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INFORMATIVE ADVERTISING IN MONOPOLISTICALLY COMPETITIVE MARKETS

ANTHONY CREANE AND AGOSTINO MANDUCHI

ABSTRACT. In their seminal paper, Grossman and Shapiro (1984) find that informative advertising is socially excessive in an oligopoly pure-strategy symmetric equilibrium (*PSSE*). However, their analysis assumed that every consumer receives at least one advertisement. Christou and Vettas (2008) present counter-examples, showing that if this assumption does not hold, the PSSE advertising may, instead, be insufficient. Christou and Vettas (2008) also show by example that quasi-concavity may not hold in Grossman and Shapiro (1984) and that there may be non-existence due to discontinuities from undercutting, although deriving existence conditions (not derived in Grossman and Shapiro (1984)) is infeasible. We revisit the question by modeling firms (like consumers) as a continuum, which mitigates the discontinuity, enables the derivation of intuitive existence conditions for a PSSE, and allows a general analysis including when some consumers receive no advertisements. More importantly, we find that advertising is always socially *insufficient* and entry is also insufficient.

Keywords: informative advertising, product differentiation, monopolistic competition, welfare.
JEL classification: L13, L15, D83

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1. Introduction

Advertising is ubiquitous in print, airwaves and digital media with over a half-trillion dollars being spent on advertising each year (wwHw.statista.com). With advertising, a firm can persuade consumers to buy its product by communicating to them its characteristics and prices. This also allows the firm to distinguish its product from its competitors' and to enter into new markets, including those in which consumers are already aware of its rivals' products.

Given the large expenditures on advertising, its potential informational role and its effect on consumption decisions, determining the welfare effect these expenditures create for society has been a focus of economists, policy makers, and the public. Seminal, classic work by Butters (1977) and Grossman and Shapiro (1984) introduced the modeling to investigate how informative (truthful) advertising benefits society through *demand creation* in the former (consumers learn of the existence of the market) and better *matching* in the latter (consumers learn a product's characteristics relative to rival products).[1] Grossman and Shapiro (1984) model the matching effect by using the Salop (1979) circular-city model: Hotelling-type transportation costs represent society's loss from poor product matches.

While society and firms incur the same cost from advertising, the benefits to society can differ from the individual firm's benefit even with truthful advertising. That is, the market could have excessive or insufficient advertising from the social perspective. First, society's benefit from the demand creation effect tends to be greater than the firm's as a firm usually does not capture the consumer's entire benefit from learning about the product (i.e., consumer surplus).[2] However, the relationship between society's benefit from the matching effect (consumers get products closer to their most preferred) versus the firm's benefit (it steals a sale from a rival whose profit is not factored into the firm's maximization problem –the *business stealing* effect) is ambiguous (Bagwell, 2007).

A key result in Grossman and Shapiro (1984) is that for a given number of firms advertising is socially excessive in a pure strategy symmetric equilibrium (*PSSE*). However, they note the demand creation effect is not present in their model because to solve the model they use an approximation (which they state is "most accurate" when there is a large number of firms) that implies all consumers receive at least one ad. Thus, their analytical results turn only on whether the business stealing effect is greater than the matching effect. Christou and Vettas (2008, Fn. 29), though, find that numerical examples show that the Grossman and Shapiro (1984) result "is not generally correct" if there are only a few firms. They also show through examples that there can be insufficient advertising in a random utility model and note that Tirole (1988) had mooted this possibility and demonstrated it in a Hotelling model.

The main point of Christou and Vettas (2008), though, is that the equilibrium in these models may not exist (Grossman and Shapiro (1984) do not derive existence conditions): "Quasi-concavity of profits may fail, as each firm may prefer to deviate to a high price, targeting consumers who only become informed about its own product."[3] Unfortunately, deriving conditions for the existence of the PSSE in the

[1] However, as these models assume unit demand and covered market, a third beneficial "quantity-demand effect" resulting from the lower prices induced by advertising does not exist.

[2] Shapiro (1980) shows that a monopolist under-advertises for this reason.

[3] Grossman and Shapiro (1984, Fn. 8) by assumption restrict the firm to prices below this price arguing the restriction "amounts to putting an upper bound on v [the consumer's value of her most

random utility model of Christou and Vettas (2008) is not feasible either. The main issue is that profits are discontinuous in price: a firm may choose to deviate from the equilibrium to a lower price that causes a discrete increase in demand.

We return to the question of informative advertising in an oligopoly, but with a different approach to resolve the problems with discontinuity, allow for the demand creation effect, and obtain existence conditions so that the PSSE can be characterized without the use of an approximation. We do this by using the Salop (1979) model but take Grossman and Shapiro (1984) implicit assumption that the number of firms is large to the limit: we consider a continuum of firms just as Butters (1977) implicitly assumes,[4] which mirrors the assumption on consumers.[5] *A priori* the assumption would not appear to change the underlying market mechanisms, and indeed our PSSE price and profits expressions are consistent with Grossman and Shapiro (1984) and have the same comparative statics.[6] With this assumption a price decrease never induces a discrete increase in demand and as a result, we can achieve our goals. Finally, this assumption allows us to focus on the effect entry has on advertising without conflating entry's effect on reducing transportation costs (as Stahl (1994, Fn. 1) noted) because with a continuum entry solely increases the mass of firms.[7]

The first gain from our approach is that we derive two explicit conditions for the existence of a (unique) PSSE. The first, expressed by Equation (16) in Section 4, is that the consumer must receive an advertisement from another firm with probability no smaller than $1/2$ - else, each firm would prefer to charge the monopoly price. This condition effectively requires that marginal advertising cost is low relative to transportation cost. The second condition, expressed by Equation (17), is that the monopoly price cannot be too much greater than the competitive price interacting with the probability that a firm is in a monopoly position. For example, if the competitive price is too close to marginal cost, then the monopoly price can be attractive even if the firm is very unlikely to be in a monopoly position. This interaction can have counter-intuitive implications as *greater* advertising costs may be needed for the PSSE to exist: with too low advertising costs, aggregate advertising is too large, driving the candidate PSSE price too low so that firms would deviate.[8]

Another gain from our approach is that we can have a monopolistically competitive PSSE in which a large fraction of consumers receives no advertisements, while Grossman and Shapiro (1984) results are derived under the assumption that all consumers receive at least one ad.[9] As discussed above, several have noted that if some consumers receive no advertisements, advertising might then be socially

preferred brand]," but they do not explicitly derive the value of v needed for this (which may conflict with their covered-market assumption).

[4]Butters (1977) assumes the number of firms approaches infinity so that the Poisson distribution can be used to approximate the probability that any given consumer receives an ad.

[5]Though we do not study this, in the limit, as transportation costs approach zero, the model approaches the Butters (1977) model.

[6]Not all of the comparative statics in the model of Christou and Vettas (2008) are the same.

[7]In Grossman and Shapiro (1984), entry reduces aggregate transportation costs directly.

[8]The second condition (v cannot be too large) also points to a potential issue with the *covered market* assumption (that v cannot be too small, so that all consumers that receive an advertisement, will choose to buy) made in all of the models. That is, the v that satisfies both could be the empty set, though it is not in our model.

[9]While Grossman and Shapiro (1984) do provide examples that have a small number of firms (n), the probability that every consumer receives at least one ad is roughly 1 in these examples. E.g., for the benchmark case, the probability is 0.9999. The lowest probability in any of the examples is 0.9945.

insufficient. We prove this in our setting. However, we find an even stronger result: that advertising is socially insufficient even if nearly all consumers receive an ad, that is, *advertising is always socially insufficient* (Proposition 5).

The intuition for our result is as follows. When aggregate advertising is low, and each customer is unlikely to receive multiple advertisements, there is a substantial demand creation effect from more advertising. In this case, as the firm does not capture all the surplus, its private return is less than the social return, a well-known effect. As aggregate advertising increases, it becomes more likely that a consumer that the firm reaches also receives other advertisements. The return to society becomes smaller as it is more about a better match (saving in transportation costs), than demand creation, but at the same time the PSSE price decreases so the firm's return is also smaller. In addition, with incomplete information, there are two effects that reduce the firm's return. First, "competition is no longer localized" (Grossman and Shapiro, 1984, p. 76); the firm potentially competes for the consumers the furthest away from it. In contrast, with complete information the firm only competes for consumers that are immediate neighbors: with incomplete information the price induces the furthest consumer to buy while with complete information the firm is not concerned with the furthest consumer. Second, as aggregate advertising increases, there is a greater chance the firm is competing with a rival who is an arbitrarily close substitute to consumers. These forces push down the price, reducing the firm's return. As a result of these effects, the social return remains greater than the firm's and so there still is socially too little advertising. In the following Subsection 1.1 we go into more detail as to how our results relate to previous results in the literature.

We also consider how social and private returns differ as the product becomes more differentiated. Christou and Vettas (2008) provide examples with a fixed number of firms in which advertising can be either insufficient or excessive from a social point of view, depending on whether product differentiation is weak or strong. In our model, we show that as product differentiation increases, the private value increases relative to the social value because the PSSE price increases.

We then endogenize entry and again find that our comparative statics are in line with Grossman and Shapiro (1984).[10] In particular, we find, as they do, that an increase in the cost of advertising could increase profits when the number of firms is fixed, and so induce entry. We show that in particular, this can occur when there is a proportional increase in the marginal cost of advertising. We present an example demonstrating that with quadratic cost of advertising, an increase in the coefficient of marginal cost increases profits for a fixed number of firms. The intuition is that advertising is a prisoner's dilemma (the cooperative solution has less advertising), so increasing sharply the cost of the last units of advertising can increase profits by inducing the cooperative goal of less advertising.

Finally, we show that socially there is too little entry: the planner would have more firms enter the market, while in Grossman and Shapiro (1984) the planner would reduce entry. This result occurs despite our model not having the social benefit of entry reducing transportation costs as it does in Grossman and Shapiro (1984). The reason for the difference partly turns on the earlier discussion regarding localized competition, but also on that inherently there is excessive entry in the Salop model. In contrast with a continuum, Dixit and Stiglitz (1977) find there can be too much or too little entry. Intuitively, since our model does not have the excess entry effect

[10]Because of intractability, the entry PSSE cannot be characterized in Christou and Vettas (2008).

nor a direct effect on transportation costs, entry is solely a question of its effect on advertising (which has an indirect effect on transportation costs). As we had already seen that for a fixed number of firms there is too little advertising, then that there is too little entry is no longer surprising. In some sense, the planner's problem is to choose the level of aggregate advertising while minimizing cost through the choice of entry and advertising levels.

1.1. Related Literature on Advertising and Welfare.

As noted above, our finding that advertising is always is socially *insufficient* in our model contrasts with Grossman and Shapiro (1984) finding that advertising is always socially *excessive* in their model. However, the differences are not quite so stark as Christou and Vettas (2008, Fn. 29) found examples in which there is insufficient advertising in the Grossman and Shapiro (1984) model. Second, the assumption that all consumers receive at least one ad in Grossman and Shapiro (1984) eliminates the demand creation effect (an effect that leads to socially insufficient advertising). Third, this assumption also weakens the welfare gain from the matching effect, as the assumption means that consumers are very likely to receive advertisements from at least *two* firms, etc.[11] Since the consumer is likely to receive multiple advertisements, the welfare gain from another advertisement is reduced. Finally, with the continuum of sellers here there is always the potential for a competing seller to be arbitrarily close, while in Grossman and Shapiro (1984) there is always a gap between sellers (as their model does not hold as the number of firms goes to infinity) giving the firm there more market power and so higher prices.

In contrast, socially insufficient advertising is consistent with models of advertising in markets with a homogeneous product, where equilibrium generally features price dispersion (Butters (1977), Stahl (1994), Stegeman (1991)). Our model can be viewed as extending this general finding to when products are differentiated. However, to account for the dominance of the incomplete surplus appropriation effect typically observed in those models, Renault (2015) notes that if the price distribution features no mass points, then the offers of the sellers charging the largest price in the support of the distribution are only accepted by the consumers who receive no other offers; hence, the business stealing effect is completely absent, and only the surplus appropriation affects the advertising decisions of these sellers. Moreover, the social value of any advertisement delivered (weakly) increases as the asking price drops, whereas the private return for the sellers is identical for all prices in the support of the distribution. Hence, equality between marginal revenue and marginal advertising cost entails under-provision of advertising, in equilibrium - except in the case of inelastic demand, considered for example in Butters (1977). In our setting with a continuous product space and a uniform distribution of sellers charging identical prices, the critical case above of offers only accepted by consumers who receive no other offers does not occur. Further, consumers always choose the products whose consumption generates the greatest surplus among those for which they receive advertisements. Thus, for any given level of the advertising chosen by the sellers, the number and the quality of the matches created in equilibrium are socially optimal, and incomplete surplus appropriation leads to the under-provision of advertising, given the elastic demand functions faced by the sellers.[12]

[11] Even in their simulations, the probability of receiving advertisements from *at least four* firms is high. For example, in the benchmark case, it is 0.96 with $n = 14$.

[12] We are thankful to Régis Renault for pointing out the similarity between the two scenarios to us.

1.2. **Plan of the Paper.** The remaining part of the paper is organized as follows. Section 2 presents the model, and Section 3 characterizes the conditions guaranteeing surplus maximization. Sections 4, 5 and 6 characterize the sellers' pricing, advertising, and entry PSSE decisions. Section 7 presents an example that highlights some important points, with particular reference to the case of free entry, and Section 8 contains some concluding remarks. The proofs of the main results are in Appendix A; the results related to the model's comparative statics are in Appendix B, along with the respective proofs.

2. The Model

We model preferences and production as in the standard Grossman and Shapiro (1984); Salop (1979) setting, with a unit continuum of buyers, endowed with a preference parameter that is uniformly distributed around a circle with unit circumference. Each buyer demands at most one unit of one of the products. The distance between any two arbitrary addresses x and s in $[0, 1)$ is defined as

$$d(s, x) = \min\{|s - x|, 1 - |s - x|\}, \tag{1}$$

where $|.|$ is the absolute value operator. A buyer with preference parameter (or "address") x who purchases and consumes the product of seller s at the price p_s receives utility

$$u(x, s) = v - t\, d(s, x) - p_s,$$

where $v \in \mathbb{R}_{++}$ and $t \in \mathbb{R}_{++}$ respectively express the payoff from consumption of the "ideal" product and the pace at which the payoff decreases as $d(s, x)$ increases. To dispense with tedious qualifications related to possible corner solutions, we follow Grossman and Shapiro (1984)[13] and make the standard "covered market" assumption, expressed, in this case, by $t < v$. Under this assumption, each buyer would be willing to pay the monopoly price $v - \frac{t}{2}$ even for her least preferred product.

There is also a set Σ of potential, ex-ante identical sellers. By default, all sellers are inactive. Each seller who decides to be active chooses an arbitrary address in the circle and produce as many units of the product as she wishes, upon demand, at the constant unit cost of 0. The buyers who do not purchase a product and the sellers who remain inactive realize a payoff equal to 0. The size of the set of the sellers who choose to be active is denoted by m.

