A Simple Model of Competitive Testing

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Abstract

A number of candidates are competing for a prize. Each candidate is privately informed about his type. The decision-maker who allocates the prize wants to give it to the candidate with the highest type. Each candidate can take a test that reveals his type at a cost. I show that if competition increases, candidates reveal more information when the cost is high, and less information when it is low. Nevertheless, the decision-maker always benefits from greater competition. If competition is large, mandatory disclosure is Pareto-dominated by voluntary disclosure. When the test is noisier, candidates are more likely to take it.

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1 Introduction

Consider a university that is looking for a new faculty member on the academic job market. The university would like to hire the candidate with the

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highest ability, but the ability of each candidate is her private information. Each candidate, however, can choose to present a paper at a conference. Doing so is costly, but it reveals the candidate's ability to the employer.

More formally, suppose that a number of candidates are competing for a prize of fixed value. Each candidate is privately informed about his type, which is drawn from the unit interval. The prize is allocated by a decision-maker, who would like to give it to the candidate with the highest type. Each candidate has access to an exogenous test which, if taken, reveals the candidate's type. The test is costly, and candidates simultaneously decide whether to take it.

Competition for jobs, as suggested above, is one setting to which this framework applies. Another is political competition: voters would like to select the most competent candidate, and candidates can invest in a media campaign to communicate their competence. Firms competing for a fixed-price procurement contract can reveal the quality of their products by asking an independent agency to certify it. Students applying to a university or competing for a scholarship can take an optional test that would demonstrate their ability, at some cost.

Since the test is costly, not all candidates take it. Instead, there is a unique symmetric equilibrium, in which a candidate takes the test if and only if his type is above some threshold. In that case, he wins the prize if the test shows him to have a higher type than any other candidate who takes the test. On the other hand, if a candidate does not take the test, the decision-maker learns that his type is below the threshold. Then the candidate can only get the prize if nobody else takes the test, in which case the decision-maker allocates the prize at random.

The first result of the paper shows that competition affects information revelation in a non-monotone way. When the cost of the test is high, increasing the number of candidates makes them weakly more likely to take the test. But when the cost is low, an increase in the number of candidates results in less information revelation. Thus, greater competition can make the decision-maker less informed. At the same time, even when the number of candidates goes to infinity, the probability that some information is revealed remains distinct from zero and from one.

To see the intuition, consider a candidate i whose type is at the threshold. Increasing competition reduces i's chance to receive the prize after taking the test, since it becomes increasingly likely that some competitor has a higher type, takes the test, and wins over i. It also reduces i's chance of getting the prize without taking the test, because the decision-maker will randomise over a larger number of candidates. But if the cost of the test is low, the threshold is low as well. Then increasing the number of candidates has a large effect on the probability that some other candidate has a type above the threshold. Hence, the first effect dominates the second, and i becomes less willing to take the test.

Second, I show that even though competition can result in less information revelation, the decision-maker always benefits from an increase in competition. On the other hand, an increase in the cost of the test hurts the decision-maker but can make candidates better off by reducing inefficient testing.

Third, the paper examines the effect of the decision-maker committing not to give the prize to any candidate who does not take the test. There is a substantial literature focusing on mandatory disclosure as a way of making decision-makers better off[1]. But does the decision-maker benefit from making disclosure mandatory when disclosure is costly and informed parties compete? On the one hand, such a move reduces the payoff of a candidate who does not take the test to zero. Hence, candidates become more willing to take it, and the decision-maker receives more information. On the other hand, if no candidate takes the test, such a commitment leaves the decision-maker unable to allocate the prize, reducing her utility. But if the number of candidates is very large, then, even without commitment, a candidate who does not take the test is very unlikely to win the prize. Thus, the first effect disappears, while the second effect remains. Hence, when competition is high, making the test mandatory strictly reduces the decision-maker's utility. Since mandatory disclosure makes candidates worse off as well, this implies that under strong competition, mandatory disclosure is strictly Pareto-dominated by voluntary disclosure. For example, when candidates' types are uniformly distributed, making the test voluntary is better whenever the number of candidates is larger than two.

Fourth, I consider what happens when the test sends a noisy signal about a candidate's type. The paper shows that candidates are more likely to take the test when it is noisy than when it is not. More generally, making the test noisier increases the probability that candidates take it. Intuitively, without noise, if a candidate whose type is at the threshold takes the test, he can only win if no other candidate has a higher type. With noise, he can also

[1] See an overview in Dranove and Jin (2010).

win if some candidate has a higher type but does worse on the test. Hence, the incentive to take the test increases.

To see the implications of these results, consider a market in which several firms compete by offering products of uncertain quality. Each firm can choose to credibly reveal the quality of its product by asking an independent body to certify it at some cost. If more firms enter the market, do buyers become more informed? The paper suggests that an increase in competition will reduce information revelation when certification is cheap relative to profit margins in the market, but not when it is costly.

Alternatively, consider an election contested by several candidates. Each candidate take a costly action to communicate her competence – for example, to take part in a public discussion that will be covered by media. The media, however, transmits information to voters with some noise. That noise is larger when the quality of journalism is lower, when voters have less trust in media, or when media penetration is low (so voters tend to learn the content of media reports through their friends, rather than directly). The paper suggests that in such situations, candidates will be more likely to invest in campaigning.

Furthermore, consider university applicants that can reveal their ability through a standardised test such as SAT or GRE. Should universities make submission of test scores optional rather than mandatory? While negative effects of highly competitive university admission tests on applicants have been noted before[2], this paper suggests that not only candidates, but also universities can be better off if submission of test scores is made optional.

Finally, consider the problem of a firm running a standardised test. The firm wants to maximise its profit, and can choose the noise level of the test. The results imply that increasing noise increases the expected number of test takers, and hence the firm's expected revenue. While it is possible that a more precise test is more costly to run, the paper suggests that the firm can intentionally make the test imprecise even in the absence of this factor[3].

The rest of this section discusses the related literature. Section 2 describes the baseline model. Section 3 examines the effect of competition on disclosure. Section 4 discusses how players' utilities are affected by competition and cost of the test. Section 5 analyses the effect of making the test

[2] See a discussion in Olszewski and Siegel (2016).

[3] Standardised tests are in fact often observed to be noisy. See, for example, The Atlantic, *"The GRE Fails at Predicting Who Will Succeed"*, March 1, 2016.

compulsory for receiving the prize. Section 6 extends the model to the case when the test is noisy, as well as to the case when a candidate's cost of taking it depends on his type. Finally, Section 7 concludes. All proofs, except for very short ones, are in the Appendix.

Related literature. A number of papers, starting with Spence (1973), look at senders who signal their types by taking costly actions. Signalling is different from testing modelled in this paper, because a test directly reveals the type to the decision-maker. Hence, in my paper a candidate with a low type cannot mimic a candidate with a high type, and the incentive compatibility constraint becomes redundant. Thus, taking the test imposes separation, and only candidates who do not take the test can pool. This ensures the existence of a unique and tractable symmetric equilibrium in a setting with a rich set of types and competing senders. Within the signalling literature, Feltovich et al. (2002), Alós-Ferrer and Prat (2012), and Daley and Green (2014) examine settings in which a Spence-type signal is complemented by an additional exogenous signal that, like test score in my paper, is correlated with the sender's type. In these papers, however, the additional signal is costless to the sender and is transmitted regardless of the sender's action – only the Spence-type signal is chosen by the sender. On the other hand, in my setup candidates choose whether to send a test score, at a cost. This allows me to examine the effect of competition and noise on their choice, as well as the welfare effects of making disclosure mandatory.

In models of auctions with costly participation[4], buyers choose whether to enter an auction. Those who enter pay the entry cost and choose their bids. The buyer with the highest bid wins. Those who do not enter receive a fixed outside option. The setup of this paper is related: candidates who decide to take the test pay the cost, their types are revealed, and the candidate with the highest type receives a common-value prize. However, types, unlike bids, are exogenous, rather than chosen by candidates[5]. Furthermore, unlike bids, types can be revealed with exogenous noise which, as Section 6.1 shows, affects the equilibrium. Finally, in this paper the outside option is not fixed:

[4]See McAfee and McMillan (1987), Levin and Smith (1994), Stegeman (1996), Menezes and Monteiro (2000), Lu (2009), Celik and Yilankaya (2009), Cao and Tian (2010), Moreno and Wooders (2011).

[5]In private-value auctions, buyers have heterogeneous valuations, which determine their bids. Here, in contrast, candidates differ in their chance of winning the prize after taking the test, but not in the valuation of the prize.

a candidate who does not take the test can still win the prize. His probability of winning depends on the number of competitors. Payoffs from taking the test and from not taking the test change at different rates as competition increases – hence, the effect of competition on the probability of taking the test is non-monotone. In contrast, in an equivalent common-value auction with costly entry, an increase in the number of potential bidders would have a monotone effect on entry[6]. This also implies that the decision-maker always gains from an increase in competition, whereas in a common-value auction with costly participation the seller wants to restrict entry (Levin and Smith, 1994).

The paper is also related to the literature on all-pay auctions or contests (see Konrad, 2009, for an overview). In this literature, contestants choose their bids or effort levels, and the contestant with the highest bid wins. In my paper, candidates are restricted to two "effort levels": taking or not taking the test. At the same time, candidates have a rich set of types, and a decision to take the test reveals the type. The type determines a candidate's chance of winning if he takes the test (but not if he does not)[7]. This structure implies very different results: for example, there is a unique pure-strategy equilibrium, whereas in a standard all-pay auction a pure strategy equilibrium typically does not exist. It also means that candidates can win the prize without taking the test (whereas in an all-pay auction, a bid of zero wins with probability zero). This underlines the non-monotone effect of competition on test participation. Furthermore, since winning depends on an exogenous type, the paper can analyse the effect of noisy type revelation. Finally, in the contest literature, the principal's aim is to maximise candidates' effort[8]. In this paper, by contrast, the decision-maker's payoff does not

[6]This happens in my paper when the decision-maker commits to give zero payoff to candidates who do not take the test (see Section 5). In fact, Proposition 6 shows that this "auction-like" setup is Pareto-dominated by the baseline setup of the paper.

