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11 July 2019

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Abstract

We provide an analytical description of possible spatial patterns in economic geography models with three identical and equidistant regions by applying results from General Bifurcation mechanisms. We then use Pflüger's (2004, Reg Sci Urb Econ) model to show what spatial patterns can be uncovered analytically. As the freeness of trade increases, a uniform distribution undergoes a direct bifurcation that leads to a state with two identical large regions and one small region. Before this bifurcation, the model encounters a minimum point above which a curve of dual equilibria with two small identical regions and one small region emerges. From further bifurcations, the equilibrium with one large region encounters agglomeration in a single region, while the equilibrium with one small region encounters a state with two evenly populated regions and one empty region. A secondary bifurcation then leads to partial agglomeration with one small region and one large region. We show that an asymmetric equilibrium with populated regions cannot be connected with other types of equilibria. Therefore, an initially asymmetric state will remain so and preserve the ordering between region sizes.

Keywords: bifurcation, economic geography, multi-regional economy, footloose entrepreneur

JEL Classification Numbers: R10, R12, R23.

*We thank Minoru Osawa for useful comments and suggestions. José M. Gaspar gratefully acknowledges financial support from Fundação para a Ciência e Tecnologia (through projects UID/GES/00731/2019, PTDC/EGE-ECO/30080/2017 and CEECIND/02741/2017) and from the Grant-in-Aid for JSPS 18K04380/18K18874/19K15108.

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1 Introduction

This paper aims to provide an analytical account of the evolution of agglomeration patterns in an increasingly globalized economy, by studying bifurcation mechanisms in economic geography in a multi-regional setting with identical and equidistant regions. That is, the economy is symmetric in all respects.

An equidistant and fully symmetric setting is interesting for two reasons. Firstly, depending on the short-run general equilibrium model borrowed from New Economic Geography (NEG), non-trivial asymmetric spatial distribution and agglomeration mechanisms may arise without the need to assume *ex-ante* exogenous asymmetries or first nature advantages (Krugman, 1993). This means that an increase in market access variability due to the existence of more regions strengthens the role of second nature advantages (increasing returns and transport costs) in determining the spatial distribution of economic activities in a more realistic space economy. Secondly, equidistance introduces a great deal of symmetry in the mathematical problems whose analysis allows us to uncover analytical results that are otherwise impossible to obtain in most of the inherently intractable NEG models.[1] In particular, we are interested in the symmetric bifurcating patterns along a smooth path where the freeness of trade (as an inverse measure of transport costs) steadily increases, in order to account for the historical increase in globalization and economic integration.

We start by introducing the long-run migration decisions faced by inter-regionally mobile agents for the general n-region model. These are modelled according to the well-known replicator dynamics, and the dynamics and the long-run spatial distribution are constrained in the $(n-1)$-dimensional simplex. We provide a brief discussion on particular one-dimensional subspaces of the simplex which are invariant for the dynamics, namely interior invariant spaces whereby all regions have positive population and boundary invariant spaces whereby at least one region has no population. We then discuss the different types of spatial long-run equilibria contained in these subspaces, particularly for the three region case.

Applying results known from General Bifurcation Symmetry Mechanisms under the replicator dynamics, we provide a complete and fully analytical description of all spatial patterns in the economy for the specific NEG model proposed by Pflüger (2004) (herein PF model) and extended by Gaspar et al. (2018) to n equidistant regions. In the former, the reduced dimensionality of two regions precludes the richness of diverse spatial patterns across multiple regions. In the latter, the study is limited to the local stability of equilibria that lie on a particular one-dimensional interior invariant space. Here, we focus on the

[1] Indeed, extensions to multiple regions often require numerical simulations in restrictive ranges of parameter space.

case of three regions but further provide a detailed analytical description of all possible bifurcations as sources of changes in the agglomeration patterns, as well as the latter's dynamic (local) stability. This is possible thanks to the tractability inherent to the PF model combined with the study of bifurcation mechanisms for an equidistant economy. In most NEG models, most results can only be achieved numerically. To the best of our knowledge, this paper is the first one to provide a full analytical description of the complex network of agglomeration patterns in a three-regional economy.

The model by Pflüger (2004) is an NEG model that belongs to the class of Footloose Entrepreneur (FE) models (e.g., Ottaviano, 2001; Ottaviano et al. 2002; Forslid and Ottaviano, 2003; Baldwin et al., 2004) which correspond to analytically solvable versions of Krugman's Core Periphery (CP) model (Krugman, 1991). Additionally, in the PF model, there is absence of income effects in the demand for manufactured goods due to the quasi-linear upper tier utility specification. As a result, the nominal wage bill paid to mobile agents does not feed back on regional income. Thus, the agents' indirect utility is obtainable as an explicit function of the spatial distribution in the economy. Since long-run migration decisions are based on regional utility differentials, this renders the PF model as one of the most tractable in the NEG literature.

We establish how the PF model with three regions undergoes successive bifurcations at critical points of the freeness of trade to reach different agglomeration patterns. When the freeness of trade is low, a state of stable total dispersion exists. As the freeness of trade increases, total dispersion undergoes a bifurcation at a break-point to encounter a state of partial dispersion.

Partial dispersion consists of two regions that are evenly populated and another region with higher (small partial dispersion) or lower (large partial dispersion) population than the other two. Since the bifurcating partial dispersion is theoretically known to be unstable, the only way to reach stable partial dispersion from the state of total dispersion is to encounter a limit point, where the stability changes. From the existence of a saddle-node bifurcation occurring at partial dispersion studied by Gaspar et. al (2018), we know that such a limit point exists and is a minimum point (of the freeness of trade) above which two small partial dispersion emerge, one dynamically stable (the one that corresponds to the largest population in one region) and the other one unstable. Moreover, the minimum point is lower than the break point above which total dispersion becomes unstable. Therefore, if partial dispersion is stable, it continues to be stable above the minimum point of the freeness of trade. Below that limit point, the system becomes unstable and may jump to some stable equilibrium dynamically (if some dynamics is considered), which may correspond to total dispersion since it is stable below the minimum point. Such jump is identical with that observed for two region CP models such as Krugman

(1991) or Forslid and Ottaviano (2003), although the latter corresponds to jumps from total dispersion to agglomeration once the freeness of trade rises above the break point. In the multi-regional PF model, there is also a jump from total dispersion to a more agglomerated spatial distribution once the freeness of trade rises above the break point. If the sustain point above which agglomeration is dynamically stable is lower than the break point, the system jumps discontinuously to agglomeration. Otherwise, it jumps to a dynamically stable small partial dispersion. This lies in contrast with its two-region counterpart, where we can observe a smooth transition from dispersion towards more agglomerated outcomes as the freeness of trade rises above the break point.

We also show that a limit point encountered by the freeness of trade is one possible way through which a possibly stable asymmetric state with populated regions can be reached.[2] However, there is no smooth continuation of asymmetric states with other more symmetric states nor with completely asymmetric states whereby (at least) one region has zero population. Thus, if the economy starts with three differently populated regions, the asymmetric state as well as the ordering of population sizes in the regions will be preserved in perpetuity.

A large partial dispersion may also encounter a sustain point for a sufficiently large freeness of trade that leads to the vanishing of population in a region (a boundary) and the emergence of a locally unstable boundary dispersion equilibrium whereby agents are evenly dispersed across two regions. Along this boundary, as the freeness of trade increases further, boundary dispersion undergoes a secondary bifurcation to reach partial agglomeration, i.e., a state with one large region, one small region, and another with zero population. Eventually, this state encounters another sustain point at a higher level of the freeness of trade above which the population in the small region vanishes and there is stable agglomeration in one single region.

Although boundary dispersion and partial agglomeration are always locally unstable, they may be stable along the boundary.[3] This means that they may be dynamically sustainable as long as no exogenous perturbation occurs that populates the empty region. We also conclude that partial agglomeration is the only completely asymmetric state that is connected to other types of equilibria.

The possibilities discussed above evidence the fair complexity in the agglomeration patterns that may arise in an equidistant symmetrical setting with just three regions. It also shows that the process of agglomeration tendencies as economies become more integrated is far from trivial.

This study could contribute to the study of economic geography, in which economic

[2]However, such a possibility seems very unlikely.

[3]In fact, partial agglomeration is stable along the boundary when it exists.

agglomeration is studied mostly for two regions. Although there are several studies for three regions, which are non-equidistant (e.g., Ago et al., 2006) or equidistant (e.g., Fujita et al., 1999; Tabuchi et al., 2005; Castro et al., 2012; Zeng and Uchikawa, 2014; Commendatore et al., 2015; Gaspar et al., 2018, 2019), this paper provides a much more complete analytical description of all spatial patterns of three equidistant regions. The paper also constitutes an important step towards the study of more complicated networks of equilibrium curves in an equidistant economy with an arbitrary number of regions.

This paper is organized as follows: Replicator dynamics in economic geography for n equidistant regions is presented in Section 2. General Bifurcation mechanism for three regions is advanced in Section 3. Spatial equilibria in the PF model is studied in Section 4. The final Section is left for concluding remarks.

2 Replicator dynamics in economic geography

We first describe the well known replicator dynamics applied to a fully symmetric n-region economic geography model. For now, the following assumptions suffice. Suppose there is an economy with $N = \{1, 2, ..., n\}$ regions and that there is a unit mass of h agents that are allowed to migrate freely among regions. The agents residing in region $i \in N$ are given by $h_i \in [0, 1]$. The spatial distribution of mobile agents $\boldsymbol{h} = (h_1, h_2, ..., h_n)$ is thus contained in the $(n-1)$-dimensional simplex defined by $\triangle = \left\{\boldsymbol{h} \in \mathbb{R}^n_+ : \sum_{i=1}^n h_i = 1\right\}$. In the long-run, agents choose to live in the region that offers them the highest pay-off (indirect utility).

