Savage's theorem with atoms

Ha-Huy, Thai

EPEE, University of Evry, Univesity Paris-Saclay, TIMAS, Thang Long University

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Thai Ha-Huy †

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Abstract

The famous theorem of Savage is based on the richness of the states space, by assuming a *continuum* nature for this set. In order to fill the gap, this article considers Savage's theorem with discrete state space. The article points out the importance the existence of pair event in the existence of utility function and the subjective probability. Under the discrete states space, this can be ensured by the intuitive *atom swarming* condition. Applications for the establishment of an inter-temporal evaluation *à la* Koopman [16], [17], and for the configuration under *unlikely atoms* of Mackenzie [24] are provided.

KEYWORDS: Savage theorem, Koopman representation, expected utility function, atom swarming.

JEL CLASSIFICATION: D11, D90.

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†EPEE, University of Evry, University Paris-Saclay

1 Introduction

"To solve problems, you don't need to look at fancy new ideas, you can look at old things with a new eye". Sir Michael Atiyah.

The *mean expected utility* maximization problem was first proposed by Bernoulli (1738) when he worked on the Saint Petersburg's paradox: although the expected value of the lottery is infinite, people are willing only a limited amount of money to pay. The hypothesis of Bernoulli is that people maximize their *mean expected utility* instead of the expected monetary gain.

Furthermore, de Finetti [6] proposed conditions under which a rational agent maximizes expected utility with respect a subjective probability. On the contrary, von Neumann - Morgenstein's theorem [29] states that the comparison of probability distributions on the set of outcomes is given by the use of an utility function.

Savage's theorem in [26] reconciled the two approaches of de Finetti and von Neumann - Morgenstein [29]. Under what later well-known as the "*Savage's axioms*", there exist a subjective probability and an utility function characterizing the behaviour of a rational agent. This surprising and powerful result does not need the mathematical structures of de Finetti or of von Neumann - Morgenstein, which are crucial for the use of separate theorem in convex analysis. The most complicated structure of Savage's world relies on the "technical axioms"[1] ensuring a *continuum* nature of the set of states.

Savage commences the proof by establishing a comparison order on the set of events satisfying the existence of a *quantitative probability* (definition of de Finetti) defined on this set. This probability plays the role of the *subjective probability.* Each act is then equivalent to a distribution on the set of outcomes. By the von Neumann - Morgenstein theorem, an utility function exists and acts are compared using theirs *expected utilities.*

Naturally, there exists a current in the literature considering the problems encom-

[1]The $P6 - P7$ axioms.

passing the possibility of *atoms*, the events which can not be divided into smaller non-null events. This consideration is not only an attempt to extend the result of Savage, but also address a fundamental question in theoretical statistics on the interpretation of probabilities. The question is, under which conditions, a comparison order according to an event is considered *more probable than* or *at least as likely thant* another can be represented by a probability measure (finitely additive or $\sigma-$additive)?

While this question has a satisfactory response for the case of atomless states space, the problem become more complicated with the possibility that atoms exist. Because of the importance of the question (theoretically and practically), numerous works have been done in this line of literature.

For the case of finite number of states, Kraft and al [21], and Scott [27] give *cancellation* as necessary and sufficient conditions for the existence of a probability measure. Kraft and al [21] also give a counterexample to prove that the additivity is not strong enough to a positive answer when the number of states is bigger or equal to 5.

For the case of infinite number of states, Chateauneuf and Jaffray [4] and Chateuneuf [3] consider the problem under the *Archimedean* property and proved that this condition is sufficient of the establishment of a probability measure. The curious readers can refer to the excellent reviews of Fisburn [8] and Mackenzie [24].

Another approach consists in enriching the set of outcomes. Gul [14] considers the finite state space, supposing that the outcomes set is connected, as Koopmans [16]. Wakker [30] assumes that outcomes set is interval of dollars.

Ascombe and Aumann [1] suppose the agent has two types of probabilities: subjective and objective ones. The arrived set of acts in the world of Ascombe & Aumann [1] is hence the set of lotteries on outcomes. Their work opens a large literature enjoying the linear structure of the set of acts, giving strong results for the configurations where the Savage's famous *sure-thing principle* is not satisfied: for example questions about ambiguity of Gilboa & Schmeidler [11], [12], objective and subjective beliefs of Gilboa and al [13], and much more works.

This articles follows the approach supposing that the set of outcomes is connected and separable, instead of the richness of the set of states.

The first part of the article considers a general space of states, which satisfies the *equal divisibility* condition: there exists a subset H which is as likely as its complement H^c. This set will play a crucial role in the establishment of a linear structure and an order on the set of probability distributions which have at most two values. Under this setup, this set satisfies the conditions imposed on von Neumann - Morgenstein's theorem and hence the existence of an utility function is established.

The second part adds the *independence* conditionto the first part's setup. This conditions states that the ranking of two acts does not change if we mix them with a third one. Under *independence* and *equal divisibility* condition, the comparison criterion can be characterized by a subjective probability and an utility function.

The third part apply these results to the case of discrete space of states. This part assumes that *atom swarming* condition is satisfied, *i.e.* every atom event is less likely that the union of events which are less likely than it. This condition implies the existence of a set H which is as likely as its complements, allowing us to invoke the results from the first and second parts.

Applying the result in the third part in the Koopman's setup [17] for inter-temporal sequences of consumptions in discrete time, the existence of an utility function and unique discount rate $\delta \geq 0.5$ is established. This result echoes Montiel Olea & Strzalecki [25] and Kochov [15].

Finally, I consider the *unlikely atom* condition in Mackenzie [24]. This condition ensures the existence of an event which does not contain atoms and is at least as likely as its complement. The richness of the outcome set allows us to relax the *third-order atom-swarming* in this work.

2 Fundamentals

2.1 Definitions

Let S be the set of states and an algebra $\mathscr{A}$ of events on S. The set S can be discrete, atomless, or even a hybrid type which contains continuum subsets as well as atoms.

