New Essentials of Economic Theory

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Abstract

This paper develops economic theory tools and framework free from general equilibrium assumptions. We describe macroeconomics as system of economic agents under action risks. Economic and financial variables of agents, their expectations and transactions between agents define macroeconomic variables. Agents variables depend on transactions between agents and transactions are performed under agents expectations. Agents expectations are formed by economic variables, transactions, expectations of other agents, other factors that impact macroeconomic evolution. We use risk ratings of agents as their coordinates on economic space and approximate description of economic and financial variables, transactions and expectations of numerous separate agents by description of variables, transactions and expectations of aggregated agents as density functions on economic space. Motion of separate agents on economic space due to change of agents risk rating induce economic flows of variables, transactions and expectations and we describe their impact on economic evolution. We apply our model equations to description of business cycles, model wave propagation for disturbances of economic variables and transactions, model asset price fluctuations and argue hidden complexities of classical Black-Scholes-Merton option pricing.

Keywords: economic theory, risk ratings, economic space, economic flows, density functions
JEL: C00, C02, C10, E00

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1. Introduction

Economic policy and regulation rely heavily on general equilibrium theory (GE) (Arrow and Debreu, 1954; Tobin, 1969; Arrow, 1974; Smale, 1976; Kydland and Prescott, 1990; Starr, 2011) and DSGE (Fernández-Villaverde, 2010; Komunjer and Ng, 2011; Negro, et al, 2013; Farmer, 2017). Existing flaws and weaknesses of GE and DSGE may bring economic authorities to unjustified decisions and add excess shocks to unsteady global economic and financial processes. Numerous papers study for pro and contra of GE (Hazlitt, 1959; Morgenstern, 1972; Ackerman, 1999; Stiglitz, 2017). A special issue of Oxford Review of Economic Policy on "Rebuilding macroeconomic theory" (Vines and Wills, Eds. 2018a) presents 14 papers of 18 authors those discuss: "What new ideas are needed? What needs to be thrown away? What might a new benchmark model look like? Will there be a 'paradigm shift'?" (Vines and Wills, 2018b).

In this paper we develop economic theory tools, models and equations that entirely differ from mainstream GE. We avoid argue here pro and contra of our approach before we explain main economic assumptions, tools, methods and equations of the model and thus move forward to introduce the model.

The sketch of our approach is based on well-known economic terms and relations. We treat macroeconomics as system of numerous economic agents. Agents have different economic and financial variables and are engaged into various economic and financial transactions with other agents. Agents perform transactions under different expectations. Agents form expectations on base of macroeconomic variables, transactions, expectations of other agents, policy, technology or regulatory changes and so on. We describe economic relations between three core economic notions - variables, transactions and expectations.

This paper has three Parts. In Part I we argue main economic assumptions, introduce economic space notion and describe economic variables and their flows on economic space. In Part II we study economic transactions and expectations on economic space and develop asset pricing model as result of equations on transactions and expectations. In Part III we apply our model equations to description of business cycles, model wave propagation for disturbances of economic variables and transactions, describe asset pricing model and price fluctuations and argue hidden complexities of classical Black-Scholes-Merton option pricing model.

We number equations independently in each Part of the paper and refer (II.4) as equation (4) in Part II. Appendixes A-D present derivation of transactions and expectations as two component functions, derive wave equations for economic variables and transaction and derive business cycle equations. We use bold italic to denote vectors and italic – scalars.

Part I. Economic Assumptions, Space and Variables

2. Economic Assumptions

Let's regard macroeconomics as a system of numerous economic agents. Under different expectations agents perform economic and financial transactions with other agents. Let's mention that our approach has almost nothing common with agent-based models (ABM) (Tesfatsion and Judd, 2005; Gaffard and Napoletano, 2012).

Agents expectations may reflect forecasts of economic growth, demand, expectations of other agents, assumptions on possible economic impact of policy, regulatory or technology changes and etc. Certain macroeconomic variables are determined as sum (without doubling) of corresponding variables of economic agents. For example, macroeconomic demand, supply, investment, credits are determined as sum of demand, supply, investment and credits of economic agents. Let's call such variables as additive. Other macroeconomic variables are determined as ratio of two additive variables and are non-additive. For example prices are determined as ratio of transactions trading values and trading volumes. Inflation, indexes are determined as ratio of prices in different moments of time and are non-additive also. We present these obvious considerations to make simple statement: agents additive variables those define additive macro variables describe all macroeconomic and financial variables.

