Learning Through Hiring: Knowledge From New Workers as an Explanation of Endogenous Growth

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Abstract

This paper develops an endogenous growth model in which the job-to-job transition of workers provides a channel for the spillover of knowledge between firms. Workers learn some of the productive knowledge used by their employer while working on the job. When a worker moves to another firm, they are able to adapt some of this knowledge for use at the hiring firm. Firms endogenously control their exposure to new knowledge by choosing the intensity that they post vacancies in a search-and-matching labor market. It is shown that under a set of assumptions regarding the initial distribution of firm types and the vacancy posting cost function, the competitive equilibrium leads to a balanced growth path that has a constant growth rate and stationary distribution of firm size.

Keywords: Endogenous growth, productivity, labor mobility, search and matching market, knowledge diffusion data.
JEL classification: O33, J60, O40

1 Introduction

Since the seminal work on endogenous growth by Romer (1986) and Lucas (1988), the spillovers (or positive externalities) of new knowledge have been viewed as an important dynamic in generating sustainable long-run growth.[1] One channel through which knowledge is thought to spillover between firms is through hiring workers with experience at other firms. Workers are exposed to their firm's production, managerial, and marketing knowledge on a daily basis. While some of this knowledge can be protected (e.g. patents) not all knowledge can. Therefore, when workers moves to another firm they are able to adapt some of the knowledge obtained at their previous employer for use at their new job, improving the stock of productive knowledge used at the hiring firm. What's more, incumbent workers at the hiring firm will be able to absorb some of the information that new worker brings, and can

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[1]See Griliches (1992) for an early review of R&D spillovers.

diffuse the knowledge further if they switch jobs, multiplying the initial social impact of the knowledge diffusion.

There is empirical support for the notion that firms learn from the new workers they hire. According to the 2013 Business Operations Survey, a nationally representative survey of New Zealand firms by Statistics New Zealand (the national statistical agency), 52 percent of businesses who reported undertaking some form of innovation in the previous two years stated that new staff were an important source of ideas for the innovation.[2] Furthermore, analysis of linked employer-employee data by Stoyanov and Zubanov (2012), Parrotta and Pozzoli (2012), and Serafinelli (2015) find correlations between firm-level hiring patterns and productivity growth that support the idea of knowledge spillover from new hires.

This paper develops an endogenous growth model in which learning from new hires is the mechanism by which knowledge diffuses between firms. Workers are assumed to passively absorb the productive knowledge of their current employer. They carry out on-the-job searching for new employment opportunities, and when they move to a less productive firms, they are able to transplant some of their knowledge into the hiring firm. The firms themselves are heterogeneous in how many workers they employ and the stock of productive knowledge they use. A firm's exposure to new knowledge is endogenously determined by the firm.

Standard endogenous growth models that feature knowledge spillover typically abstract from the mechanism by which firms learn, and usually assume an exogenous learning rate. The novel feature of this model is that it gives structure to the learning mechanism. Within the model of this paper, firms optimally choose the rate at which they learn by varying their vacancy posting rate. A higher vacancy posting rate leads to a greater inflow of new workers and hence a greater exposure to new knowledge, but also incurs a greater search cost. In addition, the rate at which vacancies posted by any firm are matched to new workers is endogenously determined by the tightness in the labor market.

The aim of this paper is to show that there is a certain set of assumptions that are sufficient for the competitive equilibrium to be a balanced growth path. Along this balanced growth path all firms improve productivity at the same rate and the distribution of firm size is constant throughout time. Such a balanced growth path is consistent with those in standard endogenous growth models featuring knowledge spillover at an exogenous rate. As a result, learning from new hires can be viewed as one possible explanation for how firms learn from other firms in standard endogenous growth models.

The rest of the paper is organized as follows. Section 2 discusses how this paper fits into the existing literature. The structure of the theoretical model is presented in Section 3. Section

[2] The types of innovation asked about were: (i) product innovation: "did this business introduce into the market any new or significantly improved goods or services?"; (ii) process innovation: "did this business implement any new or significantly improved operational processes (i.e. methods of producing or distributing goods or services)?"; (iii) organizational innovation: "did this business implement any new or significantly improved organizational/managerial processes (i.e. significant changes in this businesses strategies, structures or routines)?"; and (iv) marketing innovation: "did this business implement any new or significantly improved sales or marketing methods which were intended to increase the appeal of goods or services for specific market segments or to gain entry to new markets?"

4 presents a set of assumption on the initial distribution of firm type and the vacancy posting cost function and shows that these assumptions are sufficient to generate a balanced growth path. Section 5 concludes.

2 Related Literature

Standard endogenous growth models that rely on the spillover of knowledge typically assume that agents receive learning opportunities costlessly at an exogenous rate (see Luttmer, 2012 and Luttmer, 2015 as examples). However, some progress has been made towards adding structure to the learning process. Perla and Tonetti (2014) and Lucas and Moll (2014) develop models that introduce an endogenous search effort decision into the knowledge spillover framework. Within both models, agents optimally trade off the amount of time they spend spend producing today with the time they spend searching for new learning opportunities to become more productive in the future.

Similar to those two models, the model developed in this paper features an endogenous search effort choice embedded within the firm's vacancy posting rate decision. Firms that post more vacancies attract more workers and are therefore exposed to more knowledge spillover from new workers. However, they also incur greater search costs. Furthermore, the rate at which firms learn depends not only on their choice of vacancy postings, but also the endogenous tightness of the labor market which is related to the vacancy posting rate of all other firms.

Previous papers that have studied labor mobility as a channel for diffusing knowledge in a dynamic context primarily have used an overlapping-generations framework.[3] For example, Dasgupta (2011) and Monge-Naranjo (2012) both develop overlapping-generations models in which workers learn on the job when young and use the knowledge that they acquire to start their own firms later in life. Within these types of model, knowledge spillover is only from incumbent firms to new firms. The model developed in this paper uses the hiring of new workers as a way to explain how incumbent firms are able to learn from other incumbent firms. It also includes a search-and-matching labor market for the workers which introduces frictions into the knowledge spillover process.

The labor market used in this model draws on the modeling techniques developed in the search-and-matching literature. On one side of the labor market workers search while on the job, as in the model developed by Mortensen (2010). On the other side of the labor market, large (employing multiple workers) heterogeneous firms post job vacancies similar to the models of Mortensen (2010) and Acemoglu and Hawkins (2014). The wage setting equation used in this paper is also adopted from the approach taken in both models.

[3]Partial equilibrium analysis of this same channel has also been conducted in the area of game theory. For example see Fosfuri et al. (1998) and Glass and Saggi (2002).

3 The Model

Time is continuous. There exists a continuum of firms of measure one, and continuum of workers of measure one. Both firms and workers are infinitely lived. Firms are large (each employing a measure of workers), have different stocks of productive knowledge, and produce differentiated goods. The only active decision made by firms is the rate at which to post job vacancies in order to hire new workers. Hiring new workers benefits the firm in two ways. First, it increases the amount of labor the firm can utilize in the production process. Second, it increases the stock of productive knowledge at the firm due to knowledge spillovers from workers with previous experience at firms with superior productive knowledge.

All workers supply one unit of labor inelastically. Where workers differ is in their level of knowledge. Through their participation in the production process, workers absorb the productive knowledge of their current employer. Workers who move to a less productive firm are capable of adapting some of this knowledge for use at the hiring firm.

For simplicity, the model abstracts from both the labor-market search intensity decision of workers, and the negotiated premium firms would be willing to pay worker to entice them away from their previous employer. Instead, all workers are assumed to search for new jobs with the same search effort, and all workers will choose to move to a new firm if the wage paid by the hiring firm is at least as high as the wage paid by the worker's current employer. These simplifications remove the need to explicitly model the value of the worker.

Below, the details of the goods market, labor market, knowledge diffusion technology, and the decision choice of the firms are presented.