Denote by $\mathscr{F}_0$ the set of *finite-value* acts from S to a set of outcome X, which is endowed with a topology τ.

$$\mathscr{F}_0 = \{f : S \to X \text{ such that } f \text{ is measusable and } f(S) \text{ is finite}\}.$$

For any partition constituted by measurable subsets $A_1, A_2, \ldots, A_n$ of S, for any $x_1, x_2, \ldots, x_n \in X$, denote by $x_{1,A_1} x_{2,A_2} \ldots x_{n,A_n}$ the act $h : S \to X$ such that $h(s) = x_k$ for $s \in A_k$. For example, for some $A \in \mathscr{A}$, $x_A y_{A^c}$ denotes that act which takes value x if $s \in A$ and value y otherwise. In the same spirit, for $f, g \in \mathscr{A}$, $f_A g_{A^c}$ denotes the act h such that $h(s) = f(s)$ if $s \in A$ and $h(s) = g(s)$ if $s \in A^c$.

Let $\mathscr{P}_0$ be the set of finite support probability distributions on X. For $p_1, p_2, \ldots, p_n \in [0,1]$ such that $\sum_{k=1}^n p_k = 1$ and $x_1, x_2, \ldots, x_n \in X$, let $(p_1 : x_1, p_2 : x_2, \ldots, p_n : x_n)$ denote the random distribution on X which takes value x_k with probability p_k.

There is a binary relation, an order $\succeq$ defined on the set of finite value acts $\mathscr{F}_0$. To simply the exposition, the outcome set X can be considered as the set of constant acts, and hence be a subset of $\mathscr{F}_0$.

An event $E \in \mathscr{A}$ is called null-event if for any $x, y \in X$, any $h \in \mathscr{F}_0$, we have $x_E h_{E^c} \sim y_E h_{E^c}$.

Axiom F1. i) *The order $\succeq$ is complete and transitive.*

ii) *Non-triviality*[2]*: there exist $x, y \in X$ such that $x \succ y$.*

[2] In equivalence, the states space S is not a null-event.

iii) Monotonicity *For any* $x, y \in X$, $g \in \mathscr{F}_0$, *non-null event* $A \in \mathscr{A}$,

$$x \succeq y \text{ if and only if } x_A g_{A^c} \succeq y_A g_{A^c}.$$

iv) Weak comparative probability *For any* $A, B \in \mathscr{A}$ *and* $x \succ y$, $x' \succ y'$,

$$x_A y_{A^c} \succeq x_B y_{B^c} \text{ if and only if } x'_A y'_{A^c} \succeq x'_B y'_{B^c}.$$

v) Continuity *For any* $x \in X$, *the sets* $\{y \in X \text{ such that } y \succeq x\}$ *and* $\{y \in X \text{ such that } x \succeq y\}$ *are closed in respect with* $\tau-$*topology. Moreover, the space* (X, τ) *is connected*[3] *and separable*[4].

These conditions are the same axioms presented by Savage [26], note that for instance the famous *sure-thing principle* is not imposed. The relaxation of this condition gives rise to a large literature on ambiguity in decision theory. For a detailed review, see Etner & al [7].

This article relaxes Savage's *technical axioms* $P6 - P7$.[5] Instead of the *continuum* property of the states space, the condition (vi) ensures that the order $\succeq$ is continuous with respect to the topology τ.

For a replacement of *sure-thing principle*, I consider a version of *independence* property. In literature, *independence* property states that the comparison between two acts does not change if we mix them with the third act. Under the set up of Ascombe & Aumann [1], where the outcomes set constitutes of probabilistic distributions, the linear structure of the set of acts allows an easy definition of the mix between two different acts. In this article, since such a structure does not exit, the definition of *mixing acts* must be given using *pair-event*, the event which is equivalent to its complement.

First, observe that thanks to the *Weak comparative probability* property, we can define an order on the set of events.

[3]We can not split X into two disjoint closed subsets.
[4]There exists a countable and dense subset of X.
[5]For a detailed comments about Savage's axioms, see Gilboa [10], chapter 10.

Definition 2.1. *For any subsets $A, B \subset S$, define $A \succeq_l B$ if and only if there exist $x \succ y$ such that*

$$x_A y_{A^c} \succeq x_B y_{B^c}.$$

This comparison does not depend on the choice of x and y, i.e. $A \succeq_l B$ if and only if for any $x, y \in X$, $x_A y_{A^c} \succeq x_B y_{B^c}$. For the interpretation and the proof of the Proposition 2.1, see Savage [26] and Gilboa [10].

Proposition 2.1. *Assume that the order $\succeq$ satisfies axiom **F**. Then*

i) *The order $\succeq_l$ is total, transitive, and non-trivial: $S \succ \emptyset$.*

ii) *For $A, B \in \mathscr{A}$, $A \subset B$ implies $B \succeq A$.*

iii) Cancellation *For $A, B, C \in \mathscr{A}$ such that $(A \cup B) \cap C = \emptyset$,*

$$A \succeq_l B \text{ if and only if } A \cup C \succeq_l B \cup C.$$

Without the *continuum* nature of the state space in Savage's setup, conditions in Proposition 2.1 do not suffice for an establishment of a *quantitative probability measure*. See the counterexample provided by Kraft et al [21].

In the following subsection, under the *equal divisibility* condition and *independence* axiom, an utility function exists and we can establish a total order on the set of finite distributions in X.

2.2 Equal divisibility condition

Instead of the *technical axioms* in Savage [26], based on the atomless property of the set of states S, consider the following simplified one. For the case the states is *continuum*, this condition is always satisfied.

Definition 2.2. Equal divisibility condition *There exists an event $H \in \mathscr{A}$ such* *that for some $x, y \in X$ satisfying $x \succ y$,*

$$x_H y_{H^c} \sim x_{H^c} y_H.$$

Otherwise stated, S can be divided in two equivalent subsets:

$$H \sim_l H^c.$$

The *equal divisibility* condition has important features. First, it is clear that if measure of the set H *should* equal to $\frac{1}{2}$. Second, we can construct a subjective probability, using the connectivity of the outcomes set. And, last but not least, this condition ensures the unicity of this probability measure.

2.3 Mixing acts

In the world of Savage (with or without atoms), the possibility to construct the "mixing acts" plays an important role, for example the construction of Ascombe & Aumann [1], or the classical work in the ambiguity averse presentation of Gilboa & Schmeidler [11]. Generally, we work under the conditions ensuring that the set of acts is a convex subset included in a linear space. This linearity allows us to define the utility function and the order in the set of distributions.

Since this article does not impose such linear structure on the set of acts, we must follow another way in order to define the notion of "mixing act", which, in my knowledge, appears first in the work of Gul [14].

