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ABSTRACT. In his textbook Tirole (1988, pp. 291-294) presents a model of advertising with Hotelling duopolists. It has been inferred (e.g., Bagwell, 2007) that in the competitive equilibrium derived, there can be socially too little advertising. It is shown that given the assumptions in Tirole (1988), there cannot be socially too little advertising for this equilibrium.

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Tirole (1988, pp. 291-294) presents a model of duopolists ($i = 1, 2$) located on the ends of a Hotelling line who chose advertising (Φ_i) and price (p_i) simultaneously, and derives the symmetric pure-strategy equilibrium. The purpose is pedagogical, partly to highlight the forces that could lead to excessive or insufficient advertising. Tirole (1988) notes that "these conclusions, of course, are only valid in the competitive range," *i.e.*, only to the extent that a competitive equilibrium exists and so assumptions are made to ensure the existence of the equilibrium.[1] It has been inferred that either excessive or insufficient advertising could arise in the equilibrium presented in Tirole (1988) (e.g., Bagwell, 2007, Christou and Vettas, 2008). This note shows that given the assumptions in Tirole (1988), the competitive equilibrium derived there cannot have socially insufficient advertising.

1. Tirole's Model

Consumers are distributed uniformly along a unit length with density 1, have unit demand with gross surplus $\bar{s}$ from consuming the good. They have linear transportation cost t. They do not know of the existence of either product unless they receive an ad from a firm; then they learn that firm's location and price. Advertising Φ_i is the fraction of consumers that firm i reaches with an advertisement. Consumers have equal chances of receiving a given ad (implicitly this is independent of each firm). The cost to firm i to reach the fraction Φ_i of the consumers is quadratic: $A(\Phi_i) = a\Phi_i^2/2$. Tirole (1988, Ch.7, Fn. 27) assumes that $a > t/2$ so that the firms choose in equilibrium $\Phi < 1$.

Assumption 1. *$a > t/2$.*

Production is on demand with constant marginal cost c. Implicitly it is assumed that all potential exchanges are efficient: $\bar{s} - c - t \geq 0$.

For the consumers firm 1 reaches, a fraction $1 - \Phi_2$ are not reached by its rival and so firm 1 is a monopolist in this case. For the remaining fraction of consumers that firm 1 reaches, they are reached by firm 2, which occurs with probability Φ_2. These latter consumers are fully informed and the demand for firm 1 in this case is presented as (Tirole, 1988, p. 293, top col. 1)

$$(p_2 - p_1 + t)/2t.$$ [2]

Thus, demand for firm 1 is (Tirole, 1988, p. 293, col. 1: D_1)

$$D_1 = \Phi_1 \left[(1 - \Phi_2) \times 1 + \Phi_2 \frac{(p_2 - p_1 + t)}{2t}\right]. \tag{1}$$

1.1. Competitive Equilibrium.

Profit for firm 1 is (Tirole, 1988, p. 293)

$$\Phi_1 \left[(1 - \Phi_2) \times 1 + \Phi_2 \frac{(p_2 - p_1 + t)}{2t}\right] (p_1 - c) - a\frac{\Phi_1^2}{2} \tag{2}$$

Differentiating with respect to p_i and Φ_i and imposing symmetry yields the competitive equilibrium price (with the equation numbering as in Tirole, 1988 to ease comparison)

$$p^c = c + (2at)^{1/2}, \tag{7.15}$$

and the competitive equilibrium advertising

$$\Phi^c = \frac{2}{1 + (2a/t)^{1/2}}. \tag{7.16}$$

[1] For example, Tirole (1988, Fn. 27), notes that advertising costs cannot be "too high" in order to rule out a firm's incentive to charge a high price and focus "on one's own turf."

[2] More precisely, since the maximum demand is 1 and the minimum is 0, the demand function is $\min\left\{1, \max\left\{0, (p_2 - p_1 + t)/2t\right\}\right\}$, but this is implicit given earlier derivations in (Tirole, 1988, p. 98). For ease in following the derivations in Tirole (1988), expressions here follow those in Tirole (1988).