[7]Some authors examine contests in which there is exogenous heterogeneity among participants (e.g. Moldovanu and Sela, 2001; Liu et al., 2018). In these papers, contestants' types represent the cost of exerting effort that increases a candidate's chance of winning. In contrast, in my setup, the test has the same cost for all candidates, and a candidate's type directly determines the chance of winning the prize if the candidate takes the test. In some other all-pay auction models, e.g. Siegel (2009, 2014), contestants can have asymmetric head starts. These are quite different from types in this model, however: first, they are commonly known, and second, a head start affects a contestant's' chance of winning the prize regardless of their effort level.

[8]Typically aggregate effort, although in some all-pay contest models (e.g. Denter and

directly depend on candidates taking the test – she is only interested in correctly selecting the candidate with the highest type[9]. Hence, as Proposition 6 shows, the decision-maker strictly prefers not to make the test mandatory for receiving the prize, even though such a rule maximises the expected number of candidates who take the test (and hence would be optimal in an analogous all-pay auction model).

Another literature has looked at information disclosure by senders who cannot lie, but can choose how much information to reveal. When disclosure is costless and there is no competition, full revelation is the typical benchmark result (see Dranove and Jin, 2010, for an overview)[10]. On the other hand, when disclosure has a cost, Jovanovic (1982) shows (in a setting without competition) that full revelation is not an equilibrium. Subsequent research on costly disclosure in competitive settings has looked at the interplay between firms' decision to reveal product quality and their price-setting behaviour. In particular, Cheong and Kim (2004) and Guo and Zhao (2009) look at the effect of an increase in the number of firms[11]. In that setup, disclosure is noiseless, and the payoff of a firm that reveals information depends on endogenously determined prices. At the same time, a firm that does not reveal information receives zero profit. Since the profit of a firm that does reveal information depends negatively on competition, increasing competition has a monotone effect on disclosure. In this paper, on the other hand, the prize of the winning candidate is exogenously fixed, while a candidate who does not reveal information receives a positive expected payoff that depends on competition. This implies very different results – in particular, the effect of increasing competition on information disclosure is non-monotone and depends on the cost of the test and the number of candidates. Furthermore,

Sisak, 2016) the contest designer wants to maximise the effort of the winning candidate.

[9]In Moldovanu et al. (2007), contestants care about their relative positions, but the contest designer is still interested in maximising aggregate effort.

[10]Full revelation can fail if there is uncertainty over how much information senders have. Carlin et al. (2012) look at a setup in which competing senders are informed about their types with some probability, and those who are informed can disclose their types at no cost and without noise. In this setting, they show that competition has a monotone negative effect on disclosure, unlike this paper, which shows that the effect can be positive or negative for different costs and levels of competition.

[11]In addition, Janssen and Roy (2015), Levin et al. (2009), Board (2009) and Forand (2013) examine information disclosure in a two-firm setting, but do not focus on the effect of increasing the number of firms. Stivers (2004) and Ivanov (2013) examine a competitive market with no disclosure costs.

the exogenous nature of the test enables me to analyse the effect of test noise on information disclosure.

The result that making the test less informative increases the expected number of candidates who take it echoes some of the results in Alonso (2017). In that paper, workers sort between two firms. Each worker has a pair of types, which measures his productivity in each firm. The value of having a job is endogenously determined through bargaining. To select workers, a firm administers an interview that provides an imperfect signal about a worker's type. Participating in an interview is necessary to get the job. A key difference is that in Alonso (2017), workers are imperfectly informed about their types, but have perfect information about the realised distribution of types (since there is a continuum of workers and a continuum of vacancies). In my paper, on the other hand, each candidate i is fully informed about his type, but the realised distribution of types (and, in particular, the number of candidates who have a higher type than i does) is random (because the number of candidates is finite). With this different setup, Alonso (2017) shows that a more informative interview can, depending on the workers' information structure, encourage or discourage applications. In my paper, on the other hand, making the test more informative has a monotone negative effect on the probability that a candidate applies.

Less closely related are models of Bayesian persuasion by competing senders[12]. In these papers disclosure is costless, senders commit to a disclosure strategy before learning the state, and senders can design an information disclosure scheme rather than having to use an exogenous test with fixed parameters such as noise.

2 Model

There are $n > 1$ candidates (male) that are competing for a prize allocated by a decision-maker (female). The value of the prize to each candidate is 1. Each candidate i has a type $x_i \in [0, 1]$, which is his private information. Types are drawn independently from a distribution F with an associated density f. Each candidate can decide to take a test at a cost $c \in (0, 1)$. The

[12] Kamenica and Gentzkow (2015, 2017), Boleslavsky and Cotton (2014), Au and Kawai (2017).

test, if taken, perfectly reveals his type to the decision-maker[13].

The decision-maker receives a payoff x_i if she allocates the prize to candidate i – thus, the decision-maker would like to allocate the prize to a candidate with the highest type. If the decision-maker's posterior belief is such that several candidates have the highest expected type, she randomises between them uniformly.

The timing is as follows. First, nature draws x_i for every candidate i. Each candidate learns his type. Candidates then simultaneously decide whether to take the test. The decision-maker learns the types of candidates who took it. She then chooses a candidate that receives the prize. The paper focuses on symmetric equilibria.

3 Effect of Competition

3.1 Equilibrium

At a symmetric equilibrium, the strategy of every candidate i is a function $h : [0, 1] \to [0, 1]$ which maps the candidate's type to the probability of taking the test.

The decision-maker will allocate the prize to a candidate whose ex post expected type is the highest. At the equilibrium, then, if a candidate's type is close to zero, it is very likely that somebody has a higher type. Thus, a candidate with a very low type who takes the test is very unlikely to win. He then prefers not to take it and avoid paying the cost c. If the type is higher, the probability of winning is (weakly) larger. Then there should exist some cutoff such that a candidate takes the test if and only if his type is above it. This intuition implies the following lemma:

Lemma 1. *At every symmetric equilibrium, there exists a threshold* $b \in [0, 1]$ *such that* $h(x) = 1$ *for all* $x > b$*, and* $\Pr[h(x) > 0 \mid x \leq b] = 0$.

In words, any symmetric equilibrium is characterised by a threshold b such that candidates whose types are above b always take the test, while candidates whose types are below b never take the test – except for, possibly,

[13] Section 6 considers the case when the test is noisy, and the case when the cost is a function of a candidate's type.

some set of types whose mass is zero[14]. This last possibility is irrelevant, because the paper examines what happens in expectation. I will thus focus on the pure-strategy equilibrium in which each candidate takes the test if and only if his type is above some $b \geq 0$.

The decision-maker's expected payoff equals the expected type of the candidate whom she gives the prize. At a Bayesian equilibrium, if candidate i has a type above b (and thus takes the test), the decision-maker learns his type. Hence, if at least one candidate takes the test, the decision-maker is able to allocate the prize to the best candidate with certainty. In these situations, I will say that the decision-maker *makes an informed decision.*

If candidate i does not take the test, the decision-maker's expectation of i's type equals $\mathrm{E}_F(x \mid x < b)$, where $\mathrm{E}_F(\cdot)$ denotes expectation taken over F. This expression is well-defined whenever $b > 0$. Note that since $c > 0$, $b = 0$ cannot be an equilibrium – if it were, there would be some $\varepsilon > 0$ such that a candidate with type below ε would have such a low probability of winning the prize that he would prefer to deviate and not take the test.

Since $b > \mathrm{E}_F(x \mid x < b)$, a candidate who does not take the test has a lower ex-post expected type than any candidate who does. He can thus only win the prize if nobody else takes the test, which happens with probability $F(b)^{n-1}$. In that case, the decision-maker gives him the prize with probability $\frac{1}{n}$. Thus, if a candidate does not take the test, his overall probability of winning the prize is $F(b)^{n-1}\frac{1}{n}$. On the other hand, a candidate with type $x_i > b$ takes the test and wins the prize with certainty if every other candidate has a lower type – which happens with probability $F(x_i)^{n-1}$.

Suppose that $c \leq \frac{n-1}{n}$. At $x_i = b$, candidate i must be indifferent between taking and not taking the test, which yields the equation

$$F(b)^{n-1} - c = F(b)^{n-1}\frac{1}{n} \tag{1}$$

On the other hand, if $c > \frac{n-1}{n}$, then the left-hand side of (1) is smaller than the right-hand side for all $b > 0$. Hence, the equilibrium strategy of every candidate is to never reveal the type, so $b = 1$. Hence, the equilibrium threshold b is characterised as follows:

Lemma 2. *The unique symmetric equilibrium is given by* $F(b) = \min\left\{\left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}}, 1\right\}$.

[14] The reason for the latter possibility is that, for types between $\mathrm{E}(x \mid x < b)$ and b, the probability of winning the prize after taking the test is constant – but only as long as the mass of candidates with these types who take the test is zero.

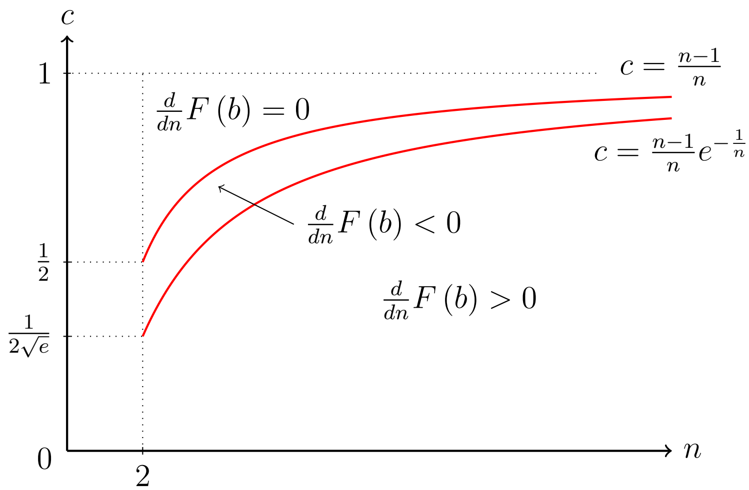

Figure 1: Effect of competition on $F(b)$.

