The folk rule through a painting procedure for minimum cost spanning tree problems with multiple sources

Bergantiños, Gustavo and Navarro, Adriana

Universidade de Vigo

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G. Bergantiños, A. Navarro-Ramos*

Economics, Society and Territory. Facultad de Economía, Campus Lagoas-Marcosende, s/n, Universidade de Vigo, Vigo, Pontevedra, Spain

Abstract

We consider minimum cost spanning tree problems with multiple sources. We propose a cost allocation rule based on a painting procedure. Agents paint the edges on the paths connecting them to the sources. We prove that the painting rule coincides with the folk rule.

Keywords: minimum cost spanning tree problems with multiple sources; painting rule.

1. Introduction

We study situations where a group of agents need services provided by several sources. Agents need to be connected, directly or indirectly, to all sources. Every connection is costly. Situations of this kind are called minimum cost spanning tree problems with multiple sources and are extensions of the classical minimum cost spanning tree problem (where there is a single source).

The first issue addressed is to find the least costly networks connecting all agents with all sources. Obviously, such a network is a tree. It can also be found in polynomial time using the same algorithms as in the classical problem (e.g., Kruskal (1956) and Prim (1957)).

The second issue addressed is how to allocate the cost of the tree obtained among the agents. Several papers have studied this issue in minimum cost spanning tree problems, but as far as we know only three have considered it in the case of multiple sources. Rosenthal (1987) and Kuipers (1997) study a situation

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*Corresponding author

Email addresses: gbergant@uvigo.es (G. Bergantiños), adnavarro@uvigo.es (A. Navarro-Ramos)

slightly different from this paper, whereas Bergantiños et al. (2017) study the same situation as we present here. Rosenthal (1987) considers situations where all sources provide the same service and agents want to be connected to at least one of them. He considers a cooperative game and studies the core of that game. Kuipers (1997) considers situations where each source offers a different service and each agent needs to be connected to a subset of the sources. He also considers a cooperative game and seeks to determine under what conditions the core is non-empty. Bergantiños et al. (2017) study the same situation as in this paper. They extend different definitions of the folk rule, defined for classical minimum cost spanning tree problems, to the case of multiple sources. They also present some axiomatic characterizations of the folk rule.

In classical minimum cost spanning tree problems the folk rule is one of the most important rules. It has been studied in several papers, including Bergantiños and Kar (2010), Bergantiños et al. (2010, 2011, 2014), Bergantiños and Vidal-Puga (2007, 2009), Branzei et al. (2004), and Tijs et al. (2006).

Our paper is closely related to that of Bergantiños et al. (2014). They study a general framework of connection problems involving a single source, which contains classical minimum cost spanning tree problems. They propose a cost allocation rule, called the *painting rule* because it can be interpreted through a painting story. The idea is the following: start with a tree t; for each agent, identify the unique path in t from that agent to the source. Agents start painting the first edge on that path. Following a protocol, an agent continues painting until all edges on her path have been painted. They also give some axiomatic characterizations of the painting rule. They prove that the painting rule coincides with the folk rule in classical minimal cost spanning tree problem. Thus, they obtain a new way of computing the folk rule and a new axiomatic characterization.

The objective of this paper is to extend the definition of the painting rule to the case of minimum cost spanning tree problems with multiple sources. The main problem that arises when doing this is that given a tree and an agent, several paths in the tree could connect the agent to a source. In order to avoid this problem, we define a two-phase procedure: In Phase 1, given a tree t, we compute a tree t^* with the same cost as t such that t^* is also a tree when it is restricted to the set of sources. Notice that for each agent there is a unique path in t^* connecting the agent with the set of all sources. In Phase 2 we apply the ideas of the painting rule to the tree t^*. This extension of the painting rule is not straightforward because it could depend on the tree t considered initially and the tree t^* computed in Phase 1, which is not determined solely by t. In Proposition 2 we prove that for each tree t and t^* considered, the painting rule always coincide with the folk rule. Thus, the painting rule is independent of the trees t and t^* considered.

The paper is organized as follows. Section 2 introduces minimum cost spanning tree problems with multiple sources. Section 3 introduces the painting rule.

2. The minimum cost spanning tree problem with multiple sources

We consider situations where a group of nodes N (called agents) wants to be connected to a set of suppliers M (called sources).

Let $N = \{1, ..., n\}$ be the finite set of agents and $M = \{a_1, ..., a_m\}$ the finite set of sources. There is a *cost matrix* $C = (c_{ij})_{i,j \in N \cup M}$ over $N \cup M$ representing the cost of the direct link between any pair of nodes, with $c_{ji} = c_{ij} \geq 0$ and $c_{ii} = 0$, for all $i, j \in N \cup M$. We denote by $\mathcal{C}^{N \cup M}$ the set of all cost matrices over $N \cup M$.

A *minimum cost spanning tree problem with multiple sources* (briefly, a problem) is a triple (N, M, C) where N is the set of agents, M is the set of sources and $C \in \mathcal{C}^{N \cup M}$ is the cost matrix. If $c_{ij} \in \{0, 1\}$, for all $i, j \in N \cup M$, then (N, M, C) is called a *simple problem*.