3.1 The Goods Market

3.1.1 The Demand for the Final Consumption Good

Each worker belongs to an identical household. The households have an inter-temporal utility function given by

$$\mathbb{E}_0\left[\int_0^\infty e^{-\kappa t} U(C(t))\,dt\right] \;=\; \mathbb{E}_0\left[\int_0^\infty e^{-\kappa t}\frac{C(t)^{1-\sigma}}{1-\sigma}\,dt\right],$$

where $\kappa > 0$ is the rate of time preference, $1/\sigma > 0$ is the elasticity of inter-temporal substitution, $U(\cdot)$ is the period utility function, and $C(t)$ is the amount of final consumption good consumed at date t.

The final consumption good comprises of a Dixit-Stigliz CES aggregation of the individual goods produced by the continuum of firms,

$$C(t) = \left[\int_{i=0}^{1} c(i,t)^{(\rho-1)/\rho}\,di\right]^{\rho/(\rho-1)}, \qquad \text{all } t,$$

where $c(i,t)$ is the consumption of the good produced by firm $i \in [0,1]$, and $\rho > 1$ is the elasticity of substitution between the output of different firms.

From the household's optimization problem, the inverse demand for the output of firm i is given by,

$$p(i,t) = P(t)\left(\frac{c(i,t)}{C(t)}\right)^{-1/\rho}, \qquad \text{all } i \in [0,1],\ t, \tag{1}$$

where $p(i,t)$ is the price of good i and

$$P(t) = \left[\int_{i=0}^{1} p(i,t)^{1-\rho}\, di\right]^{1/(1-\rho)}, \qquad \text{all } t,$$

is the aggregate price index at date t. Without loss of generality, prices will be normalized so that $P(t) = 1$ for all t.

The household's preferences also determine the real interest rate,

$$r(t) = \kappa + \sigma g(t), \qquad \text{all } t, \tag{2}$$

where

$$g(t) = \frac{d}{dt} \ln C(t), \qquad \text{all } t, \tag{3}$$

is the growth rate of the final consumption good.

3.1.2 Firms

Each firm produces a unique variety of good and is monopolistically competitive in the goods market. Firms are also heterogeneous along two other dimensions. First, each firm has a stock of productive knowledge that acts as technology in the production function. The minimum level of knowledge/productivity at any firm at date $t = 0$ is $z_{min,0} \geq 0$. Second, firms differ by the labor employed, $l \in \mathbb{R}_{\geq 0}$. Throughout this paper, the term "firm size" will always refer to the amount of employment at a firm.

Let $\mathcal{S}$ denote the joint productivity-size space for firms, and let $s \equiv (z,l)$ identify a state in $\mathcal{S}$ for a firm. The state of a particular firm will be referred to as the firm's "type". The distribution of firm types at date t is denoted by $F(s,t)$.

All firms combine their stock of productive knowledge with the labor of their workers to produce their unique good using the following production function

$$y(s) = z l^{\alpha}, \qquad \text{all } s \in \mathcal{S}, \tag{4}$$

where $0 < \alpha \leq 1$ is the labor elasticity parameter.

Because the output of all firms enter the CES aggregator for the final consumption good symmetrically, firms can be indexed by their type (s) rather than the good they produce (i). Therefore, aggregate output of the final good at date t can be written as

$$Y(t) = \left[\int_{\mathcal{S}} y(s)^{(\rho-1)/\rho}\, dF(s,t)\right]^{\rho/(\rho-1)}, \qquad \text{all } t. \tag{5}$$

Given the demand that firms faces, the revenue of an s-type firm operating at date t is

$$R(s,t) = p(s,t)y(s), \qquad \text{all } s \in \mathcal{S},\ t. \tag{6}$$

3.2 Wages

The wage setting mechanism between a firm and its workers follows the approach used by both Mortensen (2010) and Acemoglu and Hawkins (2014). It is a generalization of Nash-bargaining for one large firm negotiating with many workers, in the spirit of Stole and Zwiebel (1996).

Long term wage contracts are not feasible. Wage negotiations are costless and instantaneous, allowing the firm to renegotiate individually with all workers in as many negotiation rounds as needed before production takes place at each point in time.

If the firm and a particular worker are not able to agree to a mutually satisfactory wage, the worker sits out the production process and earns zero income at that date. The firm will then produce output using the pool of workers that it was able to negotiate a mutually satisfactory wage with. Because all workers at the firm are homogeneous in the amount of labor they supply and have the same bargaining strength, in equilibrium all workers will agree to work, and the wage for all workers at the firm, $\omega(s,t)$, will satisfy a surplus sharing rule

$$\beta\frac{\partial \pi(s,t)}{\partial l} = (1-\beta)\omega(s,t), \qquad \text{all } s \in \mathcal{S},\ t, \tag{7}$$

where $\partial\pi(s,t)/\partial l$ denotes the firm's marginal profit of labor, $0 < \beta < 1$ denotes the worker's relative bargaining strength, and the wage $\omega(s,t)$ measures the worker's marginal benefit from working.

The firm's profit function, $\pi(s,t)$, has the standard form,

$$\pi(s,t) \;=\; R(s,t) - \omega(s,t)l, \qquad \text{all } s \in \mathcal{S},\ t, \tag{8}$$

where $\omega(s,t)l$ is the total wage cost to the firm for all units of labor employed.

Substituting (8) and (6) into (7) and solving for the wage rate ω shows that the wage at an s-type firm at date t is equal to a β share of the firm's marginal revenue of labor,

$$\omega(s,t) = \beta\frac{\partial R(s,t)}{\partial l}, \qquad \text{all } s \in \mathcal{S},\ t. \tag{9}$$

Substituting (9) and (1) into (8) yields the profit function of the firm as a function of the firm's output,

$$\widehat{\pi}(y(s),t) = (1-\widehat{\beta})P(t)Y(t)^{1/\rho}y(s)^{1-1/\rho}, \qquad \text{all } s \in \mathcal{S},\, t, \tag{10}$$

where $\widehat{\beta} \equiv \beta\alpha(1-1/\rho)$.

Similarly, substituting (6) and (1) into (9) provides the wage as a function of the firm's output,

$$\widehat{\omega}(y(s),t) = \beta P(t)Y(t)^{1/\rho}\frac{\partial y(s)^{1-1/\rho}}{\partial l}, \qquad \text{all } s \in \mathcal{S},\, t. \tag{11}$$

The average wage across all workers at date t is defined as

$$\overline{\omega}(t) = \int_{\mathcal{S}} l'\omega(z',l',t)\, dF(s',t), \qquad \text{all } t. \tag{12}$$

3.3 Worker's Moving Decision

Workers seek to maximize their current wage. Therefore, when a worker is matched with another firm in the labor market, the worker will choose to change firms if the wage offered by the outside firm is equal or greater than the wage paid by their current employer.

Let $\mathcal{A}(s,t)$ be the set of type $s'=(z',l')$ firms that an $s=(z,l)$ type firm can poach workers from (i.e. workers at s'-type firms would choose to move to an s-type firm if matched). The region $\mathcal{A}(s,t)$ is defined by the set of type s' firms where

$$\mathcal{A}(s,t) \equiv \{s' \in \mathcal{S} \,:\, \omega(s,t)-\omega(s',t) \geq 0\}, \qquad \text{all } s \in \mathcal{S},\, t. \tag{13}$$

Furthermore, let $\widehat{\mathcal{A}}(s,t)$ denote the sub-region of $\mathcal{A}(s,t)$ where $z' > z$ (i.e. the set of more productive firms from which an $s=(z,l)$ type firm can hire from),

$$\widehat{\mathcal{A}}(s,t) \equiv \{s' \in \mathcal{S} \,:\, \omega(s,t)-\omega(s',t) \geq 0 \cap z' > z\}, \qquad \text{all } s \in \mathcal{S},\, t. \tag{14}$$

Similarly, let $\mathcal{B}(s,t)$ be the set of type s' firms that can poach workers from an s-type firm at time t,

$$\mathcal{B}(s,t) \equiv \{s' \in \mathcal{S} \,:\, \omega(s',t)-\omega(s,t) \geq 0\}, \qquad \text{all } s \in \mathcal{S},\, t. \tag{15}$$

3.4 The Labor Market

The labor market is modeled as a single, frictional, search-and-matching market featuring large firms and workers who search while on the job. For simplicity there is no unemployment state for workers.