Proof. If $c \leq \frac{n-1}{n}$, then (1) implies that $F(b) = \left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}} \leq 1$. If $c > \frac{n-1}{n}$, then $b = 1$ implies that $F(b) = 1$. Uniqueness follows from the fact that the expression in the lemma is in closed form. □

3.2 Competition and Disclosure

There are two natural ways of measuring the degree to which information is disclosed. One such indicator is $F(b)$, the probability that a candidate does not reveal his type. Another indicator is $F(b)^n$. This is the probability that there is no disclosure – i.e. that even the candidate with the highest type does not take the test. Thus, $1 - F(b)^n$ is the probability that the decision-maker makes an informed decision – that is, knows with certainty that the candidate who receives the prize has the highest type.

Unsurprisingly, both $F(b)$ and $F(b)^n$ weakly increase if c goes up – when the test is more costly, candidates are less likely to take it. A more interesting question is what happens if n increases. This is described in the following two propositions:

Proposition 1. *$F(b)$ increases with n if $c \in \left(0, \frac{n-1}{n}e^{-\frac{1}{n}}\right)$, decreases with n if $c \in \left[\frac{n-1}{n}e^{-\frac{1}{n}}, \frac{n-1}{n}\right]$, and stays constant as n changes if $c \in \left(\frac{n-1}{n}, 1\right)$.*

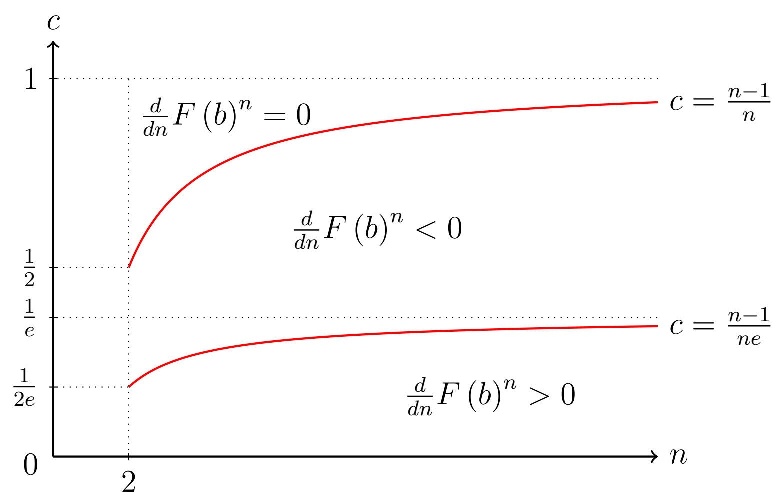

Figure 2: Effect of competition on $F(b)^n$.

This characterises the effect of increasing competition on $F(b)$ for different pairs of (c, n). Figure 1 illustrates this result. As the figure shows, increasing the number of candidates does not change the probability that a candidate takes the test when n is low (relative to a boundary that depends on c), increases that probability when n is moderate, and reduces that probability when n is high. In particular, if $c < \frac{1}{2\sqrt{e}}$, an increase in n always reduces the probability that a candidate takes the test.

Proposition 2. *$F(b)^n$ stays constant as n changes if $n < \frac{1}{1-c}$, decreases with n if $\frac{1}{1-c} \leq n \leq \frac{1}{1-ce}$, and increases with n if $n > \frac{1}{1-ce}$.*

Figure 2 illustrates this result[15]. As we can see from Figure 2, if $c \geq \frac{1}{e}$, increasing competition weakly increases the probability that the decision-maker makes an informed decision. Specifically, if $c \in \left[\frac{1}{e}, \frac{1}{2}\right]$, an increase in n strictly reduces $F(b)^n$; while if $c > \frac{1}{2}$, increasing n has no effect until n reaches $\frac{1}{1-c}$, after which any further increase in n makes the decision-maker more informed.

On the other hand, if $c \in \left(\frac{1}{e}, \frac{1}{2e}\right)$, increasing n increases the probability of an informed decision only until n reaches $\frac{1}{1-ce}$. After that, increasing n

[15]Alternatively, the result can be expressed in terms of the values of c. An increase in n reduces the probability that the best candidate reveals his type if $c \in \left(0, \frac{n-1}{ne}\right)$, increases it if $c \in \left[\frac{n-1}{ne}, \frac{n-1}{n}\right]$, and has no effect if $c \in \left(\frac{n-1}{n}, 1\right)$.

reduces that probability.

Finally, if $c \leq \frac{1}{2e}$, an increase in competition always makes the decision-maker less informed.

To summarise, increasing competition raises the amount of information available to the decision-maker when the cost of the test is high, and lowers it when the cost of the test is low.

To see the intuition behind this result, consider the marginal candidate, whose type equals b. For him, increasing n has two effects. First, the expected payoff from taking the test falls, because $F(b)^{n-1}$, the probability that no other candidate has a higher type, decreases. Second, the expected payoff from not taking the test falls as well, because $\frac{1}{n}$, the probability of being randomly selected to receive the prize when nobody takes the test, becomes smaller. But if c is low, then b is low as well. In that case, the impact of increasing the number of candidates on $F(b)^{n-1}$ is relatively large. Thus, the first effect dominates the second, and the marginal candidate becomes less willing to take the test. On the other hand, if c is high (but not so high that nobody takes the test), then $F(b)$ is close to 1. Then increasing n does not change $F(b)^{n-1}$ much, and so the second effect dominates the first. Finally, if c is very large, then no candidate takes the test, and a further increase in n has no impact on information disclosure.

3.3 The Case of Large Competition

We can also check what happens in the limit when n goes to infinity. Since $\lim_{n\to+\infty}\left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}} = \lim_{n\to+\infty}\left(\frac{c}{1-\frac{1}{n}}\right)^{\frac{1}{n-1}} = 1$, we have $\lim_{n\to+\infty} F(b) = \lim_{n\to+\infty} \min\left\{\left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}}, 1\right\} = 1$. Hence, the probability that a given candidate takes the test goes to zero. Intuitively, when $n \to +\infty$, a cutoff $b < 1$ cannot be an equilibrium – if it were, then for any type $x \in (b, 1)$, there would almost surely be a candidate with a type above x. Hence, a candidate with type x would almost surely not win the prize, and hence would strictly prefer not taking the test.

Nevertheless, the probability that the decision-maker makes an informed decision does not go to zero, as the following result shows:

Proposition 3. *When n approaches infinity, the probability that no candidate takes the test approaches c.*

Proof. $\lim_{n\to\infty}\left(\frac{cn}{n-1}\right)^{\frac{n}{n-1}} = \lim_{n\to\infty}\left(\frac{c}{1-\frac{1}{n}}\right)^{\frac{1}{1-\frac{1}{n}}} = c$. Since $F(b)^n = \min\left\{\left(\frac{cn}{n-1}\right)^{\frac{n}{n-1}}, 1\right\}$,

$\lim_{n\to\infty} F(b)^n = c.$ □

Hence, when $n \to +\infty$, the probability that at least the best candidate takes the test approaches $1-c$. Thus, even when competition is very strong, the probability that the decision-maker makes an informed decision remains distinct from zero (and also from one).

Intuitively, suppose that $F(b)^n$ approached 1 as n became large. Then in the limit the expected type of a candidate who does not take the test would equal $\mathrm{E}_F(x)$. But then any candidate with a type above $\mathrm{E}_F(x)$ would win the prize with probability 1 if he took the test. Hence, he would deviate, contradicting the initial assumption.

4 Utilities and Welfare

4.1 Effect of Competition

Does the decision-maker gain from an increase in competition? Increasing n has two effects on her payoff. On the one hand, since the type of each candidate is an independent draw from the distribution F, increasing the number of draws increases the expected type of the best candidate. In a perfect information setting, this would make the decision-maker better off. However, when c is small, greater competition can also increase the probability that even the best candidate does not take the test. If that happens, the decision-maker will have to allocate the prize at random, which means that the prize will not necessarily go to the best candidate. The reduction of information available to the decision-maker creates a negative effect of competition on her payoff. Nevertheless, the following will show that the first effect will always dominate the second, and greater competition is always better for the decision-maker.

To check this, we need to determine her expected payoff. With probability $1-F(b)^n$, at least one candidate has a type above b, and takes the test. Then the decision-maker allocates the prize to the candidate with the highest type. In that case, the decision-maker's expected payoff equals

$$\mathrm{E}(\max\{x\} \mid \max\{x\} > b) = \frac{\int_b^1 x d\left[F(x)^n\right]}{1-F(b)^n}$$

where the above expression uses the fact that the cdf of $\max\{x\}$ is $F(x)^n$. On the other hand, with probability $F(b)^n$, no candidate takes the test.