The replicator dynamics that govern migration of mobile agents in the long-run is given by:

$$f_i = \dot{h}_i \equiv h_i \left[V_i(\boldsymbol{h}) - \bar{V}(\boldsymbol{h})\right],\ \forall i \in N \setminus \{n\}, \tag{1}$$

where $V_i(\boldsymbol{h})$ is the indirect utility (pay-off) of an agent who resides in region i and $\bar{V}(\boldsymbol{h}) = \sum_{i=1}^n h_i V_i(\boldsymbol{h})$ is the weighted average indirect utility. The n^{th} region is implicitly defined, without loss of generality, such that $h_n = 1 - \sum_{i=1}^{n-1} h_i$ and thus its dynamics is residually given by $\dot{h}_n = -\sum_{i \neq n} \dot{h}_i$. The migration of mobile workers is constrained in Δ. Indirect utility $V_i(\boldsymbol{h})$ is derived from a short-run general equilibrium NEG model, taking each h_i as given, and is therefore model dependent.

A spatial distribution is said to be a steady-state (or rest point) $\boldsymbol{h} \equiv \boldsymbol{h}^* = (h_1^*, h_2^*, ..., h_n^*)$ if and only if $f_i = 0$, $\forall i \in N$ in (1). A steady-state is a spatial equilibrium if and only if the complementary condition $V_i(\boldsymbol{h}^*) - \bar{V}(\boldsymbol{h}^*) \leq 0$, $\forall i \in N$, is satisfied.[4] In other words, spatial

[4]The additional condition is given by Proposition 1.2 in Ikeda and Murota (2014, pp. 22). As noted by the latter, the definition of steady-state is standard in dynamical systems, whereas a spatial equilibrium additionally requires that no agent can get a higher indirect utility from moving to another region.

equilibria are solutions that are economically sustainable. In what follows, throughout the paper, we shall consider a parametrization such that any solution $\boldsymbol{h}^*$ always satisfies the complementary condition, i.e., $\boldsymbol{h}^*$ always corresponds to a spatial equilibrium. The equilibria in the replicator dynamics can be classified into:

- *Interior equilibria,* such that no region has zero population and it can correspond to any interior point of Δ, which may or may not lie on spaces that are invariant for the dynamics.[5]

- *Boundary equilibria,* such that at least one region has zero population and thus lie on the boundary of Δ, invariant for the dynamics.[6]

The possible agglomeration patterns (equilibria) and their stability can be analysed by applying results from *General Bifurcation mechanisms* combined with *local stability* analysis of equilibria that lie on *invariant spaces* of Δ. These are particular spaces consisting of spatial distributions $\boldsymbol{h}$ that are invariant for the dynamics. They can be classified into *interior invariant spaces* and *boundary invariant* spaces. Both admit a family of one-dimensional subspaces in Δ. For the former, these correspond to k regions with population $a \in [0, 1]$ and the other $n - k$ regions with population $b = \frac{1-ka}{n-k}$. For the latter, $k < n$ regions have zero population and the other $n - k$ regions have a population $\frac{1}{n-k}$. We provide a more detailed description of these spaces in Appendix A.

Throughout the paper, we will often distinguish between stability of equilibria in Δ and stability only along a one-dimensional invariant subspace.[7] Note that *local stability* of $\boldsymbol{h}^* \in \Delta$ requires *all* $n - 1$ eigenvalues of the Jacobian matrix of (1) evaluated at $\boldsymbol{h}^*$ to be negative, whereas stability of $\boldsymbol{h}^*$ *in* (or *along*) an invariant space of Δ refers to the restriction to a one-dimensional subspace of Δ and thus requires the negativity of a single (its corresponding) eigenvalue. The multiplicity of this eigenvalue in Δ may be higher than 1, which warrants the following Remark.

Remark 1. For an n-equidistant economy, local stability analysis of any equilibria that lie on one-dimensional invariant spaces of Δ reduces to the inspection of just two eigenvalues, one with multiplicity $\alpha \leq n - 1$ that determines stability *along* the invariant space, and the other one with multiplicity $n - 1 - \alpha$ that determines stability in the orthogonal direction to the invariant space.

This is because, as argued by Gaspar et al. (2018), mobile agents only face two types of decisions: that of choosing between any pair of evenly populated regions and that of

[5]See Appendix A.1 for more details.

[6]See Appendix A.2 for more details.

[7]Stability along a one-dimensional boundary of Δ may of particular economic interest as detailed in Section 4.3.

choosing between any pair of unevenly populated regions. The allowable permutation of regions in the two types of decisions thus amounts to their multiplicity.

Bifurcation mechanisms of interior equilibria of an n-equidistant economy are known mathematically (Elmhirst, 2004). Boundary equilibria have another kind of agglomeration mechanism. It is possible to grasp the mechanism of agglomeration of an n-equidistant economy by combining these two different mechanisms, which we detail in Section 3 for the case $n = 3$. We use the *freeness of trade* $\phi \in (0, 1)$ as a bifurcation parameter, which is standard in NEG, and search for points of the bifurcation parameter at which a qualitative change in the system in 1 occurs (emergence of equilibria and/or interchange in their stability). We define three important points:

- A *break point* $\phi_b \in (0, 1)$ is a value of ϕ at which an equilibrium branches to equilibria that interchanges the population in populated regions. The equilibrium may interchange its stability at $\phi = \phi_b$.
- A *sustain point* $\phi_s \in (0, 1)$ is a value of ϕ at which an equilibrium leads to the vanishing of population in some region(s) and thus a boundary equilibrium emerges. The equilibrium may interchange its stability at $\phi = \phi_s$.
- A *limit point* $\phi_l \in (0, 1)$ is a maximum or minimum point on a curve of equilibrium points $(\boldsymbol{h}, \phi)$. The stability of the curve may change at this point.

The existence of break points ϕ_b can be analysed through inspection of the eigenvalues of the Jacobian Matrix and their associated eigenvectors at equilibria that lie on interior invariant spaces. Limit points ϕ_l are studied in the same vein as break points, the difference being that these lead to emergence (or coalescence) of multiple equilibria. As ϕ crosses this point from below (above), the stable system becomes unstable and may jump to some locally stable equilibrium (under replicator dynamics) if ϕ_l is a maximum (minimum) point. By contrast, the existence of sustain points ϕ_s for boundary equilibria is analysed through the inspection of the eigenvalue and associated eigenvector that determines stability in the direction that is orthogonal to a boundary invariant space.

Since any invariant subspace of Δ in a model with three regions ($n = 3$) can be reduced to a one-dimensional space, studying these points in the restriction of the model to a one-dimensional case entails no loss of generality. Given symmetry, we can then choose any region i and focus on the distribution $h \equiv h_i$ and $h_j = f(h)$, $\forall j \neq i$. The one dimensional subspace is thus given by $(h, f(h)) \in \boldsymbol{h}$, where $f(h) : [0, 1] \mapsto [0, 1]$.

Merging this knowledge with General Bifurcation mechanism for three regions (Section 3), we are able to uncover the possible (stable) agglomeration patterns in a 3-region model. Analytically, we are able to determine the exact possibilities in the Pflüger (2004) (PF) model further in Section 4.

3 General Bifurcation mechanism for three regions

An equidistant economy with three regions is identical to a racetrack economy with three regions, $n = 3$, and whose state space is thus defined by the 2-dimensional simplex Δ. The mechanism of agglomeration is explained below for $n = 3$ that is most pedagogic but more realistic.

3.1 Classification of possible equilibria

Interior equilibria can be classified in accordance with the population $\boldsymbol{h}$ as

- *Total dispersion* with three identical regions:

$$\boldsymbol{h}^3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

- *Partial dispersion* with two identical regions and another region:

$$\boldsymbol{h}^2 = \left(\frac{1}{3} - 2\alpha, \frac{1}{3} + \alpha, \frac{1}{3} + \alpha\right), \quad \alpha \in \left[-\frac{1}{3}, \frac{1}{6}\right].$$

  The partial dispersion can be further classified as

  - *Large partial dispersion* with two large regions and a small region: $\alpha \in [0, 1/6]$.
  - *Small partial dispersion* with two small regions and a large region: $\alpha \in [-1/3, 0]$.

- *Asymmetric equilibrium* with three regions with different size:

$$\boldsymbol{h}^1 = (\alpha, \beta, \gamma)$$

  with three different values α, β, and γ.

Boundary equilibria can be classified as

- *Boundary dispersion* with two large regions and a region without population with

$$\boldsymbol{h}^{2*} = \left(0, \frac{1}{2}, \frac{1}{2}\right).$$

- *Asymmetric boundary equilibrium* with two regions with different positive population and a region with no population:

$$\boldsymbol{h}^{1*} = (0, \alpha, \beta), \quad 0 < \alpha < \beta < 1.$$

- *Agglomeration*: population agglomerated to a single region with

$$\boldsymbol{h}^0 = (1, 0, 0).$$

3.2 Bifurcation mechanism to change agglomeration patterns

Bifurcation is a major source of agglomeration pattern change in a symmetric system, and bifurcation mechanism of the interior equilibria can be obtained as an application of group-theoretic bifurcation theory (e.g., Ikeda and Murota, 2010) in the form of the following lemma.

Lemma 1. *Bifurcation mechanism for three regions is expressed by successive bifurcations:*

$$\text{(Three identical regions)} \Longrightarrow \text{(Two identical regions)} \longrightarrow \text{(No identical regions)}, \qquad (2)$$

where "$\Longrightarrow$" denotes the direct bifurcation and "$\longrightarrow$" indicates the secondary bifurcation.

Proof. The hierarchy of groups in (2) can be obtained as a sub-hierarchy of Figure 8.3 of Ikeda and Murota (2010). □

The six different spatial patterns presented in Section 3.1 can be classified into

- Three identical regions: $\boldsymbol{h}^3$.
- Two identical regions: $\boldsymbol{h}^2$, $\boldsymbol{h}^{2*}$, and $\boldsymbol{h}^0$.
- Three different regions: $\boldsymbol{h}^1$ and $\boldsymbol{h}^{1*}$.