3.4.1 The Firm's Side of the Labor Market

At each date, firms can choose the rate at which to post job vacancies for new workers. Denote the measure of vacancies an s-type firm posts as $n(s,t) \geq 0$. Firms are assumed to incur a real cost for posting vacancies denoted as $c(n,s,t)$.

3.4.2 The Worker's Side of the Labor Market

For simplicity, this paper abstracts from the search effort decision of workers. Instead, all workers are assumed to search on the job with unit intensity at all points in time. Any search costs to the worker are sunk costs that are constant across all workers, and therefore do not affect the the worker's decisions in any way.

3.4.3 Labor-Market Matching Technology

Posted vacancies and searching workers are matched at random. The measure of matches is given by a homogeneous of degree one Cobb-Douglas matching function. Following the notation of Mortensen (2010), it is convenient to define $\theta(t)$ as the measure of tightness in the labor market (from the firm's perspective) at date t. Because there is a unit mass of workers and each worker searches with an intensity of unity, the total search effort by workers is unity. As a result, $\theta(t)$ can be simplified to the average vacancy postings rate at date t,

$$\theta(t) = \int_{\mathcal{S}} n(s,t)\, dF(s,t), \qquad \text{all } t. \tag{16}$$

The probability that a firm's posted job vacancy is matched with a searching worker is given by the matching function

$$q(\theta) \equiv \overline{q}\theta^{-\mu}, \tag{17}$$

where $0 < \mu < 1$ is the matching elasticity, and $\overline{q} > 0$ is a normalizing scalar which can either be interpreted as capturing the efficiency of the matching function or as an exogenous (and constant) probability that an acceptable match results in the worker moving.

From the point of view of a firm, posted vacancies are matched at the Poisson rate $n(s,t)q(\theta(t))$. Because workers are infinitesimally small in size, the continuum of posted vacancies by the firm are matched in the labor market with a continuum of searching workers drawn from the entire distribution of searching workers, $lF(s',t)$. From the perspective of workers, their search effort yields a matches at the Poisson rate $\theta(t)q(\theta(t))$.

3.5 The Evolution of Firm Productivity and the Spillover of Knowledge From New Workers

Productive knowledge is equally applicable to all firms. Therefore a firm's productivity level, z, can be used to rank firms by their knowledge. When workers move between firms, they take with them their productive knowledge from their previous employer to their new employer. Only those workers hired from more productive firms contribute to the learning of new knowledge at the hiring firm. Workers from less productive firms have inferior knowledge that the hiring firm will choose to disregard in favor of their current productive practices. Therefore, when workers move to more productive firms it is the worker who learns from the firm, and the worker can potential diffuse this new knowledge in a subsequent job move.

When a worker is matched with another firm, the probability that a worker can successfully transfer knowledge from their current employer to the new firm is τ/l, where $0 < \tau \leq 1$ is a reduced-form parameter reflecting the efficiency at which knowledge can be transferred between firms. The term $1/l$ reflects the fact that workers from larger firms are less likely to diffuse productivity knowledge.

At each point in time a firm hires a continuum of workers from other firms. Each of these workers brings with them some new knowledge the hiring firm can exploit. This type of learning process is what Buera and Lucas (2018) refer to as "continuous arrival" of ideas. This represents the limiting case where each individual worker has a chance to spill over knowledge at a Poisson rate. As the size of each worker goes to zero, the number of learning opportunities increases for the firm (as the firm is hiring more workers), but this benefit is offset by a reduction in the ability of each individual worker to transfer knowledge.

Formally, the productivity level of an $s = (z, l)$ type firm evolves as

$$\frac{dz}{dt} = \tau n q(\theta(t)) z \int_{\widehat{\mathcal{A}}(s,t)} \ln(z'/z)\, dF(s',t), \qquad \text{all } z \geq z_{min,0},\ t, \tag{18}$$

where n is the number of vacancies posted which yields $nq(\theta(t))$ total matches in the labor market, and the expected improvement in productivity level from a match is given by $z\tau \int_{\widehat{\mathcal{A}}(s,t)} \ln(z'/z)\, dF(s',t)$ which depends in part upon the set of workers with superior knowledge that the firm can successfully poach, $\widehat{\mathcal{A}}(s,t)$ as defined in (14).

3.6 Value of a Firm

The value of any $s = (z, l)$ type firm at date t, denoted by $\Pi(s,t)$, satisfies the following Hamilton-Jacobi-Bellman equation

$$r(t)\Pi(s,t) = \pi(s,t) + \max_{n \geq 0} \left\{ -c(n,s,t) + \frac{\partial \Pi(s,t)}{\partial z}\frac{dz}{dt} + \frac{\partial \Pi(s,t)}{\partial l}\frac{dl}{dt} + \frac{\partial \Pi(s,t)}{\partial t} \right\}, \text{all } s \in \mathcal{S},\ t, \tag{19}$$

where $\pi(s,t) - c(n,s,t)$ is the flow of profits net of vacancy posting costs, the firm's stock of productive knowledge evolves over time according to (18), and the size of the firm evolves over time according to

$$\frac{dl}{dt} = nq(\theta(t)) \int_{\mathcal{A}(s,t)} l' \, dF(s',t) - l\theta(t)q(\theta(t)) \int_{\mathcal{B}(s,t)} dF(s',t), \qquad \text{all } l,\, t. \tag{20}$$

The first component on the right hand side (RHS) of (20) represents the inflow of new labor that the firm successfully poaches from other firms belonging to the set $\mathcal{A}(s,t)$. The second component represents the outflow of incumbent workers who are poached by other firms belonging to the set $\mathcal{B}(s,t)$.

The firm's optimal choice of vacancy postings is denoted by

$$\nu(s,t) = \arg\max_{n \geq 0} \left\{ -c(n,s,t) + \frac{\partial \Pi(s,t)}{\partial z}\frac{dz}{dt} + \frac{\partial \Pi(s,t)}{\partial l}\frac{dl}{dt} + \frac{\partial \Pi(s,t)}{\partial t} \right\},$$

and will satisfy the FOC:

$$\begin{aligned} \frac{\partial c(\nu(s,t),s,t)}{\partial \nu(s,t)} =& \frac{\partial \Pi(s,t)}{\partial z} \tau q(\theta(t)) z \int_{\widehat{\mathcal{A}}(s,t)} \ln(z'/z) \, dF(s',t) \\ &+ \frac{\partial \Pi(s,t)}{\partial l} q(\theta(t)) \int_{\mathcal{A}(s,t)} l' \, dF(s',t), \qquad \text{all } n \geq 0,\, s \in \mathcal{S},\, t, \end{aligned} \tag{21}$$

which equates the marginal cost of posting vacancies (the left hand side, LHS) with the marginal benefit of a vacancy (the RHS). The marginal benefit comprises of two terms, the marginal benefit from knowledge spillover, and the marginal benefit of additional labor.

3.7 Competitive Equilibrium and Balanced Growth Path of the Model

This section presents the definitions of a Competitive Equilibrium (CE) and Balanced Growth Path (BGP) for the model. The main focus of the paper is to examine a set of assumptions under which the CE of the model leads to a BGP along which productivity grows at a constant rate and the distribution of firm size is constant.