Then the decision-maker allocates the prize to a random candidate, and her expected payoff equals

$$\mathrm{E}(x \mid x < b) = \frac{\int_0^b xd\left[F(x)\right]}{F(b)}$$

Overall, the decision-maker's expected utility equals

$$v = \int_b^1 xd\left[F(x)^n\right] + F(b)^{n-1} \int_0^b xd\left[F(x)\right] \tag{2}$$

When $n \leq \frac{1}{1-c}$ (or, equivalently, when $c \geq \frac{n}{n-1}$), we have $b = 1$ and thus $v = \int_0^1 xd\left[F(x)\right]$, which does not depend on n. Intuitively, when no candidate takes the test, the decision-maker has to allocate the prize at random regardless of n. If $n > \frac{1}{1-c}$, then $b < 1$, and we have the following result:

Proposition 4. *When $n > \frac{1}{1-c}$, an increase in n strictly increases v.*

Hence, an increase in competition makes the decision-maker strictly better off, unless no candidates take the test (in which case the decision-maker's payoff is not affected by n). This contrasts with the standard result in the literature on auctions with endogenous entry, in which the seller typically benefits from restricting the number of potential bidders (see e.g. Levin and Smith, 1994).

Intuitively, while greater competition can reduce the probability that the decision-maker makes an informed decision, this can only occur when the cost of the test is low, as Proposition 2 states. But if c is low, then so is $F(b)$. Hence, each new candidate is likely to take the test, so the positive effect of an increase in the number of draws from F is large, outweighing the negative effect. On the other hand, in an equivalent auction with endogenous entry, a bidder who does not enter would be unable to win the good – thus, the payoff from not entering would be zero. Because of this, an increase in the number of potential bidders would always reduce entry, regardless of the cost[16], and hence greater competition can hurt the seller.

[16]This also happens if the decision-maker commits to give a payoff of zero to any candidate who does not take the test (see Section 5).

4.2 Effect of Cost

The cost c of the test can affect the decision-maker's utility by affecting b, and hence the amount of information that is revealed. If $c > \frac{n-1}{n}$, then $b = 1$, and a further increase in c does not change it. If $c < \frac{n-1}{n}$, then an increase in c increases b, making candidates less likely to reveal their types, and hence reducing the decision-maker's expected payoff.

What about candidates? Since candidates are symmetric, a randomly selected candidate wins the prize with an ex ante probability of $\frac{1}{n}$. With probability $1 - F(b)$ he also takes the test and pays the cost c. Thus, his overall expected utility equals

$$u = \frac{1}{n} - c\left[1 - F(b)\right] \tag{3}$$

This yields the following result:

Proposition 5. *An increase in c decreases u if $c < \left(\frac{n-1}{n}\right)^n$, increases u if $c \in \left(\left(\frac{n-1}{n}\right)^n, \frac{n-1}{n}\right)$, and does not affect u if $c > \frac{n-1}{n}$.*

Intuitively, for candidates the test is a deadweight loss – it only serves to reallocate the prize between candidates at a cost to those who take the test. If $c < \frac{n-1}{n}$, an increase in c has two opposite effects. On the one hand, by raising the threshold b, it reduces the expected number of candidates who take the test, thus increasing candidates' utility. On the other hand, those candidates who do take the test have to pay a higher cost. If c is sufficiently small, then the effect of increasing c on b is small as well, so the second effect dominates the first. The opposite is true when c is moderately large. Finally, if $c > \frac{n-1}{n}$, then no candidate takes the test, and increasing its cost has no effect.

Since the decision-maker always prefers a lower cost unless $c > \frac{n-1}{n}$, this implies that lowering the cost increases welfare if $c < \left(\frac{n-1}{n}\right)^n$, and has no effect on welfare if $c > \frac{n-1}{n}$. When $c \in \left(\left(\frac{n-1}{n}\right)^n, \frac{n-1}{n}\right)$, lowering the cost makes candidates worse off and the decision-maker better off.

5 Mandatory Disclosure

Can the decision-maker change the rules of the game to increase her welfare? This paper does not focus on finding a generic optimal mechanism for the decision-maker[17]. Instead, it will look at one particular avenue that the decision-maker can pursue: committing to never give the prize to a candidate who does not take the test. For example, universities often require every applicant to take the test for his or her application to be considered. Is such a commitment optimal?

By an argument similar to the one in Lemma 1, when testing is mandatory for receiving the prize, the equilibrium has a similar threshold form to the one described earlier:

Lemma 3. *Under mandatory disclosure, at every symmetric equilibrium, there exists a threshold* $\hat{b} \in [0,1]$ *such that* $h(x) = 1$ *for all* $x > \hat{b}$*, and* $\Pr\left[h(x) > 0 \mid x < \hat{b}\right] = 0$.

Proof. Identical to the proof of Lemma 1 with m and $\pi(m)$ replaced by zero. □

At the threshold, a candidate receives a payoff of $F\left(\hat{b}\right)^{n-1} - c$ if he takes the test. A candidate who does not take it receives a payoff of zero. Indifference condition gives us

$$F\left(\hat{b}\right) = c^{\frac{1}{n-1}}$$

It is easy to see that $F\left(\hat{b}\right)$ and $F\left(\hat{b}\right)^n = c^{\frac{n}{n-1}}$ are strictly increasing in n for any $c \in (0,1)$. Thus, in contrast to the case without commitment, under mandatory disclosure an increase in competition has a monotone effect on the amount of information that is disclosed.

At the same time, we can see that $F\left(\hat{b}\right) < F(b)$. Hence, a given candidate is more likely to take the test when disclosure is mandatory than when it is compulsory. Intuitively, this happens because under mandatory testing, the expected payoff of a candidate who does not take the test becomes zero, while under voluntary disclosure he is still able to win the prize.

Under voluntary testing, the expected payoff of a randomly selected candidate is given by (3). Under mandatory testing, it equals $\frac{1}{n} - c\left[1 - F\left(\hat{b}\right)\right]$.

[17]If the decision-maker has full commitment power, and if communication between her and candidates is costless, then one such possible mechanism would be to ask all candidates to report their types, and then ask the candidate with the highest reported type to take the test, promising to give him the prize if and only if the test confirms the type. In practice, costless and noiseless communication is often unavailable.

Since $F\left(\hat{b}\right) < F(b)$, mandatory testing reduces the expected payoff of a random candidate. The intuition is the same as in Section 4.2: the cost of the test is a deadweight loss, and making candidates more likely to take it reduces their utility.

What about the decision-maker's payoff? Without commitment, it is given by (2). On the other hand, when the test is mandatory, the decision-maker's payoff equals the type of the best candidate if the best candidate takes the test, which happens with probability $1 - F\left(\hat{b}\right)^n$. If the best candidate does not take the test, the decision-maker's payoff is zero. Then the expected payoff of the decision-maker equals

$$\hat{v} = \left[1 - F\left(\hat{b}\right)^n\right] \mathrm{E}\left(\max\{x\} \mid \max\{x\} > \hat{b}\right) = \int_{\hat{b}}^{1} x d\left[F\left(x^n\right)\right] \qquad (4)$$

Let $D(n) \equiv \hat{v} - v$ be the decision-maker's expected gain from making the test mandatory. When n is sufficiently large, we can show that $D(n)$ is negative, and hence the decision-maker is better off when the test is voluntary. Since voluntary testing is also better for candidates, we have the following result:

Proposition 6. *For all $c > 0$, and any F, there exists $\bar{n}$ such that voluntary testing strictly Pareto-dominates mandatory testing for all $n > \bar{n}$.*

Intuitively, a commitment to only select the winner from candidates who took the test has two effects. On the one hand, since candidates become more likely to take the test, mandatory testing increases the probability that the best candidate reveals his type. Hence, the decision-maker is more likely to make an informed decision, which increases her expected payoff. On the other hand, if nobody takes the test, this commitment leaves the decision-maker unable to allocate the prize. Since by assumption, the decision-maker always prefers to give the prize to some candidate, this reduces her payoff.

However, as n becomes large, a candidate who does not take take the test becomes increasingly less likely to win the prize even without commitment. Thus, his payoff $F(b)^{n-1}\frac{1}{n} = \frac{cn}{n-1}\frac{1}{n}$ converges to zero. Then in the limit, a candidate's incentive to take the test under voluntary disclosure becomes the same as under mandatory disclosure. Hence, the first effect of commitment disappears. On the other hand, the second, negative effect of commitment remains: as $n \to \infty$, the probability that no candidate takes the test stays strictly positive, as Proposition 3 has established. Thus, the overall gain from the commitment is negative when n is sufficiently large.

Whether mandatory testing is better than voluntary testing for small values of n depends on the shape of F. In the simple case in which types are distributed uniformly, n does not need to be very large for a voluntary test to be optimal, as the next result shows:

Corollary 1. *If $F(x)$ is uniform, then voluntary testing strictly Pareto-dominates mandatory testing if and only if $n > 2$.*

Recall that by assumption, $n \geq 2$. Hence, when the distribution of types is uniform, keeping the test voluntary is better whenever the number of candidates is larger than the minimum that is necessary for the competitive testing model to be meaningful.

6 Extensions

6.1 Noisy Test

So far we have assumed that the test perfectly reveals the candidate's type. Suppose, however, that the test is imperfect. For example, a candidate on the academic job market may invest effort into writing an additional research paper, but the paper may be a noisy signal of his quality. Similarly, an election candidate may invest in campaigning to inform voters about his competence, but the media may present her message imperfectly.

Suppose that rather than revealing the candidate i's type x_i, the test reveals a test score $s_i = x_i + z_i$, where z_i is noise. When a candidate decides to take the test, he knows his type x_i, but not the realisation of the noise. After candidate i takes the test, nature draws z_i from some distribution G with smooth logconcave density g that has full support on $\mathbb{R}$. Different candidates' noise realisations are drawn independently.