Now let's argue variables those involved into transactions between agents. Any transaction imply that seller transfer certain volume of commodities, assets, service, investment and etc., to buyer. Let's call agents variables involved into transactions between agents as additive variables of type 1. Let's call other additive variables that are defined by additive variables type 1 as additive variables type 2. For example sum of agents value-added define macroeconomic additive variable – GDP (Fox, et al, 2014). As well agents value-added variables are not subject of any transaction and are determined as difference between agents aggregate sales and expenditures. Thus we call agents value-added as additive variables type 2. Sales and expenditures are result of transactions between agents and their linear functions define agents value-added. These easy examples result second simple statement: all agents variables are determined by additive variables of type 1 those involved into transactions between agents. Hence description of transactions between agents permit model all agents variables and hence model all macroeconomic variables. This statement is well-known at least since Leontief's models (Leontief, 1941; 1955; Horowitz and Planting, 2006). Now let's present three issues that distinguish our approach from common economic treatment:

I. *We use risk ratings of economic agents as their coordinates on economic space.*

II. *We approximate description of economic and financial variables, transactions and expectations of numerous separate agents by aggregate description of variables, transactions and expectations as density functions on economic space.*

III. Motion of separate agents on economic space due to change of agents risk ratings induce economic flows of variables, transactions and expectations and we describe macroeconomic impact of such flows.

Let's discuss these issues in details.

I. Risk ratings of economic agents play role of their coordinates on economic space

Our main issue concern assessments of agents risk ratings. International rating agencies as S&P, Moody's, Fitch (Metz and Cantor, 2007; S&P, 2014; Fitch, 2018) for decades provide risk assessments for major banks, corporations, securities and etc., and deliver distributions of biggest banks by their risk ratings (Moody's, 2018; South and Gurwitz, 2018). These assessments are basis for investment expectations of biggest hedge funds, investors, traders etc. According to current risk assessment methodologies (Altman, 2010; Moody's, 2010; S&P&, 2016; Fitch, 2018) risk ratings take values of risk grades like *AAA, AA, BB, C* etc. Different rating agencies use different risk assessment methodologies and risk grades notions differs slightly.

Let's outline that risk grades *AAA, AA, BB, C* can be treated as points $x_1,\dots x_N$ of space that we call further as economic space. Risk assessment methodology use available economic statistics and determine number N of risk points. Let's propose that economic statistics and econometrics can provide sufficient data to assess risk ratings for all economic agents and for all risks that may hit macroeconomic evolution and growth. Let's assume that rating agencies may be able to estimate risk ratings for all economic agents: for large corporations and banks and for small companies, firms and even households. Now let's assume that risk assessment methodologies can define continuous spectrum of risk grades on space R. Risk methodology always can take continuous risk grades as $[0,1]$ with point 0 as most secure and 1 as most risky grades. A lot of different risks can disturb macroeconomic processes (McNeil, Frey and Embrechts, 2005;). Assessments of single risk, like credit risk, distributes agents over range $[0,1]$ of 1-dimensional economic space R. Assessments of two or three risks, like credit, exchange rate and liquidity for example, distribute economic agents over unit square or cube.

For given configuration of n macroeconomic risks, assessments of agents risk rating distribute agents by their risk coordinates $\boldsymbol{x}=(x_1,...x_n)$ over economic domain

$$0 \le x_i \le 1\,, i = 1, \dots n \tag{1.1}$$

of n-dimensional economic space R^n. Distribution of economic agents by their risk coordinates $\boldsymbol{x}=(x_1,...x_n)$ over economic domain (1.1) mean that all economic and financial variables of agents are also distributed on economic domain (1.1). Aggregation of similar variables for agents with coordinates near point $\boldsymbol{x}=(x_1,...x_n)$ of (1.1) define economic variables as functions of $\boldsymbol{x}$. Aggregations of similar transactions between agents with coordinates $\boldsymbol{x}$ and $\boldsymbol{y}$ determine transactions as functions of $\boldsymbol{x}$ and $\boldsymbol{y}$ on economic space. As we show below this helps describe dynamics of macroeconomic variables, transactions and expectations by partial differential equations on economic space.

Let's repeat our main assumptions:

1. We assume that economic statistics may provide sufficient data for risk assessment of almost all economic agents for wide range of macroeconomic risks. That permits distribute economic agents by their risk ratings as coordinates on economic space.
2. We propose that risk assessment methodologies may define continuous risk grades *[0,1]* on R for all macroeconomic risks. Ratings of n risks define risk coordinates $\boldsymbol{x}=(x_1,...x_n)$ on economic domain (1.1) of n-dimensional economic space R^n.