Definition 1. *Given the functions* $[y(s),\, c(\nu,s,t),\, q(\theta)]$, *parameters* $[\alpha,\, \rho,\, \beta,\, \overline{q},\, \mu,\, \kappa,\, \sigma,\, \tau]$, *and initial distribution function* $F(s,0)$, *the aggregate functions* $\{Y(t),\, g(t),\, r(t),\, \theta(t),$ *for all* $t \geq 0\}$, *individual functions* $\{p(s,t),\, R(s,t),\, \pi(s,t),\, \omega(s,t),\, \Pi(s,t),\, \nu(s,t),$ *for all* $s \in \mathcal{S}$ *and* $t \geq 0\}$, *correspondence* $\{\mathcal{A},\, \mathcal{B},\, \widehat{\mathcal{A}},$ *for all* $s \in \mathcal{S}$ *and* $t \geq 0\}$, *and distribution functions* $F(s,t)$ *for* $t > 0$, *are a competitive equilibrium (CE) if*

1. $Y(t)$, $p(s,t)$, $R(s,t)$, $g(t)$, *and* $r(t)$ *satisfy* (1)-(3), (5) *and* (6);
2. $\omega(s,t)$ *and* $\pi(s,t)$ *satisfy* (7) *and* (8);

3. *$\mathcal{A}(s,t)$, $\widehat{\mathcal{A}}(s,t)$, and $\mathcal{B}(s,t)$ satisfy* (13) *to* (15)*;*
4. *$\theta(t)$ satisfies* (16)*;*
5. *$\Pi(s,t)$ satisfies* (18) *to* (20)*, and $\nu(s,t)$ is the maximizing value for n;*
6. *The evolution of $F(\cdot,t)$ is consistent with both* (18) *and* (20)*, and*

$$\begin{aligned}\int_{\mathcal{S}} \tilde{l}\, dF(\tilde{z},\tilde{l},t) =& 1,\\ \int_{\mathcal{S}} dF(\tilde{z},\tilde{l},t) =& 1, \qquad \textit{all } t.\end{aligned}$$

The two conditions in the final requirement above state that at each date t, total employment is unity, and the total mass of firms is also unity. Therefore, $F(s,t)$ is a proper density function at every date.

Characterizing a competitive equilibrium involves solving a fixed point problem in the vacancy posting function $\nu(s,t)$. The vacancy posting choice of any individual firm must maximize the value of that firm taking as given the policy rules followed by all other firms.

The BGP can be defined as follows:

Definition 2. *A competitive equilibrium is a balanced growth path if*

$$F(z,l,t) = F(e^{-\gamma t}z,l,0) \equiv F_0(e^{-\gamma t}z,l), \qquad \textit{all } z \geq z_{min,0},\ l \geq 0,\ t.$$

On a BGP, aggregate output grows at a constant rate γ, and the interest rate is constant,

$$\begin{aligned}Y(t) =& e^{\gamma t}Y_0,\\ g(t) =& \gamma,\\ r(t) =& \overline{r} \equiv \eta + \sigma\gamma, \qquad \text{all } t,\end{aligned}$$

where Y_0 is the aggregate output at date $t = 0$.

The labor market tightness is constant on a BGP,

$$\theta(t) = \overline{\theta}, \qquad \text{all } t.$$

In addition, firm-level prices, revenues, profits, and wages on a BGP satisfy

$$\begin{aligned}p(z,l,t) =& p(e^{-\gamma t}z,l,0) \equiv p_0(e^{-\gamma t}z,l),\\ R(z,l,t) =& e^{\gamma t}R(e^{-\gamma t}z,l,0) \equiv e^{\gamma t}R_0(e^{-\gamma t}z,l),\\ \pi(z,l,t) =& e^{\gamma t}\pi(e^{-\gamma t}z,l,0) \equiv e^{\gamma t}\pi_0(e^{-\gamma t}z,l),\\ \omega(z,l,t) =& e^{\gamma t}\omega(e^{-\gamma t}z,l,0) \equiv e^{\gamma t}\omega_0(e^{-\gamma t}z,l), \qquad \text{all } z \geq z_{min,0},\ l \geq 0,\ t.\end{aligned}$$

As a result, the average wage rate on the BGP grows at the rate γ,

$$\overline{\omega}(t) = e^{\gamma t}\overline{\omega}(0) \equiv e^{\gamma t}\overline{\omega}_0, \qquad \text{all } t.$$

Hence the correspondences describing where firms can poach and be poached from satisfy

$$\begin{aligned}
\mathcal{A}(z,l,t) =& \mathcal{A}(e^{-\gamma t}z,l,0) \equiv \mathcal{A}_0(e^{-\gamma t}z,l), \\
\widehat{\mathcal{A}}(z,l,t) =& \widehat{\mathcal{A}}(e^{-\gamma t}z,l,0) \equiv \widehat{\mathcal{A}}_0(e^{-\gamma t}z,l), \\
\mathcal{B}(z,l,t) =& \mathcal{B}(e^{-\gamma t}z,l,0) \equiv \mathcal{B}_0(e^{-\gamma t}z,l), \qquad \text{all } z \geq z_{min,0},\ l \geq 0,\ t.
\end{aligned} \tag{22}$$

4 Balanced Growth Path Results

The main goal of this paper is to show that under a certain set of assumption, including a linear form for $c(\cdot)$, there exists a BGP. Attention is focused on the BGP of the model where the joint distribution of firm types at date $t = 0$ has the property that all firms with the same level of productive knowledge also have the same firm size. This type of distribution will be referred to as a *one-dimensional distribution*.

Define $\Phi(z,t)$ as the marginal distribution function for z under F at date t,

$$\Phi(z,t) = \int_{z_{min,0}}^{z} \int_0^\infty f(\zeta,l,t)\,dl\,d\zeta, \qquad \text{all } z \geq z_{min,0},\ t.$$

Using Φ, a one-dimensional distribution can be formally defined as follows:

Definition 3. *A distribution $F(\cdot,t)$ is one-dimensional if, for some function $\psi(z)$,*

$$\Phi(z,t) = \lim_{\varepsilon \to 0^+} \int_{z_{min,0}}^{z} \int_{-\varepsilon}^{+\varepsilon} f(\zeta,\psi(\zeta)+u)\,du\,d\zeta, \qquad \textit{all } z \geq z_{min,0},\ t.$$

Therefore, in a one-dimensional distribution all of the distribution's mass lies on the manifold $[z,\psi(z)]$. If the initial distribution is one-dimensional, then clearly it retains that property on a BGP,

$$\Phi(z,t) = \Phi(e^{-\gamma t}z,0) \equiv \Phi_0(ze^{-\gamma t}), \qquad \text{all } z \geq z_{min,0},\ t.$$

4.1 Sufficient Conditions for a BGP

This section details a set of assumptions that are sufficient to generate a BGP when the initial distribution is one-dimensional. The first two assumption relates to the initial distribution of firm type.

Assumption 1. *The initial marginal distribution function for z is a Pareto distribution with location and shape parameters $(z_{min,0},\ \chi)$, where*

$$\chi \equiv \frac{1-1/\rho}{1-\alpha(1-1/\rho)}.$$

Assumption 2. *The initial distribution of firm size satisfies*

$$\psi(z) = \psi_0 z^{\chi}, \qquad z \geq z_{min,0}, \tag{23}$$

where ψ_0 is a constant, determined by the fact that aggregate employment is unity,

$$1 = \psi_0 \int_{z_{min,0}}^{\infty} z^{\chi}\, d\Phi_0(z).$$

According to Assumption 2, the relative size of any two firms is a function of the knowledge/productivity difference between the firms. This leads to the following property in regards to the wage at each firm.

Lemma 1. *Under Assumption 2 the wage rate is the same across all firm types at date $t = 0$.*

Proof. Using (11), (4), and $P(t) = 1$, the wage rates at date $t = 0$ are

$$w_0(z) = W_0 z^{1-1/\rho} \psi(z)^{\alpha(1-1/\rho)-1}, \qquad \text{all } z \geq z_{min,0},$$

where W_0 is a constant that is the same across all firm types.

Assumption 2 implies that

$$z^{1-1/\rho}\psi(z)^{\alpha(1-1/\rho)-1} = (z')^{1-1/\rho}\psi(z')^{\alpha(1-1/\rho)-1}, \qquad \text{all } z \geq z_{min,0},\ z' \geq z_{min,0},$$

which when combined with the expression for $w_0(z)$ above shows that the wage rate is the same at all firms. Q.E.D.