We can show that the decision-maker prefers a candidate with a higher test score. Specifically, the following lemma proves that when noise is additive, logconcave g is equivalent to the monotone likelihood ratio condition, which, as Milgrom (1981) shows, implies that a higher score is a more favourable signal about the candidate's type:

Lemma 4. *If candidates i and j take the test, and $s_i > s_j$, then the decision-maker has a higher expected utility from giving the prize to candidate i then to candidate j.*

Then we can show that, as in the baseline model, the equilibrium symmetric strategy is of a threshold form:

Lemma 5. *At every symmetric equilibrium, for any G, there exists a threshold $b \in [0, 1]$ such that $h(x) = 1$ for all $x > b$, and $h(x) = 0$ for all $x < b$.*

If $b = 1$, nobody takes the test, and a small increase in noise has no effect. Consider instead the case when $b < 1$. If a candidate does not take the test, the decision-maker knows that his type is below b; while if he takes the test, the decision-maker knows that his type is above b. Thus, a candidate who does not take the test can never win over a candidate who takes it. If two or more candidates take the test, Lemma 4 ensures that the decision-maker will give the prize to the candidate with the highest test score.

Take a candidate i whose type equals b. That candidate must be indifferent between taking and not taking the test. If he does not take the test, he can only win the prize if nobody else takes the test. This happens with probability $F(b)^{n-1}$. His expected payoff is then $F(b)^{n-1} \frac{1}{n}$.

Now suppose candidate i takes the test. If the test produces noise z, the decision-maker learns i's score $b + z$. Then i wins the prize if each of his competitors either (i) does not take the test, or (ii) takes the test and receives a score below $b + z$. For a given competitor, the probability of the former event is $\Pr(x < b)$. The latter event happens if the competitor has a type $x > b$ and, after taking the test, receives a score $x + \hat{z} < b + z$. The probability of this is $\Pr(x > b \wedge \hat{z} < b + z - x)$. Since there are $n - 1$ competitors, the probability that i wins the prize equals

$$[\Pr(x < b) + \Pr(x > b \wedge \hat{z} < b + z - x)]^{n-1} = \left[F(b) + \int_b^1 f(x)\, G(b + z - x)\, dx\right]^{n-1}$$

Then ex ante, before i's test noise z is realised, his probability of winning the prize after taking the test equals

$$\int_{-\infty}^{+\infty} g(z) \left[F(b) + \int_b^1 f(x)\, G(b + z - x)\, dx\right]^{n-1} dz$$

Thus, when $b < 1$, the equilibrium is determined by the indifference condition

$$\int_{-\infty}^{+\infty} g(z)\left[F(b) + \int_{b}^{1} f(x)\, G(b+z-x)\, dx\right]^{n-1} dz - c = F(b)^{n-1}\,\frac{1}{n}$$

By varying the G, we can vary how noisy the test is. In particular, consider a family of distributions of the form $G_\lambda(z) \equiv G(\lambda z)$ for different values of $\lambda \in (0, +\infty)$. Increasing (decreasing) λ makes the noise more (less) concentrated around zero, and hence makes the test less (more) noisy[18]. How does a change in λ affect the equilibrium?

It turns out that a candidate is more likely to take the test when the test is noisier. This is summarised in the following proposition:

Proposition 7. *For any F and G, and any values of n and c, decreasing λ decreases b.*

To see the intuition behind this result, take the case when there is no noise, and consider candidate i whose type equals b. If i takes the test, he wins the prize if and only if all other candidates have types below b, and don't take the test. Now make the test noisy, and suppose that b were held constant. If the i takes the test, he still wins over anyone who does not take it, since not taking the test reveals that one's type is below b. But now i can also win over a competitor who has a higher type, if that competitor takes the test and receives a lower score than i. Hence, i's chance of winning the prize after taking the test increases. Thus, i becomes more willing to take the test, and the threshold b decreases.

Thus, candidates are more likely to take the test when the test is less informative. This may seem surprising: one could think that as $\lambda \to 0$ (i.e. as the test becomes "infinitely noisy"), the test ceases to carry any information, and candidates should be unwilling to take it. Note, however, that as $\lambda \to 0$, the test is only uninformative in the limit – for any positive λ, a higher test score still indicates that the candidate has a higher type. Hence, the test remains informative, and candidates with high types still prefer to take it.

Since each candidate is more likely to take the test when it is noisier, greater noise increases aggregate expenditure on the test, as the following result states:

[18] For example, if G is a normal distribution, then λ is proportional to the inverse of its variance.

Corollary 2. *Expected total spending on the test increases if λ decreases.*

Proof. A candidate takes the test with probability $1 - F(b)$. Hence, the expected spending on the test is $cn[1 - F(b)]$, which increases as λ falls. $\square$

Thus, if the test is run by a monopolist (as is, for example, the GRE), the monopolist has an incentive to make the test less precise.

6.2 Heterogeneous Costs

The basic model assumed that the cost of taking the test is the same for all candidates. This section will show that the basic results of the paper also hold when the cost is allowed to depend on candidate's type. For a specific application, suppose that a number of students are competing for a scholarship. The decision-maker would like to give the scholarship to the best applicant. Each applicant can take a test, and the effort required to take it is higher when the applicant's ability is lower.

Formally, suppose that for a candidate with type x, the cost of taking the test is $c(x) \in [\underline{c}, \overline{c}]$, where $0 < \underline{c} < \overline{c} < 1$. Let $c(x)$ be continuously differentiable and strictly decreasing in type. Thus, a more qualified applicant will find it easier to take the test.

In this setup, as before, we can show that the strategy of each candidate is of threshold form:

Lemma 6. *At every symmetric equilibrium, there exists a threshold $b \in [0, 1]$ such that $h(x) = 1$ for all $x > b$, and $h(x) = 0$ for all $x < b$.*

If the test is not mandatory, a candidate with type b receives an expected payoff of $F(b)^{n-1} - c(b)$ if he takes the test, and $F(b)^{n-1} \frac{1}{n}$ if he does not. If $b < 1$, the equilibrium is given by the indifference condition

$$F(b) = \left[\frac{c(b)\, n}{n-1}\right]^{\frac{1}{n-1}} \tag{5}$$

This is an equilibrium whenever $c(1) = \underline{c} \leq \frac{n-1}{n}$. If $\underline{c} > \frac{n-1}{n}$, then $F(b)^{n-1} - c(b) < F(b)^{n-1} \frac{1}{n}$ for all b, and hence the equilibrium is given by $b = 1$. Note that the left-hand side of (5) is increasing in b, and the right-hand side is decreasing in b – hence, the equilibrium is unique.

The effect of increasing the number of candidates on information revelation is then captured by the following result:

Proposition 8. *Suppose that* $\underline{c} < \frac{n-1}{n}$. *At the equilibrium,* $\frac{db}{dn} > 0$ *if and only if* $c(b) < \frac{n-1}{n} e^{-\frac{1}{n}}$.

To interpret this result, consider the case when $\overline{c}$ is low, and hence the test is cheap (relative to the value of the prize) even for candidates with low type. In particular, suppose that $\overline{c} < \frac{n-1}{n} e^{-\frac{1}{n}}$. Since $c(b) \leq \overline{c}$, the condition in the proposition is satisfied. Then $\frac{db}{dn} > 0$, and hence an increase in competition reduces the probability that a given candidate takes the test. On the other hand, suppose that $\underline{c}$ is high, and hence the test is costly for all candidates. If $\underline{c} > \frac{n-1}{n} e^{-\frac{1}{n}}$ then, since $c(b) \geq \underline{c}$, we have $\frac{db}{dn} \leq 0$, and thus an increase in competition increases the probability that a given candidate takes the test.

Hence, the basic logic of the results from Section 3 – that competition increases (reduces) information disclosure when the cost of the test is relatively high (low) holds when the cost of the test is heterogeneous.

What if the decision-maker commits not to give the prize to anyone who does not take the test? As before, the expected payoff of a candidate who does not take it equals zero, and the equilibrium threshold $\hat{b}$ is given by

$$F\left(\hat{b}\right) = c\left(\hat{b}\right)^{\frac{1}{n-1}}$$

We can verify that, as before, $b > \hat{b}$. To see that, define $z(x) \equiv \frac{F(x)^{n-1}}{c(x)}$. Then $z(b) = \frac{n}{n-1} > 1 = z\left(\hat{b}\right)$. Since $z(\cdot)$ is an increasing function, this implies that $b > \hat{b}$.

Given this, we can show that the decision-maker's expected gain from making the test mandatory is negative when competition is sufficiently strong.

Proposition 9. *For all* $c(\cdot)$, *and any* F, $\lim_{n\to\infty} D(n) < 0$.

Hence, the result from Section 5 holds in a more general setting in which the cost of the test depends on the candidate's type.

7 Conclusions

This paper developed a model of costly information disclosure by candidates competing for a prize. Disclosure was modelled as a costly test that a candidate can take to reveal his type.

Several results were derived. First, greater competition reduces information disclosure if the cost of disclosure is low, increases it if the cost is moderately high, and has no effect if the cost is very high. Second, when competition is sufficiently high, mandatory disclosure is strictly Pareto-dominated by voluntary disclosure. Third, greater test noise makes candidates more likely to take the test.

An important feature of the model was the fact that the candidates are competing for a single prize. In some settings – such as competition for political office, a company seeking to fill a single vacancy, or a university that needs to allocate a single scholarship – this assumption is naturally satisfied. In other settings, the number of prizes can be greater than one. Future work can extend the analysis to account for this possibility.

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Appendix

Proof of Lemma 1.

Let m be the expected type of a candidate who does not take the test. Denote by $\pi(x)$ the probability that a candidate with type x wins the prize. Then a candidate with type x weakly prefers to take the test whenever $\pi(x) - c \geq \pi(m)$, and weakly prefers not taking it whenever $\pi(x) - c \leq \pi(m)$.

