bertrand zero profit and monopolies) may exist and are not included in this Lemma, which is, however, sufficient for the purpose of this work; (ii) a consequence of Lemma 1 is that when $A > 0$ equilibria with positive profits for both firms do not exist. When $A > 0$ there is unique possibility, $(0, n, 0)$, in which $q_j(\Delta p)$ increases with price, thus it cannot be part of a subgame-perfect equilibrium with positive prices for both firms.

Remark 1. *When* $A > 0$*, in a subgame-perfect equilibrium at least one firm has zero profits.*

Consider the case $A < 0$. Recall the interval $(\underline{B}, \overline{B})$ referred to in Section 1 (consumers are *influenceable* if $\Delta b \in (\underline{B}, \overline{B})$), where $\underline{B} \equiv (\alpha_1 + 2\alpha_2)(n-1)$

and $\overline{B} \equiv -(2\alpha_1 + \alpha_2)(n-1)$. Given a loyalty characterization, let us now define *standard product differentiation thresholds*,

$$B_L(l_1, m, l_2; \boldsymbol{\alpha}) \equiv \underline{B} + |A| \left(3l_1 - \frac{\Delta l}{m}\right),$$

$$B_R(l_1, m, l_2; \boldsymbol{\alpha}) \equiv \overline{B} - |A| \left(3l_2 + \frac{\Delta l}{m}\right).$$

The *standard differentiation domain* (personal Benefit Domain), is

$$\mathcal{BD}(l_1, m, l_2; \boldsymbol{\alpha}) \equiv (B_L(l_1, m, l_2; \boldsymbol{\alpha}), B_R(l_1, m, l_2; \boldsymbol{\alpha})).$$

The size of the domain is $3|A|(m-1)$. Note that for all loyalty characterizations, we have $\mathcal{BD}(l_1, m, l_2; \boldsymbol{\alpha}) \subseteq (\underline{B}, \overline{B})$.

Lemma 2. *Consider a consumers preference profile* $(\Delta b, \boldsymbol{\alpha})$. *A strategy profile* $(\mathbf{p}^*, \boldsymbol{\sigma}^*(\mathbf{p}))$ *is a subgame-perfect equilibrium with positive profits for both firms, only if,* $A < 0$*, there is a loyalty characterization* (l_1, m, l_2) *with* $m > 1$ *such that* $\Delta b \in \mathcal{BD}(l_1, m, l_2; \boldsymbol{\alpha})$*, and outcome prices are*

$$p_1^* = \frac{1}{3}\left(\Delta b - \underline{B} - |A| \left(\frac{2l_1 + l_2}{m}\right)\right);$$

$$p_2^* = \frac{1}{3}\left(\overline{B} - \Delta b - |A| \left(\frac{l_1 + 2l_2}{m}\right)\right).$$

This Lemma determines, for each loyalty characterization (l_1, m, l_2), an affine correspondence between standard product differentiation Δb and outcome prices. Thus, for a given consumer preference profile $(\Delta b, \boldsymbol{\alpha})$ it defines outcome price functions $p_j^*(l_1, m, l_2; (\Delta b, \boldsymbol{\alpha}))$ for $j = 1, 2$ and an outcome price difference function $\Delta p^*(l_1, m, l_2; (\Delta b, \boldsymbol{\alpha}))$ (which was presented in Equation 1 appearing on Section 1). Furthermore, substituting to obtain

$D_1^* \equiv D_1(\mathbf{p}^*, \boldsymbol{\sigma}^*(\mathbf{p}^*))$ by using Lemma 1 together with Lemma 2, outcome demand is implicitly determined, (recall that $D_2^* = n - D_1^*$),

$$D_1^* = D_1^*(l_1, m, l_2; (\Delta b, \boldsymbol{\alpha})) = l_1 + m\frac{\Delta b - B_L(l_1, m, l_2)}{3|A|(m-1)}. \tag{4}$$

In turn, these determine profits for these outcomes

$$\Pi_j^*(l_1, m, l_2; (\Delta b, \boldsymbol{\alpha})) = p_j^*(l_1, m, l_2; (\Delta b, \boldsymbol{\alpha}))D_j^*(l_1, m, l_2; (\Delta b, \boldsymbol{\alpha})).$$

Analyzing the three terms in the expression for $p_1^*(l_1, m, l_2; (\Delta b, \boldsymbol{\alpha}))$ given by Lemma 2, we see that, the first term, Δb, reflects the straightforward intuition that equilibrium prices increase if the relative benefit consumers derive from using that service increases. In fact, this is so, proportionally to the distance to the lower bound $\underline{B}$. This second term, $\underline{B}$, is not as straightforward, seen that it already includes social power, $A = \alpha_1 + \alpha_2$: as $A < 0$, we get that $-\underline{B} = -(A + \alpha_2)(n-1)$ is always positive. Equilibrium price of firm 1 thus increases proportionally to the number of consumers according to the magnitude of social power added/ discounted by how much consumers 'like/ don't like to be together' while using service provided by firm 2 (firm 1 charges higher prices when consumers don't like to be together while using service from firm 2, $\alpha_2 < 0$). More interesting is the third term $-|A|(2l_1 + l_2)/m$. Prices decrease the more a firm transforms consumers from non-loyal to loyal (note that n is fixed). So equilibrium prices actually decrease with an increase of the number of loyal consumers. On the other side, this term is smaller when there are more non-loyal consumers, hence prices are higher. We will abbreviate the expressions for outcome prices, demand and profits by omitting the dependence on $(\Delta b, \boldsymbol{\alpha})$, which is exogeneously given.