The buyers may in principle know about the existence of the sellers and the products, but they are only able to purchase the products - if any - for which they receive advertisements from the respective sellers, The advertisements truthfully inform the buyers about the characteristics and price of the products, and the number of advertisements delivered by each seller is drawn from a Poisson distribution with parameter $\alpha \in \mathbb{R}_+$, whose value is chosen by the seller and coincides with the expected number of advertisements delivered.

In connection with the delivery process, we can envisage our circle as a projection of a vertical cylinder with unit height, with the buyers uniformly distributed on its side surface and each vertical line identifying buyers with identical preferences. The identities of the buyers who receive the advertisements are then randomly drawn from the uniform distribution on the cylinder's side-surface. The draws are thus independent, both for each seller and across sellers; the addresses of the potential trading partners faced by each seller are uniformly distributed around the circle, and

[13] See Grossman and Shapiro (1984, Fn. 9).

the probability of multiple advertisements delivered by a seller to any given buyer is equal to 0.

The advertising costs faced by each seller is linked to the level variable α by the function $c : \mathbb{R}_+ \to \mathbb{R}_+$, which is at least twice differentiable and strictly convex over $(0, \infty)$. The advertising technology features a fixed cost F - effectively an entry cost for the firms - and we identify the choice of $\alpha = 0$ with the decision to remain inactive, namely we assume that each seller will only bear the fixed cost if her advertising level is positive and set $c(0) = 0$. Because convexity is not defined over the whole domain of c, $\mathbb{R}_+$, a given advertising level α' could be implemented at a lower cost by a single seller, rather than by two sellers, and $c\left(\alpha'\right) < 2\,c\left(\frac{\alpha'}{2}\right)$ can hold if α' is sufficiently small.

$$F \equiv \lim_{\alpha\to 0}\{c(\alpha)\} > 0.$$

The first and the second derivative of $c(\alpha)$ are denoted by $c'(\alpha)$ and by $c''(\alpha)$; we conveniently assume that

$$\lim_{\alpha\to 0}\left\{c'(\alpha)\right\} = 0, \tag{2}$$

so that a suitably small, positive advertising level is always optimal, both socially and privately, if we disregard the fixed cost, and

$$\lim_{\alpha\to \infty}\left\{c'(\alpha)\right\} = \infty.$$

Our assumptions imply

$$\lim_{\alpha\to 0}\left\{\frac{c(\alpha)}{\alpha}\right\} = \lim_{\alpha\to \infty}\left\{\frac{c(\alpha)}{\alpha}\right\} = \infty, \tag{3}$$

and thereby guarantee existence of a unique positive advertising level $\underline{\alpha}$, featuring equality between the average cost

$$\underline{c} \equiv \frac{c\left(\underline{\alpha}\right)}{\underline{\alpha}} \tag{4}$$

and the marginal cost corresponding to it, which minimizes the cost per unit of advertising.[14]

Occasionally, we write the cost function as

$$c(\alpha) = \overline{c}(\alpha) + F,$$

[14]Existence of $\underline{\alpha}$ follows from continuity of $\frac{c(\alpha)}{\alpha}$ over $\mathbb{R}_{++}$ and (3). As to uniqueness, if (4) holds both if $\underline{\alpha} = \alpha'$ and if $\underline{\alpha} = \alpha''$, then strict convexity of $c(\alpha)$ over $(0, \infty)$ implies that for any $\lambda \in (0, 1)$ we must have

$$\begin{aligned} c\left(\lambda\,\alpha' + (1-\lambda)\,\alpha''\right) &\leq \lambda\, c\left(\alpha'\right) + (1-\lambda)\, c\left(\alpha''\right) \\ &= \lambda\,\alpha'\,\underline{c} + (1-\lambda)\,\alpha''\,\underline{c} \\ &= \left(\lambda\,\alpha' + (1-\lambda)\,\alpha''\right)\,\underline{c}. \end{aligned}$$

If $\alpha' \neq \alpha''$, the inequality would be strict, and the choice of any advertising level $\lambda\,\alpha' + (1-\lambda)\,\alpha''$ would be associated with an average cost lower than the minimum cost $\underline{c}$. Hence, α' must be equal to α''.

Convexity of $c(\alpha)$ over $\mathbb{R}_{++}$ is not sufficient to guarantee existence of a cost-minimizing scale of advertising. For example, a cost-minimizing scale does not exist if $c(\alpha) = b + \alpha - \frac{\alpha}{\alpha+1}$, if b is a real number no greater than 1.

where $\overline{c} : \mathbb{R}_+ \to \mathbb{R}_+$ is the function that expresses the variable cost associated with $c(\alpha_s)$. The advertising technology is assumed to be relatively efficient, in the sense that

$$\left(v - \frac{t}{4}\right) \underline{\alpha} - c\left(\underline{\alpha}\right) > 0. \tag{5}$$

This assumption guarantees that a set of active sellers with positive measure choosing the most efficient level of advertising can generate a surplus, net of the advertising costs. Given (4) and convexity of $c(\alpha)$, (5) guarantees that

$$\left(v - \frac{t}{4}\right) \alpha - c'(\alpha) > 0 \tag{6}$$

holds for values of α in a right-neighborhood of 0. Essentially, a positive margin over the variable cost at low scales of production is required to cover the fixed cost.

Trade takes place subject to mutual agreement. The timing is as follows:

(1) The sellers choose whether to be active, or not, and the active sellers choose their advertising levels. Each active seller also chooses her address in $[0, 1)$.
(2) The sellers set their prices.
(3) Consumers receive the sellers' advertisements and decide whether to purchase one of the products for which they received an offer - if any - or to exit the market without purchasing any product.

Decisions made within each stage are simultaneous. Alternatively, advertising levels and prices could be chosen simultaneously (i.e., stages (2) and (3) are merged) with the same outcomes –see the footnote after Proposition 4– but for presentation purposes the stages are assumed sequential.

All agents are risk-neutral and maximize their expected payoffs. Following Grossman and Shapiro (1984) and Christou and Vettas (2008), we focus on pure strategy symmetric Perfect Bayesian Equilibria, which for short we will refer to as *equilibria*. Specifically, we consider equilibria in which the sellers are uniformly distributed around the circle, and all active sellers choose identical advertising intensities and prices. As we show in Proposition 1, in Section 3, the uniform distribution of the sellers is a necessary condition for surplus maximization. Definition 1 provides the operational definition of equilibrium.

Definition 1. *An equilibrium is:*

(1) An acceptance strategy, used by all buyers, such that a buyer with address x who receives offers from a (finite) set of sellers S accepts the offer of an arbitrary seller in the set

$$\begin{aligned} S' \equiv \Big\{ s' \in S \text{ such that } v - t\left|s' - x\right| - p^E \geq 0 \text{ and for each } s \in S, \\ v - t\left|s' - x\right| - p^E \geq v - t\left|s - x\right| - p^E \Big\}, \end{aligned}$$

including those among the sellers in S whose offers yield the largest positive net surplus, and to forgo purchasing a product if S' is empty, namely if she does not receive any offers that would leave her with a positive net surplus.

(2a) In the case of a given mass of active sellers - *A profit maximizing ordered triple $\left(\alpha_s, i_s, p_s\right) \in \mathbb{R}_+ \times [0, 1) \times \mathbb{R}_+$, for each seller $s \in \Sigma$, such that $\alpha_s > 0$ and p_s are identical across sellers, the choices of i_s make the sellers' population uniformly distributed over $[0, 1)$, and each seller realizes a non-negative profit.*

(2b) In the case of free entry - *The same requirements as in (2a), for the active sellers, augmented with the requirement that the mass of the active sellers m makes the maximized expected profit equal to* 0*, and each seller is therefore indifferent between being active or inactive.*

In the analysis of the case of a given number of active sellers, we disregard the fixed advertising cost F; in this scenario, by (2), all firms would choose a strictly positive advertising level. In the case of free entry, we focus on the strategies of the *active* sellers. The identical choices of the advertising level and the price of the active sellers and their uniform distribution allow us to consider a simplified representation of their strategies, namely the ordered pair (α, p), where $\alpha > 0$.

3. Surplus Maximization

We first consider the problem of assigning to each address an advertising level and a set of active sellers to maximize the expected surplus generated by the exchanges. The assignments are expressed by two measurable functions $\rho : [0,1) \to \mathbb{R}_+^{[0,1)}$ and $\mu : [0,1) \to \mathbb{R}_+^{[0,1)}$. We denote by $\phi_\rho(x,y)$ the density function of the event that the closest seller from whom a buyer at x receives an advertisement - the one whose product the buyer should consume, if any - is at a distance y from her; we assume that ϕ_ρ has the extended set $\left[0, \frac{1}{2}\right] \cup \left\{\frac{v}{t}\right\}$ as its support, and has a mass of $1 - \phi_\rho\left(x, \frac{1}{2}\right)$ at $y = \frac{v}{t}$, corresponding to a utility level of 0. By convexity of $c(\alpha)$ and Jensen's inequality, all sellers operating at any address x should choose the same advertising level $\frac{\rho_x}{\mu_x}$, and hypothetical differences between the aggregate intensities at different points should only be matched by a suitable distribution of the sellers. We can then directly write the objective function as

$$W(\rho,\mu) = \int_0^1 \left(\int_0^{\frac{1}{2}} \phi_\rho(x,y)\,(v - t\,y)\,dy - \mu_x\, c\left(\frac{\rho_x}{\mu_x}\right)\right) dx, \tag{7}$$

with the convention that $c\left(\frac{\rho_x}{\mu_x}\right) = 0$ holds if $\mu_x = 0$.

Proposition 1 provides necessary conditions for surplus maximization, along with a refined expression for the total surplus with its and all others' proofs in Appendix A.

Proposition 1. *If the distribution of the sellers over the circle is a choice variable, then surplus maximization is guaranteed if the following conditions are both verified.*

I. For any given mass of active sellers m, (i) the sellers are uniformly distributed around the circle, and (ii) (almost) all sellers choose an identical advertising level.

II. The first- and second-order conditions for maximization of

$$\begin{aligned} w(\alpha,m) &= \int_0^{\frac{1}{2}} 2\,\alpha\, m e^{-2\,\alpha\, m\, x}\,(v - t x)\,dx - m\, c(\alpha) \\ &= \left(1 - e^{-\alpha\, m}\right) v + \frac{\left((\alpha\, m + 1)\, e^{-\alpha\, m} - 1\right) t}{2\,\alpha\, m} - m\, c(\alpha), \end{aligned} \tag{8}$$

w.r.t. the choice variable(s) - α and possibly m - are verified.

Proposition 2 characterizes the solution to the surplus maximization problem if the social planner can choose both m and α, and if the planner only chooses the advertising level for each seller, taking the number of the active sellers as given.[15] In both cases, considering Proposition 1, we focus directly on cases featuring a uniform distribution of the sellers.

Proposition 2.

I. *In the solution of the surplus maximization problem*

$$\max_{(\alpha,m)\in\mathbb{R}_+^2}\{w(\alpha,m)\}, \tag{9}$$

the mass of the active sellers is equal to m_w, defined as the value of m that solves

$$e^{-\underline{\alpha}\, m}\left(v+\frac{t\,\left(e^{\underline{\alpha}\, m}-\underline{\alpha}^2\, m^2-\underline{\alpha}\, m-1\right)}{2\,\underline{\alpha}\, m^2}\right)-c\left(\underline{\alpha}\right)=0, \tag{10}$$

and (almost) all active sellers choose an advertising level equal to $\underline{\alpha}$.

II. *In the solution of the surplus maximization problem*

$$\max_{\alpha\in\mathbb{R}}\{w(\alpha,m)\},$$

for a given mass $m > 0$ of active sellers, almost all sellers choose an advertising level α_m, defined as the value of α that solves

$$e^{-\alpha\, m}\left(vm+\frac{t\,\left(e^{\alpha\, m}-a^2\, m^2-\alpha\, m-1\right)}{2\,\alpha^2\, m}\right)-m\,c'(\alpha)=0. \tag{11}$$

α_m is increasing in v and is either increasing or decreasing in t, depending respectively on whether its value is smaller or greater than the value of α that solves $e^{\alpha\, m}-a^2\, m^2-\alpha\, m-1=0$; this equation is verified if $\alpha\, m\approx 1.79$, which corresponds to a probability of approximately $\frac{1}{6}$ that any given consumer receives no advertisements.

Essentially, if the planner can choose both α and m, α is set at the level that minimizes the unit cost of advertising, and the number of the sellers is chosen based on the first- and second-order conditions; in the case of an optimum conditional on a given value of m, the advertising level of each seller is chosen based on the first- and second-order conditions. In the latter case, the optimal advertising level may be smaller or greater than $\underline{\alpha}$, depending respectively on whether m is greater or smaller than m_w.

The result that α_m increases with v follows directly from the greater expected payoff from each advertisement delivered. As to the response of α_m to changes in t, there are two effects: An increase in t *reduces* the expected value of the exchanges induced by a given advertising level, because the distance between a buyer and a seller is positive with probability 1; however, it also *increases* the gain from a greater advertising level, by potentially allowing each buyer to locate a closer seller. As a result of this tension, the response of the advertising level to changes in the transportation cost t depends on the initial level of aggregate advertising. For very

[15] Grossman and Shapiro (1984) obtain a similar condition: the weighted marginal cost must be equal to the average cost. In the limit as each firm's market share goes to zero - as is always the case here - the two conditions are identical.

low levels, the consumers are unlikely to receive multiple advertisements; hence, advertising largely translates into "demand creation", and welfare is decreasing in transportation cost t. However, as aggregate advertising increases, the chances of a consumer receiving a second advertisement increases and so the better match reduces transportation costs.

The turning point corresponds to a probability of about $\frac{1}{6}$ that any given consumer receives no advertisements (at approximately $\alpha m = 1.79$). The key is whether the marginal advertisement is likely to be demand creating (so a larger value of t means a greater cost and less value from demand creation), or match enhancing (so a larger value of t means greater cost savings from better matches). This turned on whether the probability a consumer received an advertisement was less or greater than $\frac{1}{6}$ (αm is less or greater than approximately 1.79). So, for example, holding m constant, if advertising becomes costlier so that the optimal α decreases lowering optimal aggregate advertising below 1.79, the marginal welfare benefit goes from increasing in t to decreasing in t. Then, an increase in t reduces the optimal advertising level α further.

4. The Pricing Game

In this Section, we establish existence of an equilibrium with symmetric prices of the game played after that all sellers choose the same advertising level $\widehat{\alpha}$. We focus on a "reference seller" with address 0 - with no loss of generality, given symmetry - and assume that the seller's competitors all chose the same price $\widehat{p}$. The optimal pricing strategy is independent of the seller's advertising level α, as far as it is positive (any price would trivially be an optimal price if $\alpha = 0$).

We denote by

$$p_M \equiv v - \frac{t}{2}.$$

the price set by a hypothetical monopolist, under our assumption that v is sufficiently large that every buyer is potentially willing to purchase the seller's product, regardless of her address. We also introduce two further prices to which we repeatedly refer:

$$\begin{aligned} \underline{p} &\equiv \widehat{p} - \frac{t}{2}, \\ \overline{p} &\equiv \widehat{p} + \frac{t}{2}. \end{aligned} \tag{12}$$

If we disregard the possibility of values of $\underline{p}$ and $\overline{p}$ greater than the monopoly price p_M, $\underline{p}$ is the greatest price such that all buyers - except possibly the buyer with address $\frac{1}{2}$, faced with an offer for her ideal product - would prefer the offer of our reference seller to that of any other seller. By contrast, $\overline{p}\left(\leq p_M\right)$ is the lowest price such that an offer would only be possibly accepted by the buyers who receive no other offers - labeled *captive buyers*.