Definition 2.3. *For any acts $f, g \in \mathscr{F}_0$, any $A \in \mathscr{A}$, define the* mixing of f and g through H *any act $\tilde{f}$ satisfying: for any $s \in S$,*

$$\tilde{f}(s) \sim f(s)_H h(s)_{H^c}.$$

By a slightly abuse of notation, denote by $Hf + H^c h$ a mixing act of f and g through H.

It is worth noting that one must avoid to confuse $Hf + H^c h$ with $f_H h_{H^c}$. The former one can be considered as a convex combination act of f and g with weighted parameters defined using the event set H, while the later one is an act which is equal to f on H and equal to h on H^c.

The following axiom assumes the *independence* property of mixing acts.

Axiom A1. Independence *For any* $f, g, h \in \mathscr{F}_0$,

$$f \succeq g \text{ if and only if } Hf + H^c h \succeq Hg + H^c h.$$

The interpretation of this axiom is that, if we mix each element of f with an element of h, using the pair-event, and do the same for g and h, the comparison between f and g does not change after this mixing with h. The intuition is clear once we suppose that a probability measure μ on $\mathscr{A}$, the set of events, is established. Obviously, this value is equal to $\frac{1}{2}$. The mixing act between f and g through H is the act $\frac{1}{2}f + \frac{1}{2}g$. With the *independence* axiom, we get $f \succeq g$ if and only if $\frac{1}{2}f + \frac{1}{2}h \succeq \frac{1}{2}f + \frac{1}{2}h$. This is exactly the same *independence* property usually used in the literature following the set up of Ascombe - Aumann [1].

The relation between *independence* axiom and *sure-thing principle* is an important question. In Gul [14], for the states space S is finite, if the number of states of S is finite, *independence* implies *sure-thing principle*. The Proposition 2.2 states that the same conclusion is true for the general case, with the richness of outcomes set.

PROPOSITION 2.2. *Suppose that the order* $\succeq$ *satisfies axiom **F**, and* equal divisibility *condition. Then* Independence *implies the* sure-thing principle.

2.4 Utility function

For $x, y \in X$, if we consider $x_H y_{H^c}$ as an equivalence of the distribution $\left(\frac{1}{2} : x, \frac{1}{2} : y\right)$, the *independence* axiom ensures the existence of an utility function which conserve the comparison between these special distributions. The detailed proof can be found in Gul [14], using Theorem 1, chapter 9 of Debreu [5].

Proposition 2.3. *Assume that the axioms* $\boldsymbol{F}$, Independence *and the* equal divisibility *condition are satisfied. There exists unique utility function (up to a strictly increasing affine transformation)* u *such that for any* $x, y \in X$,

$$x_H y_{H^c} \succeq x'_H y'_{H^c} \text{ if and only if } \frac{1}{2}\big(u(x) + u(y)\big) \geq \frac{1}{2}\big(u(x') + u(y')\big).$$

Obviously, taking $x = y$ and $x' = y'$, the restraint of the order $\succeq$ on X is represented by function u: $x \succeq x'$ if and only if $u(x) \succeq u(x')$.

From now on, without any confusion, by a slightly abuse of notation, for any $A \in \mathscr{A}$, we define $u(x_A y_{A^c})$ the utility value of $z \in X$ such that $z \sim x_A y_{A^c}$:

$$u(x_A y_{A^c}) = u(z).$$

By the continuity property of the outcome set X, such element z always exists.

2.5 Subjective probability

The idea for the construction of a probability distribution representing the order $\succeq_l$ runs as follows.

For any $x, y \in X$, the act $x_H y_H{}^c$ can be considered equivalent to a distribution which takes value x and y with equal probability: $\big(\frac{1}{2} : x, \frac{1}{2} : y\big)$. Any $z \sim x_H y_{H^c}$ can be considered as *certainty equivalent* of this distribution. By taking $x_H z_{H^c}$, we have an equivalent for the distribution $\big(\frac{3}{4} : x, \frac{1}{4} : y\big)$, and $z_H y_{H^c}$ represents $\big(\frac{1}{4} : x, \frac{3}{4} : y\big)$, and so on. Continuing with this line of reasoning, we can have the equivalent representations of any distribution of the form $\big(\frac{k}{2^n} : x, \frac{2^n - k}{2^n} : y\big)$, for $0 \leq k \leq 2^n$. Taking the limits for n converges to infinity, we find the representation of every distribution which takes at most two values: $\big(p : x, (1-p) : y\big)$, with $x, y \in X$ and $0 \leq p \leq 1$[6].

In details, consider a construction of the following sequence $\big\{z^{k,2^n}\big\}$, with $n \geq 0$

[6] It is well known that for any $0 \leq p \leq 1$, there exists a sequence $(k_n, 2^n)$ such that $0 \leq k_n \leq 2^n$ for any n and $\lim_{n\to\infty} \frac{k_n}{2^n} = p$.

and $0 \le k \le 2^n$.

For $n = 1$, fix the elements of outcome set $z^{0,2}$, $z^{1,2}$ and $z^{2,2}$ as:

$$z^{0,2} = y,$$
$$z^{1,2} \sim x_H y_{H^c},$$
$$z^{2,2} = x.$$

For $n \ge 1$, $0 \le k \le 2^{n+1}$, fix the elements $z^{k,2^{n+1}} \in X$ as:

$$z^{k,2^{n+1}} = z^{k',2^n} \text{ if } k = 2k', \text{ with } 0 \le k' \le 2^n,$$
$$z^{k,2^{n+1}} \sim z_H^{2k',2^n} z_{H^c}^{2k'+1,2^n} \text{ if } k = 2k'+1, \text{ with } 0 \le k' \le 2^n - 1.$$

The following Lemma is intuitive and can be proven by induction. Without loss of generality, assume that $x \succeq y$.