Note that $\pi(\cdot)$ is nondecreasing. Then whenever $x \leq m$, we have $\pi(x) - c < \pi(m)$, so any candidate whose type is below m strictly prefers not to take the test. Thus, $h(x) = 0$ for all $x \leq m$. For a candidate with type $x > m$, $\pi(x)$ is the probability that every other candidate either does not take the test, or takes it and has a type below x. Hence, for all $x > m$,

$$\pi(x) = \left(\int_0^1 [1 - h(u)]\, dF(u) + \int_0^x h(u)\, dF(u) \right)^{n-1} \tag{6}$$

Let $b \equiv \sup\{x \mid h(x) < 1\}$. Then we must have $\pi(b) - c \leq \pi(m)$. Now take some $\hat{x} < b$, and suppose that $h(\hat{x}) > 0$. Then we must have $\pi(\hat{x}) - c \geq \pi(m)$. This implies that $\pi(b) \leq \pi(\hat{x})$. But $\pi(\cdot)$ is nondecreasing, so $\pi(b) = \pi(\hat{x})$. Substituting the expressions for $\pi(b)$ and $\pi(\hat{x})$ from (6) and simplifying, we get $\int_0^b h(u)\, dF(u) = \int_0^{\hat{x}} h(u)\, dF(u)$. Therefore, $\int_{\hat{x}}^b h(u)\, dF(u) = 0$. This should hold for all $\hat{x} < b$ such that $h(\hat{x}) > 0$. Hence, there exists a b such that the candidate takes the test with certainty for all types above b, and with probability zero for all types below b. □

Proof of Proposition 1.

If $c > \frac{n-1}{n}$, then $F(b) = 1$, which does not depend on n. If $c < \frac{n-1}{n}$, then $F(b) = \left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}} = e^{\frac{1}{n-1}\ln\frac{cn}{n-1}}$. Approximating n by a continuous variable and differentiating yields

$$\begin{aligned}
\frac{d}{dn}F(b) &= F(b)\left[-\frac{1}{(n-1)^2}\ln\frac{cn}{n-1} + \frac{1}{n-1}\frac{n-1}{cn}\frac{-c}{(n-1)^2}\right] \\
&= -\frac{F(b)}{(n-1)^2}\left[\ln\left(\frac{cn}{n-1}\right) + \frac{1}{n}\right]
\end{aligned}$$

which is positive whenever $\ln\left(\frac{cn}{n-1}\right) < -\frac{1}{n}$, i.e. whenever $c < \frac{n-1}{n}e^{-\frac{1}{n}}$. $\square$

Proof of Proposition 2.

If $c > \frac{n-1}{n}$, and hence $n < \frac{1}{1-c}$, then $F(b)^n = 1$, which does not depend on n. If $c < \frac{n-1}{n}$, then $F(b)^n = \left(\frac{cn}{n-1}\right)^{\frac{n}{n-1}} = e^{\frac{n}{n-1}\ln\frac{cn}{n-1}}$. Approximating n by a continuous variable and differentiating yields

$$\begin{aligned}
\frac{d}{dn}F(b)^n &= F(b)^n\left[-\frac{1}{(n-1)^2}\ln\frac{cn}{n-1} + \frac{n}{n-1}\frac{n-1}{cn}\frac{-c}{(n-1)^2}\right] \\
&= -\frac{F(b)^n}{(n-1)^2}\left[\ln\left(\frac{cn}{n-1}\right) + 1\right]
\end{aligned}$$

which is positive whenever $\ln\left(\frac{cn}{n-1}\right) < -1$, i.e. whenever $n > \frac{1}{1-ce}$. $\square$

Proof of Proposition 4.

Integrating by parts, we can transform (2) as

$$\begin{aligned}
v &= xF(x)^n\big|_b^1 - \int_b^1 F(x)^n\,dx + F(b)^{n-1}\left[xF(x)\big|_0^b - \int_0^b F(x)\,dx\right] \\
&= 1 - bF(b)^n - \int_b^1 F(x)^n\,dx + bF(b)^n - F(b)^{n-1}\int_0^b F(x)\,dx \\
&= 1 - \int_b^1 F(x)^n\,dx - F(b)^{n-1}\int_0^b F(x)\,dx
\end{aligned}$$

Substituting $F(b)^{n-1} = \frac{cn}{n-1}$ and differentiating with respect to n (treating n as a continuous variable) yields:

$$\begin{aligned} \frac{dv}{dn} &= F(b)^n \frac{db}{dn} - \int_b^1 F(x)^n \ln[F(x)]\,dx + \frac{c}{(n-1)^2}\int_0^b F(x)\,dx - F(b)^n \frac{db}{dn} \\ &= -\int_b^1 F(x)^n \ln[F(x)]\,dx + \frac{c}{(n-1)^2}\int_0^b F(x)\,dx > 0 \end{aligned}$$

□

Proof of Proposition 5.

If $c \geq \frac{n-1}{n}$, then $F(b) = 1$, and a further increase in c does not affect u. Suppose that $c < \frac{n-1}{n}$. Substituting the expression for $F(b)$ from Lemma 2 into (3) yields

$$u = \frac{1}{n} - c\left[1 - \left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}}\right] = \frac{1}{n} - c + c^{\frac{n}{n-1}}\left(\frac{n}{n-1}\right)^{\frac{1}{n-1}}$$

Differentiating with respect to c, we obtain

$$\frac{du}{dc} = -1 + \frac{n}{n-1}c^{\frac{1}{n-1}}\left(\frac{n}{n-1}\right)^{\frac{1}{n-1}} = -1 + c^{\frac{1}{n-1}}\left(\frac{n}{n-1}\right)^{\frac{n}{n-1}}$$

which is positvie if and only if $c > \left(\frac{n-1}{n}\right)^n$. □

Proof of Proposition 6.

Subtracting (2) from (4) and integrating by parts yields:

$$\begin{aligned} D(n) &= \textstyle\int_{\hat{b}}^b xd\left[F(x)^n\right] - F(b)^{n-1}\int_0^b xd\left[F(x)\right] \\ &= bF(b)^n - \hat{b}F\left(\hat{b}\right)^n - \textstyle\int_{\hat{b}}^b F(x)^n\,dx - F(b)^{n-1}\left[bF(b) - \int_0^b F(x)\,dx\right] \\ &= -\hat{b}F\left(\hat{b}\right)^n - \textstyle\int_{\hat{b}}^b F(x)^n\,dx + F(b)^{n-1}\int_0^b F(x)\,dx \end{aligned}$$

Note that $\lim_{n\to+\infty} F(b) = \lim_{n\to+\infty}\left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}} = \lim_{n\to+\infty}\left(\frac{c}{1-\frac{1}{n}}\right)^{\frac{1}{n-1}} = 1$, and $\lim_{n\to+\infty} F\left(\hat{b}\right) = \lim_{n\to+\infty} c^{\frac{1}{n-1}} = 1$. Hence, $\lim_{n\to+\infty}\hat{b} = \lim_{n\to+\infty} b = 1$, and thus

$\lim_{n\to+\infty} \int_{\hat{b}}^{b} F(x)^n\, dx = 0$. At the same time, $\lim_{n\to+\infty} F(b)^{n-1} = \lim_{n\to+\infty} \frac{cn}{n-1} = c$. Also, $\lim_{n\to+\infty} F\left(\hat{b}\right)^n = \lim_{n\to+\infty} c^{\frac{n}{n-1}} = c$. Thus,

$$\lim_{n\to+\infty} D(n) = \lim_{n\to+\infty} \left[c - c\int_0^b F(x)\, dx\right] = c\left[1 - \int_0^1 F(x)\, dx\right] < 0$$

which implies that voluntary testing is strictly better for the decision-maker when n is large enough. Together with the fact that candidates strictly prefer voluntary testing, this implies strict Pareto-dominance. □

Proof of Corollary 1.

If $F(x) = x$, then, using the transformation of $D(n)$ done in the previous proof, we have

$$\begin{aligned} D(n) &= -\hat{b}^{n+1} - \int_{\hat{b}}^{b} x^n dx + b^{n-1}\int_0^b x dx \\ &= -\hat{b}^{n+1} - \frac{1}{n+1} b^{n+1} + \frac{1}{n+1}\hat{b}^{n+1} + \frac{1}{2} b^{n+1} \\ &= \frac{n-1}{2(n+1)} b^{n+1} - \frac{n}{n+1}\hat{b}^{n+1} \end{aligned}$$

Substituting $b = F(b) = \left(\frac{cn}{n-1}\right)^{\frac{1}{n-1}}$ and $\hat{b} = F\left(\hat{b}\right) = c^{\frac{1}{n-1}}$ into the above expression yields

$$D(n) = \frac{n-1}{2(n+1)}\left(\frac{cn}{n-1}\right)^{\frac{n+1}{n-1}} - \frac{n}{n+1} c^{\frac{n+1}{n-1}} = c^{\frac{n+1}{n-1}} \frac{1}{n+1}\left[\frac{n-1}{2}\left(\frac{n}{n-1}\right)^{\frac{n+1}{n-1}} - n\right]$$

which has the same sign as $\frac{n-1}{2}\left(\frac{n}{n-1}\right)^{\frac{n+1}{n-1}} - n$, or as $\frac{1}{2}\left(\frac{n}{n-1}\right)^{\frac{2}{n-1}} - 1$. This is negative for all $n \geq 3$, and positive for $n = 2$. □

Proof of Lemma 4.

To prove the lemma, it is sufficient to show that $s_i > s_j$ implies that the conditional distribution of $x \mid s_i$ first-order stochastically dominates the conditional distribution of $x \mid s_j$. Milgrom (1981) shows that this holds if and only if the likelihood ratio $\frac{k(s|x)}{k(s|\bar{x})}$ is increasing in s for any x, $\bar{x}$ such that

$x > \bar{x}$, where $k(s \mid x)$ is a conditional distribution of s given x. Given the additive structure of noise, $k(s \mid x) = g(s - x)$, and the above monotone likelihood ratio property is equivalent to the statement that $\frac{d}{ds}\left(\frac{g(s-x)}{g(s-\bar{x})}\right) > 0$, which is equivalent to $\frac{d}{ds}\left(\ln\left[\frac{g(s-x)}{g(s-\bar{x})}\right]\right) = \frac{d}{ds}\left[\ln g(s-x) - \ln g(s-\bar{x})\right] > 0$. This holds if and only if $\frac{d}{ds}\left[\ln g(s-x)\right] > \frac{d}{ds}\left[\ln g(s-\bar{x})\right]$, $\forall x > \bar{x}$, i.e. if and only if $\frac{d}{ds}\ln g(\cdot)$ is decreasing, and hence if and only if $g(\cdot)$ is logconcave. $\square$

Proof of Lemma 5.