II. Aggregate description of economic and financial variables, transactions and expectations as density functions

Transition from description of economic properties, like variables, transactions and expectations, of separate agents to same economic properties as density functions on economic space has clear economic meaning. Risk assessment distributes agents by their ratings as coordinates on economic domain (1.1). Description of variables and transactions of numerous separate agents requires a lot of econometric data. We propose approximation that gives more rough description but requires significantly less economic data. To establish such approximation let's aggregate variables, transactions or expectations of agents with risk coordinates inside small volume dV on economic domain (1.1) and then average them. To do that let's chose economic space scale d and time scale Δ. For n-dimensional economic space R^n let's take unit volume $dV=d^n$ near point $\boldsymbol{x}$ of (1.1) and assume that space scales $d<<1$ are small to compare with scales of economic domain (1.1) but many economic agents have risk coordinates inside this unit volume dV near point $\boldsymbol{x}$. The similar requirements concern time scale: Δ should be small to compare with time scale of the problem under consideration but many transactions should be performed during Δ. For example, the number of agents in economics with population around 10^8-10^9 can be estimated as 10^8-10^9. Thus space scale $d\sim10^{-2}$ on 2-dimensional economic space defines unit volume $dV\sim 10^{-4}$ with estimate 10^4-10^5 agents inside it. Time scale $\Delta=1$ *week* is small with time term one quarter or year. Assumption - 1 transaction between agents per second gives assessment of $6*10^5$ transactions per $\Delta=1$ *week.* Thus scales $d\sim10^{-2}$ and $\Delta=1$ *week* may help approximate economic processes for time term one quarter or year. As example let's consider Credits provided by agents with coordinates inside dV near point $\boldsymbol{x}$ and average it during $\Delta=1$ *week.* Let's take that $C(t,\boldsymbol{x})$ equals sum of credits over volume dV and averaged during time Δ. Function $C(t,\boldsymbol{x})$ has meaning of density of credits provided by agents from point $\boldsymbol{x}$ at moment t. Indeed, integral of $C(t,\boldsymbol{x})$ by $d\boldsymbol{x}$ over economic domain equals total credits provided by all economic agents in economics at moment t. Averaging over time Δ reduce high frequency fluctuations of the sum of credits and makes this variable smooth. Introduction of space scale d and time scale Δ reduce accuracy of the model approximation. If one chose space scale $d=1$ then volume dV will be equal economic domain and aggregation of credits provided by agents inside economic domain equals all credits provided in macroeconomics. Thus introduction of scales $d<<1$ establishes economic approximation that is intermediate between precise description of variables of numerous separate economic agents and rough macroeconomic approximation based on aggregation of variables of all economic agents. Below we define density functions for economic and financial variables, transactions and expectations. Nevertheless expectations are not additive variables, we show in Part II how apply aggregation procedure to obtain correct form for density functions of expectations. Description of density functions of economic variables, transactions and expectations require significantly less economic data then same description with accuracy of each agent and hence simplifies the models. The same time descriptions of mutual relations between density functions of economic variables, transactions and expectations are much more informative then modeling relations between macroeconomic variables as functions of time only.

It is obvious that one may aggregate agents and their variables, transactions and expectations on economic domain (1.1) by various economic groups with section by different industry sectors, wealth, gender, age or other economic or financial conditions. Macroeconomic models based on aggregation of agents by various groups on economic domain may model relations between economic variables, transactions and expectations of different industry sectors or describe influence of any specifications those define grouping agents. For such models one may use different sets of risks and different risk measures for different groups of agents. For example risk assessment may differ for different industry sectors, for different wealthy level and etc. It is clear that any specific grouping and usage of different set of risks and risk measures induce additional complexity to the model. In current study we describe simplest framework that use aggregation of all economic agents without any additional specification and use one risk assessment measure for all agents.

The most important factor that impact evolution of density functions of variables, transactions and expectations is determined by aggregative flows of variables, transactions and expectations induced by motion of agents on economic space. Such economic flows are results of motion of agents on economic space due to change of their risk rating.