The final assumption made relates to the properties of the vacancy posting cost function.

Assumption 3. *The vacancy posting cost function, $c(n, s, t)$, has the form*

$$c(n, s, t) = l\widehat{c}(n/l)\overline{w}(t), \qquad \textit{all } n \geq 0,\ s \in \mathcal{S},\ t, \tag{24}$$

where $\widehat{c}(\cdot)$ is a strictly increasing, strictly convex, and differentiable function with $\widehat{c}(0) = \widehat{c}'(0) = 0$, and $\overline{w}(t)$ is the economy-wide average wage rate at date t as previously defined in (12).

Under Assumption 3, the vacancy posting cost does not depend upon the firm's stock of productive knowledge. In addition, $c(\cdot)$ is homogeneous of degree one in (n, l), and along the BGP the vacancy posting cost grows at the constant rate γ.

Having outlined the key assumptions, it is now possible to state the main proposition for a BGP.

Proposition 1. *If the initial distribution $F(s,0)$ satisfies Assumptions 1 and 2, and the vacancy posting cost function satisfies Assumption 3, then the competitive equilibrium is a balanced growth path.*

In order to prove Proposition 1 several properties of firms along the one-dimensional manifold of the BGP will first be established in a series of Lemmas. These Lemmas make use of the assumptions given in Proposition 1. Lemma 2 establishes all the possible moves of workers between firms on the BGP. Lemma 3 establishes the vacancy posting rate necessary for firms on the BGP to maintain their size at all dates. Lemma 4 establishes the value of firms that are on the one-dimensional manifold of the BGP and shows that the vacancy posting rate necessary for firms to maintain their size on the BGP is also the rate that causes the productivity of each firm to grow at the same constant rate.

Using the details from these Lemmas, the proof of the main proposition then proceeds in two parts. It is first shown that in a competitive equilibrium, all firm types will choose the same ratio of vacancy postings to firm size, and this rate coincides with the one shown in the Lemmas to maintain each firm's size along the BGP. It is then shown that when all firms are following this vacancy posting rule, the value of the labor market tightness in the competitive equilibrium is unique in value.

Lemma 2. *On the BGP the correspondences $\mathcal{A}$, $\widehat{\mathcal{A}}$, and $\mathcal{B}$ in (22) satisfy*

$$\begin{aligned}
\mathcal{A}(z,\psi(z),0) &\supseteq \{s' \in \mathcal{S} : s' = (z',\psi(z'))\}, \\
\mathcal{B}(z,\psi(z),0) &\supseteq \{s' \in \mathcal{S} : s' = (z',\psi(z'))\}, \\
\widehat{\mathcal{A}}(z,\psi(z),0) &= \{s' \in \mathcal{A}(z,\psi(z),0) \text{ and } z' > z\}, \qquad \text{all } z \geq z_{min,0}.
\end{aligned}$$

Proof. Immediately follows from (22) and Lemma 1. Q.E.D.

The next result, Lemma 3, shows that employment remains constant for every firm on the one-dimensional manifold if, and only if, all firms post vacancies at the rate $n(z,l) = \overline{\theta}l$, where $\overline{\theta}$ is the labor market tightness.

Lemma 3. *On a BGP, if one exists, every firm on the one-dimensional manifold must choose a vacancy posting rate that is proportional to its current employment $\nu(z,l) = \overline{\theta}l$, where $\overline{\theta}$ is the labor market tightness.*

Proof. Using (20) and Lemma 2, for a firm of initial type $(\zeta,l) = [\zeta,\psi(\zeta)]$, employment is constant over time if and only if

$$0 = \frac{dl}{dt} = \nu\left(e^{\gamma t}\zeta,\psi(\zeta)\right)\int_{z_{min,0}}^{\infty}\psi(\zeta')\,d\Phi_0(\zeta') - \overline{\theta}\psi(\zeta), \qquad \text{all } t, \tag{25}$$

where $\int_{z_{min,0}}^{\infty}\psi(\zeta')\,d\Phi_0(\zeta') = 1$ by the fact that there is measure one of workers.

From before, (25) implies that, given $\overline{\theta}$, employment remains constant at all firms if, and only if, their vacancy postings have the form

$$\nu\left(e^{\gamma t}\zeta, \psi(\zeta)\right) = \overline{\theta}\psi(\zeta), \qquad \text{all } \zeta \geq z_{min,0},\ t,$$

which is the same across all firms on the one-dimensional manifold. Q.E.D.

If each firm uses the vacancy posting rate $\nu(z,l) = \overline{\theta}l$, then market tightness is indeed $\overline{\theta}$. But Lemma 3 shows only that choosing $\nu(z,l) = \overline{\theta}l$ is required for employment to remain constant at all firms on the BGP. The last step of the proof of Proposition 1 will be to show that there exists a unique value $\overline{\theta}^*$ for which, given labor market tightness $\overline{\theta}^*$, the choice $\nu(z,l) = \overline{\theta}^* l$ is *optimal* for firms.

Lemma 4. *On a BGP, if one exists, the value function* Π *for any firm on the one-dimensional manifold satisfies*

$$\begin{aligned}\Pi(z, \psi(e^{-\gamma t}z), t) =& e^{\gamma t}\Pi(e^{-\gamma t}z, \psi(e^{-\gamma t}z), 0) \\ \equiv& e^{\gamma t}\Pi_0(e^{-\gamma t}z, \psi(e^{-\gamma t}z)), \qquad \textit{all } z \geq z_{min,0},\ t.\end{aligned} \tag{26}$$

Proof. The optimized value of a firm is the present discounted value of its operating profits (π) less vacancy posting costs (c).

For a firm of type $(e^{\gamma t}\zeta, l) = [e^{\gamma t}\zeta, \psi(\zeta)]$, relative productivity growth is constant if and only if

$$0 = \frac{d\zeta}{dt} - \gamma\zeta = \tau\nu(e^{\gamma t}\zeta, \psi(\zeta))q(\overline{\theta})\int_{\zeta}^{\infty} \ln\left(\frac{\zeta'}{\zeta}\right) d\Phi_0(\zeta') - \gamma, \qquad \text{all } t. \tag{27}$$

This condition holds for all firms on the one-dimensional manifold if, and only if, the first term on the RHS is the same across firms and equal in magnitude to the aggregate growth rate γ.

Under Assumption 1 Φ_0 has a Pareto distribution, so

$$\int_{\zeta}^{\infty} \ln\left(\frac{\zeta'}{\zeta}\right) d\Phi_0(\zeta') = \frac{1}{\chi}\left(\frac{z_{min,0}}{\zeta}\right)^{\chi}, \qquad \text{all } \zeta \geq z_{min,0}.$$

Furthermore, Lemma 3 implies that the vacancy posting rate required for constant employment is $\nu(e^{\gamma t}\zeta, \psi(\zeta)) = \overline{\theta}\psi(\zeta)$. Substituting both of these expressions into (27) and rearranging yields

$$\frac{\psi(\zeta)}{\zeta^{\chi}} = \frac{\gamma\chi}{\tau\overline{\theta}q(\overline{\theta})\left(z_{min,0}\right)^{\chi}}, \qquad \text{all } \zeta \geq z_{min,0}, \tag{28}$$

where the term on the RHS is constant across all firms on the one-dimensional manifold. By Assumption 2, the ratio on the LHS is the same for all firms on the one-dimensional manifold. Therefore, all firms on the manifold improve productivity at the same constant rate. This constant rate is γ.