Let $\tilde{\pi}$ be the probability that a candidate wins the prize after deciding not to take the test. Let $\pi(x)$ be the ex ante probability that a candidate whose type is x wins the prize. Then a candidate with type x takes the test if $\pi(x) - c > \tilde{\pi}$, and does not take the test if $\pi(x) - c < \tilde{\pi}$. Note that $\tilde{\pi}$ does not depend on x. Thus, to show that h is of a threshold form as stated in the lemma, it is sufficient to demonstrate that $\pi(x)$ is strictly increasing in x.

At the equilibrium, let $\tilde{s}$ be the score such that the expected type conditional on having score $\tilde{s}$ equals the expected type of a candidate who did not take the test. Lemma 4 implies that $\tilde{s}$ is unique.

Suppose that a candidate whose type x takes the test and receives a score $s = x + z$. Now take a competitor with type $\hat{x}$. If $s < \tilde{s}$ (and hence $z < \tilde{s} - x$), the candidate has a higher expected type than this competitor if and only if the latter takes the test and receives a score $\hat{x} + \hat{z}$ that is less than $x + z$. Given the competitor's type $\hat{x}$, this happens with probability $h(\hat{x})\,G(x + z - \hat{x})$. Thus, the probability of winning over that competitor equals

$$\int_0^1 f(\hat{x})\,h(\hat{x})\,G(x + z - \hat{x})\,d\hat{x} \equiv L(x, z)$$

and the overall probability of winning the prize equals $L(x, z)^{n-1}$. On the other hand, if $s > \tilde{s}$ (and hence $z > \tilde{s} - x$), the candidate has a higher expected type than his competitor if an only if the latter either (i) takes the test and receives a score $\hat{x} + \hat{z}$ that is less than $x + z$, or (ii) does not take the test. Given the competitor's type $\hat{x}$, the former event happens with probability $h(\hat{x})\,G(x + z - \hat{x})$, while the latter event happens with probability $1 - h(\hat{x})$. Thus, the probability of winning over that competitor equals $\int_0^1 f(\hat{x})\left[1 - h(\hat{x}) + h(\hat{x})\,G(x + z - \hat{x})\right]d\hat{x}$, and the overall probability of winning the prize equals

$$\left(\int_0^1 f(\hat{x})\left[1-h(\hat{x})+h(\hat{x})\,G(x+z-\hat{x})\right]d\hat{x}\right)^{n-1}=\left[K+L(x,z)\right]^{n-1}$$

where $K\equiv\int_0^1 f(\hat{x})\left[1-h(\hat{x})\right]d\hat{x}\geq 0$. Then we have

$$\pi(x)=\int_{-\infty}^{\tilde{s}-x} g(z)\left[L(x,z)\right]^{n-1}dz+\int_{\tilde{s}-x}^{+\infty} g(z)\left[K+L(x,z)\right]^{n-1}dz$$

Differentiating it with respect to x yields

$$\begin{aligned}\frac{d\pi(x)}{dx}&=-g(\tilde{s}-x)\left[L(x,\tilde{s}-x)\right]^{n-1}+\int_{-\infty}^{\tilde{s}-x} g(z)\frac{d}{dx}\left(\left[L(x,z)\right]^{n-1}\right)dz\\&+g(\tilde{s}-x)\left[K+L(x,\tilde{s}-x)\right]^{n-1}+\int_{\tilde{s}-x}^{+\infty} g(z)\frac{d}{dx}\left(\left[K+L(x,z)\right]^{n-1}\right)dz\\&\geq g(\tilde{s}-x)\left(\left[K+L(x,\tilde{s}-x)\right]^{n-1}-\left[L(x,\tilde{s}-x)\right]^{n-1}\right)\geq 0\end{aligned}$$

In the above, the first inequality is due to the fact that $L(x,z)$ is increasing in x while K does not change with x, which implies that $\frac{d}{dx}\left(\left[L(x,z)\right]^{n-1}\right)\geq 0$ and $\frac{d}{dx}\left(\left[K+L(x,z)\right]^{n-1}\right)\geq 0$. The last inequality holds because $K\geq 0$. Note that unless $h(\cdot)$ is zero everywhere or almost everywhere, $L(x,z)$ is strictly increasing in x, which implies that the first inequality is strict, and hence

$$\frac{d\pi(x)}{dx}>g(\tilde{s}-x)\left(\left[K+L(x,\tilde{s}-x)\right]^{n-1}-\left[L(x,\tilde{s}-x)\right]^{n-1}\right)\geq 0$$

On the other hand, if $h(\cdot)$ is zero everywhere or almost everywhere, then $K=\int_0^1 f(\hat{x})\,d\hat{x}>0$, and we have

$$\frac{d\pi(x)}{dx}\geq g(\tilde{s}-x)\left(\left[K+L(x,\tilde{s}-x)\right]^{n-1}-\left[L(x,\tilde{s}-x)\right]^{n-1}\right)>0$$

Hence, in every symmetric equilibrium we must have $\frac{d\pi(x)}{dx}>0$. Therefore, any symmetric equilibrium must be of a threshold form. □

Proof of Proposition 7.

When $b = 1$, no candidate takes the test, and a marginal change in the level of noise does not change b. Thus, $\frac{db}{d\lambda} = 0$. Now consider the case when $b < 1$. Take a distribution $G_\lambda(z) = G(\lambda z)$, and note that its pdf equals $\lambda g(\lambda z)$. The equilibrium is then given by the condition

$$\int_{-\infty}^{+\infty} \lambda g(\lambda z) \left[F(b) + \int_b^1 f(x) G(\lambda [b + z - x]) \, dx \right]^{n-1} dz - c = F(b)^{n-1} \frac{1}{n}$$

This can be written as:

$$\int_{-\infty}^{+\infty} \lambda g(\lambda z) [M(z)]^{n-1} dz - c = F(b)^{n-1} \frac{1}{n} \qquad (7)$$

where $M(z) \equiv F(b) + \int_b^1 f(x) G(\lambda [b + z - x]) \, dx$. Since (7) should hold for any λ, we can differentiate it with respect to λ to obtain

$$\begin{aligned} & \int_{-\infty}^{+\infty} g(\lambda z) [M(z)]^{n-1} dz + \int_{-\infty}^{\infty} \lambda z g'(\lambda z) [M(z)]^{n-1} dz \qquad (8) \\ & + \int_{-\infty}^{+\infty} \lambda g(\lambda z) (n-1) [M(z)]^{n-2} \frac{dM(z)}{d\lambda} dz \\ & = \frac{n-1}{n} F(b)^{n-2} f(b) \frac{db}{d\lambda} \end{aligned}$$

Note that we can write

$$\begin{aligned} & \int_{-\infty}^{+\infty} \lambda z g'(\lambda z) [M(z)]^{n-1} dz = z g(\lambda z) [M(z)]^{n-1} \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} g(\lambda z) \left[\frac{\partial}{\partial z} \left(z [M(z)]^{n-1} \right) \right] dz \\ & = - \int_{-\infty}^{+\infty} g(\lambda z) \left[\frac{\partial}{\partial z} \left(z [M(z)]^{n-1} \right) \right] dz \\ & = - \int_{-\infty}^{+\infty} g(\lambda z) [M(z)]^{n-1} dz - \int_{-\infty}^{+\infty} g(\lambda z) z (n-1) [M(z)]^{n-2} \frac{\partial M(z)}{\partial z} dz \end{aligned}$$

where the first equality is a result of differentiating by parts; the second comes from the fact that for a logconcave $g(\cdot)$, $\lim_{z \to -\infty} z g(\lambda z) = \lim_{z \to +\infty} z g(\lambda z) = 0$,[19] while $M(z)$ is bounded between zero and one; and the third comes from

[19]To see why this is the case, note that $\int_{-\infty}^{+\infty} \lambda g(\lambda x) \, dz = 1$. Together with the fact that $g(\lambda z)$ is logconcave, and hence decreasing for sufficiently high z, this means that $\forall \varepsilon > 0$

straightforward differentiation. We can substitute this into (8), which then becomes

$$-\int_{-\infty}^{+\infty} g(\lambda z)\, z\,(n-1)\,[M(z)]^{n-2}\,\frac{\partial M(z)}{\partial z}dz + \int_{-\infty}^{+\infty} \lambda g(\lambda z)\,(n-1)\,[M(z)]^{n-2}\,\frac{dM(z)}{d\lambda}dz$$
$$=\frac{n-1}{n}F(b)^{n-2}\,f(b)\,\frac{db}{d\lambda}$$

This simplifies to

$$\int_{-\infty}^{+\infty} g(\lambda z)\,[M(z)]^{n-2}\left[\lambda\frac{dM(z)}{d\lambda} - z\frac{\partial M(z)}{\partial z}\right]dz = \frac{1}{n}F(b)^{n-2}\,f(b)\,\frac{db}{d\lambda} \quad (9)$$

