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Covarrubias, Enrique

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Enrique Covarrubias

Current affiliation:
School of Mathematics
The University of Edinburgh
Edinburgh EH9 3JZ
United Kingdom

St Hugh's College
St Margaret's Road
Oxford OX2 6LE
United Kingdom

Enrique.C@ed.ac.uk

Tel. +44 (0) 131-6505079
Fax. +44 (0) 131-6506553

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Abstract

This paper deals with generic determinacy of equilibria for infinite dimensional consumption spaces. Our work could be seen as an infinite-dimensional analogue of Dierker and Dierker (1972), by characterising equilibria of an economy as a zero of the aggregate excess demand, and studying its transversality. In this case, we can use extensions of the transversality density theorem. Assuming separable utilities, we give a new proof of generic determinacy of equilibria. We define regular price systems in this setting and show that an economy is regular if and only if its associated excess demand function only has regular equilibrium prices. We also define the infinite equilibrium manifold and show that it has the structure of a Banach manifold.

JEL classification: D50, D51

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1 Introduction

Amongst the most important results when modeling competitive markets is that of generic determinacy of competitive equilibria. If we consider economies with a a finite number of agents and a finite number of commodities, it is well known that almost all initial endowments give rise to a finite number of competitive equilibria.

However, when an economy has an infinite number of commodities, determining whether equilibria is locally unique has presented us with many challenges. Araujo (1988), loosely speaking, shows that when the commodity set is a general Banach space a demand function will exist if and only the commodity space is reflexive. He also shows that even if the demand function exists, it will be C^1 if and only if the commodity space is actually a Hilbert space.

Different approaches exist then to attack this problem. Because of Araujo's results, Kehoe et al (1989) study determinacy of equilibria where the commodity set is a Hilbert space. The disadvantage of this approach, as they put it, is that the price domain (and, implicitly, the consumption set) has an empty interior. This means that they are allowing, to some extent, negative prices and consumption.

A second approach consists in using a weakened version of differentiability. Shannon (1999) and Shannon and Zame (2002) introduce the notion of quadratic concavity and demonstrate that Lipschitz continuity of the excess spending map is sufficient to yield generic determinacy. Because the nature of regularity for Lipschitz functions is weaker than for smooth economies, the set of regular economies is not open nor is it the intersection of a countable family of open sets. Instead they use a measure-theoretic analogue of full Lebesgue measure for infinite dimensional spaces.

A third approach is to assume separable utilities. In this case equilibrium conditions are described by Fredholm maps which extend differential topology to infinite dimensions. Chichilnisky and Zhou (1998) point out that the literature typically takes as the price space the natural positive cone of the dual space of the commodity set. However, with separable utility functions, only a small subset of the price space can support equilibria. There is no loss of information from discarding those elements that do not support equilibria. They show that individual demand functions are Fredholm maps and then they show that smooth infinite economies with separable utilities have locally unique equilibria. A similar approach has also been used by Balasko (1997) to study the infinite horizon model.

In this paper we extend these results. Our work could be seen as an infinite-dimensional analogue of Dierker and Dierker (1972), by characterising equilibria as a zero of aggregate excess demand. In this case, we can use extensions of the implicit function theorem and the transversality theorem. In section 2 we show that aggregate excess demand functions also are Fredholm maps and compute their index. In section 3 we define the equilibrium set and show, à la Balasko (1988), that it has the structure of a Banach manifold and that the natural projection map is smooth. We also define the notion of a regular equilibrium price system and show that an economy is regular if and only if all equilibrium prices of its associated excess demand function are regular. Finally, we give a new proof that almost all economies are regular.