III. Motion of agents on economic space due to change of their risk ratings induce economic flows of variables, transactions and expectations

Change of agents risk ratings due to their economic activity, variation of economic environment, action of risk factors and other reasons cause change of agents risk coordinates on economic space. Such change means that agents move on economic space with certain speed $\boldsymbol{v}$. Motion of agent with speed $\boldsymbol{v}$ indicates that agents carry their economic and financial variables, expectations and transactions. For example if agent provides credits C and moves with speed $\boldsymbol{v}$ then it carries flow $\boldsymbol{P}_C$ of credits as $\boldsymbol{P}_C=C\boldsymbol{v}$. Flows of variables, expectations and transactions carried by agents due to change of their risk ratings have important impact on macroeconomic evolution. To describe action of these flows on macroeconomics let's develop approximation similar to one we use to describe densities functions of variables, expectations and transactions. As we show below, aggregations of flows of separate agents define densities of economic flows of variables, transactions and expectations. Motion of different flows of variables, expectations and transactions have certain parallels to flows of fluids but all properties of economic flows are completely different from hydrodynamics. Numerous flows of economic and financial variables, expectations and transactions induce on economic domain (1.1) a great variety of mutual interactions and economic effects.

Now let's argue derivation of equations that should govern density functions of variables, transactions and expectations and their flows. These equations have similar form and we explain their derivation for credit density function $C(t,\boldsymbol{x})$ as example. Credit density function $C(t,\boldsymbol{x})$ aggregates credits of agents with coordinates inside small volume dV at point $\boldsymbol{x}$. Each agent moves on economic space with some velocity $\boldsymbol{v}$ due to change of its risk ratings. This motion of agents induces aggregate credit flows $\boldsymbol{P}_C(t,\boldsymbol{x})=C(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})$. Function $\boldsymbol{v}(t,\boldsymbol{x})$ describes velocity of flow of credit density $C(t,\boldsymbol{x})$. To describe change of credit density function $C(t,\boldsymbol{x})$ during time dt in a small volume dV on economic space let's take into account two factors of such change. The first factor describes change of $C(t,\boldsymbol{x})$ due to change of agents credits in time dt in a small volume dV. That can be presented as

$$\int dV \, \frac{\partial}{\partial t} C(t,\boldsymbol{x})$$

The second factor that impact change of credit density $C(t,\boldsymbol{x})$ is determined by credit flows $\boldsymbol{P}_C=C\boldsymbol{v}$ of agents that during time dt may flow in or flow out of small volume dV. Agents that flow in the volume dV with credit flow $\boldsymbol{P}_C=C\boldsymbol{v}$ increase credit density function $C(t,\boldsymbol{x})$ and agents that flow out of the volume dV with credit flow $\boldsymbol{P}_C=C\boldsymbol{v}$ decrease credit density function $C(t,\boldsymbol{x})$. Balance of aggregated $\boldsymbol{P}_C(t,\boldsymbol{x})=C(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})$ credit flows in and credit flows out takes form of integral of credit flows $\boldsymbol{P}_C(t,\boldsymbol{x})=C(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})$ over the surface of small volume dV:

$$\oint ds \, \boldsymbol{P}_{\boldsymbol{C}}(t,\boldsymbol{x}) = \oint ds \, C(t,\boldsymbol{x}) \, \boldsymbol{v}(t,\boldsymbol{x})$$

Due to well-known divergence theorem (Gauss' Theorem) (Strauss 2008, p.179), surface integral of the flows equals volume integral of the flows divergence. Thus balance of credit flows equals integral of the divergence of flow over small volume dV:

$$\oint ds \, C(t,\boldsymbol{x}) \, \boldsymbol{v}(t,\boldsymbol{x}) = \int dV \;\, \nabla \cdot \big(\, C(t,\boldsymbol{x}) \, \boldsymbol{v}(t,\boldsymbol{x}) \big) \qquad (1.2)$$

Hence total change of credit density function during time dt in a small volume dV equals:

$$\int dV \left[\frac{\partial}{\partial t} C(t,\boldsymbol{x}) + \nabla \cdot \big(\, C(t,\boldsymbol{x}) \, \boldsymbol{v}(t,\boldsymbol{x}) \big) \right]$$

As small volume dV is arbitrary one can take equations on density functions as:

$$\frac{\partial}{\partial t} C(t,\boldsymbol{x}) + \nabla \cdot \big(\, C(t,\boldsymbol{x}) \, \boldsymbol{v}(t,\boldsymbol{x}) \big) = F_C(t,\boldsymbol{x}) \qquad (1.3)$$