An *edge* is a non-ordered pair (i, j) such that $i, j \in N \cup M$. Sometimes we write ij instead of (i, j). A *network* g is a subset of edges. The cost associated with a network g is defined as

$$c(N, M, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there are no ambiguities, we write $c(g)$ or $c(C, g)$ instead of $c(N, M, C, g)$.

Given a network g and any pair of nodes i and j, a *path* from i to j in g is a sequence of distinct edges $g_{ij} = \{(i_{h-1}, i_h)\}_{h=1}^{q}$ satisfying that $(i_{h-1}, i_h) \in g$ for all $h = 1, ..., q$, $i = i_0$ and $j = i_q$. A *cycle* is a path from i to i with at least two edges. A *tree* is a graph without cycles that connects all the elements of $N \cup M$.

Two nodes i, j are *connected* in g if there exists a path from i to j in g. We say that $S \subseteq N \cup M$ is a *connected component* on g if every $i, j \in S$ are connected in g and S is maximal, *i.e.*, for each $T \in N \cup M$ with $S \subsetneq T$ there exist $k, l \in T$, $k \neq l$, such that k and l are not connected in g.

Let (N, M, C) be a simple problem. We denote by $g^{0,C}$ the network induced by the edges with zero cost. Namely, $g^{0,C} = \{(i, j) : i, j \in N \cup M \text{ and } c_{ij} = 0\}$. We say that S is a *C-component* if S is a connected component on $g^{0,C}$.

The first issue addressed in the literature is how to find a tree with the lowest associated cost (which is not necessarily unique). This problem is polynomial and the algorithms of Kruskal (1956) and Prim (1957) enable such a tree, which is called *minimal tree* (*mt*), to be computed. We denote by $m(N, M, C)$ the cost of any *mt* in (N, M, C).

Let (N, M, C) be a problem and t a minimal tree in (N, M, C). For each $i, j \in N \cup M$ we denote by t_{ij} the unique path in t joining i and j. Bird (1976) defines the *minimal network* associated with the minimal tree t as the problem (N, M, C^t), where $c^t_{ij} = \max_{(k,l) \in t_{ij}} c_{kl}$. It is well known that C^t is independent of the chosen t. Then, the *irreducible problem* (N, M, C^*) of (N, M, C) is defined as the minimal network associated with any minimal tree in (N, M, C).

After obtaining a minimal tree, the second issue addressed is how to divide its cost among the agents. A cost allocation rule (briefly, a rule) is a mapping

f that associates a vector $f(N,M,C) \in \mathbb{R}^N$ with each problem (N,M,C) such that $\sum_{i\in N} f_i(N,M,C) = m(N,M,C)$. The i-th element of $f(N,M,C)$ denotes the payment of agent $i \in N$.

One of the most popular rules in the classical minimum cost spanning tree problem (*mcstp*) is the folk rule. Bergantiños et al. (2017) extend the definition of the folk rule to the problem with multiple sources and provide several ways to obtain it. One of them is through cone-wise decomposition.

Norde et al. (2004) prove that every classical *mcstp* can be written as a non-negative combination of classical simple problems. What follows is an adaptation of this result to our context.

Lemma 1. *For each problem (N,M,C), there exists a positive number $m(C) \in \mathbb{N}$, a sequence $\{C^q\}_{q=1}^{m(C)}$ of simple cost matrices and a sequence $\{x^q\}_{q=1}^{m(C)}$ of non-negative real numbers satisfying two conditions:*

(1) $C = \sum_{q=1}^{m(C)} x^q C^q$.

(2) *Take $q \in \{1,\ldots,m(C)\}$ and $\{i,j,k,l\} \subset N \cup M$. If $c_{ij} \leq c_{kl}$, then $c^q_{ij} \leq c^q_{kl}$.*

Let (N,M,C) be a simple problem and $P = \{S_1,...,S_p\}$ the partition of $N \cup M$ in C-components. Bergantiños et al. (2017) define the folk rule F for simple problems as follows.

$$F_i(N,M,C) = \begin{cases} \dfrac{|S_k \in P : S_k \cap M \neq \emptyset| - 1}{|N|}, & \text{if } S(i,P) \cap M \neq \emptyset \\ \dfrac{1}{|S(i,P)|} + \dfrac{|S_k \in P : S_k \cap M \neq \emptyset| - 1}{|N|}, & \text{otherwise,} \end{cases}$$

where $S(i,P)$ is the element of P to which i belongs to. Then, the folk rule for a general problem (N,M,C) is defined as

$$F(N,M,C) = \sum_{q=1}^{m(C)} x^q F(N,M,C^q).$$

3. The painting procedure

Given a fixed tree t, Bergantiños et al. (2014) provide an algorithm to define a rule through a painting procedure in the classical *mcstp*. They motivate it as follows.

“ In order to illustrate the procedure used to obtain the rule, assume that the nodes represent the houses of the different agents and the edges are the canals which connect them to an irrigation point. These canals need painting and there is only one machine to do this for each one. The machines cannot be moved to another canal and all of them work at the same speed. At every stage, each agent is assigned to an edge while the path from his house to the source has not been completely painted. The canals in t have painters assigned to them if the painting has not been completed. In each step, the agents assigned to an edge which is not completely painted share equally the time the painting machine is in operation. This can be read as their paying the same cost in that segment. At stage 1, each agent is assigned to the first edge in the unique path in t from his house to the source. At stage s, each agent is assigned to the first unpaid edge in this unique path. If all edges in such a path have already been paid for in the previous stages, then this agent has finished his job. The procedure ends when all edges have been paid for completely."