2 The Structure of Aggregate Excess Demand Functions

2.1 The Market

In order to use differential techniques, we assume that the consumption space is a separable topological vector space for which the interior of its positive cone is non-empty. For simplicity, following Chichilnisky and Zhou (1998), we assume that the commodity space is a subset of $C(M, \mathbb{R}^n)$, where M is a compact (Riemannian) manifold. For more general consumption spaces we refer to Chichilnisky and Zhou (1995). The **consumption space** is then $X = C^{++}(M, \mathbb{R}^n)$, the positive cone of $C(M, \mathbb{R}^n)$, and the **price space** is $S = \{P \in C^{++}(M, \mathbb{R}^n) : \|P\| = 1\}$. We denote by $\langle \cdot, \cdot \rangle$ the inner product on $C(M, \mathbb{R}^n)$.

We consider a finite number I of agents. An **exchange economy** is parametrized for each agent $i = 1, \ldots, I$ by their initial endowments $\omega_i \in X$ and their individual demand functions $f_i : S \times (0, \infty) \to X$. The maps $f_i(P(t), w)$ are solutions to the optimization problem

$$\max_{\langle P(t), y \rangle = w} W_i(y)$$

where $W_i(x)$ is a separable utility function, i.e., it can be written as

$$W_i(x) = \int_M u^i(x(t), t)dt$$

We assume $u^i(x(t), t) : \mathbb{R}^n_{++} \times M \to \mathbb{R}$ is a strictly monotonic, concave, C^2 function where $\{y \in R^n_{++} : u^i(y, t) \geq u^i(x, t)\}$ is closed. Chichilnisky and Zhou (1998) have shown that this implies that $W_i(x)$ is strictly monotonic, concave and twice Fréchet differentiable.

2.1.1 Example 1

In growth models, the utility function $W_i(x)$ is a continuous-time version of a discounted sum of time-dependent utilities. Here M represents time.

2.1.2 Example 2

In finance, when the underlying parameters follow a diffusion process, $W_i(x)$ is the expectation of state-dependent utilities, where M is the state space.

2.2 Fredholm Index Theory

We will be using tools of differential topology in infinite dimensions. Therefore, we would like our maps to be Fredholm as introduced by Smale (1965).

A (linear) **Fredholm operator** is a continuous linear map $L : E_1 \to E_2$ from one Banach space to another with the properties:

1. dim ker $L < \infty$
2. range L is closed
3. coker $L = E_2/\text{range}L$ has finite dimension

If L is a Fredholm operator, then its **index** is dim ker$L -$ dim coker$L$, so that the index of L is an integer.

A **Fredholm map** is a C' map $f : M \to V$ between differentiable manifolds locally like Banach spaces such that for each $x \in M$ the derivative $Df(x) : T_xM \to T_{f(x)}V$ is a Fredholm operator. The **index** of f is defined to be the index of $Df(x)$ for some x. If M is connected, this definition does not depend on x.

2.3 Individual Demand Functions

Chichilnisky and Zhou (1998) also show that for separable utilities the individual demand functions satisfy

1. $\langle P, f_i(P, w)\rangle = w$ for any $P \in S$ and for any $w \in (0, \infty)$
2. $u_x^i(f_i(P(t), w), t) = \lambda P(t)$ for some $\lambda > 0$
3. $f_i : S \times (0, \infty) \to X$ is a diffeomorphism
4. $f_i : S \times (0, \infty) \to X$ is a Fredholm map of index zero

2.4 Aggregate Excess Demand Functions

In this paper we assume that the individual demand functions are fixed, so that the only parameters defining an economy are the initial endowments. Denote $\omega = (\omega_1, \ldots, \omega_I) \in \Omega = X^I$. For a fixed economy $\omega \in \Omega$ the **aggregate excess demand function** is a map $Z_\omega : S \to C(M, \mathbb{R}^n)$ defined by

$$Z_\omega(P) = \sum_{i=1}^{I} (f_i(P, \langle P, \omega_i\rangle) - \omega_i)$$

We also define $Z : \Omega \times S \to C(M, \mathbb{R}^n)$ by the evaluation

$$Z(\omega, P) = Z_\omega(P)$$

Definition 1. *We say that $P \in S$ is an **equilibrium** of the economy $\omega \in \Omega$ if $Z_\omega(P) = 0$. We denote the **equilibrium set***

$$\Gamma = \{(\omega, P) \in \Omega \times S : Z(\omega, P) = 0\}$$

We wish to explore the structure of aggregate excess demand functions. We first show the well-known result that the excess demand defines a vector field on the price space.

