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Abstract:

Very little is known on how traditional risk metrics behave in ultra high frequency trading (UHFT). We fill this void firstly by examining the existence of the intraday returns moments, and secondly by assessing the impact of their (non)existence in a risk management framework. We show that in the case of UHFT, the returns third and fourth moments do not exist, which entails that traditional risk metrics are unable to judge capital adequacy adequately. Hence, the use of risk management techniques such as VaR by market participants who engage with UHFT impose serious threats to the stability of financial markets, given that capital ratios may be severely underestimated.

Keywords: ultra high frequency trading, risk management, finite moments, superkurtosis

JEL Classification: C12, C54, F30, G10, G15, G17.

1. Introduction

Highly sophisticated algorithms and fast computer technology have originated a new class of trading known as Ultra High Frequency Trading (UHFT). UHFT has numerous advantages: it offers a great deal of liquidity in the market; it facilitates the instantaneous transmission of information into prices, pushing markets to be more efficient; and it creates a market place for small (retail) as well as large investors (institutions).

However, UHFT also presents unique challenges,[1] having been criticised as liable to cause large market crashes,[2] which may be amplified by the influx of algorithmic trading and the order clustering caused by unintended trading strategy coordination (Beddington et al., 2012). Hence, regulators[3] and law scholars (e.g. Yadav, 2015) have proposed measures to curb UHFT. As a consequence, market participants are required to measure and report several market risk metrics, and to take them into account when calculating their regulated capital requirements.[4,5] However, standards set by regulators are based on risk metrics that are calculated - at most - at daily frequency. Given that UHFT takes place at higher frequencies, this leaves the market risk generated by UHFT largely as a dark pool.

Very little is known about market risk associated with UHFT, and similarly little analysis has been conducted on how traditional risk metrics such as Value at Risk (VaR, hereafter) behave at such frequencies. In this note, we fill this gap by firstly investigating the existence of moments of intraday returns, and subsequently by assessing the impact of moments (non)existence in a risk management framework, employed on UHFT strategies. Specifically, we test for the existence of the first four (absolute) moments, using ultra high frequency data from the currency market. We find that the distribution of the returns of the assets traded under UHFT does not have finite moments of order higher than 2, implying that only the mean and the variance exist at intraday frequencies. This implies that the VaR is infinite when calculated with frequencies higher than daily. Hence, traditional risk measures like VaR are not a good metric for the true market risk (see also Bradley and Taqqu, 2003), and should therefore not be employed to gauge capital adequacy under UHFT. To put it differently, we find that the potential capital loss implied by VaR is unlimited in the presence of UHFT, due to the phenomenon we call *superkurtosis*.

2. Methodology

Our analysis is based on two steps. We start by verifying whether higher order moments exist (Section 2.1); we then turn to assessing the impact of potential non-existence of high order moments on VaR (Section 2.2).

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[1] See for example the Final Project Report from The Government Office for Science, London - 2012

[2] See for example Bloomberg article in April 21st, 2015 by Silla Brush, Tom Schoenberg and Suzi Rin: How a Mystery Trader With an Algorithm May Have Caused the Flash Crash and Kirilenko et al. (2017) for a suggested solution.

[3] See for example the Press Release, European Parliament, MEPs Vote Laws to Regulate Financial Markets and Curb High Frequency Trading (Apr. 15, 2014).

[4] For example, on January 16th, 2016 the Basel Committee on Banking Supervision published a document that revised standards for minimum capital requirements for Market Risk.

[5] Consistent with the policy rationale underpinning the Committee has three consultative papers on the Fundamental review of the trading book. (i) Fundamental review of the trading book, May 2012, (ii) A revised market risk framework, October 2013 and (iii) Fundamental review of the trading book: Outstanding issues, December 2014.

2.1. Testing for the existence of the asset returns' moments.

We test for the existence of up to the fourth moment of y_t; in particular, we use the test proposed by Trapani (2016) (see also Fedotenkov, 2013) for

$$\begin{cases} H_0 : E\left|y_t\right|^k = \infty \\ H_A : E\left|y_t\right|^k < \infty \end{cases}, \tag{2.1}$$

with $k = 2$, 3 and 4. In (2.1), the null hypothesis is the non-existence of the k-th absolute moment. Following the guidelines in Trapani (2016), for each k, test statistics are based on

$$\mu_k = c_k \times \frac{T^{-1}\sum_{t=1}^{T}\left|y_t\right|^k}{\left(T^{-1}\sum_{t=1}^{T}\left|y_t\right|^p\right)^{k/p}} \tag{2.2}$$

where $p = \min\{k-1, 2\}$ and

$$c_k = \begin{cases} \frac{4}{\pi} & \text{when } k = 2 \\ 1 & \text{when } k = 3 \\ \frac{1}{3} & \text{when } k = 4 \end{cases}. \tag{2.3}$$