We seek to apply the procedure described above to the case of multiple sources. The main problem that arises is that with multiple sources, given a tree t and an agent i, several paths in t could connect agent i to a source in M. Assume that in the tree t all sources are directly connected to one another (namely t_M, the restriction of t to M, is also a tree). In this case, there is only one path in t to connect each agent to the nearest source.

Our idea for extending the definition of Bergantiños et al. (2014) to the case of multiple sources is the following. First, given a problem (N, M, C) and an *mt* t in (N, M, C), we compute a tree t^* in (N, M, C^*) with the same cost as t such that t^*_M is also a tree. Second, we divide the cost of $t^* \backslash t^*_M$ using the same procedure as in Bergantiños et al. (2014) and the cost of t^*_M is divided equally among all agents.

We now give an example where we explain the above procedure intuitively. It is presented formally below.

Example 1. *Let* $N = \{1, 2, 3, 4\}$*,* $M = \{a_1, a_2, a_3, a_4\}$*,* $c_{3a_3} = 1$*,* $c_{14} = 2$*,* $c_{23} = 3$*,* $c_{4a_4} = 4$*,* $c_{34} = 5$*,* $c_{1a_1} = 6$*,* $c_{a_2a_3} = 7$ *and* $c_{ij} = 10$ *otherwise. The minimal tree* t *for this problem is represented in Figure 1.*

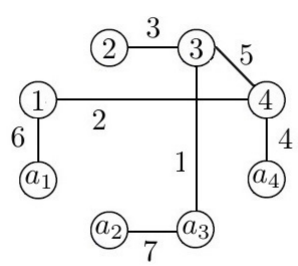

Figure 1: Minimal tree for (N, M, C).

Notice that the sources are not directly connected to one another. Every agent has several paths in t *connecting her to a source. For instance, agent* 1 *could connect to source* a_1 *through path* $\{(1, a_1)\}$ *or could connect to source* a_4 *through path* $\{(1, 4), (4, a_4)\}$*.*

We now construct the tree t^*. We first connect sources a_1 and a_3[1]*. We remove from t the most expensive edge on the unique path in t joining a_1 and a_3, which is edge $(1, a_1)$. We add to t the edge (a_1, a_3) and we change its cost from 10 to 6 (the cost of edge $(1, a_1)$).*

We now connect sources a_3 and a_4. We remove from t the most expensive edge on the unique path in t joining a_3 and a_4, which is edge $(3, 4)$. We add to t the edge (a_3, a_4) and we change its cost from 10 to 5 (the cost of edge $(3, 4)$). Figure 2 shows the modified tree.

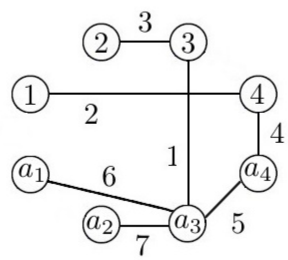

Figure 2: Alternative tree.

In this tree, each agent has a unique path to the set of sources. The path for agent 1 is $\{(1, 4), (4, a_4)\}$, for agent 2 it is $\{(2, 3), (3, a_3)\}$, for agent 3 it is $\{(3, a_3)\}$ and for agent 4 it is $\{(4, a_4)\}$. Then, the original idea of the painting procedure can be applied.

Stage 1. Agent 1 selects edge $(1, 4)$, agent 2 selects $(2, 3)$, agent 3 selects $(3, a_3)$, and agent 4 selects $(4, a_4)$. Thus, agent 3 paints edge $(3, a_3)$ completely and agents 1, 2 and 4 paint one unit of their edges. Thus, agent 3 is already connected to source a_3 and she is removed from the procedure.

Stage 2. Agents 1, 2 and 4 select the same edges as in Stage 1. Edge $(1, 4)$ is completely painted by agent 1. One more unit of edges $(2, 3)$ and $(4, a_4)$ is painted by agent 2 and 4, respectively.

Stage 3. Agent 2 keeps selecting edge $(2, 3)$ and agents 1 and 4 select edge $(4, a_4)$. Agent 2 paints one unit of edge $(2, 3)$. Agents 1 and 4 paint $\frac{1}{2}$ of edge $(4, a_4)$. Thus, edge $(2, 3)$ is completely painted and agent 2 is therefore connected to source a_3 (through agent 3) and she is removed from the procedure.

Stage 4. Agents 1 and 4 keep selecting edge $(4, a_4)$. Each agent paints $\frac{1}{2}$ of edge $(4, a_4)$, which is now completely painted. Then, both agents are connected to source a_4 and removed from the procedure.

Stage 5. The edges connecting the sources ((a_1, a_3), (a_2, a_3) and (a_3, a_4)) are painted by all agents.

[1] Note that this procedure depends on the sources chosen for connecting. For instance, instead of joining sources a_1 and a_3 it is possible to join sources a_1 and a_4. Later we prove that the cost allocation is independent of the choices made.