LEMMA **2.1.** *Assume that $x \ge y$.*

i) *For any k, n, we have*

$$x \succeq z^{2^n-1,2^n} \succeq \cdots \succeq z^{k+1,2^n} \succeq z^{k,2^n} \succeq \cdots \succeq z^{1,2^n} \succeq y.$$

ii) *For any $0 \le k \le 2^n$,*

$$u\left(z^{k,2^n}\right) = \frac{k}{2^n} u(x) + \left(1 - \frac{k}{2^n}\right) u(y).$$

iii) *If $x \succ y$, then for set $A \in \mathscr{A}$, for any n, there exists unique k_n such that:*

$$z^{k_n+1,2^n} \succ x_A y_{A^c} \succeq z^{k_n,2^n}.$$

Fix a set $A \in \mathscr{A}$, fix $x \succ y$, consider the sequence $\{(k_n, 2^n)\}_{n=0}^{\infty}$ such that for any

n,

$$z^{k_n+1,2^n} \succ x_A y_{A^c} \succeq z^{k_n,2^n}.$$

We may define the subjective probability measure of A as

$$\mu(A) = \lim_{n\to\infty} \frac{k_n}{2^n}.$$

However, the sequence $\{(k_n, 2^n)\}_{n=0}^{\infty}$ and the limit can depend on the choice of x and y. Under the satisfaction of *Independence* axiom, we can discard this possibility and prove that the value of $\mu(A)$ is independent with respect to the choice of x and y. Moreover, we obtain a simple version of Savage's theorem, applied for the set of acts which take at most two values.

PROPOSITION 2.4. *Suppose that the order $\succeq$ satisfies the axioms **F**,* Independence*, and the* equal divisibility *condition.*

i) *The measure μ is unique and independent with the choice of x and y.*

ii) *For any $A, B \in \mathscr{A}$,*

$$A \succeq_l B \text{ if and only if } \mu(A) \geq \mu(B).$$

iii) *For any $A, B \in \mathscr{A}$, any $x, y, x', y' \in X$, $x_A y_{A^c} \succeq x'_B y'_{B^c}$ if and only if*

$$\mu(A)u(x) + \big(1 - \mu(A)\big)u(y) \geq \mu(B)u(x') + \big(1 - \mu(B)\big)u(y').$$

2.6 MEAN EXPECTED UTILITY

Once the utility function and subjective probability have been established, we have the satisfaction of Savages's theorem without the continuity nature of the set of states.

Theorem 2.1. *Suppose that the order* $\succeq$ *satisfies axioms* $\boldsymbol{F}$*,* Independence *and the* equal divisibility *condition. There exists unique finitely additive probability measure* μ *and unique utility function* u *(up to a strictly increasing affine transformation) such that for any* $f, g \in \mathscr{F}_0$*:*

$$f \succeq g \text{ if and only if } \int_S u\left(f(s)\right)\mu(ds) \geq \int_S u\left(g(s)\right)\mu(ds).$$

The extension for the comparison on the set of finite acts $\mathscr{F}_0$ to the set of measurable acts $\mathscr{F}$ requires some additional properties. The events family $\mathscr{A}$ is supposed to be a $\sigma-$algebra. Arrow [2] proves that the Monotone Continuity, initiated by Villegas [28], ensured countably additive of the subjective probability.

Axiom A2. Monotone continuity *For any event* A *and sequence of events* $\{A_n\}_{n=1}^{\infty}$ *such that*

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \ldots,$$

and for any n*,* $A \succeq_l A_n$*, we have*

$$A \succeq_l \bigcup_{n=1}^{\infty} A_n.$$

Denote by $\mathscr{F}$ the set of acts which is bounded:

$$\mathscr{F} = \left\{f : S \to X \text{ measureable and } \exists\, x, y \in X \text{ such that } x \succeq f(s) \succeq y \;\forall s \in S\right\}.$$

Theorem 2.2. *Assume that* $\mathscr{A}$ *is* $\sigma-$*algebra. Suppose that the order* $\succeq$ *is defined on* $\mathscr{F}$*, and satisfies axioms* $\boldsymbol{F}$*,* equal divisibilityIndependence *and* monotone continuity*. There exists unique finitely additive probability measure* μ *and unique utility function* u *(up to a strictly increasing affine transformation) such that for any* $f, g \in \mathscr{F}$*:*

$$f \succeq g \text{ if and only if } \int_S u\left(f(s)\right)\mu(ds) \geq \int_S u\left(g(s)\right)\mu(ds).$$

3 Discrete states set

For this section, I consider the case that the states space S is discrete and has an infinite number of elements. Without loss of generality, suppose that $S = \{0, 1, 2, \dots\}$ and for any s, $\{s\}$ is non-null. The algebra $\mathscr{A}$ contains every subsets of S: $\mathscr{A} = 2^S$. Moreover, always without loss of any generality, we can assume that

$$\{0\} \succeq_l \{1\} \succeq_l \{2\} \succeq_l \cdots \succeq_l \{s\} \succeq_l \{s+1\} \succeq_l \dots .$$

Axiom A3. Atom swarming *For any $s \geq 0$, we have*

$$\{s+1, s+2, \dots\} \succeq_l \{s\}.$$

Let us discuss the *atom swarming* property. This axiom says that, every state is less likely than the set of states which are less likely than it [7].

The meaning of the *atom swarming* condition is better illustrated in the context of *time discounting.* For example, consider the setup in Koopmans [16], [17], where instead of the states, we work with discrete time. Generally, a criterion on inter-temporal consumption imposes the *impatience property*:

$$\{0\} \succeq_l \{1\} \succeq_l \{2\} \succeq_l \cdots \succeq_l \{s\} \geq \dots .$$

The *atom swarming* condition requires that the criterion is not too-impatient, *i.e.* there is no day which is more important than the union of all other days in the future[8]:

$$\{s+1, s+2, \dots\} \succeq_l \{s\}, \text{ for any } s.$$

[7] This is a weaker version of the *third-order atom-swarming* presented in Mackenzie [24], which requires that for each atom, there is a countable pairwise-disjoint collection of less-likely events that can be partitioned into three groups, each with union at least as likely as the given atom.

[8] The same idea about *not too-inpatient* property is also presented in the works of Montiel Oléa & Strzalecki [25], axiom 8 and Kochov [15], property **P**.

Since from now on we work with $\sigma-$algebra subsets of S, we need the *monotone continuity* property, which is the same as Villegas [28], to ensure that the subjective probability measure is $\sigma-$additive.

Under this axiom, the *equal divisibility* condition is satisfied.