Because under the assumptions of Proposition 1 all firms on the one-dimensional manifold improve productivity at the same rate and do not change firm size, then

$$\begin{aligned}\pi(e^{\gamma t}\zeta, \psi(\zeta), t) =& e^{\gamma t}\pi_0(\zeta, \psi(\zeta)),\\ c(\nu, e^{\gamma t}\zeta, \psi(\zeta), t) =& e^{\gamma t}c_0(\nu, \zeta, \psi(\zeta)), \qquad \text{all } \zeta \geq z_{min,0},\ t.\end{aligned}$$

Therefore both the profits and vacancy posting cost of each firm on the one-dimensional manifold grow at the constant rate γ. The present discounted value of future net profits for firms on the BGP is thus

$$\Pi(e^{\gamma t}z, \psi(z), t) = e^{\gamma t}\Pi_0(z, \psi(z)), \qquad \text{all } z \geq z_{min,0},\ t.$$

Q.E.D.

From (19),

$$r\Pi(z, l, t) = \pi(z, l, t) - l\widehat{c}(\overline{\theta})e^{\gamma t}\overline{\omega}_0 + \frac{\partial \Pi}{\partial z}\frac{dz}{dt} + \frac{\partial \Pi}{\partial l}\frac{dl}{dt} + \frac{\partial \Pi}{\partial t}, \qquad \text{all } z \geq z_{min,0},\ l \geq 0,\ t, \quad (29)$$

when the vacancy posting cost function takes on the form in Assumption 3, and on a BGP (if one exists) $\overline{\omega}(t) = e^{\gamma t}\overline{\omega}_0$ and $\nu(s) = \overline{\theta}\psi(\zeta)$.

When $l = \psi(e^{-\gamma t}z)$, (26) implies

$$\begin{aligned}\frac{\partial \Pi(z, l, t)}{\partial z} =& \frac{\partial \Pi_0(e^{-\gamma t}z, l)}{\partial z},\\ \frac{\partial \Pi(z, l, t)}{\partial t} =& e^{\gamma t}\left[\gamma \Pi_0(e^{-\gamma t}z, l) - \gamma z \frac{\partial \Pi_0(e^{-\gamma t}z, l)}{\partial z}\right], \qquad \text{all } z \geq z_{min,0},\ l \geq 0,\ t.\end{aligned}$$

On a BGP, $dz/dt = \gamma z$, so

$$\begin{aligned}\frac{\partial \Pi}{\partial z}\frac{dz}{dt} + \frac{\partial \Pi}{\partial t} =& \frac{\partial \Pi_0}{\partial z}\gamma z + e^{\gamma t}\left[\gamma \Pi_0 - \gamma z \frac{\partial \Pi_0}{\partial z}\right]\\ =& \gamma e^{\gamma t}\Pi_0, \qquad \text{all } z \geq z_{min,0},\ t.\end{aligned}$$

Also on a BGP, $dl/dt = 0$ for all z, t if $l = \psi(e^{-\gamma t}z)$.

Substituting these expressions for the derivatives of the value of a firm and dl/dt into (29), the value of firms on the one-dimensional manifold of the conjectured BGP satisfy

$$(\overline{r} - \gamma)\Pi_0(z, \psi(z)) = \pi_0(z, \psi(z)) - \overline{\omega}_0\psi(z)\widehat{c}(\overline{\theta}), \qquad \text{all } z \geq z_{min,0}. \quad (30)$$

To prove the main proposition, it remains to be shown that on the conjectured BGP the

FOC for a vacancy posting in (21) holds for all firms on the one-dimensional manifold. In normalized form that FOC is

$$\begin{aligned}\frac{\overline{\omega}_0}{q(\overline{\theta})}\widetilde{c}'\left(\nu(\zeta)/\psi(\zeta)\right) =&\frac{\partial\Pi_0(\zeta,l)}{\partial\zeta}\tau\zeta\int_{\zeta}^{\infty}\ln\left(\frac{\zeta'}{\zeta}\right)\,d\Phi_0(\zeta')+\frac{\partial\Pi_0(\zeta,l)}{\partial l}\int_0^{\infty}\psi(\zeta')\,d\Phi_0(\zeta')\\ =&\frac{\partial\Pi_0(\zeta,l)}{\partial\zeta}\frac{\tau}{\chi}z_{0,min}^{\chi}\zeta^{1-\chi}+\frac{\partial\Pi_0(\zeta,l)}{\partial l},\qquad \text{all } \zeta\geq z_{min,0},\end{aligned}\tag{31}$$

where the derivatives of Π_0 are evaluated at $(\zeta,l)=(\zeta,\psi(\zeta))$, and where the second line uses Assumption 1 (Φ_0 is a Pareto distribution) and the fact that employment across all firms integrates to unity.

Proof. Proposition 1

Recall that since $L=1$ and all workers search with unit intensity, average labor market tightness is, by definition,

$$\overline{\theta}(t)=\int_{\mathcal{S}}n(s,t)\,dF(s,t),\qquad \text{all } t,$$

where $n(s,t)$ is the vacancy posting rate chosen by the s-type firm. When a firm chooses its vacancy posting rate it take the tightness of the labor market, $\overline{\theta}(t)$, as given and respond optimally.

By Lemma 3, employment remains constant at every firm on the one-dimensional manifold of the BGP if, and only if, firms have the vacancy posting rate

$$n(s,t)=\overline{\theta}(t)l(s,t).$$

But the choices of the firms must also be *optimal*. Given the labor market tightness, a firm's choice must satisfy (31). Define

$$\lambda(\zeta;\overline{\theta})\equiv\frac{\nu(\zeta,\psi(\zeta))}{\psi(\zeta)},$$

as the solution to (31) for firms on the one-dimensional manifold of the BGP, given $\overline{\theta}$. Next, it will be shown that

1. $\lambda(\zeta;\overline{\theta})=\overline{\lambda}(\overline{\theta})$ is the same for all firms (i.e. all $\zeta\geq z_{min,0}$), and
2. there exists a unique value $\overline{\theta}^*$ with the property that $\overline{\lambda}(\overline{\theta}^*)=\overline{\theta}^*$.

To show that all firms follow the same vacancy posting strategy it is necessary to develop expressions for $\partial\Pi_0/\partial\zeta$ and $\partial\Pi_0/\partial l$. For the former, differentiate (30) with respect to ζ to get

$$(\overline{r}-\gamma)\left[\frac{\partial\Pi_0(\zeta,\psi(\zeta))}{\partial\zeta}+\frac{\partial\Pi_0(\zeta,\psi(\zeta))}{\partial l}\psi'(\zeta)\right]=\frac{\partial\pi_0(\zeta,\psi(\zeta))}{\partial\zeta}+\frac{\partial\pi_0(\zeta,\psi(\zeta))}{\partial l}\psi'(\zeta)$$
$$-\overline{\omega}_0\widehat{c}(\overline{\theta})\psi'(\zeta),\qquad \text{all } \zeta\geq z_{min,0},$$

or

$$\frac{\partial\Pi_0}{\partial\zeta}=\frac{1}{\overline{r}-\gamma}\left[\frac{\partial\pi_0}{\partial\zeta}+\left(\frac{\partial\pi_0}{\partial l}-\overline{\omega}_0\widehat{c}(\overline{\theta})\right)\psi'(\zeta)\right]-\frac{\partial\Pi_0}{\partial l}\psi'(\zeta),\qquad \text{all } \zeta\geq z_{min,0},\tag{32}$$

where the derivatives of π_0 are evaluated at $(\zeta,l)=(\zeta,\psi(\zeta))$.

Substitute (32) into (31) to write the vacancy posting FOC as

$$\begin{aligned}\frac{\overline{\omega}_0}{q(\overline{\theta})}\widehat{c}'(\lambda(\zeta;\overline{\theta}))=&\frac{1}{\overline{r}-\gamma}\left[\frac{\partial\pi_0}{\partial\zeta}+\left(\frac{\partial\pi_0}{\partial l}-\overline{\omega}_0\widehat{c}(\overline{\theta})\right)\psi'(\zeta)\right]\frac{\tau}{\chi}\zeta_{0,min}^{\chi}\zeta^{1-\chi}\\&+\frac{\partial\Pi_0(\zeta,l)}{\partial l}\left(1-\psi'(\zeta)\frac{\tau}{\chi}\zeta_{0,min}^{\chi}\zeta^{1-\chi}\right)\\=&\frac{\tau z_{0,min}^{\chi}}{\overline{r}-\gamma}\left[\frac{\partial\pi_0}{\partial\zeta}\frac{\zeta^{1-\chi}}{\chi}+\left(\frac{\partial\pi_0}{\partial l}-\overline{\omega}_0\widehat{c}(\overline{\theta})\right)\psi_0\right]\\&+\frac{\partial\Pi_0(\zeta,l)}{\partial l}\left(1-\tau\psi_0 z_{0,min}^{\chi}\right),\qquad \text{all } \zeta\geq z_{min,0},\end{aligned}\tag{33}$$

where the second part uses Assumption 2 which implies $\psi'(\zeta)\zeta^{1-\chi}/\chi=\psi_0$, and all derivative terms are evaluated at $(\zeta,l)=(\zeta,\psi(\zeta))$.