Substituting these equilibrium values into the profit expression (2) yields the competitive equilibrium profit for firm 1

$$\Pi^c = \frac{2a}{(1 + (2a/t)^{1/2})^2}. \tag{7.17}$$

This pure strategy symmetric equilibrium is the only equilibrium presented in Tirole (1988), though of course others may exist.

Returning to the demand function (1), there is another standard assumption (implicitly) made. For the consumers that firm 1 reaches, who are not reached by firm 2, firm 1 is a monopolist. Given reaching these consumers, the demand in this case is assumed equal to 1: the 1 in $(1 - \Phi_2) \times 1$ on the RHS of (1). This means that conditional on the firm reaching the consumer and its rival not reaching the consumer, the firm has a sale with probability 1, that is, all consumers accept the offer (in contrast, for the second term the firm may only sell to a fraction of the consumer it reaches). This is a variation of the "covered market" assumption and implies that the $\bar{s}$ is large enough and t is low enough so that the furthest consumer purchases. This implies that the competitive equilibrium price (denoted p^c in (7.15)) is such that the furthest consumer buys.[3]

Assumption 2. *Covered Market Assumption: $p \leq \bar{s} - t$.*

1.2. Welfare Optimum.

The planner chooses Φ (that is, the planner has both firms set the same level) to maximize (Tirole, 1988, Fn. 29, p. 294)[4]

$$\Phi^2(\bar{s} - c - t/4) + 2\Phi(1 - \Phi)(\bar{s} - c - t/2) - 2\left(a\Phi^2/2\right).$$

The first term reflects when a consumer receives ads from both firms. Their average transportation cost is $t/4$. When they receive only one ad, their average transportation cost is $1/2$. The maximization yields (Tirole, 1988, Fn. 29)

$$\Phi^* = \frac{2(\bar{s} - c) - t}{2(\bar{s} - c) - 3t/2 + 2a}. \tag{3}$$

Intuitively, Φ^* is increasing in $\bar{s}$ and straightforward calculus confirms this.

1.3. Two Implications.

Assumptions 1 and 2 have implications that by themselves are not unusual. However, by restricting $\bar{s}$ in terms of c and t, these implications imply that an equilibrium with insufficient advertising does not exist.

Lemma 1. *Given Assumptions 1 and 2, the equilibrium price (7.15) implies that $c + 2t \leq \bar{s}$.*

Proof. Combining the equilibrium price (7.15) and Assumptions 2 we have

$$c + (2at)^{1/2} \leq \bar{s} - t.$$

Solving for a obtains

$$a \leq \frac{(\bar{s} - t - c)^2}{2t}. \tag{4}$$

From Assumption 1, (4) becomes

$$\frac{t}{2} \leq \frac{(\bar{s} - t - c)^2}{2t}. \tag{5}$$

Solving for $\bar{s}$ yields

$$c + 2t \leq \bar{s}. \tag{6}$$

□

[3] The assumption could also be inferred from the statement Tirole (1988, Bottom p. 292, col. 2) "we look at equilibria with overlapping market areas for firms among the fully informed consumers."

[4] Implicitly it is assumed that the price the planner sets is such that all consumers are willing to buy since all potential exchanges are assumed efficient.

Note that Lemma 1 is a necessary condition, but not necessarily a sufficient one. Lemma 1, in turn, has an implication regarding the monopoly price.

Lemma 2. *The monopoly price is the corner solution:* $p^m = \overline{s} - t$.

Proof. From Lemma 1

$$\begin{aligned}\overline{s} &\geq c + 2t \\ \overline{s} - 2t &\geq c \\ 2\overline{s} - 2t &\geq \overline{s} + c \\ \overline{s} - t &\geq \frac{\overline{s} + c}{2},\end{aligned}$$

with the RHS of the last inequality being the solution to the monopoly profit-maximization problem assuming an interior solution to the concave problem (that is, at that price not all consumers buy). (Though straightforward, for completeness the derivation of this price is in Appendix A). Given that quantity demanded at $p = \overline{s} - t$ equals 1, so too is quantity demanded at the lower price $(\overline{s} + c)/2$ (that is, at the latter price quantity demanded is bounded by the unit length of the city). And so, profits are greater at $p = \overline{s} - t$. □

2. Nonexistence socially insufficient advertising

2.1. The condition for socially insufficient advertising. Intuitively, there is insufficient advertising when $\overline{s}$ is sufficiently large: as $\overline{s}$ increases, the planner values advertising more, but increases in $\overline{s}$ do not affect the competitive equilibrium level of advertising (7.16).