PROPOSITION 3.1. *Assume that the order $\succeq$ satisfies axioms **F**,* Independence*,* atom swarming*,* monotone continuity*. Then the* equal divisibility *property is satisfied. There exist a unique utility function u (up to a strictly increasing affine transformation) and a unique probability measure $\omega = (\omega_0, \omega_1, \dots)$ such that:*

i) *For any subsets $A, B \subset S$,*

$$A \succeq_l B \text{ if and only if } \sum_{s \in A} \omega_s \geq \sum_{s \in B} \omega_s.$$

ii) *For any $f, g \in \mathscr{F}_0$, $f \succeq g$ if and only if*

$$\sum_{s=0}^{\infty} \omega_s u\left(f(s)\right) \geq \sum_{s=0}^{\infty} \omega_s u\left(g(s)\right).$$

4 Applications

4.1 When Savage meets Koopman

Axiom A4. Time consistency *Suppose that for any, $x \succ y$ and subsets $A, B \subset S$:*

$$x_A y_{A^c} \succeq x_B y_{B^c} \text{ if and only if } x_{A+1} x_{(A+1)^c} \succeq x_{B+1} y_{(B+1)^c}.$$

This axiom, which is equivalent to the time-consistency axiom of Koopmans [17], ensures that the comparison between two sets A and B does not change under a translation to the future:

$A \succeq_l B$ if and only if $A + s \succeq_l B + s$ for any $s \geq 0$.

Theorem 4.1. *Assume that the order* $\succeq$ *satisfies axioms* $\boldsymbol{F}$*,* Independence*,* atom swarming*,* monotone continuity *and* stability*. There exist unique discount rate* $0.5 \leq \delta < 1$*, and unique (up to a strictly increasing affine transformation) utility function* u *such that for any* $f, g \in \mathcal{F}$*,*

$$f \succeq g \text{ if and only if } \sum_{s=0}^{\infty} \delta^s u\left(f(s)\right) \geq \sum_{s=0}^{\infty} \delta^s u\left(g(s)\right).$$

4.2 Atom unlikely condition

This section consider the case when the *atom unlikely* condition presented in Mackenzie [24] is satisfied. This condition establishes the existence of a set which contains no atom, and is at least as likely as its complement. In this section, I assume that $\mathcal{A}$ is a $\sigma-$algebra of events in S.

By $\sigma-$additivity, and the same arguments as in the proof of Savage's theorem, there exists a subset $H \in \mathcal{A}$ such that H contains no atoms and $H \sim_l H^c$. Pushing further this line of arguments, we can construct a sequence of sub-events in H: $\left\{A^{k,2^n}\right\}$ with $0 \leq k \leq n$, such that for any $x \succeq y \in X$, for the corresponding sequence $\left\{z^{k,2^n}\right\}$, we have

$$x_{A^{k,2^n}} y_{\left(A^{k,2^n}\right)^c} \sim z^{k,2^n}.$$

By induction, and the *sure-thing principle*, the sets $\left\{A^{k,2^n}\right\}$ do not depend on the choice of x and y. For any $B \in \mathcal{A}$ such that $B^c \succeq_l B$, we can define the sequence $\{(k_n, 2^n)\}_{n=0}^{\infty}$ such that

$$A^{k_n+1,2^n} \succ_l B \succeq A^{k_n,2^n}.$$

The measure of B can be defined as

$$\mu(B) = \lim_{n\to\infty} \frac{k_n}{2^n}.$$

For $B \in \mathscr{A}$ such that $B \succeq B^c$, define the $\{(k_n, 2^n)\}_{n=0}^{\infty}$ such that

$$H^c \cup A^{k_n+1,2^n} \succ_l B \succeq H^c \cup A^{k_n,2^n}.$$

The measure of B can be defined as:

$$\mu(B) = \frac{1}{2} + \lim_{n\to\infty} \frac{k_n}{2^n}.$$

THEOREM 4.2. *Assume that the order $\succeq$ satisfies axioms $\boldsymbol{F}$,* Independence*,* atom swarming*,* monotone continuity *and* atom unlikely *property. Then there exists a unique subjective probability and a unique (up to a strictly increasing affine transformation) utility function such that: for any $f, g \in \mathscr{F}_0$, $f \succeq g$ if and only if*

$$\int_S u\big(f(s)\big)\mu(ds) \succeq \int_S u\big(g(s)\big)\mu(ds).$$

APPENDIX

A Proof of Proposition 2.2

Consider a non-null event $A \in \mathscr{F}$. Assume that for $f, g, h \in \mathscr{F}_0$, we have $f_A h_{A^c} \succeq g_A h_{A^c}$. We must prove that for any $\tilde{h} \in \mathscr{F}_0$, $f_A \tilde{h}_{A^c} \succeq g_A \tilde{h}_{A^c}$.

First, we prove that, if there is some $\hat{h} \in \mathscr{F}_0$ such that for any $s \in S$,

$$h(s)_H \hat{h}(s)_{H^c} \sim \tilde{h}(s),$$

then for $f_A h_{A^c} \succeq g_A h_{A^c}$ if and only if $f_A \tilde{h}_{A^c} \succeq g_A \tilde{h}_{A^c}$.

Indeed, by the *Independence* axiom, $f_A h_{A^c} \succeq g_A h_{A^c}$ if and only if

$$H\left(f_A h_{A^c}\right) + H^c\left(f_A \hat{h}_{A^c}\right) \succeq H\left(g_A h_{A^c}\right) + H^c\left(f_A \hat{h}_{A^c}\right),$$

which is equivalent to

$$f_A\tilde{h}_{A^c} \succeq (Hg + H^c f)_A \tilde{h}_{A^c}.$$

Using once more the *Independence* axiom, we have $f_A\tilde{h}_{A^c} \geq g_A\tilde{h}_{A^c}$ if and only if

$$H\left(f_A\tilde{h}_{A^c}\right) + H^c\left(f_A\tilde{h}_{A^c}\right) \geq H\left(g_A\tilde{h}_{A^c}\right) + H^c\left(f_A\tilde{h}_{A^c}\right),$$

which is equivalent to

$$f_A\tilde{h}_{A^c} \succeq (Hg + H^c f)_A \tilde{h}_{A^c}.$$

Hence $f_A h_{A^c} \succeq g_A h_{A^c}$ if and only if $f_A\tilde{h}_{A^c} \succeq g_A\tilde{h}_{A^c}$.