Because (30) only holds for firms on the one-dimension manifold, it cannot be used to derive an expression for $\partial\Pi_0/\partial l$. Instead, a direct approach is required.

Let $\Delta_{l0}>0$ be a small perturbation in firm size at date $t=0$. Suppose that a firm with initial productivity z begins with $\psi(z)+\Delta_{l0}$ workers. Also suppose that at each date the firm pays the same wage and posts the same number of vacancies as a firm with initial productivity z and $\psi(z)$ workers. Then the firm with the incremental amount of additional workers has the same hiring rate, and the stock of knowledge evolve over time as if it was the firm with $\psi(z)$ workers on the one-dimension manifold. Over time, the incremental workers are poached by other firms at the rate $\overline{\theta}q(\overline{\theta})$. Thus the increment of extra workers remaining at date t is $\Delta(t)=e^{\overline{\theta}q(\overline{\theta})t}\Delta_{l0}$.

At date t, the firm's incremental profit flow per worker is

$$\left.\frac{\partial\pi(\zeta,l,t)}{\partial l}\right|_{l=\psi(e^{-\gamma t}\zeta)}=e^{\gamma t}\left.\frac{\partial\pi_0(e^{-\gamma t}\zeta,l)}{\partial l}\right|_{l=\psi(e^{-\gamma t}\zeta)},\qquad \text{all } \zeta\geq z_{min,0},\ t.$$

Hence at the interest rate $\overline{r}$, the present discounted value of the incremental profit flows resulting from the increment of workers is

$$\frac{\partial \Pi_0}{\partial l} = \frac{1}{\overline{r} - (\gamma - \overline{\theta} q(\overline{\theta}))} \frac{\partial \pi_0}{\partial l}, \qquad \text{all } \zeta \geq z_{min,0}, \tag{34}$$

where both derivative terms are evaluated at $(\zeta, l) = (\zeta, \psi(\zeta))$.

Recall from (19) that hiring has been optimized by firms. Hence the envelop theorem ensures that, to a first-order approximation, the term in braces in (19) does not change in response to the small perturbation Δ_{l0}.

Substitute (34) into (33) to write the FOC as

$$\begin{aligned}
\frac{\overline{\omega}_0}{q(\overline{\theta})} \widehat{c}'(\lambda(\zeta; \overline{\theta})) =& \frac{\tau z_{min,0}^{\chi}}{\overline{r} - \gamma} \left[\frac{\partial \pi_0}{\partial \zeta} \frac{\zeta^{1-\chi}}{\chi} + \left(\frac{\partial \pi_0}{\partial l} - \overline{\omega}_0 \widehat{c}(\lambda(\zeta; \overline{\theta})) \right) \psi_0 \right] \\
&+ \frac{1 - b_0}{\overline{r} - (\gamma - \overline{\theta} q(\overline{\theta}))} \frac{\partial \pi_0}{\partial l} \\
=& \frac{\tau z_{min,0}^{\chi}}{\overline{r} - \gamma} \left[\frac{\partial \pi_0}{\partial \zeta} \frac{\zeta^{1-\chi}}{\chi} - \overline{\omega}_0 \widehat{c}(\lambda(\zeta; \overline{\theta})) \psi_0 \right] \\
&+ \frac{\partial \pi_0}{\partial l} \left(\frac{b_0}{\overline{r} - \gamma} + \frac{1 - b_0}{\overline{r} - (\gamma - \overline{\theta} q(\overline{\theta}))} \right), \qquad \text{all } \zeta \geq z_{min,0},
\end{aligned} \tag{35}$$

where $b_0 \equiv \tau \psi_0 z_{min,0}^{\chi}$ is a constant.

To show the same value of $\lambda(\zeta; \overline{\theta})$ satisfies the FOC (35) for all firms on the one-dimensional manifold, it is suffices to show that

$$\frac{\partial \pi_0}{\partial \zeta} \zeta^{1-\chi} \qquad \text{and} \qquad \frac{\partial \pi_0}{\partial l}$$

are the same for all $(\zeta, \psi(\zeta))$.

For notational convenience, define $\eta \equiv 1 - 1/\rho$. Recall that

$$\begin{aligned}
\pi(z, l, t) =& (1 - \widehat{\beta}) R(z, l, t) \\
=& (1 - \widehat{\beta}) P(t) Y(t)^{1/\rho} (z l^{\alpha})^{\eta}, \qquad \text{all } z \geq z_{min,0},\ l \geq 0,\ t.
\end{aligned}$$

Thus on a BGP of the type conjectured,

$$\begin{aligned}
\pi(z, l, t) =& e^{\gamma t} \pi_0(e^{-\gamma t} z, l) \\
=& e^{\gamma t} x_0 [e^{-\gamma t} z l^{\alpha}]^{\eta}, \qquad \text{all } z \geq z_{min,0},\ l \geq 0,\ t,
\end{aligned}$$

where x_0 is a constant. When evaluated at $t = 0$,

$$\pi_0(\zeta, l) = x_0 \zeta^{\eta} l^{\alpha \eta}, \qquad \text{all } \zeta \geq z_{min,0},\ l \geq 0. \tag{36}$$

Taking the derivative of (36) with respect to ζ and then evaluating at $(\zeta, \psi(\zeta))$ yields

$$\begin{aligned}\left.\frac{\partial\pi_0(\zeta,l)}{\partial\zeta}\right|_{l=\psi(\zeta)} =& x_0\eta\zeta^{\eta-1}\psi(\zeta)^{\alpha\eta}\\ =& x_0\eta\psi_0^{\alpha\eta}\zeta^{\eta-1}\zeta^{\chi\alpha\eta}, \qquad \text{all } \zeta \geq z_{min,0}.\end{aligned}$$

Recall that $\chi = \eta/(1-\alpha\eta)$. Hence,

$$\begin{aligned}\left.\frac{\partial\pi_0(\zeta,l)}{\partial\zeta}\right|_{l=\psi(\zeta)}\zeta^{1-\chi} =& x_0\eta\psi_0^{\alpha\eta}\zeta^{\eta+\chi(\alpha\eta-1)}\\ =& x_0\eta\psi_0^{\alpha\eta}, \qquad \text{all } \zeta \geq z_{min,0}.\end{aligned} \tag{37}$$

Therefore, $(\partial\pi_0/\partial l)\zeta^{1-\chi}$ is constant across all firms on the one-dimensional manifold.

Similarly, taking the derivative of (36) with respect to l and then evaluated at $(\zeta, \psi(\zeta))$ yields

$$\begin{aligned}\left.\frac{\partial\pi_0(\zeta,l)}{\partial l}\right|_{l=\psi(\zeta)} =& x_0\zeta^{\eta}\alpha\eta\psi(\zeta)^{\alpha\eta-1}\\ =& \alpha\eta x_0\psi_0^{\alpha\eta-1}, \qquad \text{all } \zeta \geq z_{min,0}.\end{aligned} \tag{38}$$

So $\partial\pi_0/\partial l$ is also constant across firms on the one-dimensional manifold.