3 Existence

Given some consumers preferences $(\Delta b, \boldsymbol{\alpha})$ consider the set of strategy profiles $(\mathbf{p}^*, \boldsymbol{\sigma}^*(\mathbf{p}))$ which satisfy both Lemma 1 and 2, i.e. a set of *admissible strategies*, denoted $\mathbf{AS}(\Delta b, \boldsymbol{\alpha})$. Given $(\Delta b, \boldsymbol{\alpha})$, an admissible strategy is a strategy profile of the game, $(\mathbf{p}^*, \boldsymbol{\sigma}^*(\mathbf{p}))$, for which there is a loyalty characterization (l_1, m, l_2) with $m > 1$ such that $\mathbf{p}^* = (p_1^*(l_1, m, l_2), p_2^*(l_1, m, l_2))$ and such that demand based on consumption behavior $\boldsymbol{\sigma}^*(\mathbf{p})$ restricted to some neighborhood of Δp^* is determined by $q(\Delta p; l_1, m, l_2)$. A subgame-perfect equilibrium with a positive profits outcome for both firms must be an admissible strategy. We observe that, although associated to each loyalty characterization (l_1, m, l_2) is a unique pair of outcome prices and demand, this does not mean there is a unique strategy leading up to that outcome. However, because the loyalty characterization uniquely determines prices and demand in their neighborhood, the possible loyalty characterizations with $m > 1$ induce an equivalence relation in $\mathbf{AS}(\Delta b, \boldsymbol{\alpha})$ with respect to outcome demand and prices. We denote the strategy classes in this equivalence relation by $\underline{(l_1, m, l_2)}$. We are interested in studying and distinguishing equilibria up to these classes. All strategies within the same class produce the same outcome prices and demand. Note, however, this does not mean there cannot be equilibria from different classes with the same prices! What happens is these have different outcome demand (thus consumer behavior).

In each class $\underline{(l_1, m, l_2)}$ there is a subclass of strategies in which the consumer behavior preserves the loyalty characterization in its whole price domain, $\mathcal{PD}(l_1, m, l_2) = (T(l_1 + m), T(l_1 + 1))$. This strategy induces, in the respective price domain, continuous and linear demand (a segment in the

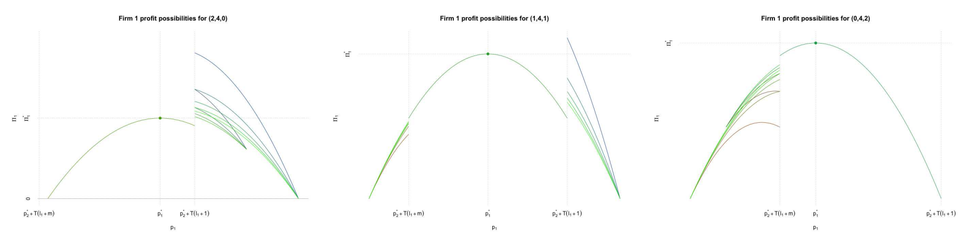
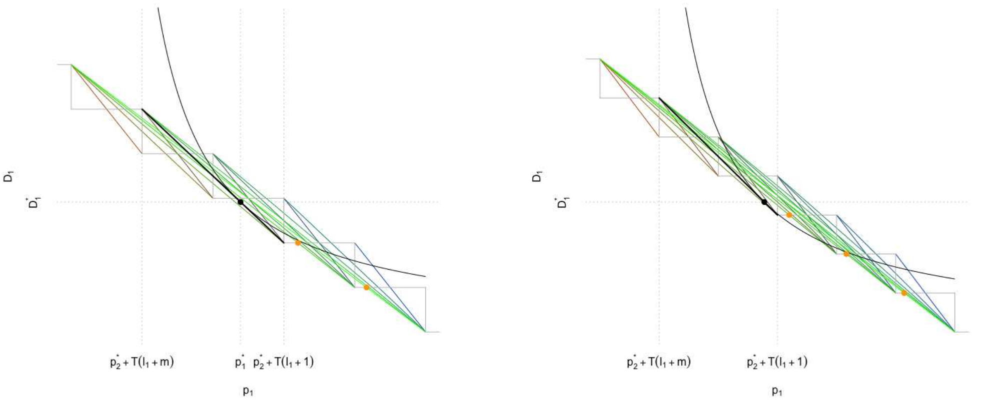


Figure 2: The possible profit functions for firm 1 induced, respectively, by loyalty preserving strategies in $\underline{(2,4,0)}$, $\underline{(1,4,1)}$, and $\underline{(0,4,2)}$ in the case $\Delta b = 0$, $\alpha_1 = \alpha_2 = -\frac{1}{2}$ with $n = 6$ consumers. We use RGB colors as before.

fiber) determined by (l_1, m, l_2) and $q(\Delta p; l_1, m, l_2)$. In particular, for these strategies there are no incentives for firms to deviate within the price domain. Moreover, the price difference interval for which there are second stage equilibria with non-monopolistic demand outcomes is $[T(n), T(1)]$. For price differences outside this interval, demand is well defined: when $\Delta p < T(n)$ the unique credible demand is $D_1 = n$ and for $\Delta p > T(1)$ it is $D_1 = 0$. As such, the main issue for existence of an equilibrium in the class $\underline{(l_1, m, l_2)}$ is how consumers behave in the intervals $(T(n), T(l_1+m))$ and $(T(l_1+1), T(1))$, i.e. what is their behavior when the loyalty characterization is no longer credible but multiple second stage equilibria exist? How is profit outside this domain?