Some comments on (2.2) and (2.3) are in order. The first statistic to be employed is μ_2, which has been designed to test for $H_0 : E\left|y_t\right|^2 = \infty$ - i.e., the non-existence of the variance. When $k = 2$, the sample second moment (at the numerator) is made scale-invariant by dividing by the square of the mean absolue value of y_t; other rescaling would be possible (chiefly, the median, which has the advantage of being well-defined), but the simulations in Trapani (2016) show that the mean absolute value yields better power and size. For $k = 2, 3$, rescaling is done using the sample variance, as is more natural. Turning to the multiplicative constants, these follow the guidelines in Trapani (2016), where each sample moment is rescaled by the corresponding sample absolute moment of a standard normal distribution.
Based on (2.2)-(2.3), we construct the test statistic

$$\psi_k = \exp\left(\mu_k\right) - 1. \tag{2.4}$$

Trapani (2016) showed that

$$P\left\{\omega : \lim_{T\to\infty} \psi_k = \infty\right\} = 1, \text{ under } H_0 : E\left|y_t\right|^k = \infty, \tag{2.5}$$

$$P\left\{\omega : \lim_{T\to\infty} \psi_k = 0\right\} = 1, \text{ under } H_A : E\left|y_t\right|^k < \infty. \tag{2.6}$$

Under H_0, ψ_k diverges to positive infinity instead of having a limiting distribution. Thus, we randomise it to produce a test statistic which has a well-defined limiting law, using the following algorithm.

Step 1 Randomly generate an *i.i.d.* $N(0,1)$ sample of size $R = \lfloor N^{1/2} \rfloor$, say $\left\{\xi_j^{(k)}\right\}_{j=1}^{R}$, independently across k, and define $\left\{\psi_k^{1/2} \times \xi_j^{(k)}\right\}_{j=1}^{R}$.

Step 2 For $u = \left\{-\sqrt{2}, \sqrt{2}\right\}$, generate $\zeta_{j,n}^{(k)}(u) = I\left(\psi_k^{1/2} \times \xi_j^{(k)} \leq u\right)$, $1 \leq j \leq r$.

Step 3 For each u, define

$$\vartheta_{n,R}^{(k)}(u) = \frac{2}{\sqrt{R}} \sum_{j=1}^{R} \left[\zeta_{j,n}^{(k)}(u) - \frac{1}{2} \right], \tag{2.7}$$

and finally the test statistic

$$\Theta_{n,R}^{(k)} = \frac{1}{2} \left[\left(\vartheta_{n,R}^{(k)} \left(-\sqrt{2} \right) \right)^2 + \left(\vartheta_{n,R}^{(k)} \left(\sqrt{2} \right) \right)^2 \right]. \tag{2.8}$$

Following the proofs in Trapani (2016) (see also Horváth and Trapani, 2017), it holds that

$$\Theta_{n,R}^{(k)} \xrightarrow{d^*} \chi_1^2, \text{ under } H_0, \tag{2.9}$$

$$R^{-1} \Theta_{n,R}^{(k)} \xrightarrow{P^*} 1, \text{ under } H_A, \tag{2.10}$$

as $T \to \infty$ for almost all realisations of $\{y_t\}_{t=1}^T$. In (2.9) and (2.10), "$\xrightarrow{d^*}$" and "$\xrightarrow{P^*}$" denote convergence in distribution and in probability, respectively, with respect to P^*, defined as the probability conditional on $\{y_t\}_{t=1}^T$.

2.2. Assessing the impact of (non)existence of moments in risk management.

We consider a representative trader with unlimited capital, who wants to calculate the VaR measure at each point in time. Each trading day t is divided in τ equidistant intraday subintervals. The observed prices at day t are denoted as P_{t_j}, for $j = 1, 2, ..., \tau$, with sample frequency defined as $m = \tau^{-1}$. We define daily log-returns as $y_t = logP_{t_\tau} - logP_{(t-1)_\tau}$, and intraday log-returns as $y_{t_j} = logP_{t_j} - logP_{t_{j-1}}$.
Hence, the VaR for a long trading position at $(1 - p)$ level of confidence, at sampling frequency m, is defined as $VaR_{(1-p)}^{(m)}$ such that $P(y_{t_j} \leq VaR_{(1-p)}^{(m)})$. We compute $VaR_{(1-p)}^{(m)}$ non-parametrically, as the p-quantile of the of the log-returns at sampling frequency m, $\{y_{t_j}, 1 \leq j \leq \tau, 1 \leq t \leq T\}$.
We measure the potential losses conditional to a VaR violation (i.e. the losses that occur when the returns are lower than the VaR measure) by constructing an evaluation function, $l_{t_j}^{(m)}$ that measures the absolute distance between actual returns, y_{t_j}, and the VaR measure:

$$l_{t_j}^{(m)} = \begin{cases} |y_{t_j} - VaR_{(1-p)}^{(m)}| & \text{if } y_{t_j} < VaR_{(1-p)}^{(m)} \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

The total potential losses over the sample period are computed as $L^{(m)} = \sum_{t=1}^{T} \sum_{j=1}^{\tau} l_{t_j}^{(m)}$. To allow comparison across the different sampling frequencies we compute the daily adjusted losses per VaR violation as $\bar{L}^{(m)} = 1361(Nm)^{-1} L^{(m)}$, for $N = \sum_{t=1}^{T} \sum_{j=1}^{\tau} I_{t_j}^{(m)}$, where:

$$I_{t_j}^{(m)} = \begin{cases} 1 & \text{if } y_{t_j} < VaR_{(1-p)}^{(m)} \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

We multiply the number of violations by the daily adjustment, $1361/m$, where 1361 reflects the 1-minute observations per day that the market is open.