Now take three elements in X such that $\overline{x} \succ x \succ \underline{x}$. We will prove that

$$f_A h_{A^c} \succeq g_A h_{A^c} \text{ if and only if } f_A x_{A^c} \succeq g_A x_{A^c}.$$

Indeed, let $h^0, h^1, \ldots, h^n, \cdots \in \mathscr{F}_0$ defined as

$$\begin{aligned}
h^0 &= h, \\
h^1 &= Hh^0 + H^c x, \\
h^2 &= Hh^1 + H^c x, \\
&\ldots, \\
h^{n+1} &= Hh^n + H^c x \text{ for any } n \geq 0.
\end{aligned}$$

Using the same arguments as the case $\tilde{h} = Hh + H^c\hat{h}$, we have $f_A h_{A^c} \succeq g_A h_{A^c}$ is equivalent to $f_A h^1_{A^c} \succeq g_A h^1_{A^c}$, which is equivalent to $f_A h^2_{A^c} \succeq g_A h^2_{A^c}$ etc.

By induction, for any n, $f_A h_{A^c} \succeq g_A h_{A^c}$ is equivalent to $f_A h^n_{A^c} \succeq g_A h^n_{A^c}$. By the construction, the sequence of acts $\{h^n\}_{n=0}^{\infty}$ converges to the constant act x, in the sense that for any $y \succ x \succ z$, there exists N such that for $n \geq N$, for any $s \in S$,

$$y \succ h^n(s) \succ z.$$

This implies for n sufficiently big, there exists $h^* \in \mathscr{F}_0$ such that $\overline{x} \succ h^*(s) \succ \underline{x}$ for any $s \in S$ and

$$h^n(s)_H h^*(s)_{H^c} \sim x,$$

for any $s \in S$. This is equivalent to $Hh^n + H^c h^* \sim x$. Hence $f_A h^n_{A^c} \succeq g_A h^n_{A^c}$ is equivalent to $f_A x_{A^c} \succeq g_A x_{A^c}$. The claim is proved.

Applying the same arguments for $\tilde{h}$, we get $f_A \tilde{h}_{A^c} \geq g_A \tilde{h}_{A^c}$ if and only if $f_A x_{A^c} \geq g_A x_{A^c}$.

The satisfaction of *sure-thing principle* is proved.

B Proof of Proposition 2.4

For the sake of simplicity, for $f, h \in \mathscr{F}_0$, and some non-null event A, the mixing act $\tilde{f}$ can be written as

$$\tilde{f} \sim Af + A^c h.$$

The axiom *Independence* axiom states that $f \succeq g$ if and only if $Af + A^c h \succeq Ag + A^c g$.

i) The proof that determination of μ is independent with the choice of x, y consists of three parts:

   a) First, consider x, y, x', y', z, z' such that

$$z \sim x_A y_{A^c},$$
$$z' \sim x'_A y'_{A^c}.$$

Fix $w, v, t \in X$ which satisfy:

$$w \sim x_H x'_{H^c},$$
$$v \sim y_H y'_{H^c},$$
$$t \sim w_A v_{A^c}.$$

We will prove that $t \sim z_H z'_{H^c}$.

Indeed, let $f = x_A y_{A^c}$ and $g = x'_A y'_{A^c}$. Since $f \sim z$ and $g \sim z'$, by *Independence* axiom, the mixture between f and g using H is equivalent to the mixture between z and z' using H. We have

$$\begin{aligned} t &\sim w_A v_{A^c} \\ &\sim Hf + H^c g \\ &\sim Hz + H^c z' \\ &\sim z_H z'_{H^c}. \end{aligned}$$

b) Now we prove the independence of $\mu^{x,y}(A)$ with respect to the choice of x, y.

Fix any $x^*, y^* \in X$ such that $x^* \succ y^*$. Fix any $A \in \mathcal{A}$. Let $p = \mu^{x^*,y^*}(A)$. We must prove that for any x, y such that $x^* \succeq x \succeq y \succeq y^*$, with $t \sim x_A y_{A^c}$,

$$u(t) = pu(x) + (1-p)u(y),$$

where u is the utility function in Proposition 2.3.

Consider the same construction of the sequence $\left\{z^{k,2^n}\right\}_{k,n=1}^{\infty}$ corresponding to x^* and y^*.

For any event $A \in \mathcal{A}$, since $x^* \succeq x^*_A y^*_{A^c} \succeq y^*$, for any n, there exists unique k_n such that

$$z^{k_n+1,2^n} \succ x^*_A y^*_{A^c} \succeq z^{k_n,2^n}.$$

We can define the measure of A, under the choice x, y as

$$\mu^{x^*,y^*}(A) = \lim_{n\to\infty} \frac{k_n}{2^n}.$$

We will prove that under any other choice of $x = z^{k,2^n}$ and $y = z^{k',2^n}$, the measure of A is the same:

$$\mu^{x,y}(A) = \mu^{x^*,y^*}(A).$$

The proof will be given by induction. Consider first the case $n = 1$.

Take for example $z^{1,2}$ and $z^{0,2}$. To simplify the presentation, let $x' = z^{1,2}$, and $y' = z^{0,2} = y^*$. Let $p = \mu^{x^*,y^*}(A)$ and $p' = \mu^{x',y'}(A)$.

Recall that by Lemma 2.1,

$$\begin{aligned}
u(z) &= pu(x^*) + (1-p)u(y^*), \\
u(z') &= p'u(x') + (1-p')u(y') \\
&= \frac{p'}{2}\big(u(x) + u(y)\big) + (1-p')u(y') \\
&= \frac{p'}{2}u(x^*) + \left(1 - \frac{p'}{2}\right) u(y^*).
\end{aligned}$$

Let $z = x_A y_{A^c}$ and $z' = x'_A y'_{A^c}$. Since $y' = y^*$, using the property proved in the part (i), we get

$$z' \sim z_H y^*_{H^c}.$$

This implies

$$\begin{aligned}
u(z') &= \frac{1}{2}(u(z) + u(y^*)) \\
&= \frac{1}{2}\big(pu(x^*) + (1-p)u(y^*)\big) + \frac{1}{2}u(y^*) \\
&= \frac{p}{2}u(x^*) + \left(1 - \frac{p}{2}\right) u(y^*).
\end{aligned}$$

Hence $p = p'$, or $\mu^{x',y'}(A) = \mu^{x,y}(A)$.

For the case of the choice x and $z^{1,2}$, we use the same arguments. For $x = x^*, y = y^*$, the conclusion is immediate.