Table 1 summarizes this procedure.

Agent → / Stage ↓	Agent 1 Edge	Agent 1 Amount	Agent 2 Edge	Agent 2 Amount	Agent 3 Edge	Agent 3 Amount	Agent 4 Edge	Agent 4 Amount
Stage 1	$(1,4)$	1	$(2,3)$	1	$(3,a_3)$	1	$(4,a_4)$	1
Stage 2	$(1,4)$	1	$(2,3)$	1			$(4,a_4)$	1
Stage 3	$(4,a_4)$	$\frac{1}{2}$	$(2,3)$	1			$(4,a_4)$	$\frac{1}{2}$
Stage 4	$(4,a_4)$	$\frac{1}{2}$					$(4,a_4)$	$\frac{1}{2}$
Stage 5	t_M^*	$\frac{6+7+5}{4}$	t_M^*	$\frac{6+7+5}{4}$	t_M^*	$\frac{6+7+5}{4}$	t_M^*	$\frac{6+7+5}{4}$
Total		$\frac{15}{2}$		$\frac{15}{2}$		$\frac{11}{2}$		$\frac{15}{2}$

Table 1: Summary of the painting procedure.

We now formally introduce the procedure explained in Example 1. We consider a two-phase procedure. In the first phase, given any *mt* t, we construct a tree t^* with the same cost as t and where all the sources are connected to one another. In the second phase we apply the painting procedure as in Bergantiños et al. (2014).

Phase 1: Constructing the tree

Given a *mcstp* with multiple sources (N, M, C) and a minimal tree t in (N, M, C), let $P(t_M) = \{S_1, ..., S_{m(t)}\}$ denote the partition of M in connected components induced by t_M.

We consider an algorithm to construct a minimal tree t^* of the irreducible problem (N, M, C^*).

We start with $t^0 = t$. Assume that stage β is defined, for all $\beta \leq \delta - 1$.

Stage δ: We have two cases,

- $P(t_M^{\delta-1}) = \{M\}$. The algorithm ends and $t^* = t^{\delta-1}$.
- $P(t_M^{\delta-1}) \neq \{M\}$. We define

$$E(t^{\delta-1}) = \{(i_{h-1}, i_h)\}_{h=1}^q$$

as the unique path from $\bigcup_{r=1}^{\delta} S_r$ to $S_{\delta+1}$ in $t^{\delta-1}$, with $i_0 \in \bigcup_{r=1}^{\delta} S_r$, $i_q \in S_{\delta+1}$, $i_1 \notin \bigcup_{r=1}^{\delta} S_r$ and $i_{q-1} \notin S_{\delta+1}$.

Let (i,j) be the most expensive edge in $E(t^{\delta-1})$ (if there are several edges, then select just one). Namely,

$$c_{ij} = \max_{(k,l) \in E(t^{\delta-1})} \{c_{kl}\}.$$

We now define,

$$t^\delta = t^{\delta-1} \backslash (i,j) \cup (i_0, i_q).$$

This process is completed in a finite number of stages (exactly at $m(t) - 1$ stages and $1 \leq m(t) \leq m$). The tree t^* is a mt for (N, M, C^*). Besides $c(C^*, t^*) = c(C, t)$ and t^*_M is also a tree.

Notice that given a tree t, several trees t^* could be obtained through this procedure.

We now formally apply Phase 1 to Example 1. We start with

$$t^0 = t = \{(1, a_1), (1, 4), (4, a_4), (2, 3), (3, a_3), (3, 4), (a_2, a_3)\}.$$

Stage 1:

- $P(t^0_M) = \{\{a_1\}, \{a_2, a_3\}, \{a_4\}\}$. Then
  $E(t^0) = \{(a_1, 1), (1, 4), (4, 3), (3, a_3)\}$.
  The most expensive edge in $E(t^0)$ is $(1, a_1)$. Thus

$$t^1 = \{(a_1, a_3), (1, 4), (4, a_4), (2, 3), (3, a_3), (3, 4), (a_2, a_3)\}.$$

Stage 2:

- $P(t^1_M) = \{\{a_1, a_2, a_3\}, \{a_4\}\}$. Then
  $E(t^1) = \{(a_3, 3), (3, 4), (4, a_4)\}$.
  The most expensive edge in $E(t^1)$ is $(3, 4)$. Thus

$$t^2 = \{(a_1, a_3), (1, 4), (4, a_4), (2, 3), (3, a_3), (a_3, a_4), (a_2, a_3)\}.$$

Stage 3:

- $P(t^2_M) = \{\{a_1, a_2, a_3, a_4\}\}$. Then the algorithm ends and $t^* = t^2$.

We know formally define the second phase of our procedure. This phase is obtained by applying the same ideas as in the painting procedure of Bergantiños et al. (2014).

Phase 2: Painting the tree.

Let t^* be an mt in (N, M, C^*) satisfying that t^*_M is a tree over M and $c(N, M, C^*, t^*) = m(N, M, C)$. By Phase 1 we know that such tree exists. We take

- $e^0_i\ (C, t^*) = \emptyset$ for all $i \in N$. In general, $e^\delta_i(C, t^*)$ denotes the edge of t^* assigned to agent i at stage δ. Agent i will pay part of the cost of this edge.