Proposition 1. *The excess demand function $Z_\omega : S \to C(M, \mathbb{R}^n)$ of economy $\omega \in \Omega$ is a vector field on S.*

Proof. Since Chichilnisky and Zhou (1998) show that $\langle P, f_i(P,y)\rangle = y$ for any $P \in S$ and for any $y \in (0,\infty)$, then

$$\begin{aligned}
\langle P, Z_\omega(P)\rangle &= \langle P, \sum_{i=1}^{I}(f_i(P,\langle P,\omega_i\rangle) - \omega_i)\rangle \\
&= \sum_{i=1}^{I}\langle P, f_i(P,\langle P,\omega_i\rangle)\rangle - \sum_{i=1}^{I}\langle P,\omega_i\rangle \\
&= \sum_{i=1}^{I}\langle P,\omega_i\rangle - \sum_{i=1}^{I}\langle P,\omega_i\rangle \\
&= 0
\end{aligned}$$

□

Denote by TS the tangent bundle of S and TS_0 its zero section. We can then interpret Z_ω as a section of TS and an equilibrium as a point where this section intersects TS_0.

2.5 The Fredholm Index of the Excess Demand

In order to use techniques of differential topology in infinite dimensions, we require our maps to be Fredholm. We now show that this is the case for the excess demand function.

Proposition 2. *The excess demand function $Z_\omega : S \to C(M,\mathbb{R}^n)$ of economy $\omega \in \Omega$ is a Fredholm map of index zero.*

Proof. Chichilnisky and Zhou (1998) have shown that the derivative Df_i of each individual demand function is a linear Fredholm operator of index zero. This is because Df_i can be written as the sum of the finite rank operator

$$-\frac{\lambda\langle P(t),(u^i_{xx})^{-1}DP(t)\rangle + \langle DP(t), f_i\rangle}{\langle P(t),(u^i_{xx})^{-1}P(t)\rangle}(u^i_{xx})^{-1}P(t)$$

and the invertible operator

$$\frac{(u^i_{xx})^{-1}P(t)}{\langle P(t),(u^i_{xx})^{-1}P(t)\rangle}Dw + \lambda(u^i_{xx})^{-1}DP(t)$$

In general the sum of two Fredholm operators of index zero is not again a Fredholm operator of index zero. However, the matrix (u^i_{xx}) is negative definite, and every negative definite matrix is invertible and its inverse is also negative definite.

$\square$

3 Determinacy of equilibrium

In this section we wish to show parametric transversal density. We first need to give a manifold structure to the equilibrium set Γ.

3.1 Regular Values

Proposition 3. *The derivative of the map $Z : \Omega \times S \to TS$ is a surjective map. In particular, it has 0 as a regular value.*

Proof. We need to compute the derivative $DZ : T(\Omega \times S) \to T(TS)$. Linearizing $Z(\omega, P)$ to first order in ϵ and letting $y_i = \langle P, \omega_i \rangle$, we get

$$
\begin{aligned}
&Z(\omega_1 + \epsilon k_1, \dots, \omega_I + \epsilon k_I, P + \epsilon h) \\
&= \sum f_i(P + \epsilon h, \langle P + \epsilon h, \omega_i + \epsilon k_i \rangle) - \sum (\omega_i + \epsilon k_i) \\
&= \sum f_i(P + \epsilon h, \langle P, \omega_i \rangle + \epsilon \langle P, k_i \rangle + \epsilon \langle h, \omega_i \rangle) - \sum \omega_i - \epsilon \sum k_i \\
&= \sum [f_i(P, \langle P, \omega_i \rangle) + \epsilon (D_{y_i} f_i)_{(P, \langle P, \omega_i \rangle)}(\langle P, k_i \rangle) + \\
&+ \epsilon (D_{y_i} f_i)_{(P, \langle P, \omega_i \rangle)}(\langle h, \omega_i \rangle) + \epsilon (D_P f_i)_{(P, \langle P, \omega_i \rangle)}(h)] - \sum \omega_i - \epsilon \sum k_i \\
&= Z(\omega_1, \dots, \omega_I, P) + \\
&+ \epsilon \sum [(D_P f_i)_{(P, \langle P, \omega_i \rangle)}(h) + (D_{y_i} f_i)_{(P, \langle P, \omega_i \rangle)}(\langle P, k_i \rangle + \langle h, \omega_i \rangle) - k_i]
\end{aligned}
$$