Now note that

$$\frac{\partial M(z)}{\partial z} = \lambda\int_b^1 f(x)\,g(\lambda[b+z-x])\,dx > 0$$

and

$$\begin{aligned}\frac{dM(z)}{d\lambda} &= f(b)\,\frac{db}{d\lambda} - f(b)\,G(\lambda z)\,\frac{db}{d\lambda} + \int_b^1 f(x)\,g(\lambda[b+z-x])\left(b+z-x+\lambda\frac{db}{d\lambda}\right)dx \\ &= f(b)\,[1-G(\lambda z)]\,\frac{db}{d\lambda} + A(z) + \frac{z}{\lambda}\frac{\partial M(z)}{\partial z} + \frac{db}{d\lambda}\frac{\partial M(z)}{\partial z} \\ &= \left(f(b)\,[1-G(\lambda z)] + \frac{\partial M(z)}{\partial z}\right)\frac{db}{d\lambda} + A(z) + \frac{z}{\lambda}\frac{\partial M(z)}{\partial z}\end{aligned}$$

where $A(z) \equiv \int_b^1 f(x)\,g(\lambda[b+z-x])\,(b-x)\,dx < 0$. Then we have:

$$\lambda\frac{dM(z)}{d\lambda} - z\frac{\partial M(z)}{\partial z} = \lambda\left(f(b)\,[1-G(\lambda z)] + \frac{\partial M(z)}{\partial z}\right)\frac{db}{d\lambda} + \lambda A(z) \quad (10)$$

---

there exists δ such that (i) $y \geq \delta$ implies $\int_y^{+\infty} g(\lambda z)\,dz < \frac{\varepsilon}{2\lambda}$, and (ii) $g(\lambda z)$ is decreasing for $z > 2\delta$. Then for any $z > 2\delta$ we have $zg(\lambda z) = 2\left(z - \frac{z}{2}\right)g(\lambda z) = 2\int_{\frac{z}{2}}^{z} g(\lambda z)\,dt < 2\int_{\frac{z}{2}}^{z} g(\lambda t)\,dt < 2\int_{\frac{z}{2}}^{+\infty} g(\lambda t)\,dt < 2\int_{\delta}^{+\infty} g(\lambda t)\,dt < \frac{\varepsilon}{2\lambda}$, where the first inequality comes from the fact that $g(\lambda z)$ is decreasing. Hence, $zg(\lambda z)$ converges to zero. The statement that $\lim_{z\to-\infty} zg(\lambda z) = 0$ can be proven analogously.

We can now substitute (10) into (9) to obtain:

$$\begin{aligned}
&\frac{db}{d\lambda}\int_{-\infty}^{+\infty}\lambda g\left(\lambda z\right)\left[M\left(z\right)\right]^{n-2}\left(f\left(b\right)\left[1-G\left(\lambda z\right)\right]+\frac{\partial M\left(z\right)}{\partial z}\right)dz\\
&+\int_{-\infty}^{+\infty}\lambda g\left(\lambda z\right)\left[M\left(z\right)\right]^{n-2}A\left(z\right)dz=\frac{1}{n}F\left(b\right)^{n-2}f\left(b\right)\frac{db}{d\lambda}
\end{aligned}$$

Then we can express $\frac{db}{d\lambda}$ as

$$\frac{db}{d\lambda}=\frac{\int_{-\infty}^{+\infty}\lambda g\left(\lambda z\right)\left[M\left(z\right)\right]^{n-2}A\left(z\right)dz}{\frac{1}{n}F\left(b\right)^{n-2}f\left(b\right)-\int_{-\infty}^{+\infty}\lambda g\left(\lambda z\right)\left[M\left(z\right)\right]^{n-2}\left(f\left(b\right)\left[1-G\left(\lambda z\right)\right]+\frac{\partial M(z)}{\partial z}\right)dz}$$

Since $A\left(z\right)<0$ and $M\left(z\right)>0$, the numerator is negative. At the same time, we can show that the denominator is strictly negative too. This is because:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\lambda g\left(\lambda z\right)\left[M\left(z\right)\right]^{n-2}\left(f\left(b\right)\left[1-G\left(\lambda z\right)\right]+\frac{\partial M\left(z\right)}{\partial z}\right)dz\\
>&\int_{-\infty}^{+\infty}\lambda g\left(\lambda z\right)\left[M\left(z\right)\right]^{n-2}f\left(b\right)\left[1-G\left(\lambda z\right)\right]dz\\
>&\int_{-\infty}^{+\infty}\lambda g\left(\lambda z\right)F\left(b\right)^{n-2}f\left(b\right)\left[1-G\left(\lambda z\right)\right]dz\\
=&F\left(b\right)^{n-2}f\left(b\right)\int_{-\infty}^{+\infty}\lambda g\left(\lambda z\right)\left[1-G\left(\lambda z\right)\right]dz\\
=&F\left(b\right)^{n-2}f\left(b\right)\left[1-\int_{-\infty}^{+\infty}\lambda g\left(\lambda z\right)G\left(\lambda z\right)dz\right]\\
=&\frac{1}{2}F\left(b\right)^{n-2}f\left(b\right)\\
\geq&\frac{1}{n}F\left(b\right)^{n-2}f\left(b\right)
\end{aligned}$$

In the above, the first inequality is due to the fact that $\frac{\partial M(z)}{\partial z}>0$. The second inequality holds because $M\left(z\right)=F\left(b\right)+\int_b^1 f\left(x\right)G\left(\lambda\left[b+z-x\right]\right)dx>F\left(b\right)$. The first equality uses a simple rearrangement of terms, while the second uses the fact that $\int_{-\infty}^{+\infty}\lambda g\left(\lambda z\right)dz=\int_{-\infty}^{+\infty}dG_{\lambda}\left(z\right)=1$. The third equality is due to the fact that $\int_{-\infty}^{+\infty}\lambda g\left(\lambda z\right)G\left(\lambda z\right)dz=\left[G\left(\lambda z\right)\right]^2\Big|_{-\infty}^{+\infty}-$

$\int_{-\infty}^{+\infty} \lambda g(\lambda z) G(\lambda z)\, dz$, and hence $\int_{-\infty}^{+\infty} \lambda g(\lambda z) G(\lambda z)\, dz = \frac{1}{2}\left[G(\lambda z)\right]^2\Big|_{-\infty}^{+\infty} = \frac{1}{2}$. Finally, the last weak inequality uses the fact that $n \geq 2$.

Since both the numerator and the denominator are negative, we conclude that $\frac{db}{d\lambda} > 0$. □

Proof of Lemma 6.

As in the proof of Lemma 1, let $\pi(x)$ be the probability that a candidate with type x wins the prize after taking the test, and let $\pi(m)$ be the probability of winning the prize without taking the test. Then a candidate with type x is indifferent between taking and not taking the test when $\pi(x) - c(x) = \pi(m)$. Since $\pi(\cdot)$ is nondecreasing and $c(\cdot)$ is strictly decreasing, this equality holds for at most one type. If such a type exists, call it b. Otherwise, if $\pi(1) - c(1) < \pi(m)$, then $b = 1$. Note that $b = 0$ cannot be an equilibrium, because $\pi(0) - c(0) = -\overline{c} < 0 \leq \pi(m)$. □

Proof of Proposition 8.

We can write (5) as $F(b)^{n-1} = \frac{n}{n-1} c(b)$. Approximating n by a continuous variable and differentiating with respect to it yields

$$F(b)^{n-1}\left[\ln F(b) + \frac{n-1}{F(b)} f(b) \frac{db}{dn}\right] = -\frac{1}{(n-1)^2} c(b) + \frac{n}{n-1} c'(b) \frac{db}{dn}$$

Hence,

$$\frac{db}{dn} = \frac{-\frac{1}{(n-1)^2} c(b) - F(b)^{n-1} \ln F(b)}{(n-1) F(b)^{n-2} f(b) - \frac{n}{n-1} c'(b)}$$

The denominator of the above is positive, since $c'(b) < 0$. Hence, $\frac{db}{dn} > 0$ whenever $F(b)^{n-1} \ln F(b) < -\frac{1}{(n-1)^2} c(b)$. Substituting $F(b)$ from (5), we find that $\frac{db}{dn} > 0$ if and only if $\frac{n}{n-1} c(b) \ln\left(\left[\frac{c(b)n}{n-1}\right]^{\frac{1}{n-1}}\right) < -\frac{1}{(n-1)^2} c(b)$, i.e. whenever $\ln\left[\frac{c(b)n}{n-1}\right] < -\frac{1}{n}$. This is true if and only if $c(b) < \frac{n-1}{n} e^{-\frac{1}{n}}$. □

Proof of Proposition 9.

Since the expressions for v and $\hat{v}$ are unchanged, using the same steps as in the proof of Proposition 6, we can write $D(n) = -\hat{b} F\left(\hat{b}\right)^n - \int_{\hat{b}}^{b} F(x)^n\, dx +$

$F(b)^{n-1}\int_0^b F(x)\,dx$. We have $\lim_{n\to\infty} F(b) = \lim_{n\to\infty}\left[\frac{c(b)n}{n-1}\right]^{\frac{1}{n-1}} = \lim_{n\to\infty}[c(b)]^{\frac{1}{n-1}} = 1$, and $\lim_{n\to\infty} F\left(\hat{b}\right) = \lim_{n\to\infty} c\left(\hat{b}\right)^{\frac{1}{n-1}} = 1$, where the last equality in each case uses the fact that $c(b), c\left(\hat{b}\right) \in [\underline{c}, \overline{c}] \subseteq (0,1)$ at all values of n. Hence, $\lim_{n\to\infty} b = \lim_{n\to\infty} \hat{b} = 1$, and $\lim_{n\to\infty}\int_{\hat{b}}^b F(x)^n\,dx = 0$. At the same time, $\lim_{n\to\infty} F\left(\hat{b}\right)^n = \lim_{n\to\infty} c\left(\hat{b}\right)^{\frac{n}{n-1}} = c(1) = \underline{c}$, and $\lim_{n\to\infty} F(b)^{n-1} = \lim_{n\to\infty}\frac{c(b)n}{n-1} = c(1) = \underline{c}$. Thus, $\lim_{n\to+\infty} D(n) = -\underline{c} + \underline{c}\int_0^1 F(x)\,dx < 0$. □