Then the above lemma can be restated as the following proposition:

Proposition 1. *The interior equilibria undergo successive bifurcations:*

$$\boldsymbol{h}^3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \Longrightarrow \boldsymbol{h}^2 = \left(\frac{1}{3} - 2\alpha, \frac{1}{3} + \alpha, \frac{1}{3} + \alpha\right) \longrightarrow \boldsymbol{h}^1 = (\alpha, \beta, \gamma). \qquad (3)$$

The boundary equilibria undergo the secondary bifurcation:

$$\boldsymbol{h}^{2*} = \left(0, \frac{1}{2}, \frac{1}{2}\right) \longrightarrow \boldsymbol{h}^{1*} = (0, \alpha, \beta). \qquad (4)$$

Proof. The hierarchy (2) can be restated in terms of the change of population distribution $\boldsymbol{h}$. □

The direct bifurcation $\boldsymbol{h}^3 \Longrightarrow \boldsymbol{h}^2 = \left(\frac{1}{3} - 2\alpha, \frac{1}{3} + \alpha, \frac{1}{3} + \alpha\right)$ is asymmetric in the sense that $\alpha > 0$ and $\alpha < 0$ correspond to different spatial patterns: large and small partial dispersions, respectively. This bifurcation, accordingly, produces two kinds of patterns, which are unstable just after the bifurcation.

Each of the secondary bifurcations of two kinds presented above is pitchfork and produces a single spatial pattern. In $\boldsymbol{h}^2 \longrightarrow \boldsymbol{h}^1 = (\alpha, \beta, \gamma)$, there appear (α, β, γ) and (α, γ, β) that can be identified as a single pattern. Such is also the case for $\boldsymbol{h}^{2*} \longrightarrow \boldsymbol{h}^{1*} = (0, \alpha, \beta)$.

Figure 1 describes the general bifurcation mechanism for any fully symmetric model with three regions. The circles denote the size of mobile population and a region without a circle has no population. Thus, by the direct bifurcation in the state of total dispersion, there emerge two direct branches for $\alpha > 0$ and $\alpha < 0$ with two identical regions and another region (*partial dispersion*). By the secondary bifurcation (called secondary bifurcation A in Figure 1) from these branches, there possibly emerge other interior equilibria, i.e., the asymmetric equilibrium with three different regions.[8] The secondary bifurcation B. from the *boundary dispersion* $\boldsymbol{h}^{2*}$ leads to the asymmetric boundary equilibrium (*partial agglomeration*), a state with cities with large, small, and no populations, i.e., $\boldsymbol{h}^{1*} = (0, \alpha, \beta)$.

The bifurcation mechanisms associated with (3) and (4), which exhaust mathematical possibilities, are quite insightful in the study of agglomeration of three regions. However, the existence of a specific bifurcation is dependent on the model and its parameters. In fact, it is shown for the PF model in this paper that the secondary bifurcation A engendering the asymmetric states does not exist and the secondary bifurcation B engendering the partial agglomeration exists for any parameter values and for some $\phi \in (0, 1)$.

3.3 Emergence of boundary equilibria via sustain points

The interior equilibria other than *total dispersion* can possibly transform into corner equilibria by the vanishing of population at one or two region(s) at a sustain (bifurcation) point, as was studied in Ikeda et al. (2012).[9] We have the following proposition for the present case.

Proposition 2. *For the three regions, there are sustain points of four kinds, called I*

[8]Note that the term "secondary bifurcation" was used by Gaspar et al. (2018) to determine the existence of a saddle-node (fold) bifurcation along the branches with partial agglomeration that analyses their emergence and stability. It thus should not be confused with secondary bifurcations in this paper.

[9]Here we employ an extended use of a *sustain point*, whereas it is customary to call the sustain point as the limit of sustainability of the agglomeration in a two-region economy.

through IV:

$$\begin{aligned}
&\textit{Sustain I:} && \boldsymbol{h}^2 \ \textit{(small partial dispersion)} && \dashrightarrow \boldsymbol{h}^0 \ \textit{(agglomeration)}, && (5)\\
&\textit{Sustain II:} && \boldsymbol{h}^2 \ \textit{(large partial dispersion)} && \dashrightarrow \boldsymbol{h}^{2*} \ \textit{(boundary dispersion)}, && (6)\\
&\textit{Sustain III:} && \boldsymbol{h}^1 \ \textit{(asymmetric)} && \dashrightarrow \boldsymbol{h}^{1*} \ \textit{(asym. boundary eq.)}, && (7)\\
&\textit{Sustain IV:} && \boldsymbol{h}^{1*} \ \textit{(asymmetric boundary eq.)} && \dashrightarrow \boldsymbol{h}^0 \ \textit{(agglomeration)}. && (8)
\end{aligned}$$

Proof. The existence of these four sustain points is apparent from the dashed arrows in Figure 1. □

There are two kinds of transitions from interior to boundary equilibria. The small partial dispersion may encounter the sustain point I, with the population in the two small regions vanishing simultaneously, to arrive at the *agglomeration.*[10] The large partial dispersion with two large regions and one small region may encounter the sustain point II to arrive at the *boundary dispersion* $\boldsymbol{h}^{2*} = \left(0, \frac{1}{2}, \frac{1}{2}\right)$.

There are two kinds of transitions between two different boundary equilibria. The secondary bifurcation from the *boundary dispersion* leads to other boundary equilibria (*partial agglomeration*), a state with cities with large, small, and no populations, i.e., $\boldsymbol{h}^{1*} = (0, \alpha, \beta)$ $(0 < \alpha < \beta < 1)$. This state may encounter the sustain point III to recover the population of a region to become the asymmetric state or encounter the sustain point IV to lose the population of a smaller region to become the state of *agglomeration.*

3.4 Emergence of asymmetric equilibrium

In general, an asymmetric state with three differently populated regions $h_1 > h_2 > h_3$ has three kinds of critical points:

- Break point which is a connection with $h_1 > h_2 = h_3$ or $h_1 = h_2 > h_3$
- Sustain point which is a connection with $h_1 > h_2 > h_3 = 0$
- A limit (maximum or minimum point) of the trade freeness ϕ

For the PF model, it is to be proved that we may only have a limit (maximum or minimum point) of the trade freeness ϕ for the asymmetric state.

[10]This may also be referred to as a *corner equilibrium* as corresponds to a single circle placed at any vertex of the simplex.

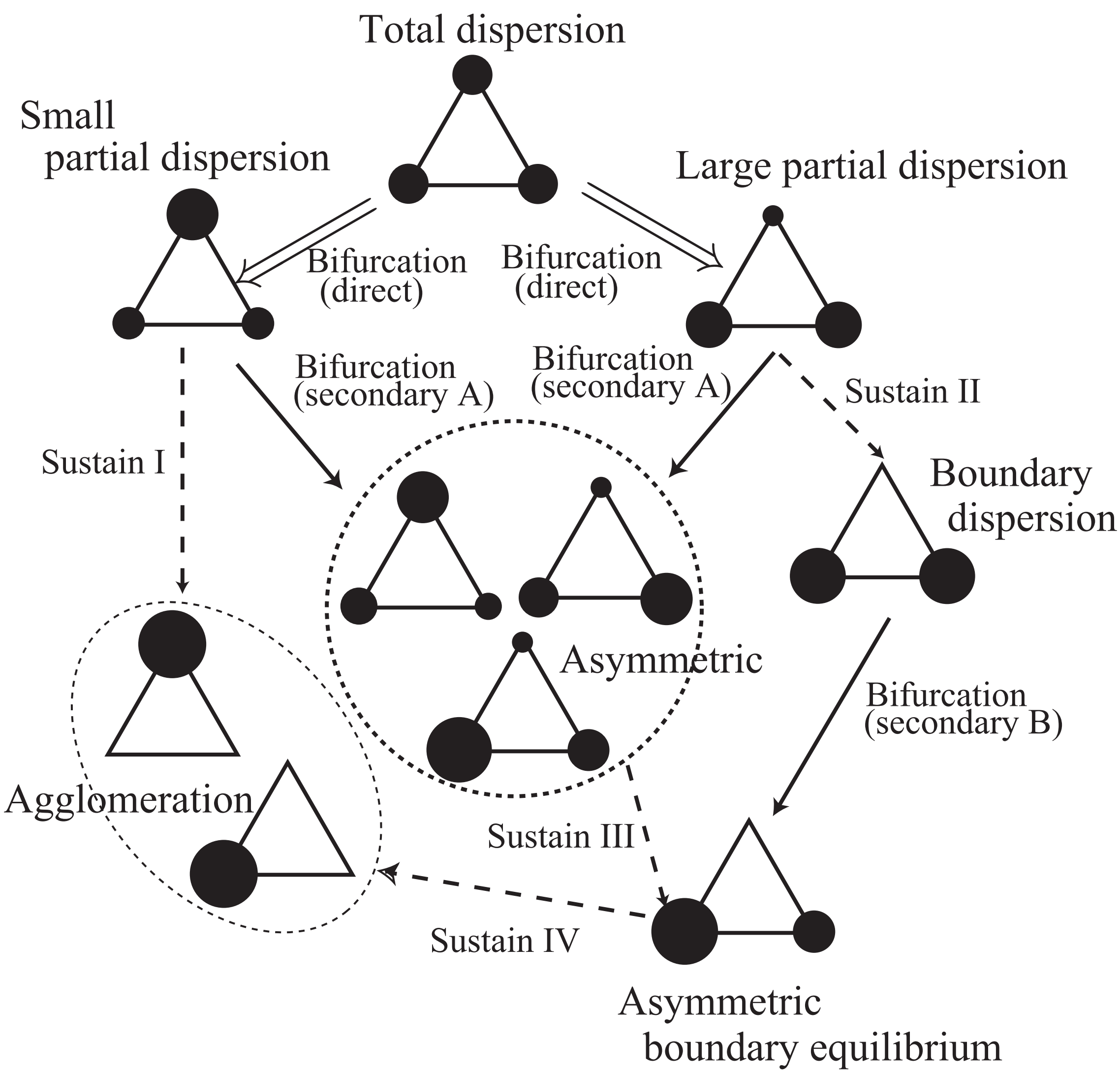

Figure 1 – Description of the general bifurcation mechanism for three regions.

4 Spatial equilibria in the PF model

Bearing the general bifurcation mechanism in mind, we study all possible spatial equilibria for the PF model with three regions.