In Lemma 1, we establish that the price chosen by the reference seller must be strictly positive and is also subject to a further, possibly tighter lower bound.

Lemma 1. *The price that maximizes the expected profit of a seller whose competitors choose the advertising level $\widehat{\alpha}$ and the price $\widehat{p}$, is strictly positive and no smaller than $\underline{p}$ in* (12).

The results in Lemma 1 essentially follow from the fact that each buyer is a captive buyer with a positive probability, and from the inelastic demand faced by the seller for prices lower than $\underline{p}$.

It is readily verified that the buyers with address $\beta > \frac{v-p}{t}$, where p is the price of the reference seller, would not be willing to purchase the seller's product, regardless of whether or not they receive other offers. If $p \geq v$, then the set of the buyers who do purchase the seller's product has measure 0, and we can further restrict attention to prices strictly lower than v. Let us then consider any buyer with address $\beta \in \left[0, \frac{v-p}{t}\right]$ - and thus focus on one of the two semi-circles delimited by the address of the reference seller and by its antipodal address, to fix ideas - and set

$$x_{\beta}^{L}\left(p, \widehat{p}\right) = \frac{\widehat{p}-p}{t},$$
$$x_{\beta}^{R}\left(p, \widehat{p}\right) = 2\,\beta - \frac{\widehat{p}-p}{t}.$$

If $p \leq \widehat{p}$, then a buyer would accept an offer from the reference seller if she received no offers from sellers with addresses in the set

$$T_{\beta}^{-}\left(p, \widehat{p}\right) = \begin{cases} \emptyset, & \text{if } \beta < \frac{\widehat{p}-p}{t}, \\ \left(x_{\beta}^{L}\left(p, \widehat{p}\right), x_{\beta}^{R}\left(p, \widehat{p}\right)\right), & \text{if } \beta \geq \frac{\widehat{p}-p}{t}. \end{cases} \tag{13a}$$

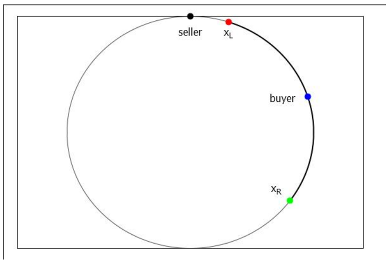
Basically, if the reference seller charges a price lower than that of her competitors, the buyers with higher valuations for her product would choose her offer regardless of what other offers, if any, they receive (first line of (13a)). For values of p greater than $\underline{p}$, some buyers would rather purchase the products of sellers closer to them, the price differential notwithstanding (second line of (13a)); the number of these sellers increases with the buyer's distance from the reference seller. In the case of buyers with address β in $\left(\frac{1}{2}, 1\right)$, the values of the variables corresponding to x_{β}^{L} and x_{β}^{R} are the same as those in (13a), except for the sign. Similar remarks apply in the case of (13b) below. Notice that because the probability of an offer from a seller with any given address is equal to 0, we consider open intervals of addresses with no consequence on the results.

If $p \geq \widehat{p}$, then a buyer with address β would accept the offer of our reference seller if she received no offers from sellers with addresses in the set

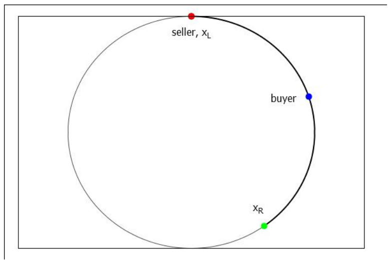
$$T_{\beta}^{+}\left(p, \widehat{p}\right) = \begin{cases} \left(x_{\beta}^{L}\left(p, \widehat{p}\right), x_{\beta}^{R}\left(p, \widehat{p}\right)\right), & \text{if } \beta \leq \frac{1}{2} - \frac{p-\widehat{p}}{t}, \\ [0, 1), & \text{if } \beta > \frac{1}{2} - \frac{p-\widehat{p}}{t}. \end{cases} \tag{13b}$$

The interval in the first line of (13b) is identical to that in the second line of (13a). In the case of the second line of (13b), only the captive buyers would accept an offer from the reference seller, given the distance and the high price of the product.

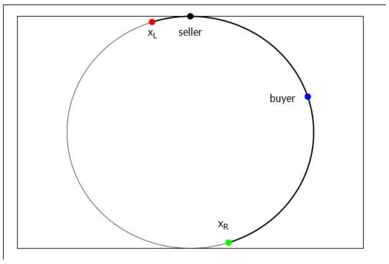
Figures 1a, 1b, and 1c illustrate the different possible scenarios. Unless the interval with the potentially preferred sellers engulfs the whole circle, its boundaries are symmetric around the buyer's address. If $p = \widehat{p}$, then the length of the interval is equal to twice the distance between the buyer and the reference seller. If $p < \widehat{p}$, the interval is shorter, and its length is equal to 0 for buyers suitably close to the seller; conversely, if $p > \widehat{p}$, the interval is generally longer, and fully covers the circle in the case of buyers far away from the seller.



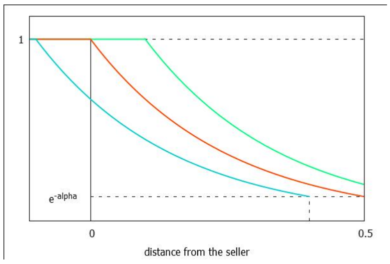
(A) The case of $p < \widehat{p}$.



(B) The case of $p = \widehat{p}$.



(C) The case of $p > \widehat{p}$.



(D) Probability of trade as a function of the distance between the seller and the buyer.

FIGURE 1. Parts 1a, 1b, and 1c illustrate the sets of the addresses of the sellers whose products, offered at $\widehat{p}$, would be preferred to the product of a seller located at the top of the circle. Part 1d illustrates the probability of trade as a function of the distance between the seller and the buyer, in an example with $\alpha = 2$, $m = 1$ and $t = 1$.

The probability q_y that our reference buyer will receive no offers from sellers in an interval $(-y, 0]$ or $[0, y)$, where $y \in \left[0, \frac{1}{2}\right)$, obeys the differential equation

$$\frac{d q_y}{d y} = -\alpha\, m\, q_y.$$

By integrating both sides of the equation and using the boundary condition $q_0 = 1$, we can immediately establish that the complementary event of receiving advertisements from one or more sellers at a distance smaller than y is exponentially distributed, with parameter $\alpha\, m$.

By using (13), we can then write the "twin" demand functions for positive prices satisfying $\widehat{p} < p_M$ and $p \in \left(\underline{p}, \overline{p}\right)$ - so that the reference seller can potentially sell her product even to the antipodal buyers who do not receive better offers - as:

$$\begin{aligned} D^{-}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right) &= 2\alpha \left(\int_0^{\frac{\widehat{p}-p}{t}} d\beta + \int_{\frac{\widehat{p}-p}{t}}^{\frac{1}{2}} e^{-2\widehat{\alpha}\, m \left(\beta - \frac{\widehat{p}-p}{t}\right)} d\beta \right) \\ &= \alpha \left(\frac{2}{t} \left(\widehat{p} - p\right) + \frac{1}{\widehat{\alpha}\, m} \left(1 - e^{\frac{2\widehat{\alpha}\, m}{t}(\widehat{p}-p) - \widehat{\alpha}\, m}\right)\right), \qquad (14a) \end{aligned}$$

if $p \in \left[\overline{p}, \widehat{p}\right]$, and as

$$D^{+}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right) = 2\alpha\left(\int_{0}^{\frac{1}{2}-\frac{p-\widehat{p}}{t}} e^{-2\widehat{\alpha}\, m\left(\beta-\frac{\widehat{p}-p}{t}\right)} d\beta + \int_{\frac{1}{2}-\frac{p-\widehat{p}}{t}}^{\frac{1}{2}} e^{-\widehat{\alpha}\, m} d\beta\right)$$
$$= \alpha\left(\frac{1}{\widehat{\alpha}\, m}\left(e^{\frac{2\widehat{\alpha}\, m}{t}\left(\widehat{p}-p\right)} - e^{-\widehat{\alpha}\, m}\right) + \frac{2e^{-\widehat{\alpha}\, m}}{t}\left(p-\widehat{p}\right)\right). \quad (14b)$$

if $p \in \left[\widehat{p}, \min\left\{\overline{p}, p_M\right\}\right]$. As in the case of the profit equations (15) below, both expressions cover the case of $p = \widehat{p}$; the candidate optimal, symmetric price that we are targeting coincides with $\widehat{p}$.

Figure 1d can help us to interpret the expressions for the probability of acceptance of an advertisement as a function of the distance between the seller and the buyer, used in the integrands of (14a) and in (14b). The blue, the orange and the green curve are referred to the cases of $p > \widehat{p}$, $p = \widehat{p}$ and $p < \widehat{p}$. The expression for the expected profit of our reference seller at an interior equilibrium is then

$$\pi\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right) = \begin{cases} \pi^{-}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right), & \text{if } p \in \left[\overline{p}, \widehat{p}\right), \\ \pi^{+}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right), & \text{if } p \in \left[\widehat{p}, \min\left\{\overline{p}, p_M\right\}\right], \end{cases}$$

where

$$\pi^{-}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right) = p\, D^{-}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right), \quad (15a)$$
$$\pi^{+}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right) = p\, D^{+}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right), \quad (15b)$$

and the problem correspondingly faced by the seller is

$$\max_{p\in\mathbb{R}}\left\{\pi\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right)\right\},$$

where α, $\widehat{\alpha}$, m and $\widehat{p}$ are taken as given.

Lemma 2 establishes that the optimal price set by a seller responding to the choice of $\widehat{p}$ by her competitors is also bounded above by the monopoly price p_M.

Lemma 2. *A seller who was forced to choose a price no smaller than* $\min\left\{\overline{p}, p_M\right\}$, *if all other sellers choose an advertising level* $\widehat{\alpha}$ *and a price* $\widehat{p} < p_M$, *would maximize her profit by choosing the monopoly price* p_M.

Lemma 2 allows us to conclude that a symmetric equilibrium price can be equal either to p_M or to a price strictly lower than p_M. It also justifies both the assumption of prices no greater than the monopoly price, and therefore the procedure used to derive (13), (14) and (15).

We are now ready to characterize the symmetric price equilibrium, in Proposition 3.

Proposition 3. *If*

$$\widehat{\alpha}\, m > \log(2) \quad (16)$$

and $v \leq \mathring{v}\left(\widehat{\alpha}, m, t\right)$, *where*[16]

$$\mathring{v}\left(\widehat{\alpha}, m, t\right) = \frac{t}{2}\left(1 + \frac{e^{\widehat{\alpha}\, m} - 1}{\widehat{\alpha}^2\, m^2}\right), \quad (17)$$

[16] It is straightforward to show there always exist values of v such that $t \leq v \leq \mathring{v}$, i.e., that the covered market assumption plus the upper bound on v does not eliminate the monopolistically competitive equilibrium. The condition simplifies to $\left(\widehat{\alpha}\, m\right)^2 + 1 < e^{\widehat{\alpha}\, m}$, which is always verified.

then the symmetric equilibrium price and profit gross of advertising costs are

$$\mathring{p}\left(\widehat{\alpha}, m, t\right) = \frac{t}{2\,\widehat{\alpha}\,m}, \tag{18}$$

$$\mathring{\pi}\left(\alpha, \widehat{\alpha}, m, t\right) = \frac{\alpha\,t\,\left(1 - e^{-\widehat{\alpha}\,m}\right)}{2\,\widehat{\alpha}^2\,m^2}. \tag{19}$$

Condition (16) is perhaps more easily interpreted as the probability a given consumer receives an advertisement is no smaller than $e^{-2\,\frac{\log(2)}{2}} = \frac{1}{2}$. As in Salop (1979); Grossman and Shapiro (1984), the monopolistically competitive price does not increase with v, albeit the monopoly price does.

The conditions for a monopolistically competitive equilibrium (16) and (17) rule out deviating to the monopoly price as a best response. Condition (16) ensures a sufficiently large probability that any customer receives multiple advertisements, and thereby a sufficiently intense level of competition. Condition (17) guarantees that the largest valuation v and therefore the monopoly price is not too much greater than the equilibrium price (since in the Salop model the equilibrium price does not depend on v). Aggregate advertising αm has two opposite effects on this condition. First, increases in aggregate advertising increase competition thereby decreasing the equilibrium price making a deviation to the monopoly price more attractive; however, it also makes it less likely that a consumer reached by a firm is not reached by any other firm, and thus less likely a captive customer, making the monopoly price less attractive. As it turns out, the largest value of v compatible with an interior equilibrium in (17) is at first decreasing and then increasing in aggregate advertising $\alpha\,m$, depending respectively on whether $\alpha\,m$ is smaller or greater than approximately 1.59.

As one goal of this paper is to extend Grossman and Shapiro (1984) and gain new insights, a critical question is whether or not our assumptions are introducing spurious results via effects that do not exist in Grossman and Shapiro (1984). One check is how the equilibrium prices compare and indeed the price in (18) has the same limiting properties. As the advertising level α grows, and we ideally approach perfect information, the price approaches the competitive price; an increase in the mass of the active sellers produces the same effect. The profit *gross* of advertising costs in (19) shares the same properties.

We close the present Section by recalling that as in previous work, the covered market and unit demand assumptions make the symmetric equilibrium price in (18) consistent with surplus maximization.

5. The Advertising Game

5.1. Equilibrium Existence. We now move back to the stage in which the active sellers, representing a set with mass m, simultaneously choose their advertising intensities, correctly anticipating a monopolistically competitive equilibrium in the pricing stage. Using (19), we can express each seller's problem as

$$\max_{\alpha \in \mathbb{R}_+} \left\{\Pi\left(\alpha, \widehat{\alpha}, m, t\right)\right\},$$

where

$$\Pi\left(\alpha, \widehat{\alpha}, m, t\right) = \mathring{\pi}\left(\alpha, \widehat{\alpha}, m, t\right) - c(\alpha). \tag{20}$$

By convexity of $c(\alpha)$, the first order condition

$$\frac{\partial \Pi\left(\alpha, \widehat{\alpha}, m, t\right)}{\partial \alpha} = \frac{t\left(1 - e^{-\widehat{\alpha}\, m}\right)}{2\left(\widehat{\alpha}\, m\right)^2} - c'(\alpha) = 0. \tag{21}$$

guarantees an interior optimum. Moreover, for any given positive m, (2) and positivity of $\frac{t\left(1-e^{-\widehat{\alpha}\, m}\right)}{2(\widehat{\alpha}\, m)^2}$ for any $\widehat{\alpha} > 0$ guarantee existence of a best response to any choice of $\widehat{\alpha}$ by the competitors. A candidate symmetric equilibrium advertising level is thus a value of $\widehat{\alpha}$ such that[17]

$$H\left(\widehat{\alpha}, m, t\right) = \left.\frac{\partial \Pi\left(\alpha, \widehat{\alpha},\, m,\, t\right)}{\partial \alpha}\right|_{\alpha=\widehat{\alpha}} = 0. \tag{22}$$

Letting the first partial derivative of $H\left(\widehat{\alpha}, m, t\right)$ w.r.t. $\widehat{\alpha}$ be denoted by $H_{\widehat{\alpha}}\left(\widehat{\alpha}, m, t\right)$, we have

Lemma 3. *The partial derivative $H_{\widehat{\alpha}}\left(\widehat{\alpha}, m, t\right)$ is negative.*

Negativity of $H_{\widehat{\alpha}}\left(\widehat{\alpha}, m, t\right)$ guarantees uniqueness of the value of $\widehat{\alpha}$ defined by (22), and thus generally allows us to use the Implicit Function Theorem in our analysis.