Now assume that the assertion is true for any number n. We will prove that it is also true for $n+1$. Consider any $0 \leq k' \leq k \leq 2^{n+1}$. By the construction of the sequence $\{z^{k,2^n}\}_{n=0}^{\infty}$, there exist $x, x', y, y' \in \left\{z^{k,2^n}\right\}_{k=0}^{2^n}$ such that $x \succeq y$, $x' \succeq y'$ and

$$\begin{aligned} z^{k,2^{n+1}} &= x_H x'_{H^c} \\ z^{k',2^{n+1}} &= y_H y'_{H^c}. \end{aligned}$$

Define $t = z_A^{k,2^{n+1}} z_{A^c}^{k',2^{n+1}}$, $w = x_A y_{A^c}$, $w = x'_A y'_{A^c}$. By the part (i), the equivalence $t \sim w_H v_{H^c}$ is satisfied. Hence

$$\begin{aligned} u(t) &= \frac{1}{2}\big(u(w) + u(v)\big) \\ &= \frac{1}{2}\left(pu(x) + (1-p)u(y) + pu(x') + (1-p)u(y')\right) \\ &= p\left(\frac{1}{2}\big(u(x) + u(x')\big)\right) + (1-p)\left(\frac{1}{2}\big(u(y) + u(y')\big)\right) \\ &= pu\left(z^{k,2^{n+1}}\right) + (1-p)u\left(z^{k',2^{n+1}}\right). \end{aligned}$$

This implies

$$\mu^{z^{k,2^{n+1}}, z^{k',2^{n+1}}}(A) = \mu^{x,y}(A) = \mu^{x^*,y^*}(A).$$

Consider now any x, y such that $x^* \succeq x \succeq y \succeq y^*$. Let $\left\{z^{k_n,2^n}\right\}_{n=0}^{\infty}$ and $\left\{z^{k'_n,2^n}\right\}_{n=0}^{\infty}$ be sequences such that

$$\begin{aligned} \lim_{n\to\infty} u\left(z^{k_n,2^n}\right) &= u(x), \\ \lim_{n\to\infty} u\left(z^{k'_n,2^n}\right) &= u(y). \end{aligned}$$

By continuity property,

$$\begin{aligned}u\left(x_A y_{A^c}\right) &= \lim_{n\to\infty} u\left(z_A^{k_n,2^n} z_{A^c}^{k'_n,2^n}\right)\\ &= pu(x) + (1-p)u(y),\end{aligned}$$

which is equivalent to

$$\mu^{x,y}(A) = \mu^{x^*,y^*}(A).$$

For any $x \succeq y$ and $x' \succeq y'$, fix $x^* \succeq x, x'$ and $y^* \succeq y'$, we have

$$\mu^{x,y}(A) = \mu^{x^*,y^*}(A) = \mu^{x',y'}(A).$$

Hence the choice of value $\mu(A)$ does not depend on the choice of x, y.

c) *μ is a probability measure.*

In order to complete the proof, we must prove that for $A, B \in \mathscr{A}$ such that $A \cap B = \emptyset$,

$$\mu\big(A \cup B\big) = \mu(A) + \mu(B).$$

Define $C = \big(A \cup B\big)^c$. Since $x_A x_{B\cup C} = x \succeq x_{A\cup B} y_c \succeq x_A y_{B\cup C}$, there exists $w \in X$ such that

$$x_A x_B y_C = x_{A\cup B} y_C \sim x_A w_{B\cup C}.$$

Applying the *sure-thing principe* by replacing x by y on the event A, we get

$$x_B y_{A\cup C} = y_A x_B y_C \sim y_A w_{B\cup C}.$$

From $x_{A\cup B}y_C \sim x_A w_{B\cup C}$ and $x_B y_{A\cup C} \sim y_A w_{B\cup C}$ we get

$$\mu\big(A\cup B\big)u(x) + \big(1-\mu\big(A\cup B\big)\big)\,u(y) = \mu(A)u(x) + (1-\mu(A))\,u(w),$$
$$\mu(B)u(x) + (1-\mu(B))\,u(y) = \mu(A)u(y) + (1-\mu(A))\,u(w).$$

Subtracting the second equation by the first equation, we obtain

$$\big(u(x)-u(y)\big)\mu\big(A\cup B\big) = \big(u(x)-u(y)\big)\big(\mu(A)+\mu(B)\big),$$

which implies

$$\mu\big(A\cup B\big) = \mu(A)+\mu(B).$$

The proof is completed.

C Proof of Theorem 2.2

By Proposition 2.4, there exists unique probability measure μ and a utility function (up to a strictly increasing affine transformation) such that for any events $A, B \in \mathscr{A}$, outcomes $x, y, x', y' \in X$, the act $x_A y_{A^c} \succeq x'_B y'_{B^c}$ if and only if:

$$\mu(A)u(x) + \mu(A^c)u(y) \geq \mu(B)u(x') + \mu(B^c)u(y').$$

Suppose that the assertion of the theorem is true for the acts which take almost $n-1$ different values. We will prove that it is verified for n different values.

Let $f = x_{1,A_1}x_{2,A_2}\cdots x_{n,A_n}$, with $\{A_k\}_{k=1}^n$ a partition of S. Fix any constant $v \in X$. For $1 \leq k \leq n$, define $p_k = \mu(A_k)$.

We will prove that

$$f \succeq v \text{ if and only if } \sum_{k=1}^{n} p_k u(x_k) \geq u(v).$$

Fix $w \in X$ such that

$$x_{1,A_1} x_{2,A_2} x_{3,A_3} \cdots x_{n,A_n} \sim x_{1,A_1} w_{\cup_{k=2}^n A_k}.$$

By the *sure-thing principle* property, this implies

$$x_{2,A_1} x_{2,A_2} \cdots x_{n,A_n} \sim x_{2,A_2} w_{\cup_{k=2}^n A_k},$$

which is equivalent to

$$p_1 u(x_2) + p_2 u(x_2) + p_3 u(x_3) + \cdots + p_n u(x_n) = p_1 u(x_2) + (p_2 + p_3 + \cdots + p_n) u(w).$$

Hence

$$u(w) = \frac{1}{\sum_{k=2}^n p_k} \sum_{k=2}^n p_k u(x_k).$$

This allows us to deduce the value of f:

$$\begin{aligned} u\big(x_{1,A_1} w_{\cup_{k=2}^n A_k}\big) &= p_1 u(x_1) + \left(\sum_{k=2}^n p_k \right) u(w) \\ &= \sum_{k=1}^n p_k u(x_k). \end{aligned}$$

We have $f \succeq v$ if and only if $x_{1,A_1} w_{\cup_{k=2}^n A_k} \succeq v$, which is equivalent to

$$\sum_{k=1}^n p_k u(x_k) \geq u(v).$$

For any $f = x_{1,A_1} x_{2,A_2} \cdots x_{n,A_n}$ and $g = y_{1,B_1} y_{2,B_2} \cdots y_{m,B_m}$, by considering v such that $v \sim g$, one has

$$f \succeq g \text{ if and only if } \sum_{k=1}^n p_{A_k} u(x_k) \geq \sum_{k=1}^m p_{B_k} u(y_k).$$

The proof is completed.