So

$$
\begin{aligned}
&DZ_{(\omega, P)}(k_1, \dots, k_I, h) \\
&= \sum_{i=1}^{I} \left[(D_P f_i)_{(P, \langle P, \omega_i \rangle)}(h) + (D_{y_i} f_i)_{(P, \langle P, \omega_i \rangle)}(\langle P, k_i \rangle + \langle h, \omega_i \rangle) - k_i \right]
\end{aligned}
$$

Or in matrix form, $DZ_{(\omega,P)} =$

$$\left(\begin{array}{c|c} \overbrace{0\ldots0}^{I} & 1 \\ \hline \underbrace{(D_{y_i}f_i)_{(P,\langle P,\omega_i\rangle)}(\langle P,-\rangle) - Id}_{i=1,\ldots,I} & \sum_i (D_P f_i)_{(P,\langle P,\omega_i\rangle)}(-)+ \\ & \\ & +\sum_i (D_{y_i}f_i)_{(P,\langle P,\omega_i\rangle)}(\langle -,\omega_i\rangle) \end{array}\right)$$

To compute the cokernel let

$$DZ_{(\omega,P)}(k_1,\ldots,k_I,h) = (Q,\dot{Q}) \in T(TS)$$

We need to solve for $(k_1,\ldots,k_I,h)$. We first observe that $h = Q$. The second row would then be,

$$\sum\{[(D_{y_i}f_i)(\langle P,k_i\rangle) - (k_i)] + [(D_P f_i)(Q)] + [(D_{y_i}f_i)(\langle Q,\omega_i\rangle)]\} = \dot{Q}$$

Then

$$\sum [(D_{y_i}f_i)(\langle P,k_i\rangle) - (k_i)] = H(Q,\dot{Q}) \tag{1}$$

where

$$H(Q,\dot{Q}) = \dot{Q} - \sum\{[(D_P f_i)(Q)] + [(D_{y_i}f_i)(\langle Q,\omega_i\rangle)]\}$$

But for every $i = 1,\ldots,I$, $(D_{y_i}f_i)(\langle P,k_i\rangle) - (k_i)$ is onto. And, therefore, so is DZ.

□

3.2 The Infinite Equilibrium Manifold

Knowing that 0 is a regular value of Z we would like to give the equilibrium set Γ the structure of a Banach manifold (Abraham and Robbin, 1967).

Quinn (1970) defines a C^∞ **representation of maps** $\rho : A : M \to N$ consisting of a Banach manifold A together with a function $\rho : A \to C^\infty(M,N)$ such that the evaluation map

$$Ev_\rho : A \times M \to N; (a,m) \mapsto \rho_a(m)$$

is C^∞. In our situation, $Ev_\rho : A \times M \to N$ corresponds to $Z : \Omega \times S \to TS$.

Suppose we have a C^∞ map $F : W \to N$ which is transversal to Ev_ρ. Quinn (1970) shows that if we form the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{g} & W \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle F}\big\downarrow \\ A \times M & \xrightarrow{Ev_\rho} & N \\ {\scriptstyle \pi_A}\big\downarrow & & \\ A & & \end{array} \tag{2}$$

where $P = (Ev_\rho \times F)^{-1}(\Delta_N)$, then P is a C^∞ Banach manifold, and $\pi_A \circ h$ is a C^∞ map.