4.1 Model description with n equidistant regions

We succinctly introduce the Pflüger (2004) model (PF) which was extended by by Gaspar et al. (2018) to n equidistant regions. We shall omit most derivations and write down just the main assumptions and results.

The set of regions in the economy is defined by $N = \{1, 2, ..., n\}$. There is a unit mass of H (skilled) inter-regionally mobile workers and a unit mass of L (unskilled) immobile workers which are assumed to be evenly distributed across all regions, i.e., $L_i = L/n$,

$\forall i \in N$. The number of mobile workers in region $i \in N$ is given by $h_i \in [0,1]$.

There is an homogeneous good A_i, which is produced one-for-one under perfect competition using immobile workers and is freely traded across all regions. We take it as the numéraire, setting both its price and the wage paid to immobile workers to unity. The other good is a CES composite of manufactures produced under monopolistic competition and increasing returns to scale. Manufacturing firms require one unit of mobile workers to start production and one unit of immobile worker per unit of good that is produced (i.e., production is footloose). This good is subject to trade barriers in the form of iceberg costs, $\tau \in (1, +\infty)$. A firm in region i ships τ units of a good to a foreign country for each unit that arrives in region $j \neq i$, i.e., $\tau_i = \tau$ if $i \neq j$ and $\tau_i = 1$ if $i = j$, with $j \in N$.

Preferences of workers in region $i \in N$ are described by the following utility function:

$$U_i = \mu \ln C_i + A_i, \tag{9}$$

where $\mu > 0$ and:

$$C_i = \left[\int_{s \in S} c_i(s)^{\frac{\sigma-1}{\sigma}} ds \right]^{\frac{\sigma}{\sigma-1}}$$

is the CES composite, and $c_i(s)$ is the consumption of variety s that is manufactured by a firm in region i. The parameter $\sigma > 1$ corresponds to the elasticity of substitution. Agents maximize (9) subject to the budged constraint $P_i C_i + A_i = y_i$, where P_i is the regional price index and y_i is an agent's nominal income. The indirect utility of a worker is given by $V_i(\boldsymbol{h}) = y_i(\boldsymbol{h}) - \mu \ln P_i(\boldsymbol{h}) - \mu$.

On the supply side, manufacturing firms set the usual profit maximizing price that is a constant mark-up over the marginal cost. Free entry implies zero profits at equilibrium implying that the wage bill of mobile workers completely absorbs marginal profits. This yields a nominal wage that is an explicit function of the spatial distribution $\boldsymbol{h}$. After Gaspar et al. (2018, pp. 867-870), indirect utility boils down to:

$$V_i(\boldsymbol{h}) = \frac{\mu}{\sigma} \sum_{j=1}^{n} \frac{\phi_{ij} \left(\lambda/n + h_j \right)}{\phi + (1-\phi) h_j} + \frac{\mu}{\sigma - 1} \ln \left[\phi + (1-\phi) h_i \right] + \eta, \tag{10}$$

where $\eta = \mu \left(\ln \mu - 1 \right) - \mu (1-\sigma)^{-1} \ln \left[\beta (\sigma - 1)^{-1} H / \alpha \right]$ is a constant.

For the 3-region case, interior equilibria that are not completely asymmetric lie on the one-dimensional interior invariant spaces $\mathcal{I} \in \Delta$ (see Appendix A.1) and correspond to: total dispersion and (large or small) asymmetric dispersion. A boundary $\mathcal{B} \in \Delta$ (see Appendix A.2) is a set whereby at least one region has no population (mobile workers), i.e., $h_i = 0$, for some $i = \{1, 2, 3\}$.[11] There are three types of equilibria on $\mathcal{B}$: boundary

[11]For simplicity of exposition, we will often use the term 'population' which denotes only the mobile

dispersion, asymmetric boundary equilibria and agglomeration.

4.2 Interior equilibria

For $n = 3$, interior equilibria that are not completely asymmetric are contained in the one-dimensional invariant space $\mathcal{I}$. Suppose, without loss of generality, that regions 2 and 3 share the same population while region 1 has a different population. Then, from Appendix A.1, the corresponding interior invariant subspace $\mathcal{I}$ can be restated as $(h_1, h_3) \equiv (h, f(h)) = \left(h, \frac{1-h}{2}\right)$. Interior equilibria $\boldsymbol{h} \in \mathcal{I}^* = \{\boldsymbol{h} \in \mathcal{I} : f_i = 0, \forall i \in N\}$ are classified as total dispersion $\boldsymbol{h}^3$ and (large or small) partial dispersion $\boldsymbol{h}^2$. Since some results along this one-dimensional invariant space are already known from the n-region case studied by Gaspar et al. (2018), we shall refer to them more succinctly.

Total dispersion corresponds to a spatial distribution whereby all regions are evenly populated:

$$\boldsymbol{h}^3 = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right).$$

In our one-dimensional invariant space $\mathcal{I}$, it is given simply by $h^* = \frac{1}{3}$. For $n = 3$, total dispersion is stable if:[12]

$$\phi < \phi_b \equiv \frac{\sigma(1-\lambda)+\lambda}{\lambda+3-\sigma(\lambda+5)}, \tag{11}$$

where ϕ_b is the break point from the direct bifurcation in the state of total dispersion that leads to emergence of states with the two identical regions 2 and 3 and the asymmetric region 1.

Partial dispersion corresponds to a spatial distribution $\boldsymbol{h} \in \mathcal{I}^*$ such that two regions have identical population and the other one has a different population:

$$\boldsymbol{h}^2 = \left\{\left(\tfrac{1}{3} - 2a, \tfrac{1}{3} + a, \tfrac{1}{3} + a\right), \left(\tfrac{1}{3} + a, \tfrac{1}{3} - 2a, \tfrac{1}{3} + a\right), \left(\tfrac{1}{3} + a, \tfrac{1}{3} + a, \tfrac{1}{3} - 2a\right)\right\},$$

with $a \in \left(-\frac{1}{3}, \frac{1}{6}\right)$. In the restriction $(h_1, h_3) = \left(h, \frac{1-h}{2}\right)$, it is simply given by $h \equiv h^* \in (0, 1)$. If $h > \frac{1}{3}$, region 1 is larger and there is *small partial dispersion*. When $h < \frac{1}{3}$, region 1 is smaller and there is *large partial dispersion*. The direct bifurcation from total dispersion to partial dispersion is transcritical, which means that a branch of partial dispersion equilibria crosses total dispersion at $\phi = \phi_b$. The criticality of the bifurcation (see Gaspar et al., 2018) determines that, along $\mathcal{I}$ and in a neighbourhood of (ϕ_b, h^*), these equilibria correspond to unstable small partial dispersion for $\phi < \phi_b$ and stable

agents in the economy.

[12]See Gaspar et al. (2018, pp. 872).

large partial dispersion for $\phi > \phi_b$. Although the latter is stable in $\mathcal{I}$, it is locally unstable in Δ.[13]

We further investigate the possibility of a secondary bifurcation A described in Section 3.2 from the state of partial dispersion (either large or small) to asymmetric equilibria with three different regions.

Proposition 3. *No break point exists that connects partial dispersion with completely asymmetric equilibria.*

Proof. The condition for existence of a break point engendering the secondary bifurcation is that partial agglomeration interchanges stability in the direction orthogonal to $\mathcal{I}$. This requires the corresponding eigenvalue to be zero for some $\phi \in (0,1)$, i.e., $\gamma \equiv \frac{\partial f_i(\boldsymbol{h})}{\partial h_i} = 0$,for $i \neq 1$. As shown by Gaspar et al. (2018, pp. 892-893), we have $\gamma > 0$ for $h^* < \frac{1}{3}$ and $\gamma < 0$ for $h^* > \frac{1}{3}$ and hence the eigenvalue is non-zero and does not change its sign. $\square$

This result establishes that there exists no break point that engenders the secondary bifurcation A from partial dispersion (either large or small) such that asymmetric equilibria with three different regions emerge.

However, more can be said about small partial dispersion. In fact, the PF model exhibits dual small partial dispersion equilibria for a sufficiently high range of the freeness of trade. Since the bifurcating partial dispersion from total dispersion is unstable, the only way to reach stable partial dispersion is for the freeness of trade to encounter a limit point, where the stability changes. This leads to the following statement.

Proposition 4. *Along the primary branch of partial dispersion, the PF model undergoes a saddle-node bifurcation at a limit (minimum) point of the freeness of trade, $\phi_l \in (0,1)$, above which two small dispersion equilibria emerge. The one with more population in region* 1 *is stable in Δ, whereas the one with less population in region* 1 *is unstable in Δ.*

Proof. The existence of a saddle-node bifurcation at $\phi \equiv \phi_l \in (0,1)$ is demonstrated by Gaspar et al. (2018) for $h^* > \frac{1}{n}$. For $h^* < \frac{1}{n}$, no bifurcation exists because large partial dispersion is always stable in $\mathcal{I}$. As a result, there is a limit point of the freeness of trade that leads to a curve of small partial dispersion equilibria, one stable and the other unstable in $\mathcal{I}$. The limit point is a minimum point because the curve of partial dispersion equilibria along the primary branch is tangent to the line $\phi = \phi_l$ and lies to its right. Since the bifurcating small partial dispersion is unstable in $\mathcal{I}$, for $\phi > \phi_l$, the other small partial dispersion with more population in region 1 is stable in $\mathcal{I}$. Finally, from Proof of Proposition 3, we have that the small partial dispersion with more population in region 1 is locally stable in Δ. $\square$

[13] The eigenvalue that determines the stability in the direction orthogonal to $\mathcal{I}$ is always positive.

From the existence of a saddle-node bifurcation occurring at partial dispersion, we know that a limit point exists and is a minimum point (of the freeness of trade) above which two small partial dispersion states emerge, one dynamically stable (the one with more population in the large region) and the other one unstable. Moreover, the minimum point is lower than the break point above which total dispersion becomes unstable. Therefore, if partial dispersion is stable, it continues to be stable above the minimum point of the freeness of trade. Below that limit point, the system becomes unstable and may jump to some stable equilibrium under replicator dynamics, which may correspond to total dispersion since $\phi_l < \phi_b$, i.e., total dispersion is certainly stable below the minimum point.