The idea behind existence of an equilibrium with characterization (l_1, m, l_2) is whether the equilibrium price difference $\Delta p^*(l_1, m, l_2)$ is sufficiently away from its price domain's boundaries $(T(l_1+m), T(l_1+1))$. As this is the domain of credibility for (l_1, m, l_2), when Δp^* is too close to one of the thresholds, one of the firms may have an incentive to force a change of loyalty characterization with a small price deviation, which is either a jump or non-differentiable point in demand. In terms of firm 1, this means the price p_1^*



(a) $\underline{(2,3,1)}$ satisfies condition (i). (b) $\underline{(3,3,1)}$ does not satisfy condition (i).

Figure 3: Ilustration of Lemma 3, $\Delta b = 0$, $\alpha_1 = \alpha_2 = -\frac{1}{2}$. Marked in orange are the respective points $P_1(k;(l_1,m,l_2))$.

found in Lemma 2 is only guaranteed as a profit maximum in the interval between $p_2^*+T(l_1+m)$ and $p_2^*+T(l_1+1)$ and three situations may occur: the outcome may be an equilibrium independently of the continuation chosen, it may depend on that choice, or it may not be an equilibrium independently of that choice. An example of the three possible situations is depicted in Figure 2, referring to the undifferentiated case.

Consider a consumers preference profile $(\Delta b, \boldsymbol{\alpha})$. Given a loyalty characterization (l_1,m,l_2), let us define for some $k \in \{0, n-3\}$ the following

$$P_1(k;(l_1,m,l_2)) \equiv p_2^*(l_1,m,l_2) + T(k+2) + \frac{-A}{n-k};$$

$$P_2(k;(l_1,m,l_2)) \equiv p_1^*(l_1,m,l_2) - T(n-k-1) + \frac{-A}{n-k}.$$

Lemma 3. *Consider a consumers preference profile* $(\Delta b, \boldsymbol{\alpha})$*. A subgame-perfect equilibrium belonging to a* $\underline{(l_1, m, l_2)}$ *class with* $m > 1$ *exists if, and only if,* $A < 0$*,* $\Delta b \in \mathcal{BD}(l_1, m, l_2; \boldsymbol{\alpha})$*, and*

(i) when $l_1 > 0$*, for every* $k \in \{0, \ldots, l_1 - 1\}$*,*

$$\Pi_1^*(l_1, m, l_2) \geq (k+1)P_1(k; (l_1, m, l_2));$$

(ii) when $l_2 > 0$*, for every* $k \in \{0, \ldots, l_2 - 1\}$*,*

$$\Pi_2^*(l_1, m, l_2) \geq (k+1)P_2(k; (l_1, m, l_2)).$$

Interestingly, the only actual deviation incentives for firms are an increase in price, maintaining or even reducing the number of loyal consumers (hence the variable k), and possibly capturing more non-loyal consumers. Increasing the number of loyal consumers decreases equilibrium prices. In the right hand side of the inequalities in this Lemma are possible profits for credible consumer strategies with demand adding up to $k+1$, credible at $p_1 = P_1$. Figure 3 contains an ilustration of this Lemma.

Proof outline. We start by showing that credible demand is bounded below by a piecewise linear (credible demand) function. The proof then follows by establishing conditions under which the isoprofit function is above. □

Recall that, given a consumers preference profile with $A < 0$, for all (l_1, m, l_2), $\mathcal{BD}(l_1, m, l_2; \boldsymbol{\alpha}) \subseteq (\underline{B}, \overline{B})$ and that $\mathcal{BD}(0, n, 0; \boldsymbol{\alpha}) = (\underline{B}, \overline{B})$.

Theorem 1. *Consider a consumers preference profile* $(\Delta b, \boldsymbol{\alpha})$ *with* $A < 0$*. If* $\Delta b \in (\underline{B}, \overline{B})$*, a subgame-perfect equilibrium belonging to the strategy class* $\underline{(0, n, 0)}$ *exists.*

Proof. Observe that if $\Delta b \in (\underline{B}, \overline{B})$, the class $\underline{(0, n, 0)}$ trivially satisfies Lemma 3. □

We have taken the liberty to rewrite Theorem 1 (which was presented in Section 1) with the terminology and concepts introduced in these latter sections, but the underlying result is exactly the same: it establishes existence by construction of an equilibrium. Outcome demand corresponds to the case where all consumers are non-loyal.

A corollary of Theorem 1, also presented in Section 1, is that in the case products are not differentiated, existence is guaranteed when $A < 0$.