Proposition 4. *For any positive value of m, (21) defines a unique value of $\widehat{\alpha}$, denoted by $a(m)$. If $a(m)$ and m, along with t and v, satisfy the conditions in (16) and (17), then there exists a unique symmetric monopolistically competitive pair $\left(a(m), \mathring{p}(a(m), m, t)\right)$, for $\mathring{p}\left(\widehat{\alpha}, m, t\right)$ defined in (18).*[18]

The comparative statics of our model are all consistent with those in Grossman and Shapiro (1984), including the possibility that an increase in advertising costs can *increase* profit, suggesting that the underlying mechanisms at work are the same; we thus refer to Grossman and Shapiro (1984) for a general discussion on the comparative statics and present the derivations in Appendix B. However, the comparative statics allow the existence conditions (16) and (17) to be expressed in terms of the exogenous advertising and transportation costs. First, an increase in transportation cost t increases both the equilibrium price and aggregate advertising, and thereby relaxes the equilibrium conditions (16) and (17).[19] Note that while by itself an increase in aggregate advertising would decrease the equilibrium price, the direct effect of transportation cost t dominates. As a result, $\mathring{v}\left(\widehat{\alpha}, m, t\right)$ in (17) increases monotonically in t. Second, the effect of an increase in advertising costs tightens condition (16) as it reduces aggregate advertising ($a(m)m$). The effect on condition (17) is more complex, reflecting the non-monotonic effect of aggregate advertising. That is, because increases in advertising costs decrease aggregate

[17] This is equivalent to condition (11b) in Grossman and Shapiro (1984), once one substitutes for their price in (11a).

[18] As noted in Section 2, although advertising and then price levels are chosen sequentially, this is also the equilibrium with the levels chosen simultaneously since the expected revenue per unit of advertising level is independent of α. As a result, the cross-partial derivative of the profit evaluated at $\widehat{\alpha}$ and $\mathring{p}$ is zero, and so profits are locally concave in the two variables.

[19] Of course, it eventually binds against the covered market assumption.

advertising, and decreases in aggregate advertising first tighten and then relaxes the second condition (17), so too do increases in advertising costs first tighten and then relaxes the second condition (17). Both points are illustrated by the example in Section 7 below.

5.2. **Welfare Comparison in the Advertising Stage.** Having established that our model is consistent with Grossman and Shapiro (1984), we turn to examine how the monopolistically competitive outcome for a given mass of firms m compares to the social outcome.

We find that advertising is always socially insufficient (recall that with unit demand and the market covered assumption, the price has no welfare effect though in that sense the following proposition is a second-best statement). That is, the welfare gain from better matching always outweighs the capture effect even when the demand creation effect is negligible.

Proposition 5. *For a fixed value of m, the symmetric monopolistically competitive equilibrium has* too little *advertising compared to the social optimum.*

Remark 1. *The proof of Proposition 5 establishes a more general result: for a fixed value of m, the marginal social benefit per firm from increasing advertising is greater than the marginal private benefit of each firm. Thus, e.g., at the socially optimal level of entry, there is too little advertising.*

Given the findings in Grossman and Shapiro (1984) – who find that there is always *excessive* adverting – our result of *insufficient* advertising is initially surprising. As Grossman and Shapiro (1984) make clear, though, their results are for when *every* consumer receives a least one advertisement so there is no demand creation effect there (as there is here). As a firm does not capture the entire surplus from demand creation, having the demand creation effect could be sufficient to reverse the results and have society wanting the firms to advertise more and indeed Christou and Vettas (2008) report that when there is demand creation in Grossman and Shapiro (1984) (i.e., some consumers receive no advertisements), then there can be socially insufficient advertising.

However, our result is for all equilibria including ones in which there is very little demand creation (i.e., each consumer receives at least one advertisement with probability close to one). There are intuitive reasons for this result, since as Grossman and Shapiro (1984) noted, with incomplete information each firm is potentially competing not only for nearby consumers as in the complete information Salop model, but also for far away consumers who would find the firm's product a bad match. This competitive pattern forces down the price and the firms' private return to advertising. In addition, with the continuum of sellers here there is always the potential for a competing seller to be arbitrarily close, while in Grossman and Shapiro (1984) there is always a gap between sellers so the firms here have less market power and so lower prices. Second, the assumption in Grossman and Shapiro (1984) that every consumer receives at least one ad implies that every consumer is very likely to receive ads from two or more firms. As a result, the marginal advertisement does not have a large improvement in match/expected transportation costs and the marginal reduction in transportation costs from another advertisement is smaller in the approximation than the exact expression when their assumption holds (i.e., the number of firms is large or the probability of receiving an advertisement is large). That is, even ignoring the elimination of the demand creation effect, their assumption makes the social return to advertising *smaller* than it is. Finally, the analysis in Grossman and Shapiro (1984) restricts the set of possible deviation prices, *i.e.*, it is assumed that a firm cannot deviate up to the monopoly price or down to the supercompetitive price. As we have shown here, the condition for the former to hold here is not irrelevant and so including it in the analysis of the Grossman and Shapiro (1984) model could affect the conclusions there. Likewise, allowing the firm to deviate to the supercompetitive price in Grossman and Shapiro (1984) could affect their conclusions. For example, in the benchmark case of the numerical analysis in Grossman and Shapiro (1984), it seems that a deviation to the supercompetitive price (which they do not allow) is more profitable than their equilibrium price.

6. The Entry Game

6.1. Equilibrium Existence. In this section we move back to the beginning of the game to determine the measure m of the set of the active firms under free entry. Given our assumption of a large set of potential sellers, equilibrium requires the size of this set to make the expected profit of all sellers - at the optimal advertising level $a(m)$ and net of the advertising costs, if active - equal to 0. Formally, we must have[20]

$$\Pi^*(F, m, t) = \frac{t\,(1 - e^{-a(m)\,m})}{2\,a(m)\,m^2} - \overline{c}(a(m)) - F = 0, \tag{23}$$

where the advertising costs are expressed as the sum of the fixed cost F and the variable cost $\overline{c}\,(a(m))$.

We start the equilibrium analysis with Lemma 4, focusing on the effects of the measure of the set of the active sellers on the level of advertising chosen by each firm in equilibrium. In Lemma 4, as in the case of the other results stated in the present Section, we refer to scenarios featuring a positive mass of active seller m. Notice also that (2) guarantees a strictly positive level of $a(m)$, in any scenario.

Lemma 4. *The equilibrium advertising level $a(m)$ is a strictly decreasing and differentiable function of the size of the set of the active sellers m.*

Lemma 4 is intuitively straightforward: if the incumbents maintained a constant level of advertising in the face of entry of new firms, then each incumbent's marginal benefit from advertising would decrease due to increased competition; the advertising level must thus drop to achieve a new equilibrium.

Lemma 5 states that the expected profit of each firm decreases as the mass of the active sellers increases.

Lemma 5. *Each firm's expected profit is a strictly decreasing and differentiable function of the size of the set of the active sellers m.*

The result in Lemma 5 expresses the balance of two opposite effects. A greater number of sellers have a direct, negative effect on the profit of each firm, via more intense competition; however, it also has an indirect, positive effect via the lower advertising level chosen by each firm, given Lemma 4. As it turns out, strict convexity of $c(\alpha)$ is a sufficient condition for a negative net effect. Lemma 6 states that an increase in m results in an increase in the aggregate level $a(m)\,m$, the decrease in each firm's advertising level notwithstanding.

[20]This is condition (11c) in Grossman and Shapiro (1984).

Lemma 6. *The aggregate advertising level $a(m)\,m$ is strictly increasing and differentiable in m.*

Lemma 7 focuses on the response of the measure of the set of the active sellers to changes in the fixed cost F.

Lemma 7. *The size of the set of the active sellers is a strictly decreasing and differentiable function of the fixed cost F, denoted by $m^*(F)$.*

We can then state our existence result for the case of free entry in Proposition 6.

Proposition 6. *Let $\widehat{\alpha}'$ denote the unique value of the advertising level $\widehat{\alpha}$ that solves*

$$\frac{t}{4\log(2)^2} - c'\left(\widehat{\alpha}\right) = 0. \tag{24}$$

Then if the fixed cost F satisfies $F < F'$, where

$$F' = \frac{\widehat{\alpha}'\,t}{4\,\log(2)^2} - \overline{c}\left(\widehat{\alpha}'\right) > 0, \tag{25}$$

the model with free entry has a unique, symmetric monopolistically competitive equilibrium.

As with exogenous m, the comparative statics with endogenous m are in line with those of Grossman and Shapiro (1984); the general characteristics of the model are in line with theirs. The proofs are in Appendix B. In particular, an increase in costs, specifically a multiplicative increase, may boost profits and thereby boost entry. This possibility is illustrated in the example considered in Section 7.

6.2. **Welfare Comparison in the Entry Stage.** Given the market level of entry and the corresponding equilibrium advertising level, would the planner encourage or discourage entry? Despite entry not having the positive welfare effect of reducing the average distance for consumers, whereas in Grossman and Shapiro (1984) it does have this positive effect, we find that there is always *insufficient* entry, much like there is insufficient advertising for fixed m in Proposition 5.

Proposition 7. *The symmetric monopolistically competitive equilibrium has* too little entry *compared to the social optimum.*

As in the case of Proposition 5, the result is more general: for any fixed α (including the one selected by the planner), the planner would choose a level of entry greater than that corresponding to the market equilibrium. Alternatively, if the firms were forced to advertise at the efficient level, there would be fewer active firms than is socially optimal.

The proof of the Proposition reveals that the underlying mechanism is the same as for Proposition 5. This is because the planner's problem could be viewed as choosing the optimal *aggregate* advertising $\alpha\,m$ in (9) subject to minimizing costs through entry and advertising levels since entry has no direct effect on the average distance consumers incur in buying their product. As the costs are the same to the firms as to society, the question is only if the social benefit of the marginal increase in aggregate advertising is greater or smaller than the private benefit. Since the social benefit was greater when the choice was advertising level for fixed entry, then it is greater when the choice is entry for a fixed advertising level, as that too increases aggregate advertising. Proposition 7 may initially seem surprising since generally in strategic models there is excessive rather than insufficient entry (Mankiw and

Whinston, 1986). However, in the Dixit and Stiglitz (1977) model there could be too little entry depending on the consumers' preferences.

7. An Example

To provide some further understanding of the results, we investigate an example in which the advertising costs, at the firm level, is expressed by the function

$$c^{\chi}(\alpha) = \frac{\chi\,\alpha^2}{4} + F,$$

for given parameters χ and F in $\mathbb{R}_{++}$. In a symmetric equilibrium, each firm chooses its advertising level α to maximize its profit

$$\Pi^{\chi}\left(\alpha, \widehat{\alpha}, m, t\right) = \frac{\alpha\, t}{2\widehat{\alpha}^2 m^2}\left(1 - e^{-\widehat{\alpha}\, m}\right) - \frac{\chi\,\alpha^2}{4} - F.$$

Convexity of the cost function guarantees that the first order condition for an optimum in (21), which reads

$$\begin{aligned}\frac{\partial \Pi^{\chi}\left(\alpha, \widehat{\alpha}, m, t\right)}{\partial \alpha} &= \frac{t\,\left(1 - e^{-\widehat{\alpha}\, m}\right)}{2\widehat{\alpha}^2 m^2} - \frac{\chi\,\alpha}{2}\\ &= 0,\end{aligned}$$

identifies an optimal choice of α for each firm. Setting $\alpha = \widehat{\alpha}$, we can then obtain the condition for the candidate monopolistically competitive equilibrium value of the advertising level:

$$\frac{t}{2\widehat{\alpha}^2 m^2}\left(1 - e^{-\widehat{\alpha}\, m}\right) - \frac{\chi\,\widehat{\alpha}}{2} = 0. \tag{26}$$

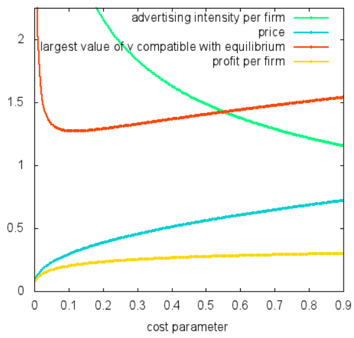
Inspection of the LHS of (26) reveals that for any given value of m, there exists a unique, positive value of $\widehat{\alpha}$ which is potentially compatible with an equilibrium, and which does qualify as an equilibrium if (16) and (17) are verified.

In Figure 2a, we plot the values of the main endogenous variables for values of the cost parameter χ between 0 and 0.9 - which approximately coincides with the largest value of χ for which the aggregate advertising level is no smaller than log(2), as required by (16). Larger values of the cost parameter correspond to a lower advertising level and thus to higher prices, given the greater probability that any buyer reached by a seller will not receive other advertisements. The balance of the effect of the reduced competition and the lower marked coverage is such that the sellers' profits increase with the advertising cost parameter. The greatest value of the preference parameter υ compatible with equilibrium existence, $\mathring{\upsilon}$, is only affected by changes in the cost parameter via the equilibrium advertising level, and its response follows the pattern indicated in the discussion of (17).

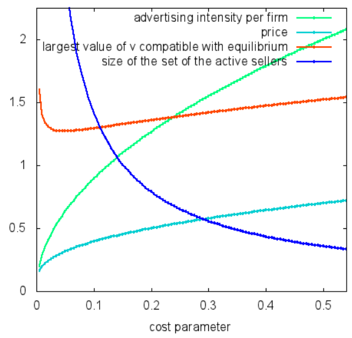
In the case of free entry, under our assumption that the number of potential sellers is sufficiently large that some potential sellers choose not to be active, an equilibrium is a pair $\left(\widehat{\alpha}, m\right)$ that solves the system comprised of (26) and the zero profit-condition

$$\Pi^{\chi}\left(\alpha, \widehat{\alpha}, m, t\right)\Big|_{\alpha=\widehat{\alpha}} = 0,$$

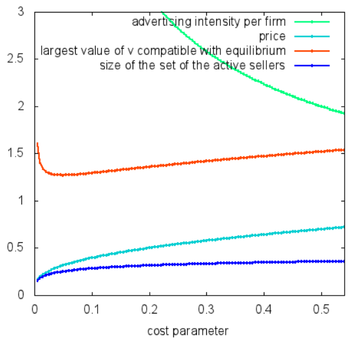
Figures 2d, 2b, and Figure 2c respectively illustrate the responses of the main endogenous variables in cases of changes in the fixed cost F, changes in the variable cost parameter χ and proportional changes in the two parameters. As in the case of Figure 2a, the plot covers a set of values of the parameters such that (16) is verified.