Figure 2 illustrates a numerical example of the bifurcation mechanism along $\mathcal{I}$ for $h \in [0,1]$ (i.e., including its limit points $h = \{0,1\}$ which belong to a boundary $\mathcal{B}$) which fits the discussion above. These results were confirmed analytically by Gaspar et al. (2018) for n regions with one asymmetric region with population $h \in (0,1)$ and $n-1$ regions with population $h_j = \frac{1-h}{n-1}$. The point $\phi_s \in (0,1)$ is the sustain point of kind I defined in Section 3.3 (analysed in more detail in Section 4.3) and corresponds to the threshold of the freeness of trade above which agglomeration is stable. As discussed by Gaspar et al. (2018), its ordering regarding the break point ϕ_b depends on parameters λ and/or σ. If $\phi_b < \phi_s$, a dynamic jump from total dispersion to stable small partial dispersion may occur once $\phi > \phi_b$. Otherwise, the jump is towards stable agglomeration just as in the FE model (Forslid and Ottaviano, 2003) or the CP model (Krugman, 1991; Fujita et al., 1999).

4.3 Boundary equilibria

Without loss of generality, we study the boundary $\mathcal{B}$ such that $h_1 = 0$, which is equivalent to the one-dimensional subspace given by $(h_2, h_3) \equiv (h, f(h)) = (h, 1-h)$, with $h \in [0,1]$.[14]

The boundary $\mathcal{B}$ may be of particular economic interest because it is invariant for the dynamics. Since the replicator dynamics in (1) describes dynamics 'driven by imitation', this captures the intuition that it may be harder for regions that have no industry to attract potential migrants. If the system starts at a solution in $\mathcal{B}$ that is unstable in Δ, it must jump into a direction other than $\mathcal{B}$, thereby gaining population in the region with no population to exit from a boundary solution to an interior one.

Boundary dispersion corresponds to a spatial distribution $\boldsymbol{h} \in \mathcal{B}^*$,[15] whereby two regions

[14] Due to symmetry, we can choose any boundary $\mathcal{B}$.

[15] Where $\mathcal{B}^*$ is the set of boundary dispersion equilibria for n regions defined in Appendix A.2.

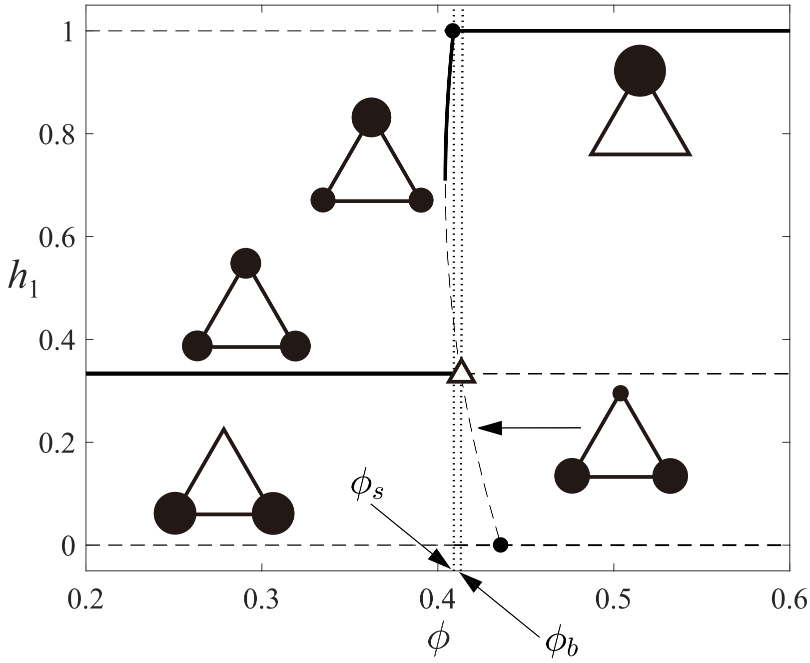

Figure 2 – Bifurcation diagram for h along an invariant space where $h_1 = a$ and $h_2 = h_3 = \left(\frac{1-a}{2}\right)$ with $a \in [0,1]$ ($\mu = 0.4$, $\sigma = 5$, $\lambda = 6$; $\triangle$: break point, $\bullet$: sustain point, $\bigcirc$: limit point; solid curve: stable, dashed curve: unsustainable and/or unstable in Δ).

are evenly populated and the other region has no population:

$$\boldsymbol{h}^{2*} = \left\{ \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right), \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right), \left(0, \tfrac{1}{2}, \tfrac{1}{2}\right) \right\}.$$

In the restriction $h_1 = 0$, boundary dispersion $\boldsymbol{h}^{2*}$ defined in Section 3.1 is given simply by $h \equiv h^* = \frac{1}{2}$. From Gaspar et al. (2018, pp. 890) and setting $n = 3$, boundary dispersion is stable along $\mathcal{B}$ if:

$$\alpha \equiv 3(3\sigma\phi + \sigma - 2\phi) - 2\lambda(\sigma - 1)(1 - \phi) < 0. \tag{12}$$

Solving $\alpha = 0$ for ϕ, we get that boundary dispersion is stable in $\mathcal{B}$ if:

$$\phi < \phi_b^{II} \equiv \frac{2\lambda(\sigma - 1) - 3\sigma}{2\lambda(\sigma - 1) + 9\sigma - 6}, \tag{13}$$

where ϕ_b^{II} is the break point for the secondary bifurcation described in Proposition 1 (secondary B in Figure 1). By stable along $\mathcal{B}$, it is meant that, after any perturbation that shifts populations strictly between regions 2 and 3 (i.e., region 1 gains no population exogenously), the spatial distribution will return to boundary dispersion dynamically.

Notice that the denominator in (13) is always positive. Therefore, $\phi_b^{II} > 0$ if and only if:

$$\lambda > \lambda_b^{II} \equiv \frac{3\sigma}{2(\sigma-1)}.$$

Therefore, if λ is high enough, then $\phi_b^{II} > 0$ and there exists $\phi \in (0, \phi_b^{II})$ for which boundary dispersion is stable in $\mathcal{B}$.[16] Note also that $|2\lambda(\sigma-1) - 3\sigma| < 2\lambda(\sigma-1) + 9\sigma - 6$, which means that $\phi_b^{II} < 1$. We conclude that boundary dispersion interchanges stability in $\mathcal{B}$ at ϕ_b^{II} if $\lambda > \lambda_b^{II}$, which we assume henceforth. Thus, boundary dispersion undergoes a secondary bifurcation at ϕ_b^{II}.

Partial agglomeration corresponds to spatial distributions $\boldsymbol{h} \in \mathcal{B}^*$ whereby one region has zero population and the other two have different positive populations:

$$\boldsymbol{h}^{1*} = \left\{ \left(\tfrac{1}{2}+a, \tfrac{1}{2}-a, 0\right), \left(\tfrac{1}{2}-a, 0, \tfrac{1}{2}+a\right), \left(0, \tfrac{1}{2}+a, \tfrac{1}{2}-a\right) \right\},$$

and $a \in \left(\frac{1}{2}, 1\right)$. In the restriction $h_1 = 0$, the asymmetric boundary equilibrium $\boldsymbol{h}^{1*}$ defined in Section 3.1 is given by $h \equiv h^* \in (0,1)$. From (1), one can easily check that $\boldsymbol{h}^{1*}$ is an equilibrium if and only if $V_2 = V_3$, where:

$$V_2(h) = \frac{\mu}{\sigma}\left[\frac{h+\frac{\lambda}{3}}{\phi+(1-\phi)h} + \phi\frac{\frac{\lambda}{3}+1-h}{\phi+(1-\phi)(1-h)} + \frac{\lambda}{3}\right] + \frac{\mu}{\sigma-1}\ln\left[\phi+(1-\phi)h\right] + \eta, \quad (14)$$

and:

$$V_3(h) = \frac{\mu}{\sigma}\left[\phi\frac{h+\frac{\lambda}{3}}{\phi+(1-\phi)h} + \frac{\frac{\lambda}{3}+1-h}{\phi+(1-\phi)(1-h)} + \frac{\lambda}{3}\right] + \frac{\mu}{\sigma-1}\ln\left[1-h(1-\phi)\right] + \eta. \quad (15)$$

Due to symmetry, we can focus on the case where region 2 has more population than region 3, i.e., $h \in \left(\frac{1}{2}, 1\right)$. We have the following result regarding existence and uniqueness of asymmetric boundary equilibria.

Proposition 5. *There exists at most one partial agglomeration* $h \equiv h^* \in \left(\frac{1}{2}, 1\right)$.

Proof. See Appendix B. □

Figure 3 illustrates the existence of partial agglomeration $h^* \in \left\{\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)\right\}$ by plotting $F(h) \equiv V_2(h) - V_3(h)$ for a set of parameter values.[17]

We now investigate whether the sustain point of III defined in Section 3.3 exists, i.e., a freeness of trade $\phi \equiv \phi_s^{III} \in (0,1)$, such that the eigenvalue associated with the eigenvector

[16] Notice that since $\lambda_b^{II} > \sigma/(\sigma-1)$, we have $\phi_b^{II} < 0$ if the no black-hole condition is not satisfied. In other words, if dispersion is unstable for all $\phi \in (0,1)$, then boundary dispersion is unstable in $\mathcal{B}$ for all $\phi \in (0,1)$ and no bifurcation occurs at ϕ_b^{II}.

[17] By symmetry, along $\mathcal{B}$ if an asymmetric boundary equilibrium exists it consists of two symmetric points around $h = 1/2$, i.e., $\boldsymbol{h^*} \in \left\{\left(0+a, \frac{1}{2}-a\right) \cup \left(\frac{1}{2}+a, 1-a\right)\right\}$, with $a \in (0,1)$.