Corollary 1. *Every undifferentiated duopoly with homogeneous consumers and negative social power has a pure price subgame-perfect equilibrium with positive profits for both firms.*

The question of whether asymmetric pure price equilibria exist is particularly troublesome in this case. We recall that the outcome price difference comprises three components which drive the price asymmetry effect,

$$3\Delta p^*(l_1, m, l_2) = \underbrace{2\Delta b + (n-1)\Delta\alpha}_{\substack{\text{product}\\ \text{differentiation}\\ \text{effect}}} + \underbrace{A\frac{\Delta l}{m}}_{\substack{\text{loyalty}\\ \text{behavior}\\ \text{effect}}} .$$

In the case with no product differentiation, the difference in price relies on Δl. As such, the existence based on Theorem 1 does not produce different prices. Therefore, we want to prove that at least an equilibrium with asymmetric loyal consumption exists. This is done in the next theorem.

Theorem 2. *If $A < 0$, every undifferentiated duopoly has an asymmetric pure price subgame-perfect equilibrium with positive profits for both firms. Furthermore, in such an equilibrium firms have different prices and profits.*

4 Concluding remarks

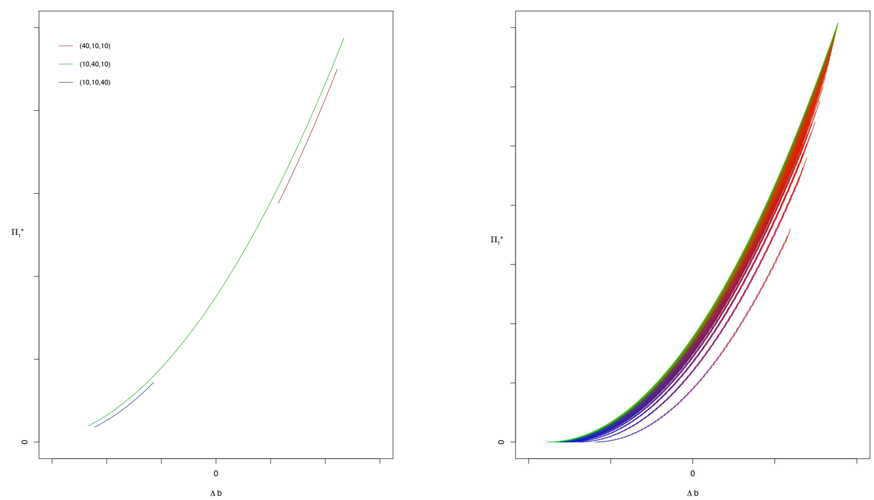
We have shown that when consumers have negative social power, a pure price subgame-perfect equilibrium with positive profits for both firms exists, and even in the case products are not differentiated, asymmetric pure price equilibria always exist. There are some natural extensions to the model, namely: (i) considering consumer heterogeneity by allowing for more than one atom in the distribution of types; (ii) extending the duopoly to an oligopoly; and/ or (iii) removing mandatory consumption and introducing a reservation price. In fact, following the work in [7], one can use social power as another alternative microeconomic foundation to deduce the general form linear demand,

$$D_1(p_1, p_2) = \lambda - \beta p_1 + \gamma p_2,$$

where λ, β, γ are given, in this case, for each class $\underline{(l_1, m, l_2)}$, by

$$\lambda = l_1 + \frac{m}{-A(m-1)} T(l_1 + 1) > 0 \quad \text{and} \quad \beta = \gamma = \frac{m}{-A(m-1)} > 0.$$

Note that own effects do not dominate cross effects due to mandatory consumption forcing $D_1 = n - D_2$. That can also be relaxed by introducing in the action space of consumers a third option, without creating any particular obstacle. Nevertheless, these extensions, although being with no doubt interesting for further study, do not contribute or add to the point in question here. The main idea of smoothing demand by allowing consumers to use behavior strategies in an interval, which is created by network effects, is in itself, a general idea. If one intends to study price competition with restored demand continuity, possibly even coupled with other strategic variables, this approach to stabilize pure price solutions lends itself to fairly straightforward incorporation into any scenario, or model.



(a) Three particular characterizations (l_1, m, l_2).

(b) All possible loyalty characterizations.

Figure 4: Equilibrium outcome profits of firm 1 in a case with 60 consumers. We have set $\alpha_1 = \alpha_2 = -1$, and on the left are depicted equilibrium profits in terms of Δb for the classes $\underline{(40, 10, 10)}$, $\underline{(10, 40, 10)}$, $\underline{(10, 10, 40)}$. On the right, all possible classes (loyalty characterizations) with $m > 1$. We use RGB colors as before. We observe that the greener lines (equilibria with higher number of non-loyal consumers), have a larger domain and produce higher profits. Naturally, equilibria with higher number of loyal consumers for one firm exist in domains where that firm is preferred.