Function $F_C(t,x)$ in the right side (1.3) describes action of any factors defined by variables, transactions and expectations and their flows on credit density function $C(t,\boldsymbol{x})$. Equation (1.3) depends on flow $\boldsymbol{P}_C(t,\boldsymbol{x})=C(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})$ and hence one should derive equation on this flow. Completely same considerations as above cause equations on flows $\boldsymbol{P}_C(t,\boldsymbol{x})=C(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})$ as:

$$\frac{\partial}{\partial t} \boldsymbol{P}_C(t,\boldsymbol{x}) + \nabla \cdot \big(\, \boldsymbol{P}_C(t,\boldsymbol{x}) \, \boldsymbol{v}(t,\boldsymbol{x}) \big) = \boldsymbol{G}_C(t,\boldsymbol{x}) \qquad (1.4)$$

Function $\boldsymbol{G}_C(t,\boldsymbol{x})$ describes action of any factors defined by variables, transactions and expectations and their flows on credit flows $\boldsymbol{P}_C(t,\boldsymbol{x})$. Let's underline that equations (1.3; 1.4) define "simple" relations for macroeconomic variables as functions of time only. Indeed, integral by $d\boldsymbol{x}$ of credit density $C(t,\boldsymbol{x})$ over economic domain (1.1) equals macroeconomic credits $C(t)$ issued by all agents:

$$C(t) = \int d\boldsymbol{x} \;\; C(t,\boldsymbol{x}) \qquad (1.5)$$

Integral by $\boldsymbol{dx}$ for equations (1.3) over economic domain (1.1) equals

$$\frac{d}{dt}C(t) + \int d\boldsymbol{x}\,\nabla\cdot\big(\,C(t,\boldsymbol{x})\,\boldsymbol{v}(t,\boldsymbol{x})\big) = \int d\boldsymbol{x}\;F_C(t,\boldsymbol{x}) = F_C(t) \tag{1.6}$$

Due to (1.2) integral in left side (1.6) equals zero as no in- or out- flows exist outside of economic domain (1.1) and no economic agents exist outside economic domain (1.1). Thus (1.6) takes simple form of ordinary differential equation:

$$\frac{d}{dt}C(t) = F_C(t) \tag{1.7}$$

The problems of (1.7) are hidden by function $F_C(t)$ determined by integral in the right side of (1.6). Function $F_C(t,\boldsymbol{x})$ may depend on other variables, transactions, expectations and their flows and integral in (1.6) may define $F_C(t)$ as very complicated function. Thus time evolution of macroeconomic variables like macro credits $C(t)$ may depend on properties of hidden dynamics of variables, transactions and expectations and their flows on economic space. Integral by $\boldsymbol{dx}$ for equations (1.4) over economic domain (1.1) define ordinary differential equation on new macroeconomic variables $\boldsymbol{P}_C(t)$:

$$\boldsymbol{P}_C(t) = \int d\boldsymbol{x}\;\boldsymbol{P}_C(t,\boldsymbol{x}) = C(t)\boldsymbol{v}(t) \tag{1.8}$$

$$\frac{d}{dt}\boldsymbol{P}_C(t) = \int d\boldsymbol{x}\;\boldsymbol{G}_C(t,\boldsymbol{x}) = \boldsymbol{G}_C(t) \tag{1.9}$$

Integral (1.8) define macroeconomic flows $\boldsymbol{P}_C(t)$ of credits $C(t)$ (1.5) with velocity $\boldsymbol{v}(t)$ and equation (1.9) describes evolution of macroeconomic credit flows $\boldsymbol{P}_C(t)$ determined by function $\boldsymbol{G}_C(t)$ in the right side of (1.9). Similar equations are valid to macroeconomic flows of other additive variables as demand and supply, investment and GDP and etc. Economic meaning of equations (1.9) is following. Velocity $\boldsymbol{v}(t)$ of macroeconomic flow $\boldsymbol{P}_C(t)$ of credits $C(t)$ describes motion on economic domain (1.1) that is bounded along each risk axes by most secure and most risky grades [0,1]. Thus for each axis motion from secure to risky direction with velocity $\boldsymbol{v}(t)$ should change by opposite motion from risky to secure area of (1.1). Thus velocity $\boldsymbol{v}(t)$ and macroeconomic flow $\boldsymbol{P}_C(t)$ of credits $C(t)$ should fluctuate in time and such fluctuations describe credit cycles of macroeconomics. Similar fluctuations of flows model business cycles of GDP, investment and etc. Description of correlations between cycles of different macro variables and particular models that define forms of functions $F_C(t)$ and $\boldsymbol{G}_C(t)$ should be studied for each economic case. In Part III we present one simple model of business cycles caused by interactions between transactions.