Substituting (37) and (38) into (35), the FOC can be rewritten as

$$\begin{aligned}\frac{\overline{\omega}_0}{q(\overline{\theta})}\widehat{c}'(\lambda(\zeta;\overline{\theta})) =& \frac{b_0}{\overline{r}-\gamma}\left[\frac{x_0\eta\psi_0^{\alpha\eta-1}}{\chi} - \overline{\omega}_0\widehat{c}(\lambda(\zeta;\overline{\theta}))\right]\\ &+ \alpha\eta x_0\psi_0^{\alpha\eta-1}\left(\frac{b_0}{\overline{r}-\gamma} + \frac{1-b_0}{\overline{r}-(\gamma-\overline{\theta}q(\overline{\theta}))}\right), \qquad \zeta \geq z_{min,0},\end{aligned}$$

or

$$\begin{aligned}\frac{\overline{\omega}_0}{q(\overline{\theta})}\widehat{c}'(\lambda(\zeta;\overline{\theta})) + \frac{b_0}{\overline{r}-\gamma}\overline{\omega}_0\widehat{c}(\lambda(\zeta;\overline{\theta})) =& \frac{b_0}{\overline{r}-\gamma}\frac{x_0\eta\psi_0^{\alpha\eta-1}}{\chi}\\ &+ \alpha\eta x_0\psi_0^{\alpha\eta-1}\left(\frac{b_0}{\overline{r}-\gamma} + \frac{1-b_0}{\overline{r}-(\gamma-\overline{\theta}q(\overline{\theta}))}\right), \zeta \geq z_{min,0}.\end{aligned} \tag{39}$$

Note that the value on the RHS of (39) is positive and independent of ζ. By Assumption 3, the LHS is strictly increasing in λ, takes the value zero at $\lambda = 0$, and diverges as $\lambda \to \infty$. Therefore, for any fixed $\overline{\theta}$, there is a unique value of λ that satisfies (39). Furthermore, this value of λ is independent of ζ. Hence all firms will choose vacancy postings to satisfy

$$\lambda(\zeta;\overline{\theta}) = \overline{\lambda}(\overline{\theta}), \qquad \text{all } \zeta \geq z_{min,0}.$$

The second step in the proof is to show that there exists a unique value of $\overline{\theta}^*$ for which $\overline{\lambda}(\overline{\theta}^*) = \overline{\theta}^*$.

From Lemma 3, when all firms choose the same value of λ then $\overline{\theta} = \overline{\lambda}(\overline{\theta})$. The FOC for all firms on the BGP then becomes

$$\begin{aligned}\frac{\overline{\omega}_0}{q(\overline{\theta})}\widehat{c}'(\overline{\theta}) + \frac{b_0}{\overline{r}-\gamma}\overline{\omega}_0\widehat{c}(\overline{\theta}) =& \frac{b_0}{\overline{r}-\gamma}\frac{x_0\eta\psi_0^{\alpha\eta-1}}{\chi} \\ &+ \alpha\eta x_0\psi_0^{\alpha\eta-1}\left[\frac{b_0}{\overline{r}-\gamma} + \frac{1-b_0}{\overline{r}-(\gamma-\overline{\theta}q(\overline{\theta}))}\right].\end{aligned} \tag{40}$$

Define two functions $Z_L(\overline{\theta})$ and $Z_R(\overline{\theta})$ by the LHS and RHS of (40) respectively.

It will be shown that

$$\begin{aligned} Z_L(0) &= 0, & Z_L'(\overline{\theta}) &> 0, & &\text{and } \lim_{\overline{\theta}\to\infty} Z_L(\overline{\theta}) = \infty, \\ Z_R(0) &= \overline{Z}_R, & Z_R'(\overline{\theta}) &< 0, & &\text{and } \lim_{\overline{\theta}\to\infty} Z_R(\overline{\theta}) = \underline{Z}_R, \end{aligned}$$

where $0 < \underline{Z}_R < \overline{Z}_R$ are finite. It then follows immediately that there exists a unique value of $\overline{\theta}^*$ such that $\overline{\lambda}(\overline{\theta}^*) = \overline{\theta}^*$.

First consider $Z_L(\overline{\theta})$. The function $q(\overline{\theta}) = \overline{q}\overline{\theta}^{-\mu}$ is strictly decreasing in $\overline{\theta}$ and diverges as $\overline{\theta} \to 0^+$. By Assumption 3, $\widehat{c}(0) = \widehat{c}'(0) = 0$. Hence $Z_L(0) = 0$, and $Z_L(\overline{\theta})$ diverges as $\overline{\theta} \to \infty$. The derivative of Z_L is given by

$$Z_L'(\overline{\theta}) = \widehat{c}'(\overline{\theta})\overline{\omega}_0\left(\frac{\overline{\theta}^{\mu}}{\overline{q}}\left[\frac{\widehat{c}''(\overline{\theta})}{\widehat{c}'(\overline{\theta})} + \mu\overline{\theta}\right] + \frac{b_0}{\overline{r}-\gamma}\right) > 0, \qquad \text{all } \overline{\theta} \geq 0.$$

By Assumption 3, the term in the braces is positive implying that the value of $Z_L(\overline{\theta})$ is monotonically increasing for all $\overline{\theta} \geq 0$.

For the function $Z_R(\overline{\theta})$, since $\mu \in (0,1)$, the term $\overline{\theta}q(\overline{\theta}) = \overline{q}\overline{\theta}^{1-\mu}$ takes the value zero at $\overline{\theta} = 0$. At $\overline{\theta} = 0$ the value of the Z_R is

$$\overline{Z}_R \equiv Z_R(0) = \frac{x_0\eta\psi_0^{\alpha\eta-1}}{\overline{r}-\gamma}\left(\frac{b_0}{\chi} + \alpha\right) > 0.$$

As $\overline{\theta}$ diverges, the the last term in the brackets of value of (40) vanishes, so Z_R converges to the finite value

$$\underline{Z}_R \equiv \lim_{\overline{\theta}\to\infty} Z_R(\overline{\theta}) = \frac{x_0 \eta \psi_0^{\alpha\eta-1}}{\overline{r}-\gamma}\left(\frac{b_0}{\chi}+\alpha b_0\right) < \overline{Z}_R.$$

Finally, the derivative of $Z_R(\overline{\theta})$ is

$$Z_R'(\overline{\theta}) = -X_0 \frac{(1-\mu)\overline{q}\overline{\theta}^{-\mu}}{\left[r-\gamma+\overline{q}\overline{\theta}^{1-\mu}\right]^2} < 0 \qquad \text{all } \overline{\theta} \geq 0,$$

where X_0 is a constant. Q.E.D.

Let the unique value of $\overline{\theta}$ that satisfies (40) be denoted by $\overline{\theta}^*$. The proof of the proposition implies that when the initial distribution of firms is one-dimensional and satisfies the assumptions made in Proposition 1, all firms in the competitive equilibrium follow the optimal vacancy posting rule

$$\nu(s,t) = \overline{\theta}^* \psi(e^{-\gamma t} z), \qquad \text{all } s \in \mathcal{S}, t.$$

In such an environment, each firm type maintains its initial firm size, and the productivity level of all firms grows at the constant rate γ, generating a BGP.

5 Conclusions

The endogenous growth model developed in this paper uses learning from new hires to provide a channel for the spillover of knowledge between firms. Workers passively absorb knowledge while working and are able to take some of this knowledge to their new employer. Firms optimally choose their vacancy posting rate, which determined their exposure to new knowledge. This structure endogenizes the learning process that is usually treated as exogenous in traditional endogenous growth models with knowledge spillover.

The main result of this paper shows that there is a set of assumptions regarding the initial distribution of firm types and the vacancy posting cost function that are sufficient for the competitive equilibrium to be a balanced growth path. Along this balanced growth path aggregate output grows at a constant rate and the size/employment of each firm is constant. Therefore, while knowledge spillovers between firms are likely to occur through many different channels, the diffusion of knowledge through new hires is capable of explaining long-run sustainable growth and could be used as a structural explanation of the exogenous learning process in standard models of endogenous growth.

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