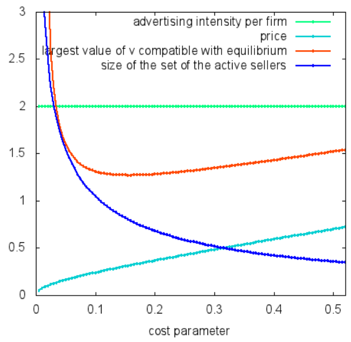
(A) Different values of the (variable) cost parameter with an exogenous mass of active sellers.



(B) Different values of the fixed cost parameter with an endogenous mass of active sellers.



(C) Different values of the variable cost parameter with an endogenous mass of active sellers.



(D) Different values of *both* the fixed and the variable cost parameter with an endogenous mass of active sellers.

FIGURE 2

The non-monotonic response of $\mathring{v}$ persists in the face of changes of both α and m induced by changes in the cost parameters with free entry. If only one of the cost parameters changes, then the changes in α and m take opposite directions. Identical proportional increases of both parameters do not affect the advertising level, in the specific case of a quadratic cost function, and reduce the size of the set of the active firms.

8. Conclusion

We have introduced a tractable model that captures the essence of Grossman and Shapiro (1984), without approximations or the requirement that essentially every consumer receives at least one advertisement. The model also accounts for the possibility of non-existence of the monopolistically competitive equilibrium (Christou and Vettas, 2008) from either deviating up to the monopoly price or deviating down to the supercompetitive price of Salop. Tractability, achieved by modeling both the buyers' *and* the sellers' population as a continuum, allowed us to provide explicit conditions for the monopolistically competitive equilibrium to hold.

In addition to the theoretical contributions, we obtain new insights. In particular, one inference of Grossman and Shapiro (1984) is that when in equilibrium nearly all consumers receive at least one ad, there must be excessive advertising. However, we show that in our model even when nearly all consumers receive at least one ad, there is still socially too little advertising. In fact, there always is insufficient socially insufficient advertising in the pure strategy symmetric monopolistically competitive equilibrium rather than excessive advertising, a result that could be viewed as extending a general finding in models with homogeneous goods (Butters (1977), Stahl (1994), Stegeman (1991)) to heterogeneous goods. The intuition comes from Grossman and Shapiro (1984): with incomplete information competition is no longer localized, as is the case in the full information setting of Salop (1979). That is, here the firm not only competes for consumers with its nearest neighbor (as in Salop) but also for consumers far away, and so may face a rival with a good match for the consumer. This extra competition reduces the equilibrium price. As our model (like its predecessors) assumes unit demand and covered market, it eliminates a potentially positive effect from advertising: more advertising results in lower prices (what Grossman and Shapiro (1984) call the quantity demanded effect). This suggests, that, if anything, the models may be underestimating the extent informative advertising is socially insufficient.

We then endogenize entry, finding that there is socially insufficient entry rather than socially excessive entry as found in Grossman and Shapiro (1984) even though here entry does not have the direct social benefit of reducing transportation costs (i.e., better matches) that exists in Grossman and Shapiro (1984). The reason there is too little entry is, as advertising has convex costs, entry is a second tool to "produce" advertising and so the planner chooses entry and advertising levels to minimize costs for a given level of aggregate advertising. That is, just like the firm does not capture the full social value of the marginal increase in advertising, the firm does not capture the full social value of entry on aggregate advertising. This result is in contrast to the standard result of excessive entry even though *business stealing* –one firm's sale often comes at the expense of another firm– is present. However, in the seminal monopolistic-competitive model Dixit and Stiglitz (1977) find there can be too little or too much entry.

Finally, the tractability of our model allows for new questions to be asked within the framework, which, can therefore, be a valuable tool for further analysis. From this point of view, the framework can allow us to extend the analysis of the paper in future research without encountering the difficulties that are present in Grossman and Shapiro (1984), whereby the main results are derived by using approximations that among other things rule out demand creation. Likewise, although Christou and Vettas (2008) provide many insights, their analysis could not be extended to endogenizing entry nor were explicit conditions derived for the equilibrium.

Appendix A. Proofs

A.1. Proof of Proposition 1.

Part I. We first show that if the assignment ρ were subject to the constraint of a *given* total advertising level $R \in \mathbb{R}_{++}$, written as

$$\int_0^1 \rho(s)\, ds \leq R, \tag{A.1}$$

and we disregarded the advertising costs, then surplus maximization would require an identical advertising level across (almost) all addresses in $[0, 1)$. Let $\Psi_\rho(x, s)$ denote the probability that a buyer with address x will *not* receive an advertisement from any sellers with address in $(x - s, x + s)$, under a generic assignment ρ; if necessary, all distributions considered here can be completed by including a point with positive mass that corresponds to a utility level of 0, as in the case of ϕ_ρ in (7). For each buyer, receiving advertisements from multiple sellers at the same distance s has probability 0, and the dynamics of $\Psi_\rho(x, s)$ is expressed by the differential equation

$$\frac{d\Psi_\rho(x, s)}{ds} = -(\rho(x - s) + \rho(x + s))\, \Psi_\rho(x, s).$$

By integrating both sides of the previous equation and using the initial condition $\Psi_\rho(x, x) = 1$, we obtain

$$\Psi_\rho(x, \xi) = e^{-\int_x^\xi (\rho(x-s)+\rho(x+s))\, ds}. \tag{A.2}$$

Under the given uniform distribution of the buyers, convexity of $\Psi_\rho(x, \xi)$, viewed as a function of the integral in (A.2), implies via Jensen's inequality that the probability of receiving no advertisements from sellers at distances no greater than ξ, averaged across buyers, is minimized if $\Psi_\rho(x, \xi)$ is independent of the buyer's address x. As (A.1) should hold as an equality, this requirement is in turn satisfied if the advertising level is equal to R for (almost) every $s \in [0, 1)$. Setting $\overline{\Psi}_\rho(\xi) = \int_0^1 \Psi_r(x, \xi)\, dx$, and using $\rho = r$ to denote the uniform assignment, we thus have

$$\overline{\Psi}_r(\xi) \leq \overline{\Psi}_\rho(\xi) \tag{A.3}$$

where strict inequality holds in the presence of differences with positive measure between ρ and r over any subinterval of $\left[0, \frac{1}{2}\right)$.

Because receiving and not receiving an advertisement from a seller within any distance y are complementary events, we can set $\phi_\rho(x, s) = -\psi_\rho(x, s)$, where $\psi_\rho(x, s) = \frac{d\Psi_\rho(x,s)}{ds}$ is the density function associated with $\Psi_\rho(x, s)$, and $\overline{\psi}_\rho(s) =$

$\int_0^1 \psi_\rho(x,s)\,dx$, and rewrite the expected total surplus as

$$\begin{aligned}
\int_0^1 \left(\int_0^{\frac{1}{2}} \phi_\rho(x,s)\,(v - t\,s)\,ds \right) dx &= \int_0^{\frac{1}{2}} \left(\int_0^1 \phi_\rho(x,s)\,(v - t\,s)\,dx \right) ds \\
&= \int_0^{\frac{1}{2}} \left(\int_0^1 \left(-\psi_\rho(x,s)\right)\,(v - t\,s)\,dx \right) ds \\
&= \int_0^{\frac{1}{2}} \left(-(v - t\,s) \int_0^1 \psi_\rho(x,s)\,dx \right) ds \\
&= -\int_0^{\frac{1}{2}} (v - t\,s)\,\overline{\psi}_\rho(s)\,ds.
\end{aligned}$$

Integration by parts allows us to further work out the previous expression as follows:

$$\begin{aligned}
-\int_0^{\frac{1}{2}} (v - t\,s)\,\overline{\psi}_\rho(s)\,ds &= -\,(v - t\,s)\,\overline{\Psi}_\rho(s)\Big|_0^{\frac{1}{2}} - t\int_0^{\frac{1}{2}} t\,\overline{\Psi}_\rho(s)\,ds \\
&= v - \left(v - \frac{t}{2}\right)\overline{\Psi}_\rho\left(\frac{1}{2}\right) - t\int_0^{\frac{1}{2}} \overline{\Psi}_\rho(s)\,ds,
\end{aligned}$$

where $\overline{\Psi}_\rho(s) = \int \overline{\psi}_\rho(s)\,ds$. We can then establish that the surplus under the uniform assignment r is no smaller than the surplus under a generic assignment ρ by noting that $\overline{\Psi}_r\left(\frac{1}{2}\right) \le \overline{\Psi}_\rho\left(\frac{1}{2}\right)$ and (A.3) imply

$$\begin{aligned}
&\int_0^1 \left(\int_0^{\frac{1}{2}} \phi_r(x,s)\,(v - t\,s)\,ds \right) dx - \int_0^1 \left(\int_0^{\frac{1}{2}} \phi_r(x,s)\,(v - t\,s)\,ds \right) dx \\
&= \left(v - \frac{t}{2}\right)\left(\overline{\Psi}_\rho\left(\frac{1}{2}\right) - \overline{\Psi}_r\left(\frac{1}{2}\right)\right) + t\left(\int_0^{\frac{1}{2}} \overline{\Psi}_\rho(s)\,ds - \int_0^{\frac{1}{2}} \overline{\Psi}_\rho(s)\,ds\right) \\
&\ge 0.
\end{aligned}$$

To minimize the total advertising costs with a given mass m of active sellers, all sellers should choose the same advertising level. The minimized total cost of any admissible assignment ρ for a given mass m of active sellers is then achieved by setting $\mu_x = \frac{m\,\rho_x}{R}$ for each address $x \in [0,1)$, and is equal to $R\,c\left(\frac{R}{m}\right)$. Optimality of the uniform assignment, therefore, persists even if the surplus levels *net* of the minimized cost are considered.

Part II. As we know from Part I, the cost of implementing the uniform assignment r is minimized if $\mu_x = \frac{m\,R}{R} = m$ holds for every $x \in [0,1)$. Setting $\rho_x = \alpha\,m$ in (A.2), for every $x \in [0,1)$, we can express $1 - \Psi_r^*(x,\xi)$, the cumulative distribution of the distance from the seller with whom any buyer is matched, as an exponential distribution with parameter $2\,\alpha\,m$, which in turn allows us to obtain (8) from (7).

A.2. **Proof of Proposition 2.** As mentioned in the text, we build on Proposition 1 and focus directly on cases in which the sellers are uniformly distributed over $[0,1)$.

Part I. (10) expresses the first order condition for maximization of $w\left(\underline{\alpha}, m\right)$ w.r.t. m if each seller chooses the optimal scale of production $\underline{\alpha}$, which is independent of m. The LHS of (5) is the limit of the LHS of (10) as $m \to 0$, and its positivity guarantees existence of a value of m at which both (10) and the second order-condition for an optimal choice are verified.

Part II. (11) expresses the first order condition for maximization of $w(\alpha, m)$ w.r.t. α for a given mass m of sellers, and (6) guarantees positivity of the limit of its LHS for conveniently small values of α. Continuity of $\frac{\partial w(\alpha,m)}{\partial \alpha}$ allows then us to conclude that there must then exist a positive value of α for which the first and second order conditions for the planner's problem are verified.

A.3. **Proof of Lemma 1.** As each buyer who receives an advertisement is a captive buyer with probability $e^{-\widehat{\alpha}\, m}$, the reference seller can realize an expected profit equal to $\alpha\, e^{-\widehat{\alpha}\, m}\, p' > 0$ by charging a price $p' \in \left(0, p_M\right)$. Both the optimized profit and the price allowing the seller to achieve it must, therefore, be positive as well.

Moreover, if the reference seller set her price equal to $\underline{p} > 0$, then each advertisement delivered would lead to a transaction, and the seller's expected profit would be equal to $\alpha\, \underline{p}$. By contrast, any alternative (positive) price $p'' < \underline{p}$ would yield an expected profit of $\alpha\, p'' < \alpha\, \underline{p}$, and its choice would, therefore, be dominated by the choice of $\underline{p}$.

A.4. **Proof of Lemma 2.** We consider separately the cases of $\overline{p} \le p_M$ and $\overline{p} > p_M$. If $\overline{p} \le p_M$, then a price no lower than $\overline{p}$ can only possibly be accepted by the captive buyers. Hence, the result follows from the fact that p_M is the unique solution to the monopoly pricing problem that the seller would correspondingly face.

If $\overline{p} > p_M$, then the reference seller could out-compete the more distant sellers with a positive probability, albeit the buyers at a distance greater than $\frac{v-p}{t}$ would not purchase her product even if they were captive buyers. The expected demand if $p \in \left(p_M, \overline{p}\right)$ and its first derivative are then

$$\begin{aligned}\overline{D}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right) &= 2\alpha\left(\int_0^{\frac{1}{2}-\frac{p-\widehat{p}}{t}} e^{-2\,\widehat{\alpha}\, m\left(\beta-\frac{\widehat{p}-p}{t}\right)} d\beta + \int_{\frac{1}{2}-\frac{p-\widehat{p}}{t}}^{\frac{v-p}{t}} e^{-\widehat{\alpha}\, m} d\beta\right)\\ &= \frac{\alpha}{\widehat{\alpha}\, m\, t}\left(2\,\widehat{\alpha}\, m\, e^{-\widehat{\alpha}\, m}\left(v-\widehat{p}\right) + t\left(e^{\frac{2\,\widehat{\alpha}\, m}{t}\left(\widehat{p}-p\right)} - e^{-\widehat{\alpha}\, m}\left(\widehat{\alpha}\, m+1\right)\right)\right),\end{aligned} \tag{A.4}$$

$$\frac{\partial \overline{D}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right)}{\partial p} = -\frac{2\,\alpha\, e^{\frac{2\,\widehat{\alpha}\, m}{t}\left(\widehat{p}-p\right)}}{t}.$$

Since the derivative is negative, increases of p above p_M lead to a lower profit, and also, in this case, p_M is an optimal choice in the interval $\left[p_M, \overline{p}\right]$.