A Proofs

Let us define for ease of notation the function

$$Q_1(\Delta p; l_1, m, l_2) \equiv l_1 + mq(\Delta p; l_1, m, l_2).$$

Lemma 1

Proof. Let $(\mathbf{p}^*, \boldsymbol{\sigma}^*(\mathbf{p}))$ be a subgame-perfect equilibrium with positive profits for both firms, hence, where $p_1^*, p_2^* \neq 0$. We will do the proof in terms of firm 1, firm 2 is analogous. Note that whether demand is credible or not for prices $\mathbf{p}$ depends only on Δp, and not on the particular prices being charged. Thus we can write $D_1(\Delta p, \boldsymbol{\sigma}^*)$. Let us denote equilibrium outcome demand by $D_1^* \equiv D_1(\Delta p^*, \boldsymbol{\sigma}^*)$. Because D_1^* is a result of profit maximization for both firms, it must lie at the intersection of isoprofit functions, and be tangent to both. In particular, as both firms can change their price in both directions, D_1^* must have the same right and left limit (jumps provide incentive to price deviations), and the same slope to the right and to the left (else demand crosses at least one of the isoprofits). Recall that credible demand is contained in a fiber space formed by the union of horizontal ($m = 0$), vertical ($m = 1$) and oblique ($m > 1$) line segments. Having the same right and left limit means that either D_1^* is located at a point whose neighborhood has constant loyalty characterization (demand stays in the same segment), or at an intersection point (if it changes segment). If demand stays in the same segment for an interval containing Δp^*, then $m > 1$ (because for $m = 1$ the domain is a single point and if $m = 0$ a change in price would produce no change in demand (horizontal line) and there would be an incentive), which means demand is on an oblique segment, thus given by $Q_1(\Delta p; l_1, m, l_2)$. So we are left with the case where D_1^* lies at an intersection point, possibly leading to a change of loyalty characterization. If $A > 0$ there is a unique oblique line $m = n$. If $A < 0$, as the slope of each line is (completely) determined by m (for obliques it is $\frac{m}{A(m-1)}$) and oblique lines with the same m start at $T(l_1 + m)$, thus depending on l_1, different loyalty characterizations (line segments) which intersect, have different slopes (note that $l_1 + m = n - l_2$ and n is fixed). Furthermore, the number of possible loyalty characterizations is finite, hence, all the intersection points of these segments are isolated, that is, there is as neighborhood (of Δp), for which they are unique. As such there is a neighborhood of Δp^* for which, at most, there is a unique change of demand's loyalty characterization, in this case happening at Δp^*. In both cases $A > 0$ and $A < 0$, seen that at D_1^* the right and left slope must be the same, this means that the loyalty characterization on the left and right of Δp^* must be the same. Therefore, there must be an interval containing Δp^* for which the loyalty characterization is constant, except possibly at Δp^*. But if Δp^* is an intersection point, demand is nevertheless given by $Q_1(\Delta p; l_1, m, l_2)$. In particular, this means the equilibrium must lie at an interior point of the price domain, as at the boundary there must be a change to a different loyalty characterization (the previous is no longer credible). $\square$

Lemma 2

Proof. Let $(\mathbf{p}^*, \boldsymbol{\sigma}^*(\mathbf{p}))$ be a subgame-perfect equilibrium with positive profits for both firms. For each loyalty characterization consider the function $F_1 \equiv F_1(l_1, m, l_2) : (p_2^* + T(l_1 + m), p_2^* + T(l_1 + 1)) \to \mathbb{R}$, given by $F_1(p_1) =$

$p_1Q_1(p_1, p_2^*; l_1, m, l_2)$. According to Lemma 1 there is (l_1, m, l_2) with $m > 1$ and a neighbourhood $V(\Delta p^*) \in (T(l_1+m), T(l_1+1))$ such that, for $p_1 - p_2^* \in V(\Delta p^*)$ we have $\Pi_1(p_1, p_2^*, \boldsymbol{\sigma}^*(p_1, p_2^*)) = F_1(p_1)$. Using the expression for Q_1 (see equation 3) we get

$$F_1(p_1) = \frac{m}{A(m-1)}p_1^2 + \left(l_1 + m\left(\frac{p_2^* + T(l_1+1)}{-A(m-1)}\right)\right)p_1.$$

Note that as (l_1, m, l_2) is credible we have $\Delta p^* \in (T(l_1+m), T(l_1+1))$, hence $p_1^* < p_2^* + T(l_1+1)$, and as $p_1^* > 0$ by assumption, so is $p_2^* + T(l_1+1)$. A first question is whether the underlying quadratic function maximum is interior to the domain of F_1 or if it is outside, thus making the maximum of F_1 be at the boundary of its domain. Suppose the maximum of F_1 is p^M at the boundary. Then $p^M - p_2^* = \Delta p$ would not be an interior point of $(T(l_1+m), T(l_1+1))$, and according to Lemma 1, $(\mathbf{p}^*, \boldsymbol{\sigma}^*(\mathbf{p}))$ cannot be a subgame-perfect equilibrium. So the maximum of F_1 must be interior. As such it is given (using the first order condition) by

$$P_1(p_2^*) = \frac{p_2^* + T(l_1+1)}{2} + \frac{-A(m-1)l_1}{2m}.$$

Analougously we get

$$P_2(p_1) = \frac{p_1 - T(l_1+m)}{2} + \frac{-A(m-1)l_2}{2m}.$$

Noting that $T(l_1+m) = \Delta b + A(l_1+m-1) - \alpha_2(n-1)$ and that $T(l_1+1) = \Delta b + Al_1 - \alpha_2(n-1)$ (see equation 2) we get that

$$P_1(P_2(p_1)) = \frac{p_1}{4} + \frac{2T(l_1+1) - T(l_1+m)}{4} + \frac{-A(m-1)}{4m}(l_2 + 2l_1),$$

$$P_1(P_2(p_1)) = \frac{p_1}{4} + \frac{\Delta b - \alpha_2(n-1) + Al_1 - A(m-1)}{4} + \frac{-A(m-1)}{4m}(l_2 + 2l_1).$$