3. Data and empirical results

We use 1-minute data of the front-month futures contracts for the EUR/USD exchange rate, obtained from TickData. The period of the study spans from August 1, 2003 to August 5, 2015. We focus on the exchange rate market as it is considered continuously trading, so that we do not have to take into consideration significant data alterations. The choice of the specific exchange rate is justified by the fact that *(i)* it is the most liquid currency pair, and *(ii)* it represents the most heavily traded exchange rate for financial transactions. Our final sample consists of 3028 trading days, which contain more than 16 million 1-minute data.
We start our analysis with results on the existence of moments in Table 1.[6] The results show that only the second moment exists across all sampling frequencies. By contrast, the null hypothesis for the non-existence of the third and fourth moments cannot be rejected for all intraday sampling frequencies, suggesting that these moments do not exist. On the contrary, moving to lower sampling frequencies, i.e. daily, weekly and biweekly, the existence of the relevant moments is statistically valid.

TABLE 1
Tests for moments existence

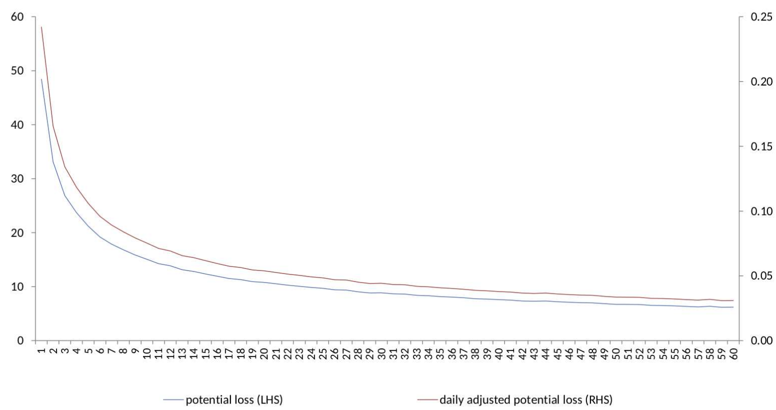
Series	**Size**	μ_2		μ_3		μ_4	
		$\Theta^{(2)}_{n,R}$	*p-value*	$\Theta^{(3)}_{n,R}$	*p-value*	$\Theta^{(4)}_{n,R}$	*p-value*
1m	$5,350,729$	0.122	0.00	2313	0.95	1.78×10^6	0.66
2m	$2,675,323$	0.245	0.00	1635	0.28	8.91×10^5	0.67
5m	$1,070,149$	0.613	0.00	1034	0.42	3.56×10^5	0.75
10m	$535,033$	1.223	0.00	731	0.79	1.78×10^5	0.85
15m	$356,719$	1.839	0.00	597	0.36	1.18×10^5	0.71
20m	$269,464$	2.434	0.00	519	0.57	8.98×10^4	0.80
30m	$178,318$	3.679	0.00	422	0.62	5.94×10^4	0.84
60m	$95,140$	6.896	0.05	308	0.56	3.17×10^4	0.90
daily	$4,035$	0.042	0.00	1.56	0.00	0.99	0.00
weekly	815	1.230	0.00	0.044	0.00	1.73	0.00
biweekly	375	0.041	0.00	1.508	0.00	0.836	0.00

Table 1 suggests that it is important to study the impact of the non-existence of higher order moments in a risk management setting. To this end, we consider a scenario where traders assume that moments do exist at the intraday frequencies. Under this scenario, we calculate the potential losses ($L^{(m)}$ and $\bar{L}^{(m)}$), conditional to VaR violation, as shown in Section 2.2 (see Figure 1).

In all cases considered, potential losses are decidedly higher for the higher sampling frequencies, while they decrease gradually as the sampling frequency decreases. This holds for both the total potential losses ($L^{(m)}$) and the daily adjusted potential losses ($\bar{L}^{(m)}$). For instance,

[6] We have also considered lower frequencies than the biweekly and the results show that all moments continue to exist

FIG 1. *Potential losses under VaR*



in the 1-minute sampling frequency we observe that a trader would have lost 48 times more capital than anticipated by the VaR, whereas the daily adjusted losses, for the same frequency, are about 25% more.

4. Conclusions

The findings of this study are reported for the first time in the literature. We maintain that these results stem from *superkurtosis* - i.e., from the non-existence of higher order moments at high frequencies. We note that these extreme losses in the higher frequencies are calculated assuming that a trader believes that the higher moments do exist. Hence, we can deduct that real losses are significantly higher (even infinite) since we have already established that these moments do not exist.
Therefore, employing traditional risk measures for market participants who engage with UHF imposes serious threats to the stability of the financial markets, given that capital ratios may be severely underestimated.

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