A.5. **Proof of Proposition 3.** The first derivatives of the profit equations in (15) are

$$\frac{\partial \pi^-\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right)}{\partial p} = \alpha\left(\frac{2}{t}\left(\widehat{p}-2p\right) + \frac{1}{\widehat{\alpha}\, m} + e^{\frac{2\widehat{\alpha}\, m}{t}\left(\widehat{p}-p\right)-\widehat{\alpha}\, m}\left(\frac{2p}{t} - \frac{1}{\widehat{\alpha}\, m}\right)\right), \tag{A.5}$$

$$\frac{\partial \pi^+\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right)}{\partial p} = \alpha\left(e^{-\widehat{\alpha}\, m}\left(\frac{2}{t}\left(2p-\widehat{p}\right) - \frac{1}{\widehat{\alpha}\, m}\right) + e^{\frac{2\widehat{\alpha}\, m}{t}\left(\widehat{p}-p\right)}\left(\frac{1}{\widehat{\alpha}\, m} - \frac{2p}{t}\right)\right). \tag{A.6}$$

(A.5) and (A.6), evaluated at $p = \widehat{p}$, are respectively the left- and the right-derivative of the seller's profit at the symmetric equilibrium price, and the necessary conditions for $p = \widehat{p}$ to be a best response to the competitors' choice of $\widehat{p}$ are

$$\left.\frac{\partial \pi^{-}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right)}{\partial p}\right|_{p=\widehat{p}} = 0, \tag{A.7}$$

$$\left.\frac{\partial \pi^{+}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right)}{\partial p}\right|_{p=\widehat{p}} = 0. \tag{A.8}$$

Both (A.7) and (A.8) are verified iff $\widehat{p}$ is expressed by $\mathring{p}\left(\widehat{\alpha}, m, t\right)$ in (18). It is also readily verified that $\widehat{\alpha}\, m > \log(2)$ is equivalent to negativity of the second derivative of $\pi^{+}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right)$ at $p = \mathring{p}\left(\widehat{\alpha}, m, t\right)$; $\widehat{\alpha}\, m > \log(2)$ also guarantees a negative second derivative of $\pi^{-}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right)$, and thereby ensures local profit maximization.

As to other possible profit maximizing prices, separately considered, if the seller's competitors set their prices equal to $\mathring{p}$, (A.7) also holds at $p = p^{-}$, where

$$p^{-} = \frac{t}{2\widehat{\alpha}\, m}\left(1 - \widehat{\alpha}\, m - \log(2)\right), \tag{A.9}$$

This price can however be disregarded, as (A.9) and $\widehat{\alpha}\, m > \log(2)$ imply

$$p^{-} < \frac{t}{2\widehat{\alpha}\, m}\left(1 - 2\,\log(2)\right) < 0.$$

(A.8) also admits a further real solution at $p = p^{+}$, where

$$p^{+} = \frac{t}{2\widehat{\alpha}\, m}\left(1 + \widehat{\alpha}\, m - \log(2)\right), \tag{A.10}$$

However, evaluation of the second derivative reveals that if $\widehat{\alpha}\, m > \log(2)$, then $p = p^{+}$ yields a local *minimum* of $\pi^{+}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right)$.

Even if $\widehat{\alpha}\, m > \log(2)$, we must still consider the possibility of non-interior best responses to the competitors' choice of $p = \mathring{p}$ before concluding that $\mathring{p}$ is a symmetric equilibrium price. For prices below $\mathring{p}$, the fact that p^{-} in (A.9) is negative and differentiability of $\pi^{-}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right)$ w.r.t. p imply that $\frac{\partial \pi^{-}\left(\alpha, \widehat{\alpha}, m, p, \widehat{p}, t\right)}{\partial p}$ is negative between 0 and $\mathring{p}$, and therefore rules out any price lower than $\mathring{p}$ as a best response to $\mathring{p}$.

For prices above $\mathring{p}$, it is readily verified that the minimum point p^{+} in (A.10) satisfies $p^{+} \in \left(\mathring{p}, \overline{p}\right)$. If $\overline{p} \leq p_M$, and the profit function is expressed by (15b) over $\left[\mathring{p}, \overline{p}\right]$, then the seller's profit achieves a local maximum over the same interval at $p = \overline{p}$. If $\overline{p} > p_M$, then the profit is expressed by (15b) over $\left[\mathring{p}, p_M\right]$, and by (A.4) over $\left[p_M, \overline{p}\left(\overline{\alpha}, m, t\right)\right]$; if $p_M > p^{+}$, then p_M could in principle dominate $\mathring{p}$, from the seller's point of view. In both scenarios, Lemma 2 guarantees that the choice of p_M dominates the choice of $\overline{p}$, regardless of the ranking of the two prices. The critical value of v in (17) is then obtained by comparing the expected profits realized by charging $\mathring{p}$ and by charging p_M, respectively expressed by $\mathring{\pi}\left(\widehat{\alpha}, m, t\right)$ in (19) and by

$$\widehat{\pi}_M = \alpha e^{-\widehat{\alpha}\, m}\left(v - \frac{t}{2}\right).$$

A.6. **Proof of Lemma 3.** The expression for the given partial derivative is

$$H_{\widehat{\alpha}}\left(\widehat{\alpha},t,m\right)=t\,\frac{(2+\widehat{\alpha}\,m)\,e^{-\widehat{\alpha}\,m}-2}{2\,\widehat{\alpha}^{3}m^{2}}-c''\left(\widehat{\alpha}\right)$$

The numerator of the ratio on the RHS, $(2+\widehat{\alpha}\,m)\,e^{-\widehat{\alpha}\,m}-2$, is decreasing in $\widehat{\alpha}$, and is therefore maximized at $\widehat{\alpha}=0$, where its value is equal to 0. Hence, the first term on the RHS is always non-positive, and strict convexity of $c(\alpha)$ implies that the entire expression is negative for any value of $\widehat{\alpha}$.

A.7. **Proof of Proposition 4.** If $\widehat{\alpha}=0$, then our reference seller would only possibly face captive buyers, and the optimal price and expected profit per buyer contacted, at any level of advertising, would both be equal to $v-\frac{t}{2}>0$. Because the derivative $c'\left(\widehat{\alpha}\right)$ approaches 0 as $\widehat{\alpha}\to 0$, the equilibrium advertising level, if it exists, is necessarily positive. Existence follows then from the fact that $\lim_{\widehat{\alpha}\to 0}\left\{H\left(\widehat{\alpha},t,m\right)\right\}=\infty$ and $\lim_{\widehat{\alpha}\to\infty}\left\{H\left(\widehat{\alpha},m,t\right)\right\}=0$, as revealed by direct calculation, whereas $\lim_{\widehat{\alpha}\to 0}\left\{c'\left(\widehat{\alpha}\right)\right\}=\infty$, and uniqueness follows from negativity of $H_{\widehat{\alpha}}\left(\widehat{\alpha},m,t\right)$, established in Lemma 3.

A.8. **Proof of Proposition 5.** By multiplying both sides of the condition for an equilibrium value in in (22) by m, we obtain

$$\frac{m\,t}{2\,\left(\widehat{\alpha}\,m\right)^{2}}(1-e^{-\widehat{\alpha}\,m})-m\,c'\left(\widehat{\alpha}\right)=0.$$

The marginal effect of increased advertising on welfare is expressed by the LHS of (11). As the private and social cost are identical, whether the social optimum features a greater or a smaller advertising level than the symmetric monopolistically competitive equilibrium, depends on whether the social marginal benefit is greater than the private benefit at $\widehat{\alpha}$, namely on the direction of the inequality

$$e^{-\widehat{\alpha}\,m}\left(m\,v+\frac{t\left(e^{\widehat{\alpha}\,m}-\left(\widehat{\alpha}\,m\right)^{2}-\widehat{\alpha}\,m-1\right)}{2\,\widehat{\alpha}^{2}\,m}\right)-\frac{m\,t}{2\,\left(\widehat{\alpha}\,m\right)^{2}}\left(1-e^{-\widehat{\alpha}\,m}\right)\gtreqless 0\quad\text{(A.11)}$$

However, for any t such that the above expression is negative, there does not exist an equilibrium associated with such a t. First, the LHS of (A.11) is positive at $t=0$ and decreasing in t with a root at $t=\frac{2\,\widehat{\alpha}\,m\,v}{1+\widehat{\alpha}\,m}$. Yet, for any t greater than $\frac{2\,\widehat{\alpha}\,m\,v}{1+\widehat{\alpha}\,m}$ the price associated with such a t would be greater than the monopoly price $v-\frac{t}{2}=v-\frac{1}{2}\frac{2\,\widehat{\alpha}\,m\,v}{1+\widehat{\alpha}\,m}=\frac{v}{1+\widehat{\alpha}\,m}=\frac{t}{2\widehat{\alpha}\,m}$ (and so firms would deviate from that price).

A.9. **Proof of Lemma 4.** By Lemma 3, we have $H_{\widehat{\alpha}}\left(\widehat{\alpha},m,t\right)<0$ for any $m>0$. Hence, $a(m)$ does admit a derivative expressed by $\frac{da(m)}{dm}=-\left.\frac{H_m(\widehat{\alpha},m,t)}{H_{\widehat{\alpha}}(\widehat{\alpha},m,t)}\right|_{\widehat{\alpha}=a(m)}$, where

$$\begin{aligned}H_{m}\left(\widehat{\alpha},m,t\right)&=\frac{\widehat{\alpha}\,t\,e^{-\widehat{\alpha}\,m}}{2\,\left(\widehat{\alpha}\,m\right)^{2}}-\frac{4\,\widehat{\alpha}^{2}\,m\,t}{\left(2\,\left(\widehat{\alpha}\,m\right)^{2}\right)^{2}}\left(1-e^{-\widehat{\alpha}\,m}\right)\\&=t\,\frac{\left(2+\widehat{\alpha}\,m\right)\,e^{-\widehat{\alpha}\,m}-2}{2\,\widehat{\alpha}^{2}\,m^{3}}\\&<0;\end{aligned}$$

the last inequality follows from the fact that $\left(2+\widehat{\alpha}\, m\right)\, e^{-\widehat{\alpha}\, m}-2$ is negative if $\widehat{\alpha}>0$, noted in the proof of Lemma 3. We can then conclude that $\frac{d\,a(m)}{d\,m}<0$ holds for any positive value of m.

A.10. **Proof of Lemma 5.** To make a better use of previous results, it is convenient to focus on the profit expression

$$\widehat{\Pi}\left(\widehat{\alpha},m,t\right)=\frac{t\,\left(1-e^{-\widehat{\alpha}\, m}\right)}{2\,\widehat{\alpha}\, m^2}-c\left(\widehat{\alpha}\right),$$

which is readily obtained from (20), evaluated at $\widehat{\alpha}=a(m)$. The expression for the partial derivative of the profit in (23) w.r.t. the mass of the active sellers can then be written as

$$\begin{aligned}\Pi_m^*(F,m,t)&=\frac{\partial\widehat{\Pi}\left(\widehat{\alpha},m,t\right)}{\partial m}+\frac{\partial\widehat{\Pi}\left(\widehat{\alpha},m,t\right)}{\partial\widehat{\alpha}}\times\frac{d\,a(m)}{d\,m}\\&=-\frac{t\,\left(2-\left(2+\widehat{\alpha}\, m\right)\, e^{-\widehat{\alpha}\, m}\right)}{2\,\widehat{\alpha}\, m^3}\times\frac{t\,\left(1-e^{-\widehat{\alpha}\, m}\right)+2\,\left(\widehat{\alpha}\, m\right)^2\,\left(\widehat{\alpha}\, c''(\alpha)-c'\left(\widehat{\alpha}\right)\right)}{t\,(2-(2+\widehat{\alpha}\, m)\, e^{-\widehat{\alpha}\, m})+2c''(\alpha)\,\widehat{\alpha}^3\, m^2}.\end{aligned}$$

Using the first order condition (21), written as $t\,(1-e^{-\widehat{\alpha}\, m})-2\,(\widehat{\alpha}\, m)^2\, c'(\alpha)=0$, we can then establish the claim by writing

$$\Pi_m^*(F,m,t)=-\frac{t\,\left(2-(2+\widehat{\alpha}\, m)\, e^{-\widehat{\alpha}\, m}\right)\,\widehat{\alpha}^2\, c''\left(\widehat{\alpha}\right)}{m\,\left(t\,\left(2-(2+\widehat{\alpha}\, m\right)\, e^{-\widehat{\alpha}\, m})+2\,\widehat{\alpha}^3\, m^2\, c''\left(\widehat{\alpha}\right)\right)}<0, \tag{A.12}$$

where the inequality is guaranteed by $c''(\alpha)>0$.

A.11. **Proof of Lemma 6.** Direct calculation yields

$$\begin{aligned}\frac{d(a(m)\, m)}{d\,m}&=\frac{d\,a(m)}{d\,m}\, m+a(m)\\&=-\left.\frac{2c''\left(\widehat{\alpha}\right)\,\widehat{\alpha}^4\, m^2}{t\,\left(\left(2+\widehat{\alpha}\, m\right)\, e^{-\widehat{\alpha}\, m}-2\right)-2c''\left(\widehat{\alpha}\right)\,\widehat{\alpha}^3\, m^2}\right|_{\widehat{\alpha}=a(m)}.\end{aligned} \tag{A.13}$$

Given convexity of $c(\alpha)$, the negativity of $\left(2+\widehat{\alpha}\, m\right)\, e^{-\widehat{\alpha}\, m}-2<0$, established in the proof of Lemma 3, allows us to conclude that the denominator of the previous expression is certainly negative, and to thereby establish the result.

A.12. **Proof of Lemma 7.** By using once again the Implicit Function Theorem, we obtain:

$$\frac{d\,m}{d\,F}=-\frac{\Pi_F^*(F,m,t)}{\Pi_m^*(F,m,t)}.$$

The result is then established by using (A.12) and noting that $\Pi_F^*(F,m,t)=-1$.

A.13. **Proof of Proposition 6.** (24) expresses the equilibrium condition of the entry game in (22) for any pair $\left(\widehat{\alpha},m\right)$ such that $\widehat{\alpha}\, m=\log(2)$. Existence of a value of $\widehat{\alpha}$ that solves (24) follows from positivity of $\frac{\widehat{\alpha}'\, t}{4\,\log(2)^2}$ and the fact that the range of $c'(\alpha)$ is assumed to be the whole interval $(0,\infty)$. Moreover, strict convexity of $c(\alpha)$ implies that the solution is unique, and that there is thus a unique pair $\left(\widehat{\alpha},m\right)=\left(\widehat{\alpha}',m'\right)$, with $m'=\frac{\log(2)}{\widehat{\alpha}'}$, that satisfies (24). Notice also that convexity of $c(\alpha)$ and optimality of $\widehat{\alpha}$ imply positivity of the overhead margin for any pair $\left(\widehat{\alpha},m\right)$, and thus also positivity of F' in (25).

To identify the pairs $\left(\widehat{\alpha}, m\right)$ that solve both (16) and (22), if any such pairs exist, we then consider the derivative

$$\left.\frac{d(a(m)\,m)}{dF}\right|_{m=m^*(F)} = \left.\frac{d(a(m)\,m)}{dm}\right|_{m=m^*(F)} \times \frac{dm^*(F)}{dF},$$

whose RHS is the product of a positive and a negative factor, by Lemmas 6 and 7. The sign of $\frac{d((\widehat{\alpha},m,t)\,m)}{dF}$ at $m = m^*(F)$ is therefore negative, and we can conclude that the pairs $\left(\widehat{\alpha}', m'\right)$ that solve (16) and that do therefore qualify as equilibria for the respective values of F, if $v \geq \mathring{v}$ in (17), are those corresponding to values of F smaller than F'.