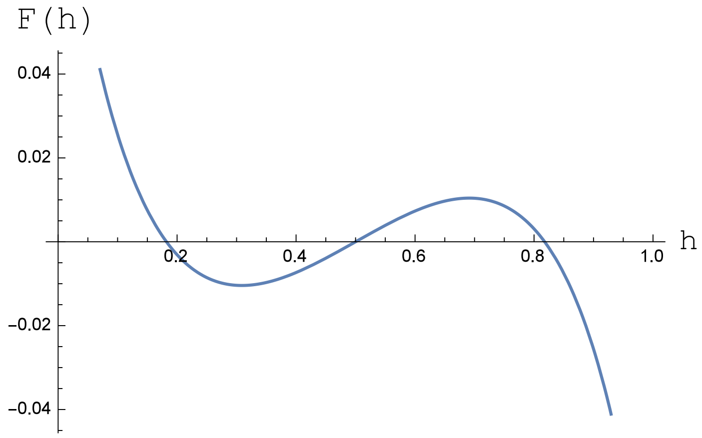

Figure 3 – Existence of asymmetric boundary equilibria. We plot $F(h)$ with parameter values $(\sigma, \phi, \lambda) = (4, 0.25, 4)$.

$\begin{bmatrix} 0 & 1 \end{bmatrix}^T$ of the Jacobian matrix of (1) evaluated at $h^* \in \left\{ \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \right\}$ changes sign for some $\phi \in (0, 1)$. The sustain point must thus satisfy the following condition:[18]

$$V_1(h^*) = \bar{V}(h^*) \equiv hV_2 + (1 - h)V_3. \tag{16}$$

Condition (16) determines the possibility in the change of stability of partial agglomeration in the direction that is orthogonal to the invariant boundary $\mathcal{B}$. We have the following result regarding local stability of partial agglomeration and the possibility of a sustain point of kind III described in Section 3.3.

Proposition 6. *No sustain point exists for any partial agglomeration $\boldsymbol{h}^{1*}$, which is locally unstable in Δ.*

Proof. See Appendix B. □

From this Proposition, it follows that there exists no sustain point such that the state of partial agglomeration engenders the state of partial agglomeration. Moreover, any partial agglomeration $h^* \in \left\{ \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \right\}$ is locally unstable in Δ. However, it is stable along the invariant boundary $\mathcal{B}$ when it exists. In this case, the economy will always converge asymptotically to partial agglomeration after exogenous perturbations that shift population solely between populated regions. The only way to exit to an interior solution is if the empty region gains population exogenously (i.e., the system must jump to a direction other than $\mathcal{B}$).

We further investigate the type of secondary bifurcation from boundary dispersion $\boldsymbol{h}^{2*}$ that leads to partial agglomeration $\boldsymbol{h}^{1*}$. The next result analytically defines the precise qualitative behaviour of equilibria along the boundary.

[18] If exogenous migration to region 1 occurs, it will go back to zero population (continue to increase) if the equality sign is changed to '$<$' ('$>$'):

Proposition 7. *As ϕ increases, boundary dispersion undergoes a supercritical pitchfork bifurcation at ϕ_b^{II}. Before the bifurcation, boundary dispersion is stable along the boundary. After the bifurcation, partial agglomeration with two differently populated regions exists and is stable along the boundary. This state eventually encounters a sustain point $\phi_s \in (0,1)$ that leads to the state of agglomeration which is stable there in after.*

Proof. See Appendix B. □

The implications from Proposition 3 are illustrated in Figure 4, which depicts the bifurcation diagram along $\mathcal{B}$ as ϕ increases. Parameter values are $\mu = 0.4$, $\lambda = 1$, and $\sigma = 5$.[19]

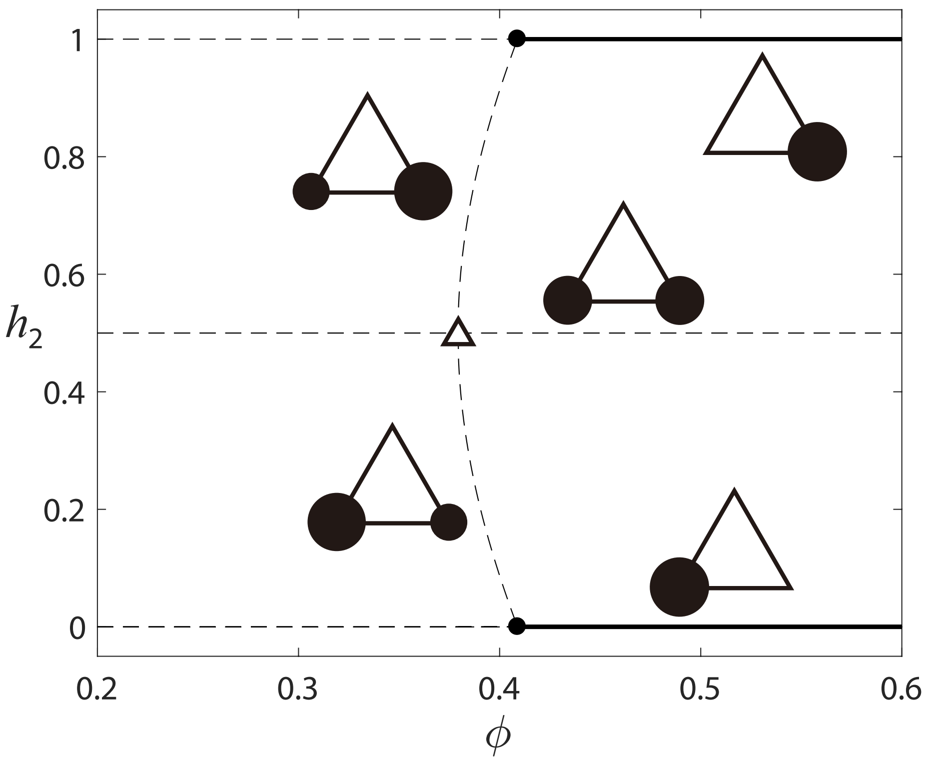

Figure 4 − Bifurcation diagram for $h \in \mathcal{B}$ ($\mu = 0.4$, $\sigma = 5$, $\lambda = 6$; △: break point, •: sustain point, ◯: limit point; solid curve: stable, dashed curve: unsustainable and/or unstable in Δ).

When $\phi < \phi_b^{II}$, boundary dispersion is stable in $\mathcal{B}$ (but a saddle in Δ).[20] Once $\phi > \phi_b^{II}$, boundary dispersion becomes unstable along $\mathcal{B}$ and undergoes the secondary bifurcation B in Figure 1 that leads to partial agglomeration. Two asymmetric boundary equilibria $\left(h_-^*, h_+^*\right) = \left(\frac{1}{2} + \alpha, \frac{1}{2} - \alpha\right)$ for $\alpha \in \left(0, \frac{1}{2}\right)$, stable along $\mathcal{B}$ (but locally unstable), emerge and become more asymmetric (i.e., α increases) as ϕ increases further. Agglomeration

[19]The results are robust under a wider range of parameter values.

[20]We know this because it is locally unstable as shown by Gaspar et al. (2018, Theorem 1 pp. 873) for $n \geq 3$ regions, and because of Proposition 2.

$h^* = \{1, 0\}$ is unstable during this process. Finally, once $\phi > \phi_s$, boundary equilibria coalesce into agglomeration and the latter becomes stable. Asymmetric boundary equilibria are stable in $\mathcal{B}$ if $\phi_b^{II} < \phi < \phi_s$,where ϕ_s is the sustain point above which agglomeration becomes stable.

We conclude that, along the boundary, asymmetric boundary dispersion is always stable when it exists. We have also shown that an asymmetric boundary equilibrium $h \in (0, 1)$ cannot encounter a level for the freeness of trade that leads to the state of asymmetric interior equilibria. Finally, note that any agglomeration configuration is a limit point of an interior space $\mathcal{I}$ and a boundary $\mathcal{B}$, and connects both spaces because it lies on a vertex of Δ. Continuity of (1) in $\boldsymbol{h}$ thus establishes that the sustain points I and IV defined in Section 3.3 coincide and are given by $\phi \equiv \phi_s \in (0, 1)$ given by:[21]

$$\frac{(1-\phi_s)\left[\lambda(1-\phi_s) - 3\phi_s\right]}{3\sigma\phi_s} - \frac{\ln\phi_s}{\sigma - 1} = 0.$$

4.4 Accessibility to asymmetric equilibrium

We discuss the accessibility to an asymmetric equilibrium with all differently populated regions for the PF model. We have the following Proposition showing that such an equilibrium is not connected to equilibria of other types even if it exists. Its existence is not guaranteed and is quite doubtful.

Proposition 8. *Asymmetric equilibria for the PF model are aloof equilibria that are not connected to equilibria of other types.*

Proof. There are two routes to arrive at an asymmetric equilibrium: the secondary bifurcation B and the sustain point III. There two routes are denied by the discussion above (Propositions 3 and 6). □

If an asymmetric equilibrium exists and is stable, it continues to be stable and asymmetric until reaching a limit point (maximum or minimum point) of the freeness of trade. Beyond this point, the system becomes unstable and may jump to some other stable equilibrium dynamically.

[21] See (18) in Proof of Proposition 5.

4.5 General Bifurcation mechanism and agglomeration patterns: the whole picture

The analyses conducted in the previous sections now allow us to provide a complete gallery of the spatial patterns in the PF model for three regions that collects the results from previous sections. Figure 5 illustrates the general bifurcation diagram for the 3-region model, where $h = h_{max}$ corresponds to the mobile agents in the region with the largest population.

Total dispersion is the only stable equilibrium when the trade freeness ϕ is low, and encounters the break point A when ϕ increases. From this point, there emerge a pair of branches for two partial dispersion states; an unstable large partial dispersion develops into a boundary dispersion at the sustain point B, and an unstable small partial dispersion becomes stable at the limit point C and develops into agglomeration at the sustain point D. In addition, there is a curve DE of asymmetric boundary equilibria (partial agglomeration) connecting agglomeration and boundary dispersion.