- $c^0(C, t^*) = 0$ and $c^\delta(C, t^*)$ represents the part of the cost of each edge that it is paid at stage δ.

- $p^0_i(C, t^*) = 0$ for all $i \in N$. In general, $p^\delta_i(C, t^*)$ is the cost that agent i pays at stage δ.

- $E^0(C,t^*) = t^* \backslash t^*_M$ and $E^\delta(C,t^*)$ is the set of unpaid edges of $t^* \backslash t^*_M$ at stage δ.

When no confusion arises we will write e^δ_i, $e^\delta_i(C)$ or $e^\delta_i(t^*)$ instead of $e^\delta_i(C,t^*)$. We will do the same with $c^\delta(C,t^*)$, $p^\delta_i(C,t^*)$ and $E^\delta(C,t^*)$. Assume that stage β is defined, for all $\beta \leq \delta - 1$.

Stage δ:

- For each $i \in N$, let e^δ_i be the first edge in the unique path in t^* from i to M belonging to $E^{\delta-1}$. If all edges in such path are not in $E^{\delta-1}$, take $e^\delta_i = \emptyset$.
- For each $(i,j) \in E^{\delta-1}$ we define

$$N^\delta_{ij} = \{k \in N : e^\delta_k = (i,j)\}$$

and

$$c^\delta = \min\left\{ c_{ij} - \sum_{r=0}^{\delta-1} c^r : (i,j) \in E^{\delta-1} \right\}.$$

- For each $i \in N$, we define

$$p^\delta_i = \begin{cases} \dfrac{c^\delta}{\left|N^\delta_{e^\delta_i}\right|}, & \text{if } e^\delta_i \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

- We define

$$E^\delta = \left\{ (i,j) \in E^{\delta-1} : \sum_{r=0}^{\delta} c^r < c_{ij} \right\}.$$

This procedure ends when we find a stage $\gamma(C,t^*)$ ($\gamma(C)$, $\gamma(t^*)$ or γ when no confusion arises) such that $E^\gamma = \emptyset$. Since $E^0 = t^* \backslash t^*_M$, $E^{\delta+1} \subset E^\delta$ and $E^{\delta+1} \neq E^\delta$, γ is finite.

Stage $\gamma + 1$. The cost of all edges on t^*_M, $c(t^*_M) = \sum_{(i,j)\in t^*_M} c^*_{ij}$, is divided equally among all agents. Then,

$$p^{\gamma+1}_i = \frac{c(t^*_M)}{|N|}.$$

For each problem (N,M,C), each *mt* t, and each $i \in N$, we define the painting rule $f^{P,t}_i$ as

$$f^{P,t}_i(N,M,C) = \sum_{\delta=1}^{\gamma+1} p^\delta_i(C,t^*).$$

Note that this definition depends on trees t and t^* considered.

Remark 1. *Suppose that $|M| = 1$, i.e., there is a unique source and then we have a classical minimum cost spanning tree problem $(N, 0, C)$. Let t be a minimal tree in $(N, 0, C)$. In this case, we do not need to apply Phase 1 in our procedure. Thus, we go directly to Phase 2 where $t^* = t$. Applying Phase 2 in our procedure is the same than applying the procedure followed in Bergantiños et al. (2014) to the problem (N_0, C, t). Then, given a classical minimum cost spanning tree problem $(N, 0, C)$ and a minimal tree t, the allocation obtained by applying our procedure to $(N, 0, C)$ and t coincides with the allocation obtained by applying the procedure of Bergantiños et al. (2014) to (N_0, C, t). As a consequence we can see our procedure as a generalization of Bergantiños et al. (2014) to the case of multiple sources.*

Now we formally apply Phase 2 to Example 1. We start with:

- e_1^0, e_2^0, e_3^0, $e_4^0 = \emptyset$.
- $c^0 = 0$.
- p_1^0, p_2^0, p_3^0, $p_4^0 = 0$.
- $E^0 = \{(1,4), (4, a_4), (2,3), (3, a_3)\}$.

Stage 1:

- $e_1^1 = (1,4)$, $e_2^1 = (2,3)$, $e_3^1 = (3, a_3)$ and $e_4^1 = (4, a_4)$.
- $N_{14}^1 = \{1\}$, $N_{23}^1 = \{2\}$, $N_{3a_3}^1 = \{3\}$ and $N_{4a_4}^1 = \{4\}$.
- $c^1 = \min\{c_{14}, c_{23}, c_{3a_3}, c_{4a_4}\} = \min\{2, 3, 1, 4\} = 1$.
- p_1^1, p_2^1, p_3^1, $p_4^1 = 1$.
- $E^1 = \{(1,4), (4, a_4), (2,3)\}$.

Stage 2:

- $e_1^2 = (1,4)$, $e_2^2 = (2,3)$, $e_3^2 = \emptyset$ and $e_4^2 = (4, a_4)$.
- $N_{14}^2 = \{1\}$, $N_{23}^2 = \{2\}$ and $N_{4a_4}^2 = \{4\}$.
- $c^2 = \min\{c_{14} - 1, c_{23} - 1, c_{4a_4} - 1\} = \min\{2 - 1, 3 - 1, 4 - 1\} = 1$.
- $p_1^2 = 1$, $p_2^2 = 1$, $p_3^2 = 0$ and $p_4^2 = 1$.
- $E^2 = \{(2,3), (4, a_4)\}$.