D Proof of Proposition 3.1

First, we prove that under axioms **F1** and **A2**, **A3**, the *equal divisibility* property is satisfied: there exists an event $H \subset S$ such that $H \sim_l H_c$.

Withou loss of generality, consider a permutation of elements of S: $\{s_0, s_1, s_2 \ldots, \}$ such that

$$\{s_0\} \succeq_l \{s_1\} \succeq_l \{s_2\} \succeq_l \cdots \succeq_l \{s_k\} \succeq \ldots.$$

Define

$$A_0 = \{s_0\},$$
$$B_0 = \emptyset.$$

For any k, if $A_k \succeq_l B_k$ then

$$A_{k+1} = A_k,$$
$$B_{k+1} = B_k \cup \{s_{k+1}\},$$

Otherwise, if $B_k \succ_l A_k$, then

$$A_{k+1} = A_k \cup \{s_{k+1}\},$$
$$B_{k+1} = B_k.$$

Define

$$A = \bigcup_{k=0}^{\infty} A_k.$$

Observe that

$$B = \bigcup_{k=0}^{\infty} B_k.$$

We will prove that $A \sim_l B$.

First, consider the case there exists k^* such that $A_{k^*} \succeq_l B_{k^*}$ and $B_k \succ A_k$ for any $k \geq k^* + 1$.

This implies

$$B = B_{k^*+1} = B_{k^*} \cup \{s_{k^*+1}\} \succ_l A_{k^*+1} = A_{k^*} \succeq_l B_{k^*}.$$

Since for any $k \geq k^* + 1$, $B_k \succ_l A_k$, we have

$$A_k = A_{k^*} \cup \{s_{k^*+2}, s_{k^*+3}, \dots, s_k\}.$$

This implies

$$\begin{aligned}
A &= \bigcup_{k=0}^{\infty} A_k \\
&= A_{k^*} \cup \{s_{k^*+2}, s_{k^*+3}, \dots\} \\
&\succeq_l A_{k^*} \cup \{s_{k^*+1}\} \\
&\succeq_l B_{k^*} \cup \{s_{k^*+1}\} \\
&= B.
\end{aligned}$$

For the case where there exist an infinite number of k such that $A_k \succeq_l B_k$, by the axiom **A2**, we have $A \succeq_l B$.

Now we prove that $B \succeq_l A$.

If $\{s_0\} \sim_l \{s_1, s_2, \dots\}$, then $A = \{s_0\} \sim_l B = \{s_1, s_2, \dots, s_k, \dots\}$.

If $\{s_1, s_2, \dots\} \succ \{s_0\}$, then there exists k such that $B_k \succ A_k$. Using the same arguments as the first part, we get $B \succeq_l A$.

Hence $A \sim_l B$. Obviously, $A \cup B = S$. Let $H = A$ and $H^c = B$, the *equal divisibility* condition is satisfied. By the Proposition 2.4, there exists a probability measure defined on the $\sigma-$algebra of all subsets of S and an utility function u

such that for any $f, g \in \mathscr{F}_0$, $f \succeq g$ if and only if

$$\sum_{s=0}^{\infty} \omega_s u\big(f(s)\big) \geq \sum_{s=0}^{\infty} \omega_s u\big(f(s)\big).$$

By the *monotone continuity* condition, the measure ω is $\sigma-$additive. Take $\omega_s = \mu(\{s\})$, we get $\sum_{s=0}^{\infty} \omega_s = 1$ and $\mu(A) = \sum_{s \in A} \omega_s$ for any $A \subset S$.

E Proof of Theorem 4.1

Using Theorem 2.2, there exists a probability $\omega = (\omega_0, \omega_1, \dots)$ and an utility function such that for any $f, g \in \mathscr{F}_0$, $f \succeq g$ if and only if

$$\sum_{s=0}^{\infty} \omega_s u\big(f(s)\big) \geq \sum_{s=0}^{\infty} \omega_s u\big(g(s)\big).$$

For $T \geq 0$, define ω^T as

$$\omega_s^T = \frac{\omega_{T+s}}{\sum_{s'=0}^{\infty} \omega_{T+s'}}, \forall\, s \geq 0.$$

By the *stability* property, $\omega = \omega^T$, and hence:

$$\omega_s = \frac{\omega_{T+s}}{\sum_{s'=0}^{\infty} \omega_{T+s'}} \text{ and } \omega_{s+1} = \frac{\omega_{T+s+1}}{\sum_{s'=0}^{\infty} \omega_{T+s'}}.$$

This implies

$$\frac{\omega_{s+1}}{\omega_s} = \frac{\omega_{T+s+1}}{\omega_{T+s}},$$

for every T, s.

But this is equivalent to

$$\frac{\omega_{s+1}}{\omega_s} = \delta,$$

for some $\delta > 0$ and for every $s \geq 0$, or $\omega_s = \delta^s \omega_0$ for every $s \geq 0$.

Since $\sum_{s=0}^{\infty} \omega_s = 1$, one has $0 < \delta < 1$ and $\omega_s^* = (1-\delta^*)\delta^s$ for $s \geq 0$.

For $f, g \in \mathscr{F}$, $f \succeq g$ is equivalent to

$$(1-\delta) \sum_{s=0}^{\infty} \delta^s u\left(f(s)\right) \geq (1-\delta) \sum_{s=0}^{\infty} \delta^s u\left(g(s)\right).$$

The common term $1-\delta$ can be relaxed, for the sake of simplicity.

The condition *atom swarming* is equivalent to

$$1-\delta \leq \sum_{s=1}^{\infty} (1-\delta)\delta^s = \delta,$$

which is equivalent to $\delta \geq 0.5$.

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