A.14. **Proof of Proposition 7.** We compare the marginal effect of entry, expressed by the LHS of (10), and the monopolistically competitive free entry equilibrium condition (23), both evaluated at $m = m^*(F)$ and $\widehat{\alpha} = a\left(m^*(F)\right)$. As the private and the social fixed cost are identical, the social marginal benefit is greater than the private benefit at the symmetric monopolistically competitive equilibrium if and only if

$$\left. e^{-\widehat{\alpha}\,m}\left(v + \frac{t\,\left(e^{\widehat{\alpha}\,m} - \widehat{\alpha}^2\,m^2 - \widehat{\alpha}\,m - 1\right)}{2\,\widehat{\alpha}\,m^2}\right) - \frac{t\,\left(1 - e^{-\widehat{\alpha}\,m}\right)}{2\,\widehat{\alpha}\,m^2}\right|_{m=m^*(F),\widehat{\alpha}=a(m^*(F))} > 0.$$

$$e^{-a\left(\widehat{m}\right)m}\left(v + \frac{t\,\left(e^{a\left(\widehat{m}\right)m} - a\left(\widehat{m}\right)^2 m^2 - a\left(\widehat{m}\right)m - 1\right)}{2\,a\left(\widehat{m}\right)m^2}\right) - \frac{t\,\left(1 - e^{-a\left(\widehat{m}\right)m}\right)}{2\,a\left(\widehat{m}\right)\,m^2} > 0. \tag{A.14}$$

As with expression (A.11), expression (A.14) is decreasing in t with a root at $t = v\frac{2,\widehat{\alpha}\,m}{1+\widehat{\alpha}\,m}$. However, as in the proof of Proposition 5, for any t greater than $v\frac{2\,\widehat{\alpha}\,m}{1+\widehat{\alpha}\,m}$ there is no longer a symmetric monopolistically competitive equilibrium as the price required for such an equilibrium would be greater than the monopoly price $v - \frac{t}{2}$ (and so firms would deviate from that price).

Appendix B. Comparative Statics

In B.1 below, we consider the responses of the endogenous variables to exogenous changes in the transportation cost t, in advertising costs, and in the size of the set of the active sellers m. In B.2, m is endogenized. For presentational simplicity, though the equilibrium level of advertising depends on t, etc., it will be subsumed in the notation and only made explicit in the derivatives.

B.1. **Exogenous m.** In this section, m, t and the function $c(\alpha)$ are exogenous. As noted in the main body, the results here are the same as in Grossman and Shapiro (1984). In particular, as Grossman and Shapiro (1984) find, a change in cost has an ambiguous effect on profits. To highlight which type of change in cost can lead to an increase or a decrease in profit, we focus on two specific, standard types of changes. We first consider an *additive* change in marginal cost by considering a new cost function that can be decomposed into two parts: $\tilde{c}(\alpha) \equiv c(\alpha) + \sigma\,\alpha$, where σ is a real number, so $\tilde{c}\,'(\alpha) = c'(\alpha) + \sigma$. One could also interpret this as asking the effect of a small per-unit tax on advertising. An increase in this type of cost always decreases profits. Second, we consider a *multiplicative* change in marginal cost by considering a different cost function that can be decomposed into the form $\check{c}(\alpha) \equiv kc(\alpha) + F$, where k is a real number, so $\check{c}\,'(\alpha) = kc'(\alpha)$. This case includes quadratic cost functions, which we consider in Section 7. By convexity of $c(\alpha)$, the effect of a unit increase in k on the (marginal) cost is *greater* at higher levels of advertising. This could also view this as the effect of a small ad-valorem (based on the marginal cost) tax on each unit of advertising when marginal cost is increasing. An increase in this type of cost increases profits when marginal cost is large relative to total costs. A summary of the results is presented in Table 1:

		Endogenous variables			
		p	α	$\alpha\, m$	π
Exogenous	F	0	0	0	$-$
	t	$+$	$+$	$+$	$+$
variables	σ	$+$	$-$	$-$	$-$
	k	$+$	$-$	$-$	?
	m	$-$	$-$	$+$	$-$

TABLE 1. Comparative statics analysis with exogenous values of m.

B.1.1. *Comparative statics on equilibrium advertising.*

We generally rely on negativity of the partial derivative of the condition for a symmetric equilibrium ($H(\widehat{\alpha}, m, t)$, equation (22)) w.r.t. $\widehat{\alpha}$, $H_{\widehat{\alpha}}\left(\widehat{\alpha}, m, t\right)$, stated in Lemma 3. One useful fact from the proof of Lemma 3 is that $(2 + \widehat{\alpha}\, m)e^{-\widehat{\alpha}\, m} - 2$ is decreasing in $\widehat{\alpha}$, is maximized at $\widehat{\alpha} = 0$ and so is negative. Two comparative statics were already shown: entry (m) decreases advertising levels ($a(m)$), Lemma 4, and increases aggregate advertising ($a(m)m$), Lemma 6. A third is immediate: as entry costs F are fixed, they do not affect advertising levels. Turning to t, we have

Lemma B.1. *With exogenous m, an increase in transportation cost t increases the equilibrium level of advertising:* $\frac{da(m)}{dt} > 0$.

Proof. As $\frac{da(m)}{dt} = -\frac{H_t}{H_{\widehat{\alpha}}}|_{\widehat{\alpha}=a(m)}$, we need to calculate H_t from (22):

$$H_t(\widehat{\alpha}, m, t) = \frac{1 - e^{-\widehat{\alpha}\, m}}{2\,(\widehat{\alpha}\, m)^2} > 0.$$

We then have

$$\frac{da(m)}{dt} = -\frac{H_t}{H_{\widehat{\alpha}}}\bigg|_{\widehat{\alpha}=a(m)} = \frac{-\widehat{\alpha}\,(1 + \widehat{\alpha}\, m)e^{-\widehat{\alpha}\, m})}{t((2 + \widehat{\alpha}\, m)e^{-\widehat{\alpha}\, m} - 2)) - 2\widehat{\alpha}^3 m^2 c''\left(\widehat{\alpha}\right)} > 0. \quad \text{(B.1)}$$

□

We next consider the effect of change in cost on the level of advertising.

Lemma B.2. *An additive or multiplicative increase in marginal cost decreases the equilibrium level of advertising:* $\frac{da(m)}{d\sigma} < 0$ *and* $\frac{da(m)}{dk} < 0$.

Proof. As $\frac{a(m)}{d\sigma} = -\frac{H_\sigma}{H_{\widehat{\alpha}}}|_{\widehat{\alpha}=a(m)}$, we need to calculate H_σ from (22) using $c(\alpha) \equiv \tilde{c}(\alpha) + \sigma\,\alpha$:

$$H_\sigma\left(\widehat{\alpha}, m, t\right) = -1 < 0.$$

We then have

$$\frac{da(m)}{d\sigma} = -\frac{H_\sigma}{H_{\widehat{\alpha}}}\bigg|_{\widehat{\alpha}=a(m)} = \frac{2\,\widehat{\alpha}^3 m^2}{t((2+\widehat{\alpha}\,m)e^{-\widehat{\alpha}\,m}-2)) - 2\widehat{\alpha}^3 m^2 c''\left(\widehat{\alpha}\right)} < 0. \tag{B.2}$$

For when, instead, it is a change in k for cost function $\check{c}(\alpha) \equiv kc(\alpha) + F$, $H_k = -c'(\alpha) < 0$ and the proof follows analogously (or any cost change that increases marginal cost). □

B.1.2. *Comparative statics on price.* The effect of either type of change in marginal cost on the equilibrium price, $\mathring{p}\left(\widehat{\alpha}, m, t\right) = \frac{t}{2\widehat{\alpha}m}$ at $\widehat{\alpha} = a(m)$, is obvious: since by Lemma B.2 advertising is decreasing in marginal cost, then we have

Lemma B.3. *With exogenous m, an additive or multiplicative increase in marginal cost increases the equilibrium price:* $\frac{d\mathring{p}}{d\sigma} > 0$ *and* $\frac{d\mathring{p}}{dk} > 0$.

For the effect of transportation cost t on the equilibrium price, there is an indirect and direct effect. The direct effect raises price, but the indirect effect reduces it: an increase in t causes $\widehat{\alpha}$ to increase, which causes the price to fall. However, the direct effect dominates:

Lemma B.4. *If m is exogenous, an increase in transportation cost t increases the equilibrium price:* $\frac{d\mathring{p}}{dt} > 0$.

Proof.

$$\begin{aligned}\frac{d\mathring{p}}{dt} &= \frac{1}{2\left(\widehat{\alpha}\,m\right)} - \frac{t}{2\,\widehat{\alpha}^2 m}\frac{da(m)}{dt}\\ &= \frac{1}{\widehat{\alpha}\,m}\frac{t(1-(1+\widehat{\alpha}\,m)e^{-\widehat{\alpha}\,m}) + 2c''(\alpha)\widehat{\alpha}^3 m^2}{t(2-(2+\widehat{\alpha}\,m)e^{-\widehat{\alpha}\,m}) + 2c''(\alpha)\widehat{\alpha}^3 m^2}\\ &> 0,\end{aligned}$$

where the inequality follows because $1-(1+\widehat{\alpha}\,m)e^{-\widehat{\alpha}\,m}$ is increasing in either α or m and at zero value for either is zero, i.e., the expression is non-negative. Hence, the numerator and denominator are positive: increased transportation cost t raises the equilibrium price. □

The effect of entry (m) on the equilibrium price, like with transportation cost t, has a direct and indirect effect. By the direct effect, entry reduces the price – there is more competition. However, there is an indirect effect: entry also reduces a firm's advertising level, and lower advertising increases price. However, it was shown in Lemma 6 that aggregate advertising increases with entry, and therefore the direct effect dominates: price decreases.

Lemma B.5. *An increases in the size of the set of the active sellers decreases the equilibrium price:* $\frac{d\mathring{p}}{dm} < 0$.

Proof.

$$\frac{d\mathring{p}}{dm} = \frac{t}{2(a(m)\,m)^2}\frac{-d[a(m)\,m]}{dm} < 0,$$

where $\frac{d(a(m)\,m)}{dm} > 0$ by Lemma 6. □

B.1.3. *Comparative statics on equilibrium profit.* For consistency, consider the profit expression evaluated at $\widehat{\alpha} = a(m)$ from Appendix A:

$$\widehat{\Pi}\left(\widehat{\alpha}, m, t\right) = \frac{t\,\left(1 - e^{-\widehat{\alpha}\,m}\right)}{2\,\widehat{\alpha}\,m^2} - c\left(\widehat{\alpha}\right). \tag{B.3}$$

Changes in entry or transportation cost, etc. affect profits not only directly, but indirectly through the *equilibrium* level of advertising $(a(m))$ of the other firms. Starting with the effect of transportation cost t, differentiating equilibrium profits (B.3) obtains

$$\frac{d\widehat{\Pi}\left(\widehat{\alpha}, m, t\right)}{dt} = \frac{\partial\widehat{\Pi}\left(\widehat{\alpha}, m, t\right)}{\partial t} + \frac{\partial\widehat{\Pi}\left(\widehat{\alpha}, m, t\right)}{\partial\widehat{\alpha}}\frac{d\,a(m)}{dt}. \tag{B.4}$$

In this case, it is easy to see from examining the profit expression (B.3) that the direct effect on the RHS of (B.4) is positive. For the indirect effect, its second term was shown in (B.1) to be positive: transportation cost t increases the equilibrium levels of advertising. However, intuitively the first part of the second term is negative as greater equilibrium advertising reduces the demand from a contacted consumer (as there is a greater possibility they received an advertisement from a better match) and raises cost (this can be easily checked by differentiating (B.3) with respect to $\widehat{\alpha}$). Thus, again the indirect effect runs contrary to the direct effect. However, the expressions simplify, and an increase in transportation cost t increases profits: the direct effect dominates the indirect effect.

Lemma B.6. *For exogenous m, an increase in transportation cost t increases equilibrium profits:* $\frac{d\widehat{\Pi}}{dt} > 0$.

Proof. Using (B.1) for $\frac{da(m)}{dt}$, and differentiating equilibrium profits with respect to $\widehat{\alpha}$ and t, the expression simplifies to

$$\frac{d\widehat{\Pi}\left(\widehat{\alpha}, m, t\right)}{dt} = \frac{(1 - e^{-\widehat{\alpha}\,m})\widehat{\alpha}^2\,c''(\alpha)}{t(2 - (2 + \widehat{\alpha}\,m)e^{-\widehat{\alpha}\,m}) + 2c''(\alpha)\,\widehat{\alpha}^3 m^2} > 0, \tag{B.5}$$

where again the first order condition (21) helps to simplify the numerator of the second term on the RHS of B.4. ☐

We now turn to the effect of the two types of change in the marginal cost of advertising on profits. Considering first an additive change, (σ) and differentiating equilibrium profits obtains

$$\frac{d\widehat{\Pi}\left(\widehat{\alpha}, m, t\right)}{d\sigma} = \frac{\partial\widehat{\Pi}\left(\widehat{\alpha}, m, t\right)}{\partial\sigma} + \frac{\partial\widehat{\Pi}\left(\widehat{\alpha}, m, t\right)}{\partial\widehat{\alpha}}\frac{d\,a(m)}{d\sigma}. \tag{B.6}$$

From inspection the direct effect is negative; a firm is worse off from an additive increase in its marginal cost. For the indirect effect, the second term was shown above to be negative: marginal cost decreases the equilibrium levels of advertising. However, as noted when examining the effect of transportation cost t on equilibrium profits (Lemma B.6), the first part of the second term intuitively is also negative. Once again, the indirect effect runs contrary to the direct effect. However, again the expression simplifies, and it is straightforward to show that an additive increase in marginal cost decreases profits:

Lemma B.7. *For exogenous m, an additive increase in marginal cost of advertising decreases equilibrium profits:* $\frac{d\widehat{\Pi}}{d\sigma} < 0$.