Stage 3:

- $e_1^3 = (4, a_4)$, $e_2^3 = (2,3)$, $e_3^3 = \emptyset$ and $e_4^3 = (4, a_4)$.
- $N_{23}^3 = \{2\}$ and $N_{4a_4}^3 = \{1, 4\}$.

- $c^3 = \min\{c_{23} - 2, c_{4a_4} - 2\} = \min\{3-2, 4-2\} = 1$.
- $p_1^3 = \frac{1}{2}$, $p_2^3 = 1$, $p_3^3 = 0$ and $p_4^3 = \frac{1}{2}$.
- $E^3 = \{(4, a_4)\}$.

Stage 4:

- $e_1^4 = (4, a_4)$, $e_2^4 = \emptyset$, $e_3^4 = \emptyset$ and $e_4^4 = (4, a_4)$.
- $N_{4a_4}^4 = \{1, 4\}$.
- $c^4 = \min\{c_{4a_4} - 3\} = \min\{4-3\} = 1$.
- $p_1^4 = \frac{1}{2}$, $p_2^4 = 0$, $p_3^4 = 0$ and $p_4^4 = \frac{1}{2}$.
- $E^4 = \emptyset$. Thus, $\gamma = 4$.

Stage 5: For each $i \in N$,

$$p_i^5 = \frac{c(t_M^*)}{4} = \frac{18}{4} = \frac{9}{2}.$$

Then,

$$\begin{aligned}
f_1^P(N, M, C) &= 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{9}{2} = \frac{15}{2}, \\
f_2^P(N, M, C) &= 1 + 1 + 1 + 0 + \frac{9}{2} = \frac{15}{2}, \\
f_3^P(N, M, C) &= 1 + 0 + 0 + 0 + \frac{9}{2} = \frac{11}{2}, \\
f_4^P(N, M, C) &= 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{9}{2} = \frac{15}{2}.
\end{aligned}$$

We now show that the solution does not actually depend on the minimal tree t considered initially and the tree t^* defined in Phase 1. To that end, we introduce two propositions.

Proposition 1. *Let (N, M, C) and (N, M, C') be two mcstp with multiple sources satisfying that there is an order σ over the set of edges of $N \cup M$ such that for all $i, j, k, l \in N \cup M$ satisfying that $\sigma(i, j) < \sigma(k, l)$, then $c_{ij} \leq c_{kl}$ and $c'_{ij} \leq c'_{kl}$. Let t be a minimal tree in C, C', and $C + C'$. Then,*

$$f^{P,t}(N, M, C + C') = f^{P,t}(N, M, C) + f^{P,t}(N, M, C').$$

Proof. Applying Phase 1 to t, we can obtain a common *mt* t^* for (N, M, C^*), (N, M, C'^*) and $(N, M, C^* + C'^*)$.

We now compute Phase 2. First, consider the case when for all $i, j, k, l \in N \cup M$ satisfying that $\sigma(i, j) < \sigma(k, l)$, then $c_{ij} < c_{kl}$ and $c'_{ij} < c'_{kl}$. Thus $c_{ij} + c'_{ij} < c_{kl} + c'_{kl}$. For all $i \in N$, let $i^M \in N \cup M$ denote the immediate successor of i in the unique path from i to M in t^*. Without loss of generality, we assume that $c_{ii^M} < c_{jj^M}$ when $i < j$, for all $i, j \in N$. Then,

Stage 1:

- $\forall i \in N$, $e_i^1(C) = e_i^1(C') = e_i^1(C + C') = (i, i^M)$.
- $\forall i \in N$, $N_{ii^M}^1(C) = N_{ii^M}^1(C') = N_{ii^M}^1(C + C') = \{i\}$.
- $c^1(C) = \min_{i \in N}\{c_{ii^M}\} = c_{11^M}$,
  $c^1(C') = \min_{i \in N}\{c'_{ii^M}\} = c'_{11^M}$ and
  $c^1(C + C') = \min_{i \in N}\{c_{ii^M} + c'_{ii^M}\} = c_{11^M} + c'_{11^M}$.
- $\forall i \in N$, $p_i^1(C) = c_{11^M}$, $p_i^1(C') = c'_{11^M}$ and $p_i^1(C + C') = c_{11^M} + c'_{11^M}$.
- $E^1(C) = E^1(C') = E^1(C + C') = \{(i, i^M)\}_{i=2}^{|N|}$.

  Then, for all $i \in N$, $p_i^1(C + C') = p_i^1(C) + p_i^1(C')$.