The fixed point, $p_1^* = P_1(P_2(p_1^*))$ is

$$p_1^* = \frac{\Delta b - \alpha_2(n-1) + Al_1 - A(m-1)}{3} + \frac{-A(m-1)}{3m}(l_2 + 2l_1),$$

noting that $l_1 + l_2 = n - m$, we get

$$p_1^* = \frac{\Delta b - \alpha_2(n-1) - A(m-1) - A(n-m)}{3} + \frac{A(l_2 + 2l_1)}{3m},$$

$$p_1^* = \frac{\Delta b - (A + \alpha_2)(n-1)}{3} + \frac{A(l_2 + 2l_1)}{3m},$$

thus leading to

$$p_1^* = \frac{1}{3}\left(\Delta b - \underline{B} - |A|\left(\frac{2l_1 + l_2}{m}\right)\right).$$

From $P_2(p_1^*)$ we get p_2^*. Now, from both expressions we get

$$\Delta p^* = \frac{1}{3}\left(2\Delta b + (n-1)\Delta\alpha + A\frac{\Delta l}{m}\right).$$

As we have seen, for the consumers behavior to be credible, we must have $\Delta p^* \in (T(l_1+m), T(l_1+1))$ leading to an upper and lower bound for Δb and the remaining to be proved necessary condition $\Delta b \in \mathcal{BD}(l_1, m, l_2; \boldsymbol{\alpha})$. The upper bound is determined by $\Delta p^* > T(l_1 + m)$, leading to

$$\frac{1}{3}\left(2\Delta b + (n-1)\Delta\alpha + A\frac{\Delta l}{m}\right) > \Delta b + A(l_1 + m - 1) - \alpha_2(n-1),$$

$$\Delta b < (n-1)\alpha_1 + A\frac{\Delta l}{m} - 3A(l_1 + m - 1) + 2\alpha_2(n-1).$$

Recalling that $\underline{B} = (\alpha_1 + 2\alpha_2)(n-1)$ and $m = n - l_1 - l_2$ we get

$$\Delta b < \underline{B} - 3A(n-1) + 3Al_2 + A\frac{\Delta l}{m},$$

thus

$$\Delta b < \overline{B} + A\left(3l_2 + \frac{\Delta l}{m}\right).$$

Similarly, the lower bound is

$$\frac{1}{3}\left(2\Delta b + (n-1)\Delta\alpha + A\frac{\Delta l}{m}\right) < \Delta b + Al_1 - \alpha_2(n-1),$$

thus

$$\Delta b > \underline{B} - A\left(3l_1 - \frac{\Delta l}{m}\right).$$

To conclude the proof we will show that this guarantees positive prices. Note that this can be rewritten as

$$\Delta b - \underline{B} > -A\left(\frac{(3m-1)l_1 + l_2}{m}\right).$$

which for $m > 1$ implies that

$$\Delta b - \underline{B} > -A\left(\frac{2l_1 + l_2}{m}\right),$$

hence $p_1^* > 0$. Analougously, we can use the upper bound for the price of firm 2. So, $\Delta b \in \mathcal{BD}(l_1, m, l_2; \boldsymbol{\alpha})$ in particular guarantees that $p_1^* > 0$ and $p_2^* > 0$. □

Lemma 3

Proof. Let $A < 0$, $\Delta b \in \mathcal{BD}(l_1, m, l_2)$ and consider an outcome with loyalty characterization (l_1^*, m^*, l_2^*) and prices $\mathbf{p}^* = (p_1^*(l_1, m, l_2), p_2^*(l_1, m, l_2))$. Our aim is to find sufficient and necessary conditions for existence of a strategy profile $(\mathbf{p}^*, \boldsymbol{\sigma}^*(\mathbf{p}))$ which is a subgame-perfect equilibrium and produces the above mentioned outcome.

We will do the analysis in terms of firm 1, firm 2 is analogous. It is helpful to keep in mind Figure 3 throughout the proof. Recall that credible non-monopolistic demand exists only for $p_1 \in [p_2^* + T(n), p_2^* + T(1)]$, an interval which is partioned in $n-1$ blocks of size A by the thresholds for each $l_1 \in \{1, 2, \dots, n\}$. At each threshold point, say $p_1 = p_2^* + T(l_1)$, the minimum and maximum value for credible demand are, respectively, $l_1 - 1$ and l_1. As such credible demand is bounded above by the line $\overline{d}_1(p_1) \equiv n + \frac{p_1 - (p_2^* + T(n))}{A}$. Let us denote $Q_1^*(p_1) \equiv Q_1(p_1 - p_2^*; l_1^*, m^*, l_2^*)$. Recall that $D_1^* = Q_1(p_1^*)$, and that a credible strategy exists for which $D_1(p_1, p_2^*) = Q_1^*(p_1)$ in the whole domain $(p_2^* + T(l_1^* + m^*), p_2^* + T(l_1^* + 1))$, which in particular preserves (l_1^*, m^*, l_2^*). The isoprofit demand curve for firm 1, $h_1(p_1; (\mathbf{p}^*, \boldsymbol{\sigma}^*(\mathbf{p}^*))) = \frac{p_1^* D_1^*}{p_1}$, which we abbreviate to $h_1(p_1)$, is tangent to $Q_1^*(p_1)$ at p_1^*. (Note that Q_1^*, h_1 and $\overline{d}_1$ have negative slopes.) As Q_1^* is linear, the question is thus, are there strategies for price deviations outside $(p_2^* + T(l_1^* + m^*), p_2^* + T(l_1^* + 1))$ which lead to demand continuations of Q_1^* below h_1?