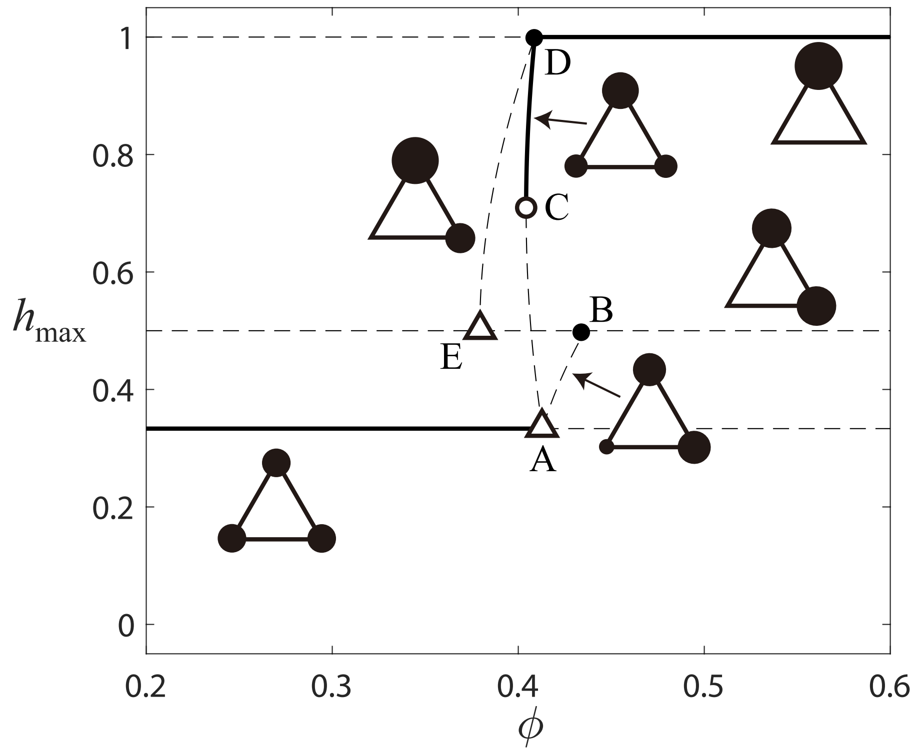

Figure 5 – Bifurcation diagram for $h = h_{max}$, where h_{max} corresponds to the region with largest population ($\mu = 0.4$, $\sigma = 5$, $\lambda = 6$; $\triangle$: break point, $\bullet$: sustain point, $\bigcirc$: limit point; solid curve: stable, dashed curve: unsustainable and/or unstable in Δ).

Notice that the break-point ϕ_b^{II} for boundary dispersion appears to lie to the left of the break-point ϕ_b for total dispersion. Comparing (11) with (13), it can easily be shown that $\phi_b > \phi_b^{II}$. Therefore, if symmetric dispersion is unstable ($\phi > \phi_b$), boundary dispersion is also unstable in $\mathcal{B}$.

We have seen that, after the bifurcation in the state of total dispersion that gives rise to two branches of partial dispersion, no sustain point exists, and therefore no secondary bifurcation will lead to the emergence of interior equilibria with three different regions. Moreover, no sustain point exists that preserves population in three different regions if partial dispersion is perturbed in the direction that populates the empty region. Moreover, depending on the freeness of trade, we find that only total dispersion (low ϕ), partial dispersion (intermediate ϕ), or agglomeration (high ϕ) can be locally stable in Δ. We summarize our results and main contributions in the following Theorem.

Theorem 1. *The only connected spatial equilibria in the PF model are agglomeration, partial dispersion, total dispersion, boundary dispersion and partial agglomeration. Only three of them are possibly locally stable: (i) full agglomeration for high enough economic integration; (i) partial dispersion for intermediate economic integration; and (iii) symmetric dispersion for low economic integration. If the economy starts at a spatial configuration with* $h_1 > h_2 > h_3 > 0$*, it will remain completely asymmetric and the ordering between region sizes will be preserved for any level of economic integration.*

We recall that from the set of possible spatial distributions described in Theorem 1, it is possible that two sets of different types of equilibria are simultaneously stable for some intermediate ranges of regional integration, namely: (i) full agglomeration and symmetric dispersion; or (ii) partial dispersion and symmetric dispersion. Therefore there are two types of locational hysteresis. The former is possible if inter-regional mobility is low, and the latter if it is high (Gaspar et al., 2018). In either case, an increase in integration above ϕ_b will lead to a discontinuous jump towards a small partial dispersion state with a significantly populated region.

5 Concluding remarks

This paper has provided a much more complete analytical description of all spatial patterns of three equidistant regions. Possible courses of the progress of spatial agglomeration presented herein would be insightful in the study of agglomeration. For instance, the knowledge of the possible bifurcations that lead to changes in agglomeration patterns could be extended to other well-known NEG models.

The more natural candidates are the earlier NEG models with only global agglomeration forces (Akamatsu et al., 2017),[22] such as the original CP model by Krugman (1991)

[22]The distinction between global and local dispersion forces is that the former act between regions and are dependent on the distance structure, whereas the latter act within regions and are independent of the distance structure. See Akamatsu et al. (2017) for more details.

and similar ones: the modified version with land instead of immobile workers by Puga (1999), the FE model by Forslid and Ottaviano (2003) or its logarithmic upper-tier utility version developed by Ottaviano (2001), or the quasi-linear upper tier utility setting with quadratic utility over manufactured goods and additive transport costs as in Ottaviano et al. (2002).[23] Examples of well established NEG models in the literature with local agglomeration/dispersion forces also worth exploring in a three-region equidistant context are for instance Helpman (1998), Murata and Thisse (2005), Redding and Sturm (2008) and Allen and Arkolakis (2015). Finally, a number of more complex models with both global and local agglomeration/dispersion forces are e.g. Tabuchi (1998) or Pflüger and Südekum (2008).

This paper is also an important step towards the study of NEG in an equidistant economy with an arbitrary number of regions. Bifurcation mechanisms and stability of spatial patterns for multi-regional equidistant economies have been studied by Aizawa et al. (2019), who show complicated networks of bifurcating equilibrium curves that connect several invariant equilibria (i.e., equilibria that are preserved for any values of the freeness of trade). For instance, our results of stable small partial agglomeration provide an analytical confirmation of the theoretically possible star-like pattern (Prop. 8, pp. 11) for the PF model. Merging this knowledge with further applications from NEG models in the guise of those previously mentioned would certainly provide further insights on the complex spatial agglomeration patterns that may arise in a multi-regional economy.

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[23]As the latter's analytical possibilities largely outweigh the formers' (Picard and Tabuchi, 2010), it could potentially allow for further insights on the (in)existence of completely asymmetric equilibria.

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Appendix A - Invariant spaces

In this Appendix, we provide a description of the main properties regarding both interior invariant spaces and boundary invariant spaces for a symmetric equidistant n-region NEG model. We shall focus mainly on subspaces of Δ that are one-dimensional.

A.1 Interior invariant spaces

For simplicity, we shall refer only to interior spaces whereby $k \in \{0, ..., n-1\}$ regions have a population a and the other $n-k>0$ regions have a population b, which can be defined as:

$$\mathcal{I} = \left\{ \boldsymbol{h} \in \Delta : (\exists i \in N)(\exists j \in N)\left[h_i \in (0,1), h_j = \frac{1-h_i k}{n-k}, i \neq j \right] \right\}.$$

The set $\mathcal{I}$ is invariant for the dynamics, i.e., any orbit passing through $\boldsymbol{h}_0 \in \mathcal{I}$ lies in $\mathcal{I}$.[24] The subset of spatial interior equilibria contained in $\mathcal{I}$ is described as:

$$\mathcal{I}^* = \{ \boldsymbol{h} \in \mathcal{I} : f_i = 0, \forall i \in N \}.$$

Each one-dimensional subspace $\mathcal{I}$ can alternatively be described by $(h, f(h)) = \left(h, \frac{1-hk}{n-k}\right)$, with $h \in (0,1)$. The equilibria in $\mathcal{I}^*$ are classified as *total dispersion*, with $k=0$ such that $h = \frac{1}{n}$, and *partial dispersion* with $k \in \{1, ..., n-1\}$. For $n=3$, the one-dimensional invariant subspace $\mathcal{I}^*$ simplifies to $(h, f(h)) = \left(h, \frac{1-h}{2}\right)$ and the equilibria of total dispersion, large partial dispersion and small partial dispersion defined in Section 3.1 are contained in $\mathcal{I}^*$.

A.2 Boundary invariant spaces

A boundary of Δ is an invariant space whereby at least one region has no mobile agents. Such a set can be defined by:

$$\mathcal{B} = \{ \boldsymbol{h} \in \Delta : (\exists K \subset N)[h_i = 0, \forall i \in K] \}.$$

The set $\mathcal{B}$ is invariant for the dynamics, since, from (1), starting with $h_i = 0$ implies $\dot{h}_i = 0$ so that no orbit will leave the boundary. The subset of $\mathcal{B}$ containing any boundary equilibria is given by:

$$\mathcal{B}^* = \{ \boldsymbol{h} \in \mathcal{B} : f_i = 0, \forall i \in N \}.$$

[24] To see this, note that, from (1), we get $\dot{h}_j = -\dfrac{k}{n-k}\dot{h}_i$ as required for $\mathcal{I}$ to be invariant under the action of the replicator dynamics.

Let us now focus on a boundary $\mathcal{B}$ such that there are $k \in \{0, ..., n-1\}$ regions with zero population, $m \in \{0, ..., n-1\}$ regions with $h_i \in [0, 1]$ and $n-k-m > 0$ regions with $h_j = \frac{1-h_i m}{n-k-m}$. Then $\mathcal{B}$ can be rewritten as the one-dimensional subspace $(h, f(h)) = \left(h, \frac{1-hm}{n-k-m}\right)$. Equilibria in $\mathcal{B}^*$ can be classified as follows. When $h = 1$ (or $h = 0$ and $n-k-m = 1$), the equilibria in $\mathcal{B}^*$ correspond to *agglomeration*. When $m = \{2, ..., n-2\}$ and $h = \frac{1}{m}$ (or $h = 0$ and $n-k-m > 1$), agents are evenly dispersed among some of the regions and the equilibria in $\mathcal{B}^*$ are called *boundary dispersion*. Otherwise, the equilibria in $\mathcal{B}^*$ are called *asymmetric*. It is easy to notice that both boundary dispersion and asymmetric boundary equilibria require $n \geq 3$.