Stage 2:

- $\forall i \in N \backslash \{1\}$, $e_i^2(C) = e_i^1(C)$, $e_i^2(C') = e_i^1(C')$ and $e_i^2(C+C') = e_i^1(C+C')$. If $1^M \in M$ then $e_1^2(C) = e_1^2(C') = e_1^2(C + C') = \emptyset$. If $1^M \notin M$ then $e_1^2(C) = e_1^2(C') = e_1^2(C + C') = e_{1^M}^1(C)$.
  Then, $\forall i \in N$, $e_i^2(C) = e_i^2(C') = e_i^2(C + C')$.
- $N_{ii^M}^2(C) = N_{ii^M}^2(C') = N_{ii^M}^2(C + C')$, for all $i \in N \backslash \{1\}$.
- $c^2(C) = \min_{i \in N \backslash \{1\}}\{c_{ii^M} - c^1(C)\} = c_{22^M} - c_{11^M}$,
  $c^2(C') = \min_{i \in N \backslash \{1\}}\{c'_{ii^M} - c^1(C')\} = c'_{22^M} - c'_{11^M}$ and
  $c^2(C + C') = \min_{i \in N \backslash \{1\}}\{c_{ii^M} + c'_{ii^M} - c^1(C + C')\} = c_{22^M} + c'_{22^M} - (c_{11^M} + c'_{11^M})$.
- $\forall i \in N \backslash \{1\}$, $p_i^2(C) = \dfrac{c_{22^M} - c_{11^M}}{|N_{e_i^2}^2(C)|}$, $p_i^2(C') = \dfrac{c'_{22^M} - c'_{11^M}}{|N_{e_i^2}^2(C')|}$ and
  $p_i^2(C + C') = \dfrac{c_{22^M} + c'_{22^M} - (c_{11^M} + c'_{11^M})}{|N_{e_i^2}^2(C + C')|}$.
  If $1^M \in M$ then $p_1^2(C) = p_1^2(C') = p_1^2(C + C') = 0$. If $1^M \notin M$ then $p_1^2(C) = \dfrac{c_{22^M} - c_{11^M}}{|N_{e_1^2}^2(C)|}$, $p_1^2(C') = \dfrac{c'_{22^M} - c'_{11^M}}{|N_{e_1^2}^2(C')|}$ and
  $p_1^2(C + C') = \dfrac{c_{22^M} + c'_{22^M} - (c_{11^M} + c'_{11^M})}{|N_{e_1^2}^2(C + C')|}$.

- $E^2(C) = E^2(C') = E^2(C + C') = \{(i, i^M)\}_{i=3}^{|N|}$.

  Then $\forall i \in N$, $p_i^2(C + C') = p_i^2(C) + p_i^2(C')$.

Repeating this argument, we can prove that $\gamma(C) = \gamma(C') = \gamma(C + C')$ and that for each stage $\delta = 1, ..., \gamma$ and for every $i \in N$, we have that $p_i^\gamma(C + C') = p_i^\gamma(C) + p_i^\gamma(C')$. Besides, for every $i \in N$, $p_i^{\gamma+1}(C) = \dfrac{c(t_M^*)}{|N|}$, $p_i^{\gamma+1}(C') = \dfrac{c'(t_M^*)}{|N|}$ and $p_i^{\gamma+1}(C + C') = \dfrac{c(t_M^*) + c'(t_M^*)}{|N|}$. Thus,

$$f^{P,t}(N, M, C + C') = f^{P,t}(N, M, C) + f^{P,t}(N, M, C').$$

Now, consider the general case when, if $\sigma(i, j) < \sigma(k, l)$, then $c_{ij} \leq c_{kl}$ and $c'_{ij} \leq c'_{kl}$. Let C^ε and C'^ε be two cost functions such that:

- For each $i, j \in N \cup M$, $c_{ij} - \varepsilon \leq c_{ij}^\varepsilon \leq c_{ij} + \varepsilon$ and $c'_{ij} - \varepsilon \leq c'^\varepsilon_{ij} \leq c'_{ij} + \varepsilon$
- If $\sigma(i, j) < \sigma(k, l)$ then $c_{ij}^\varepsilon < c_{kl}^\varepsilon$ and $c'^\varepsilon_{ij} < c'^\varepsilon_{kl}$.
- t is a minimal tree in C^ε, C'^ε, and $C^\varepsilon + C'^\varepsilon$.

Notice that C^ε and C'^ε satisfy the condition in the first case studied. So, $f^{P,t}(N, M, C^\varepsilon + C'^\varepsilon) = f^{P,t}(N, M, C^\varepsilon) + f^{P,t}(N, M, C'^\varepsilon)$.

Finally, taking into account the definition of the rule $f^{P,t}$, we have that $\lim_{\varepsilon\to 0} f^{P,t}(N, M, C^\varepsilon) = f^{P,t}(N, M, C)$, $\lim_{\varepsilon\to 0} f^{P,t}(N, M, C'^\varepsilon) = f^{P,t}(N, M, C')$ and $\lim_{\varepsilon\to 0} f^{P,t}(N, M, C^\varepsilon + C'^\varepsilon) = f^{P,t}(N, M, C + C')$. Thus,

$$f^{P,t}(N, M, C + C') = f^{P,t}(N, M, C) + f^{P,t}(N, M, C'). \qquad \square$$

We now prove that for each problem (N, M, C) and every minimal tree t the painting rule associated with t coincides with the folk rule. Thus, the painting rule is well defined and is independent of the minimal tree t and the tree t^* computed in Phase 1.