Let us start with the case $p_1 \leq p_2^* + T(l_1^* + m^*)$. As $Q_1^*(p_2^* + T(l_1^* + m^*)) = \overline{d}_1(p_2^* + T(l_1^* + m^*))$, h_1 crosses $\overline{d}_1$ at a point $p_1 > p_2^* + T(l_1^* + m^*)$, thus, because $\overline{d}_1 \geq D_1$, any credible demand continuation of Q_1^* for $p_1 \leq p_2^* + T(l_1^* + m^*)$ is below the isoprofit h_1. There are no incentives for firm 1 to undercut.

Let us now look at the case $p_1 \geq p_2^* + T(l_1^* + 1)$. When $p_1 > p_2^* + T(1)$, the unique credible demand is $D_1 = 0$, therefore, we need only look at the interval $[p_2^* + T(l_1^* + 1), p_2^* + T(1))$, which is partitioned into l_1^* blocks by thresholds $T(l_1^* + 1), T(l_1^*), T(l_1^* - 1), \dots, T(1)$. Note that, by Lemma 2, we must have $m^* > 1$ and thus, at least two loyal consumers, i.e. $0 \leq l_1^* < n - 2$.

Claim 1. *In every interval (block)* $[p_2^* + T(l_1 + 2), p_2^* + T(l_1 + 1)]$, *where*

$l_1 \in \{0, \ldots, n-2\}$, *demand for firm 1 is bounded below by*

$$\underline{d_1}(p_1) \equiv \begin{cases} l_1 + 1 & \text{if} \quad p_2^* + T(l_1+2) \leq p_1 \leq P_1^I(l_1) \\ \\ Q_1(p_1 - p_2^*; l_1, n-l_1, 0) & \text{if} \quad P_1^I(l_1) \leq p_1 \leq p_2^* + T(l_1+1) \end{cases}.$$

where

$$P_1^I(l_1) = p_2^* + T(l_1+2) + \frac{-A}{n-l_1}.$$

Taking into account the above claim, we need only guarantee that $h_1(p_1) \geq \underline{d_1}(p_1)$ for $p_1 \in [p_2^* + T(l_1^*+1), p_2^* + T(1))$. As Q_1 decreases with price, this amounts to show that $h_1(P_1^I(l_1)) \geq \underline{d_1}(P_1^I(l_1))$ in the aforementioned blocks (intervals), i.e. for every $l_1 \in \{0, \ldots, l_1^* - 1\}$. If this is the case, then at least the strategy producing $\underline{d_1}$ is a credible continuation for demand in which there are no incentives for firm 1 to deviate. We have thus to show that for every $l_1 \in \{0, \ldots, l_1^* - 1\}$, it holds that $\frac{\Pi_1^*}{P_1^I(l_1)} \geq l_1 + 1$. As $P_1^I(l_1) > p_1^* > 0$, we can rewrite this as $\Pi_1^* \geq P_1^I(l_1)(l_1+1)$. □

Proof of Claim. Recall that the credible demand fiber space is formed by the union of oblique line segments determined by Q_1 for characterizations with $m > 1$, and horizontal and vertical segments for $m = 0$ and $m = 1$. (Observe figure 1.) For $m > 1$, each Q_1 segment is completely determined by l_1 and m (note that $l_2 = n - l_1 - m$), with price domain $[p_2^* + T(l_1+m), p_2^* + T(l_1+1)]$. Consider now a block $(p_2^* + T(l_1+2), p_2^* + T(l_1+1))$ for some $l_1 \in \{0, \ldots, n-2\}$. The minimum value of credible demand at $p_1 = p_2^* + T(l_1+2)$ is l_1+1 and the maximum value at $p_1 = p_2^* + T(l_1+1)$ is l_1+1, as such, all oblique lines of the equilibrium fiber space, whose domain contains this block, cross the horizontal line $l_1 + 1$ (the unique credible non-oblique line in the mentioned interval). The first intersection happens at some point, call it p_1^I, such that $Q_1(p_1^I - p_2^*; l_1', m', l_2') = l_1 + 1$ for some characterization (l_1', m', l_2'). We will now show that this characterization is of a particular form determined by l_1. Given a fixed $\bar{l}_1$, for every possible value of m', i.e. $1 < m' \leq n - \bar{l}_1$, the segments determined by $(\bar{l}_1, m', n - \bar{l}_1 - m')$ intersect (*end*) at $p_2^* + T(\bar{l}_1 + 1)$. The slope of lines determined by Q_1 is $\frac{m'}{A(m'-1)}$, hence, among characterizations with $\bar{l}_1$, the first to cross is of the form $(\bar{l}_1, n - \bar{l}_1, 0)$ (maximum m' for lowest slope because $A < 0$). As such, characterizations with $l_2' = 0$ are the candidates to originate the first crossing at p_1^I. For all l_1', characterizations of the form $(l_1', n - l_1', 0)$ intersect (*start*) at $p_2^* + T(n)$. As such the first crossing is provided by the line which *ends* first, that is, where $p_2^* + T(l_1' + 1)$ is smallest (the steepest line is where l_1' is higher because $m' = n - l_1'$). Taking into account that it must contain the domain, (thus *end* after $p_2^* + T(l_1 + 2)$) we must have $l_1' < l_1 + 1$, and so we get $l_1' = l_1$. The first crossing in the interval $(p_2^* + T(l_1 + 2), p_2^* + T(l_1 + 1))$ is thus determined by solving $Q_1(p_1^I - p_2^*; l_1, n - l_1, 0) = l_1 + 1$, which is $p_1^I = P_1^I(l_1)$. From $p_2^* + T(l_1 + 1)$ to $P_1^I(l_1)$ minimum demand is $l_1 + 1$, then it follows $Q_1(p_1 - p_2^*; l_1, n - l_1, 0)$. □