Lemma 3. *For there to be socially insufficient advertising,* $\overline{s}$ *must be greater than*

$$\overline{s}^* \equiv \frac{(c + t/2)((2a/t)^{1/2} - 1) + 2a - t/2}{(2a/t)^{1/2} - 1}. \tag{7}$$

Proof. Differencing the socially optimal level of advertising (3) from (7.16) yields

$$\Phi^* - \Phi^c = 2\frac{(2a/t)^{1/2}(2s - 2c - t) - 2(s - c - t) - 4a}{(4s - 4c - 3t + 4a)[1 + (2a/t)^{1/2}]}, \tag{8}$$

which, since Φ^* is increasing in $\overline{s}$ while Φ^c is constant in $\overline{s}$, is increasing in $\overline{s}$. Solving for $\overline{s}$ such that the above is zero obtains

$$\frac{(c + t/2)((2a/t)^{1/2} - 1) + 2a - t/2}{(2a/t)^{1/2} - 1} \equiv \overline{s}^*.$$

Note that since by Assumption 1 $a > t/2$, then $(2a/t)^{1/2} - 1 > 0$. This ensures that $\overline{s}^* > 0$. □

2.2. A necessary condition for the existence of a competitive equilibrium. When considering a candidate equilibrium of price and advertising levels, as Tirole (1988) noted, one possible deviation for a firm is to set a higher price. Such a deviation must be ruled out for the equilibrium to exist.[5] For ease, the analysis here focuses on if the monopoly price is more profitable than the competitive price, even though the monopoly price may be dominated by a lower price,[6] as the objective here is to show the non-existence of the equilibrium, not the optimal deviation. From Lemma 2 the monopoly price is $p^m = \overline{s} - t$. As $\overline{s}$ increases, the monopoly price becomes more profitable, while the competitive equilibrium price does

[5] "Roughly, the competitive case arises...[when] charging a high price and focusing on one's own turf does not yield enough demand..." Tirole (1988, Fn. 27).

[6] Any price greater than the monopoly price would not maximize profits for when the firm is in a monopoly position. When the firm is in the duopoly position p^c maximizes its profit. As $p^c < p^m$ and the duopoly profit expression is quasi-concave, prices greater than p^m would reduce duopoly profits further.

not change (7.15) and so the competitive equilibrium profit (7.17) does not change. Thus, there exists a sufficiently large $\bar{s}$ at which the firm would deviate so long as $\Phi^c < 1$, which is ensured by Assumption 1. Specifically, if firm i deviates to $\bar{s} - t$ its profit, given Assumption 2, is

$$\Pi^m \equiv \Phi^c(1-\Phi^c)(\bar{s}-t-c)+\Phi^c\Phi^c \min\left\{1, \max\left\{0, \frac{p^c-(\bar{s}-t)+t}{2t}\right\}\right\}(\bar{s}-t-c). \quad (9)$$

The second term is positive and reflects that at the monopoly price it is possible that there are consumers sufficiently close to the deviating firm i such that they buy from firm i even though they receive an ad from firm j. Let $\bar{s}^m$ denote the $\bar{s}$ such that deviating to the monopoly price is profitable (and so the competitive equilibrium does not exist). That is, for $\bar{s} \geq \bar{s}^m$, $\Pi^m \geq \Pi^c$. However, because of the second term in (9), it is more practical to focus on the profit expression without the second term:

$$\underline{\Pi}^m = \Phi^c(1-\Phi^c)(\bar{s}-t-c) = 2\frac{(2a/t)^{1/2}-1}{(1+(2a/t)^{1/2})^2}(\bar{s}-t-c) \leq \Pi^m, \quad (10)$$

where the underline indicates that this is a lower limit to what monopoly profit could be and the inequality is because the second term in (9) is positive. Clearly, for $\bar{s}$ such that $\underline{\Pi}^m > \Pi^c$, then $\Pi^m > \Pi^c$ and so the monopoly price is more profitable than the competitive price, that is, the Tirole (1988) competitive equilibrium does not exist. That is, a necessary, but not necessarily sufficient condition for the competitive equilibrium is that