Proof. It is useful to rewrite the equilibrium profit expression as

$$\widehat{\Pi}\left(\widehat{\alpha}, m, t\right) = \frac{t}{2\widehat{\alpha}m^2}(1 - e^{-\widehat{\alpha}(\sigma)m}) - \int_0^{\widehat{\alpha}(\sigma)} (c'(\alpha) + \sigma)d\alpha$$

Then (B.7), the derivative of the equilibrium profit w.r.t. σ, becomes

$$\frac{d\widehat{\Pi}\left(\widehat{\alpha}, m, t\right)}{d\sigma} = -\int_0^{\widehat{\alpha}(\sigma)} d\alpha + \left[\frac{te^{-\widehat{\alpha}\, m}}{2m\widehat{\alpha}} - \frac{t(1 - e^{-\widehat{\alpha}\, m})}{2(\widehat{\alpha}\, m)^2} - c'(\widehat{\alpha}(\sigma))\right]\frac{da(m)}{d\sigma} \tag{B.7}$$

Using (B.2) for $da(m)/d\sigma$ and collecting yields

$$\frac{d\widehat{\Pi}\left(\widehat{\alpha}, m, t\right)}{d\sigma} = -\frac{2\widehat{\alpha}^4 m^2\, c''(\alpha)}{t(2 - (2 + \widehat{\alpha}\, m)e^{-\widehat{\alpha}\, m}) + 2\widehat{\alpha}^3\, m^2\, c''(\alpha)} < 0. \tag{B.8}$$

□

This case of an additive increase in marginal cost is equivalent to the case of $\beta = 1$ in Grossman and Shapiro (1984). Specifically, if their β (the elasticity of the marginal effect of the shift parameter on cost with respect to the proportion of consumers reached) equals one (unit elastic), then an increase in the shift parameter has zero effect on the advertising level (Grossman and Shapiro, 1984, Table 1), which is the case here.[21] Since $c''(\alpha) > 0$ implies $\eta > 0$ in Grossman and Shapiro (1984), we have the same effect as they do: an additive increase in marginal cost reduces equilibrium profits. It should also be clear that a different type of increase in cost could result in equilibrium profits increasing. Indeed, a multiplicative increase in cost (k) could do this. In this case, equilibrium profits are

$$\widehat{\Pi}\left(\widehat{\alpha}, m, t\right) = \frac{t}{2\widehat{\alpha}m^2}(1 - e^{-\widehat{\alpha}m}) - kc(\widehat{\alpha})$$

Differentiating profits with respect to k instead of σ is a slight modification to (B.7) with the same general direct (higher k increases costs) and indirect (higher costs (k) reduces equilibrium advertising which benefits the firm) effects. Specifically,

$$\frac{d\widehat{\Pi}\left(\widehat{\alpha}, m, t\right)}{dk} = -c(\widehat{\alpha}) + \left[\frac{te^{-\widehat{\alpha}\, m}}{2m\widehat{\alpha}} - \frac{t(1 - e^{-\widehat{\alpha}\, m})}{2(\widehat{\alpha}\, m)^2} - kc'(\widehat{\alpha})\right]\frac{da(m)}{dk}$$

This simplifies to

$$\frac{d\widehat{\Pi}\left(\widehat{\alpha}, m, t\right)}{dk} = -c(\widehat{\alpha}) + \frac{t(1 - (1 + \widehat{\alpha}\, m)e^{-\widehat{\alpha}\, m}) + 2\widehat{\alpha}^2 m^2 kc'}{t(2 - (2 + \widehat{\alpha}\, m)e^{-\widehat{\alpha}\, m}) + 2\widehat{\alpha}^3 m^2\, k\, c''(\alpha)}\widehat{\alpha}c', \tag{B.9}$$

and its sign depends on the relative size of the marginal cost to total costs. If the former is sufficiently large, then profits could increase with an increase in k, as was noted by Grossman and Shapiro (1984). In Example 7, we show that with cost of the form $k\alpha^2$ and "large" $\alpha\, m$ in equilibrium, then an increase in k increases equilibrium profits.

[21] Specifically, here an additive increase in marginal cost increase advertising costs in our model by α, and the derivative of that with respect to α is of course 1.

		Endogenous variables								
		Exogenous m				Endogenous m				
		p	$\widehat{\alpha}$	$\widehat{\alpha}\, m$	π	p	$\widehat{\alpha}$	m	$\widehat{\alpha}\, m$
Exogenous	F	0	0	0	$-$	$+$	$+$	$-$	$-$
	t	$+$	$+$	$+$	$+$	$+$	0	$+$	$+$
variables	σ	$+$	$-$	$-$	$-$	$+$	0	$-$	$-$
	k	$+$	$-$	$-$	$?^a$	$+?$	$?^b$	$?^a$	$-?$
	m	$-$	$-$	$-$	$-$				

TABLE 2. Comparative statics analysis with endogenous values of m. Question mark indicates that if marginal cost is relatively large to total cost, then profit increases (a possibility noted by Grossman and Shapiro (1984)); otherwise they decrease. For example, with a quadratic variable cost, an increase in k *increases* profits.
[a]If marginal cost is relatively large to total cost, then positive.
[b]If marginal cost is relatively large to total cost, then negative. For the case of k on p and $\widehat{\alpha}$, even examples with profits increasing have aggregate advertising decreasing (and so price increasing). For changes in costs, if the marginal cost of advertising does not increase uniformly for all levels of α (the k case), then it is possible in which an increase in advertising cost increases profits (as Grossman and Shapiro (1984) note)

.

B.2. **Endogenous m.** We now consider the effect of transportation cost t, advertising costs c, and entry costs F when the size of the set of the active sellers m is endogenous. Table 2 summarizes the effects, which include the results already reported in Table 1 for exogenous values of m.

B.2.1. *Comparative statics on the set of the active sellers.* We begin by determining the effect of transportation and entry costs on the entry equilibrium defined in (23): $\Pi^*(F, m, t)$. Using the Implicit Function Theorem requires that $\Pi^*_m(F, m, t) < 0$, that profits are decreasing in m, which has already been shown in (A.12).

The negative effect of the entry costs F on entry is established in Lemma 7. For the effect of transportation cost t on entry, from (B.5) an increase in transportation cost t increases profit ($\Pi^*_t(F, m, t) > 0$), and so its effect one entry is immediate:

Lemma B.8. *An increase in transportation cost t increases the size of the set of the active sellers:* $\frac{dm}{dt} > 0$.

Proof. From (B.5) we have $\Pi^*_t(F, m, t) > 0$ and from (A.12) we have $\Pi^*_m(F, m, t) < 0$ and so $dm/dt = -\Pi^*_t/\Pi^*_m > 0$, or specifically using the expressions (B.5,A.12),

$$\frac{dm}{dt} = -\frac{\Pi^*_t}{\Pi^*_m} = \frac{(1 - e^{-\widehat{\alpha}\, m})m}{t(2 - (2 + \widehat{\alpha}\, m)e^{-\widehat{\alpha}\, m})} > 0. \tag{B.10}$$

□

Turning to the effect of an additive increase in marginal cost σ on entry, it too follows almost immediately.

Lemma B.9. *An additive increase in marginal cost of advertising decreases the size of the set of the active sellers:* $\frac{dm}{d\sigma} > 0$.

Proof. From expression (B.8) we have $\Pi^*_\sigma < 0$ and from (A.12) we have $\Pi^*_m < 0$ and so $\frac{dm}{d\sigma} = -\frac{\Pi^*_\sigma}{\Pi^*_m} < 0$, or explicitly

$$\frac{dm}{d\sigma} = -\frac{2\widehat{\alpha}^2 m^3}{t\,(2-(2+\widehat{\alpha}\,m)e^{-\widehat{\alpha}\,m})} < 0. \tag{B.11}$$

□

As noted in the preamble this could also be interpreted as the effect of a per-unit tax on advertising, with it decreasing entry.

For the effect of a multiplicative increase in marginal cost, we have already seen from (B.9) that such an increase could increase equilibrium profits if marginal cost is relatively large (i.e., steeply sloped) on the margin. As a result, the effect of a change in k could either increase or decrease entry, and in the example of Section 7 with quadratic cost, increases entry. Lemma B.10 expresses the net result.

Lemma B.10. *A multiplicative increase in marginal cost of advertising (e.g., when* $c(\alpha) = k\alpha^2, dm/dk > 0$*), increases the size of the set of the active sellers if the marginal cost of the last unit is relatively large to total cost.*

B.2.2. *Long run effects on advertising.* With the effects on entry from transportation, marginal advertising cost, and entry cost, we can establish the entry equilibrium effects of these costs (transport, advertising and entry) on advertising. Specifically, we can use the first order condition in equilibrium (22), but now with entry level m as a function of transportation and advertising costs from the free entry condition (23). We begin with the effect of transportation cost t on advertising levels. There is, as usual, two opposing effects. First, the direct effect of increasing transportation cost t for fixed m (positive, established in Lemma B.1) leads to more advertising. However, there is also the indirect effect: increases in t induce entry, which has a negative effect on advertising levels. These two effects cancel out.

Lemma B.11. *With endogenous m, an increase in transportation cost t does not affect a firm's advertising level:* $da(m^*(F))/dt = 0$.

Proof. In Lemma 3 it was established that $H_{\widehat{\alpha}} < 0$. With the calculation of dm/dt (B.10),

$$H_t = \frac{t(e^{-\widehat{\alpha}\,m}(2+\widehat{\alpha}\,m)-2)}{2\widehat{\alpha}^2 m^3}\frac{dm(t,F)}{dt} + \frac{1-e^{-\widehat{\alpha}\,m}}{2\widehat{\alpha}^2 m^2} = 0.$$

Thus,

$$\frac{da(m^*(F))}{dt} = -\frac{H_t}{H_{\widehat{\alpha}}}\bigg|_{m=m^*(F),\widehat{\alpha}=a(m^*(F))} = 0.$$

□

Since increases in transportation cost t do not change any one firm's advertising in the long run, but does induce entry, it follows that

Corollary B.1. *With endogenous m, an increase in transportation cost t increases aggregate advertising.*

The effect of entry costs F is immediate: since entry costs F reduce entry m, but does not affect advertising directly, we have

Lemma B.12. *With endogenous m, an increase in entry costs increases advertising level per firm:* $\frac{da(m^*(F))}{dF} > 0$.

As it has already been shown in Lemma 6 that entry increases aggregate advertising, and since entry costs do not affect advertising levels directly, it follows

Corollary B.2. *With endogenous m, an increase in entry cost decreases aggregate advertising.*

We next turn to the effect of changes in the costs of advertising on advertising in the long run. Beginning with an additive increase in marginal cost for advertising (or equivalents a per-unit tax on advertising), there are two counter effects. On one hand, for a fixed number of firms an increase in marginal cost, reduces equilibrium advertising. However, it also reduces profits which induces exit, which has a positive effect on advertising, so the net effect is unclear. It turns out that the effects exactly offset themselves, and so an additive increase in marginal cost does not affect the long run equilibrium advertising per firm.

Lemma B.13. *With endogenous m, an additive increase in the marginal cost of advertising does not affect a firm's advertising level:* $\frac{da(m^*(F))}{d\sigma} = 0$.

Proof. In Lemma 3, it was established that $H_{\widehat{\alpha}} < 0$. With the calculation of $\frac{dm}{d\sigma}$ in (B.11),

$$H_\sigma = \frac{t(e^{-\widehat{\alpha}\, m}(2 + \widehat{\alpha}\, m) - 2)}{2\widehat{\alpha}^2 m^3}\frac{dm}{d\sigma} - 1 = 0,$$

Thus,

$$\frac{d\widehat{\alpha}}{d\sigma} = -\frac{H_\sigma}{H_{\widehat{\alpha}}}\Big|_{m=m^*(F),\widehat{\alpha}=a(m^*(F))} = 0.$$

□

While the result may initially seem surprising, it too is in line with what Grossman and Shapiro (1984) found, because as discussed after Lemma B.7, $\beta = 1$ in this case. Finally, although the advertising per firm does not change, since the increase in marginal cost induced exit, aggregate advertising has decreased - witness Lemma B.3.

Corollary B.3. *With endogenous m, an increase in a per unit-tax on advertising (an additive increase in marginal cost) decreases aggregate advertising.*

Thus, a per-unit tax on advertising level would have the expected effect of reducing aggregate advertising.

Finally turning to the effect of a multiplicative increase in marginal cost k, not surprisingly given Lemma B.10, such a change in cost could increase or decrease a firm's advertising level in the long run, depending on the relative size of the marginal cost on the margin: if the marginal cost is relatively small, then it is possible a firm's advertising increases. The intuition follows from Lemma B.10: if on the margin, marginal cost is relatively small, this induces exit, which increases the equilibrium level of advertising (Lemma 4). This effect has to be large enough to offset the direct effect from the increase in marginal cost. For aggregate advertising, this becomes even more muddled as exit on its own would decrease aggregate advertising. However, for the quadratic cost example presented in the example in Section 7, for "large" $\widehat{\alpha}\, m$, a multiplicative increase in marginal cost, decreases aggregate advertising. That is, the direct effect on an individual firm in reducing its advertising overwhelms the entry effect on aggregate advertising.

B.2.3. *Long run effects on price.* The long-run effect of different variables on price is primarily through the effect on aggregate advertising $a(m)\,m$, and so the above corollaries on the long-run effects on aggregate advertising determine the effect on price (except for transportation cost t). We begin by considering the effect of entry costs F on the equilibrium price $\mathring{p}$ in (18) at $\widehat{\alpha} = a(m^*(F), m = m^*(F)$. Because entry increases aggregate advertising (which decreases price), increases in fixed cost increase the price.

Lemma B.14. *With endogenous m, an increase in entry cost increases price:* $\frac{d\mathring{p}}{dF} > 0$.

Proof. Differentiation of the expression for the equilibrium price (18) with respect to the fixed cost F yields

$$\left.\frac{d\mathring{p}}{dF}\right|_{\widehat{\alpha}=a(m^*(F),m=m^*(F)} = -\frac{t}{2}\frac{d\,a(m)\,m}{dm}\frac{dm}{dF}\frac{1}{a(m)^2} > 0.$$

The term $\frac{da(m)m)}{dm}$ was established as positive in lemma 6. The term $\frac{dm}{dF}$ was determined to be negative in Lemma 7. Thus, the total effect is for fixed cost to increase the equilibrium price because aggregate advertising decreases. □

We next consider the effect of an additive increase in marginal cost (σ). From Corollary B.3, we know that an additive increase in marginal cost decreases aggregate advertising, and so from (18) it increases the long-run price $\frac{d\mathring{p}}{d\sigma} > 0$.

Corollary B.4. *With endogenous m, an increase in an additive increase in marginal cost (or a small per-unit tax on advertising) increases the price.*

Turning to the effect of a multiplicative increase in marginal cost, as the effect on aggregate advertising is ambiguous, the effect on the long-run price is ambiguous. Again, for the example of Section 7, with "large" $\widehat{\alpha}\,m$, given that aggregate advertising decreases with a multiplicative increase in marginal cost, the long-run price is increasing in this case.

The final comparative static is of transportation cost t on the equilibrium price.

$$\left.\frac{dp}{dt}\right|_{\widehat{\alpha}=a(m^*(F),m=m^*(F)} = \frac{1}{2\widehat{\alpha}\,m} - \frac{t}{2(\widehat{\alpha}\,m)^2}\left(\frac{\partial[\widehat{\alpha}m]}{\partial m}\frac{dm}{dt} + \frac{\partial\widehat{\alpha}}{\partial t}m\right). \tag{B.12}$$

The first term is the direct effect from an increase in transportation cost t on the price and is positive. The second term is the indirect effect from transportation cost t increasing. This induces more entry, which we have seen increases aggregate advertising thereby reducing the price, running counter to the direct effect. There is also the effect transportation cost t has directly on equilibrium advertising levels, but that was shown to be zero in Lemma B.11. The sum of terms proves to be positive.

Lemma B.15. *With endogenous m, an increase in transportation cost t increases the long run price.*

Proof. Substituting A.13 and B.10 into B.12 and collecting the terms we have

$$\left.\frac{dp}{dt}\right|_{\widehat{\alpha}=a(m^*(F)),m=m^*(F)} = \frac{t(e^{-\alpha\, m}(2+\alpha\, m)-2)^2 - 2\,c''(\alpha)\,\widehat{\alpha}^3 m^2 t(e^{-\alpha\, m}(1+\alpha\, m)-1))}{(e^{-\alpha\, m}(2+\alpha\, m)-2)[t(e^{-\alpha\, m}(2+\alpha\, m)-2)-2c\widehat{\alpha}^3 m^2]} > 0.$$

The inequality follows because as shown before both $e^{-\alpha\, m}(1+\alpha\, m)-1$ and $e^{-\alpha\, m}(2+\alpha\, m)-2$ are non-positive. □

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