Theorem 2

Proof. Consider a consumers profile for an undifferentiated duopoly with $A < 0$. We have $b_1 = b_2$, thus $\Delta b = 0$, and $\alpha_1 = \alpha_2 = -a/2$ for some $a > 0$, thus $A = -a$ and $\Delta\alpha = 0$. The proof follows by showing that there are classes $\underline{(l_1, m, l_2)}$ with $\Delta l \neq 0$ which satisfy Lemma 3. Note that, in this case, $\underline{B} = -3a(n-1)/2$ and $\overline{B} = 3a(n-1)/2$. We get $p_1^* = a(n-1)/2 - a\left(\dfrac{2l_1 + l_2}{3m}\right)$ and $p_2^* = a(n-1)/2 - a\left(\dfrac{l_1 + l2_2}{3m}\right)$. Furthermore,

$B_L(l_1, m, l_2) = -3a(n-1)/2 + 3al_1 - a\dfrac{\Delta l}{m}$, and, replacing in the expression for equilibrium demand in Equation 4, we get,

$$\Pi_1^*(l_1, m, l_2) = \frac{am}{m-1}\left((n-1)/2 - \left(\frac{2l_1 + l_2}{3m}\right)\right)^2. \tag{5}$$

Now, we also get that

$$P_1(k; (l_1, m, l_2)) = a\left(n - 1 - \left(\frac{l_1 + 2l_2}{3m} + k + 1 - \frac{1}{n-k}\right)\right).$$

In order to satisfy condition (i) in Lemma 3, we must have for all $k \in \{0, \ldots, l_1 - 1\}$,

$$\frac{am}{m-1}\left((n-1)/2 - \left(\frac{2l_1 + l_2}{3m}\right)\right)^2 \geq$$
$$a(k+1)\left(n - 1 - \left(\frac{l_1 + 2l_2}{3m} + k + 1 - \frac{1}{n-k}\right)\right).$$

Note that, as $a > 0$, this does not depend on a, and, moreover, the left hand side can be rewritten as a quadratic in $k+1$,

$$-(k+1)^2 + (k+1)\left(n - 1 - \frac{l_1 + 2l_2}{3m} + \frac{1}{n-k}\right),$$

with maximum at $\dfrac{1}{2}\left(n - 1 - \dfrac{l_1 + 2l_2}{3m} + \dfrac{1}{n-k}\right)$, which is attained 'inside' the domain. For simplicity, let us consider a classes with $l_2 = 0$ and $m \geq 2l_1$, with $l_1 > 0$, so that $\Delta l = l_1$. Such classes trivially satisfy condition (ii) of Lemma 3. Furthermore, as $k+1$ is an integer, $1/(n-k) < 1/m$, and $1/m < (l_1 + 2l_2)/3m$, we may remove the contribution of $1/(n-k)$ from the above expression (if the maximum is not at an integer, it does not matter). As such, for an equilibrium in these classes to exist, it is sufficient to show that $\Delta b \in \mathcal{BD}(l_1, m, l_2)$ and

$$\frac{m}{m-1}\left(\frac{n-1}{2} - \frac{2l_1 + l_2}{3m}\right)^2 \geq \left(\frac{n-1}{2} - \frac{l_1 + 2l_2}{6m}\right)^2.$$

Substituting $l_2 = 0$ and developing the squared expressions, we can get to

$$l_1^2 \left(\frac{15m + 1}{6m(n-1)} \right) + \frac{m}{2} 3(n-1) \geq l_1(3m+1).$$

As $m \geq 2l_1$, we have that $\frac{m}{2}3(n-1) \geq l_1 3(n-1) > l_1(3m+1)$. To conclude the proof of existence for an equilibrium in these classes, note that, because $l_1 < m/2 < (n-1)/2$, we get $B_L(l_1, m, l_2) < 0 < B_R(l_1, m, l_2)$, and as such $\Delta b = 0 \in \mathcal{BD}(l_1, m, l_2)$.

From the expressions in equations 1 and 5 it is clear that for each firm prices and profits in these classes are different. $\square$

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