Lemma 4. *For a firm not to deviate from the competitive price, $\bar{s}$ must be less than*

$$\hat{s} \equiv \frac{a+(c+t)((2a/t)^{1/2}-1)}{(2a/t)^{1/2}-1}. \quad (11)$$

Proof. The competitive equilibrium price is dominated by the monopoly price whenever (10) is greater than (7.17), or subtracting the latter from the former, when the following is positive

$$\underline{\Pi}^m - \Pi^c = 2\frac{[(2a/t)^{1/2}-1](\bar{s}-c-t)-a}{[1+(2a/t)^{1/2}]^2},$$

which is increasing in $\bar{s}$. Solving for the $\bar{s}$ such this is zero obtains

$$\hat{s} \equiv \frac{a+(c+t)((2a/t)^{1/2}-1)}{(2a/t)^{1/2}-1}.$$

For $\bar{s} > \hat{s}$, $\Pi^m \geq \underline{\Pi}^m > \Pi^c$, and the firm would deviate from the competitive price. □

Proposition 1. *For Tirole's competitive equilibrium in a Hotelling model of advertising, advertising cannot be socially insufficient.*

Proof. Subtracting (11) from (7) yields

$$s^* - \hat{s} = \frac{2a-(2at)^{1/2}}{2[(2a/t)^{1/2})-1]} > 0.$$

The inequality follows as the numerator and denominator are positive since by Assumption 1, $a > t/2$. Thus, $s^* > \hat{s} \geq \bar{s}^m$. □

That is, the $\bar{s}$ needed for socially insufficient advertising in the competitive equilibrium is greater than the maximum $\bar{s}$ possible for a firm not to deviate from the competitive equilibrium price.

The intuition for the result is straightforward. The competitive equilibrium price and advertising levels are independent of $\bar{s}$. However, larger $\bar{s}$ increases the social return from advertising; as $\bar{s}$ increases, the planner would increase the level of advertising. Thus, there is a threshold $\bar{s}$, s^*, such that if $\bar{s}$ is greater than this, then there would be insufficient advertising in the candidate equilibrium if it exists. However, as $\overline{s}$ increases, the incentives to deviate to the monopoly price increase. For this model and its assumptions, the $\overline{s}$ at which the firm would deviate from the equilibrium price is less than s^*.

It has been shown that the competitive equilibrium presented in Tirole (1988) cannot have socially insufficient advertising. However, this was the pure-strategy competitive equilibrium presented in Tirole (1988) and there may also exist a mixed-strategy equilibrium in prices, or prices and advertising. Likewise, relaxing various assumptions could change the characteristics of the equilibrium. The result here does imply the same for other equilibria that may exist.

Appendix A. Monopoly Price

In the Hotelling model, given a monopolist at 0 that sets a price p, a consumer located at x is willing to buy if $\overline{s} - p - tx \geq 0$. If the $\tilde{x}$ such that $\overline{s} - p - t\tilde{x} = 0$ is less than one ($\tilde{x} < 1$), then the demand the firm faces is $D_m = (\overline{s} - p)/t$ and its profit is $(p - c)(\overline{s} - p)/t$. Maximizing this with respect to p yields $\mathring{p} = (\overline{s} + c)/2$, which is the profit-maximizing price so long as the quantity demanded associated with $\mathring{p}$ is less than 1; else the profit-maximizing price is $\overline{s} - t$ (since the firm could then raise its price to $\overline{s} - t$ without any change in the demand in its product).

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