Proposition 4. *The equilibrium set Γ is a C^∞ Banach manifold. We shall call it the **equilibrium manifold**. Furthermore the natural projection map $pr_\Omega : \Omega \times S|_\Gamma \to \Omega$ is a C^∞ map.*

Proof. Notice that the inclusion $0 \to C(M, \mathbb{R}^n)$ is a C^∞ map. We also know from Proposition 3 that DZ is surjective, so it has 0 as a regular value. Then, we can form the pullback diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow \\ \Omega \times S & \xrightarrow{Z} & TS \\ {\scriptstyle pr_\Omega}\big\downarrow & & \\ \Omega & & \end{array}$$

and as in diagram (2) we get that Γ is a C^∞ Banach manifold and the natural projection map is a C^∞ map. $\square$

3.3 Regular Economies

Definition 2. *We say that an economy is **regular** (resp. **critical**) if and only if ω is a regular (resp. critical) value of the projection $pr : \Gamma \to \Omega$.*

Definition 3. *Let Z_ω be the excess demand of economy ω. A price system $P \in S$ is a **regular equilibrium price** system if and only if $Z_\omega(P) = 0$ and $DZ_\omega(P)$ is surjective.*

We would like to compare the set of regular economies with those economies whose excess demand function has only regular prices. In finite dimensions these two sets are equal (Dierker, 1982).

Quinn (1970) will tell us that these two sets coincide; precisely, in diagram (2), $\rho_a \pitchfork F$ if and only if a is a regular value of $\pi_A \circ h$. And so we get,

Proposition 5. *The economy $\omega \in \Omega$ is regular if and only if all equilibrium prices of Z_ω are regular.*

Proof. Consider the diagram

$$\begin{tikzcd}
\Gamma \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Omega \times S \arrow[r, "Z"] \arrow[d, "pr_\Omega"'] & TS \\
\Omega &
\end{tikzcd}$$

Quinn's result says that the excess demand Z_ω is transversal to zero if and only if 0 is a regular value of pr_Ω. $\square$

3.4 Determinacy of equilibrium

We would now like to understand how big is the set of economies that give an excess demand function with all equilibrium prices being regular. For that, we need a couple of definitions.

A **left Fredholm map**[1] is a map of Banach manifolds of class at least C^1 whose derivative at each point has closed image and finite dimensional kernel.

A map is $\sigma-$**proper** if its domain is the countable union of sets, restricted to each of which the function is proper.

Quinn has also proved that a transversal density theorem holds in infinite dimensions.

Theorem 1. *(Quinn, 1970) Let $\rho : A : M \to N$ be a C^∞ representation of left Fredholm maps, M separable, and $F : W \to N$ a C^∞ σ-proper left Fredholm map. If further*

[1] some authors call it a semi-Fredholm map

1. *F is transversal to Ev_ρ, and*

2. *each ρ_a satisfies that for each $m \in M$ and $w \in W$ such that $\rho_a(m) = F(w)$, then $(imT_m\rho_a) \cap (imT_wF)$ is finite dimensional*

then the set of a with $\rho_a \pitchfork F$ is residual in A.

The infinite-dimensional transversal density theorem can be used to give us an alternative proof that almost all economies are regular.

Proposition 6. *Almost all economies are regular. That is, the set of economies $\omega \in \Omega$ that give rise to an excess demand function Z_ω with only regular equilibirum prices, are residual in Ω.*

Proof. We have seen that $Z_\omega : S \to TS$ is a Fredholm map. In particular it is a left Fredholm map. Observe that the inclusion $0 \to C(M, \mathbb{R}^n)$ is σ-proper.

We also know that $Z(\omega, P)$ has 0 as a regular value since $DZ(\omega, P)$ is surjective. Finally, for each $P \in S$ such that $Z_\omega(P) = 0$ we have

$$(\mathrm{im}T_P Z_\omega) \cap (\mathrm{im}T_0 0)$$

is finite dimensional since obvioulsy 0 is finite. Therefore, theorem 1 implies the result. □

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