For $n = 3$, the one-dimensional subspace $\mathcal{B}$ simplifies to $(h, f(h)) = (h, 1-h)$ and the equilibria of agglomeration, boundary dispersion and asymmetric boundary equilibria defined in Section 3.1 are contained in $\mathcal{B}^*$.

Appendix B

This Appendix contains the most extensive and formal proofs of our analytical results regarging the PF model with 3 regions.

Proof of Proposition 5
The solution to $V_2(h^*) = V_3(h^*)$, using (14) and (15) yields:

$$F \equiv \frac{(2h^*-1)(1-\phi)[\lambda(1-\phi)-3\phi]}{3\sigma\left[h^*(1-\phi)-1\right]\left[h^*(1-\phi)+\phi\right]} + \frac{1}{\sigma-1}\ln\left[\frac{\phi+(1-\phi)h^*}{1-h^*(1-\phi)}\right] = 0. \tag{17}$$

We know that $h^* = 1$ is always an equilibrium. Moreover, $h^* = 1/2$ is always a solution to (17). We thus restrict our analysis to the open interval $h \in \left(\frac{1}{2}, 1\right)$. We have

$$\begin{aligned}\frac{dF}{dh} =& h^2(\phi-1)^2\left[2\lambda(\sigma-1)(\phi-1)+3\sigma(\phi-1)-6\phi\right]+\\ &+h(\phi-1)^2\left[-2\lambda(\sigma-1)(\phi-1)-3\sigma(\phi-1)+6\phi\right]+\\ &+\lambda(\sigma-1)\left(\phi^3-\phi^2+\phi-1\right)+3\phi\left[\sigma\left(\phi^2+\phi+2\right)-\phi^2-1\right].\end{aligned}$$

The second derivative is given by:

$$\frac{d^2F}{dh^2} = (2h-1)(\phi-1)^2(2\lambda(\sigma-1)(\phi-1)+3\sigma(\phi-1)-6\phi),$$

which is negative if $h > \frac{1}{2}$. Therefore, $F(h)$ is strictly concave for $h \in \left(\frac{1}{2}, 1\right)$. Thus, there exists at most one zero for $h \in \left(\frac{1}{2}, 1\right)$. Since $F(h)$ is continuous and $F\left(\frac{1}{2}\right) = 0$, a necessary and sufficient condition for $F(h) = 0$ is that $F'\left(\frac{1}{2}\right) > 0$ and $F(h=1) < 0$. We have that $F(h=1)<0$ if and only if:

$$\frac{(1-\phi)\left[\lambda(1-\phi)-3\phi\right]}{3\sigma\phi}-\frac{\ln\phi}{\sigma-1}<0, \tag{18}$$

which yields:

$$\lambda>\lambda_{-}\equiv\frac{3\phi\left[(\sigma-1)(1-\phi)-\sigma\ln\phi\right]}{(\sigma-1)(\phi-1)^2}>0.$$

We have $F'\left(\frac{1}{2}\right)>0$ if and only if:

$$2\lambda(\sigma-1)(\phi-1)+3(3\sigma\phi+\sigma-2\phi)>0, \tag{19}$$

which yields:

$$\lambda<\lambda_{+}\equiv\frac{3(3\sigma\phi+\sigma-2\phi)}{2(\sigma-1)(1-\phi)},$$

with $\lambda_{+}>0$. We have that:

$$\lambda_{+}-\lambda_{-}=-\frac{3\sigma\left(\phi^2-2\phi\ln\phi-1\right)}{2(\sigma-1)(\phi-1)^2}>0.$$

We conclude that a boundary equilibrium $h^*\in\left(\frac{1}{2},1\right)$ exists if $\lambda_{-}<\lambda<\lambda_{+}$ and $\lambda>\lambda_b$. $\square$

Proof of Proposition 6
Condition (16) determines the change in stability of asymmetric boundary equilibria in the orthogonal direction to the boundary $\mathcal{B}$. Following a similar approach to Gaspar et al. (2018), existence of an asymmetric boundary equilibrium can be implicitly defined in terms of λ, using (17), as:

$$\lambda=\lambda^*\equiv\frac{3}{(1-\phi)^2}\left\{-\frac{\sigma\left[h(1-\phi)-1\right]\left[h(1-\phi)+\phi\right]\ln\left[\frac{h(1-\phi)+\phi}{h(\phi-1)+1}\right]}{(2h-1)(\sigma-1)}+(1-\phi)\phi\right\},$$

which is positive for all $h\in(0,1)$. Evaluating (16) at $\lambda=\lambda^*$, we get:

$$\begin{aligned}\Omega\equiv\Bigg\{&(2h-1)\phi\left\{\ln\phi-(1-h)\ln\left[1-h(1-\phi)\right]-h\ln\left[h(1-\phi)+\phi\right]\right\}-\\&-\left[(1-h)h(3\phi-1)-\phi\right]\ln\left[\frac{h(1-\phi)+\phi}{1-h(1-\phi)}\right]\Bigg\}\frac{1}{(2h-1)}<0.\end{aligned}$$

One can easily check that: $\Omega(h=0)=\Omega(h=1)=0$. We also have that:

$$\frac{\partial^2\Omega}{\partial h^2}(h)=\left(\frac{(2h-1)(1-\phi)\left\{1-\phi\left[4(1-h)h(1-\phi)-3\right]\right\}}{\left[h(1-\phi)-1\right]\left[h(1-\phi)+\phi\right]}+\right.$$
$$\left.+2(\phi+1)\ln\left(\left[\frac{h(1-\phi)+\phi}{1-h(1-\phi)}\right]\right)\right)\frac{1}{(2h-1)^3},$$

which is always negative for $h\in[0,1]$. Therefore, $\Omega(h)$ is strictly concave and is thus positive for $h\in(0,1)$, which means that no freeness of trade $\phi\equiv\phi_s^{III}\in(0,1)$ exists that satisfies (16) and thus the sustain point of kind III does not exist. Moreover, the fact that $\Omega(h)>0$ implies that asymmetric boundary equilibria are always locally unstable. $\square$

Proof of Proposition 7

Bifurcation at boundary dispersion

Along the invariant boundary $\mathcal{B}$, we investigate the existence of a bifurcation occurring at boundary dispersion. We can use $f_2(h)$ from (1) because $f_2(h):(0,1)\mapsto\mathbb{R}$:

$$\begin{aligned}f_2(h)&=h\left[V_1(h)-\bar{V}(h)\right]\\&=h\left[V_1(h)-hV_1(h)-(1-h)V_3(h)\right]\iff\\f_2(h)&=h(1-h)\left[V_1(h)-V_3(h)\right],\end{aligned}$$

where we have used the fact that $h_1=0$ is constant regarding $\frac{\partial f}{\partial h}(h)$. We have the following results:

(i). (Non-hyperbolicity) We have $\frac{\partial f}{\partial h}\left(\frac{1}{2};\phi_b\right)=0$. This condition is necessary for bifurcation to occur and is trivial given that solving $\frac{\partial f}{\partial h}\left(\frac{1}{2};\phi_b\right)=0$ yields the break-point $\phi=\phi_b$ in (13).

(ii). We have $\frac{\partial^2 f}{\partial h^2}\left(\frac{1}{2};\phi_b\right)=0$; $\frac{\partial f}{\partial r}\left(\frac{1}{2};\phi_b\right)=0$; and

$$\frac{\partial^2 f}{\partial\phi\partial h}\left(\tfrac{1}{2};\phi_b\right)=\frac{\mu(2\sigma-1)(2\lambda(\sigma-1)+9\sigma-6)^2}{(2\lambda+3)^2(\sigma-1)^3\sigma}>0.$$

(iii). We have:

$$\frac{\partial^3 f}{\partial h^3}\left(\tfrac{1}{2};\phi_b\right)=-\frac{432\mu(2\sigma-1)^3}{(2\lambda+3)^3(\sigma-1)^4}<0.$$

Conditions (i)-(iii) show that, along $\mathcal{B}$, the PF model with 3 regions undergoes a pitchfork bifurcation (see Guckenheimer and Holmes 2002; pp. 150). From (iii), we conclude that the pitchfork is supercritical.

Existence and stability of asymmetric boundary equilibrium

(Existence). The LHS of (18) is the negative of the LHS for the condition for stability of agglomeration (see Gaspar et al. 2018, pp. 871). This means that equating the LHS of (18) to zero yields the sustain point $\phi \equiv \phi_s \in (0,1)$, above which agglomeration is stable. In other words, the sustain points of kinds I and IV defined in Section 3.3 coincide. Therefore, a necessary condition for existence of asymmetric boundary equilibria is $\phi < \phi_s$. Moreover, solving (19) for ϕ we get $\phi > \phi_b^{II}$. As a result, a boundary equilibrium $h^* \in \left(\frac{1}{2}, 1\right)$ exists if $\phi_b^{II} < \phi < \phi_s$, i.e., if both boundary dispersion and agglomeration are simultaneously unstable.

(Stability). Since the PF model with $n = 3$ undergoes a supercritical pitchfork bifurcation at boundary dispersion, as $\phi > \phi_b^{II}$, two symmetric branches $\{h_1^*, h_3^*\} = \left\{\frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon\right\}$ arise in a neighbourhood of boundary dispersion, that is, $h^* \in \left(\frac{1}{2}, 1\right)$ emerges. For $\phi_b^{II} < \phi < \phi_s$, it is stable in $\mathcal{B}$ because it is unique and both boundary dispersion and agglomeration are unstable. Once $\phi > \phi_s$, agglomeration becomes stable and no boundary equilibrium exists hereinafter.

Given symmetry of the model, the results for $h_2 = 0$ apply to any $h_i = 0$, for $i = \{1, 2, 3\}$. This concludes the proof. $\square$