Proposition 2. *For every problem (N, M, C) and every minimal tree t for (N, M, C),*

$$f^{P,t}(N, M, C) = F(N, M, C).$$

Proof. By Lemma 1, we know that $C = \sum_{q=1}^{m(C)} x^q C^q$ where for each q, (N, M, C^q) is a simple problem. Besides t is a minimal tree for each (N, M, C^q). By Proposition 1 and the definition of the folk rule F, it is enough to prove that $f^{P,t}(N, M, C^q) = F(N, M, C^q)$ when (N, M, C^q) is a simple problem and t is a minimal tree in (N, M, C^q).

Let t^* be a tree obtained on Phase 1. For all $i \in N$, let $i^M \in N \cup M$ denote the immediate successor of i in the unique path from i to M in t^*. Now, we apply the procedure of Phase 2:

Stage 1: Take $i \in N$.

- $\forall i \in N$, $e_i^1(C^q, t^*) = (i, i^M)$.
- $\forall i \in N$, $N_{ii^M}^1(C^q, t^*) = \{i\}$.

Let $P = \{S_1, ..., S_p\}$ be the partition of $N \cup M$ in C^q-components. We consider several cases:

Case 1: $S(i, P) \cap M \neq \emptyset$, for all $i \in N$. Then,

- $c^1(C^q, t^*) = 0$.
- $\forall i \in N$, $p_i^1(C^q, t^*) = 0$.
- $E^1(C^q, t^*) = \emptyset$.

Then, $\gamma = 1$ and $\forall i \in N$,

$$p_i^2(C^q, t^*) = \frac{|S_k : S_k \cap M \neq \emptyset| - 1}{|N|}.$$

Thus, $\forall i \in N$,

$$f_i^{P,t}(N, M, C^q) = \frac{|S_k : S_k \cap M \neq \emptyset| - 1}{|N|} = F(N, M, C^q).$$

Case 2: $|S(i, P)| = 1$, for all $i \in N$. Then $S(i, P) \cap M = \emptyset$, $\forall i \in N$. Now

- $c^1(C^q, t^*) = 1$.
- $\forall i \in N$, $p_i^1(C^q, t^*) = 1 = \dfrac{1}{|S(i, P)|}$.
- $E^1(C^q, t^*) = \emptyset$.

As in the first case, $\gamma = 1$ and $\forall i \in N$

$$p_i^2(C^q, t^*) = \frac{|S_k : S_k \cap M \neq \emptyset| - 1}{|N|}.$$

Therefore

$$f_i^{P,t}(N, M, C^q) = \frac{1}{|S(i, P)|} + \frac{|S_k : S_k \cap M \neq \emptyset| - 1}{|N|} = F(N, M, C^q).$$

Case 3: Otherwise.

- $c^1(C^q, t^*) = 0$.
- $\forall i \in N$, $p_i^1(C^q, t^*) = 0$.
- $E^1(C^q, t^*) = \{(i, i^M) \in E^0 : c_{ii^M}^q = 1\} \neq \emptyset$.

Stage 2:

- Let $i \in N$. If $S(i,P) \cap M \neq \emptyset$, then $e_i^2(C^q, t^*) = \emptyset$. If $S(i,P) \cap M = \emptyset$, there exists a unique $j \in S(i,P)$ such that $(j, j^M) \in E^1$. Thus $e_i^2(C^q, t^*) = (j, j^M)$.

- $N^2_{e_i^2}(C^q, t^*) = S(i,P)$.

- $c^2(C^q, t^*) = 1$.

- For each $i \in N$,

$$p_i^2(C^q, t^*) = \begin{cases} 0, & \text{if } S(i,P) \cap M \neq \emptyset \\ \dfrac{1}{|S(i,P)|}, & \text{otherwise.} \end{cases}$$

- $E^2(C^q, t^*) = \emptyset$.

In this case, $\gamma = 2$ and $\forall i \in N$

$$p_i^3(C^q, t^*) = \frac{|S_k : S_k \cap M \neq \emptyset| - 1}{|N|}.$$

Then,

$$f_i^{P,t}(N, M, C^q) = \begin{cases} \dfrac{|S_k : S_k \cap M \neq \emptyset| - 1}{|N|}, & \text{if } S(i,P) \cap M \neq \emptyset \\ \dfrac{1}{|S(i,P)|} + \dfrac{|S_k : S_k \cap M \neq \emptyset| - 1}{|N|}, & \text{otherwise.} \end{cases}$$

Therefore, $f_i^{P,t}(N, M, C^q) = F_i(N, M, C^q)$, for all $i \in N$. $\square$

Since the rule coincides with the folk rule, which does not depend on the tree t chosen, the rule can be denoted by f^P instead of $f^{P,t}$.

Bergantiños et al. (2017) extend the folk rule for *mcstp* with multiple sources using four approaches: As the Shapley value of the irreducible game (Bergantiños and Vidal-Puga (2007)), as an obligation rule (Tijs et al. (2006) and Bergantiños and Kar (2010)), as a partition rule (Bergantiños et al. (2010, 2011)), and through a cone-wise decomposition (Branzei et al. (2004) and Bergantiños and Vidal-Puga (2009)). Thus, the painting rule is a new way of calculating the extension of the folk rule to this context. The main advantage of this approach is that it makes it very clear that the allocation of an agent given by the folk rule depends only on her path to the sources and the connection cost between them in the irreducible problem.

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