In Part II we show that equations on transactions have form similar to (1.3; 1.4) taking into account that transactions density functions depend on two coordinates $\boldsymbol{x}$ and $\boldsymbol{y}$. In Part II we argue that expectations of agents can't be treated as additive variables and derivation of equations on aggregated expectations requires further considerations. We propose that economic value or economic importance of agents expectations should be taken proportional to value of transactions approved by this particular expectation. In Part II we introduce additive factors that we call – expected transactions – that are proportional to product of transactions and expectations. Our approach permits define density functions of expected transactions and flows of expected transactions. Further we derive equations on expected transactions and their flows that have form similar to (1.3; 1.4). That permits derive definitions and equations for density functions of expectations and their flows. Further in Part II we show that considerations similar to those we use for description of expectations can be applied for description of prices as densities functions on economic space and we derive definitions and equations for price density functions and their flows. That allows model dynamics of asset pricing determined by corresponding transactions. It is well-known that asset pricing is one of the most important problems of economics and finance and papers by (Cochrane and Hansen, 1992; Cochrane and Culp, 2003; Hansen, 2013; Campbell, 2014; Fama, 2014; Cochrane, 2017) refer only few but important studies on asset pricing. These studies argue models that determine "correct" price of assets. In our paper we don't argue "correct" price and don't study why asset price should take certain value. We describe prices as results of transactions performed by agents in economy. In Part II we study different definitions of prices caused by different aggregations of transactions and show how economic equations on transactions, expectations and their flows determine equations on prices caused by transactions.

Let's argue some consequences of our macroeconomic approximations. As we mention above equations similar to (1.3; 1.4) describe density functions and flows of numerous economic and financial variables, transactions and expectations. Thus equations (1.3; 1.4) define macroeconomic approximations for each selected set of variables, transactions and expectations. Let's take a model that describes macroeconomics by set of k different transactions. As such transactions one can study for example credit transactions, investment, buy-sell transactions and etc. Each type k of transactions defines change of variables of sellers and buyers. For example credit transaction change value of credits provided by Creditor (seller) and amount of loans received by Borrowers (buyers). Hence each type of transactions can change only two additive variables of type 1 – one for seller and one for buyer and their prices. Thus k types of transactions can change $2k$ additive variables of type 1 and their prices. Transactions of each type can be performed under different expectations. Let's assume that k types of selected transactions are performed under W expectations. To develop self-consistent macroeconomic model that describe $2k$ additive variables of type 1 determined by k types of selected transactions one should assume that all W expectations are formed by endogenous $2k$ additive variables, k selected transactions and their flows. If part of W expectations can depend on exogenous variables, transactions, expectations and their flows or exogenous shocks and etc., then one describes macroeconomic model in presence of exogenous factors. Expectations approve transactions and thus impact change of economic and financial variables. Hence expectations may transfer impact of exogenous variables, transactions or shocks on macroeconomic evolution, transactions and variables.

Importance of expectations is not reduced by their role as transfer of exogenous shocks on macroeconomic dynamics. As we argue above expectations can depend on economic flows of variables, transactions and other expectations. Dependence of expectations on economic flows makes them key factors that determine impact of economic flows on macroeconomics. Dynamics of economic flows like credit flows $\boldsymbol{P}_C(t,\boldsymbol{x})=C(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})$, flows of variables, transactions and expectations and their mutual interactions on economic domain (1.1) establish very complex picture. For example economic flows on economic domain (1.1) generate business cycles that describe slow oscillations of macroeconomic variables. On the other hand perturbations of economic flows cause wave propagation of disturbances and shocks of economic variables, transactions and expectations those induce fast oscillations of economic parameters. Consistent macroeconomic model on base of economic equations (1.3; 1.4) that describe dynamics of $2k$ variables that depend on k transactions under action of W expectations establish a really tough problem. Reductions of complete system of equations permit study various approximations of macroeconomic evolution. In Part III we study approximations of equations (1.3; 1.4) that describe "simplified" model interactions between two variables, between two transactions, between transactions and expectations. Such approximations help describe model examples of business cycles and different examples of wave propagation of disturbances of economic variables and transactions inside economic domain and on surface of economic domain (1.1). Similar approximations permit develop model of price fluctuations induced by interactions between transactions and numerous expectations.

3. Economic Space

Notion of economic agents is a basic economic term (Giovannini, 2008): "One of the fundamental characteristics of activities defined as economic processes is that they involve relations between various agents. The definition of economic agent is therefore absolutely fundamental in determining the nature of the economic processes: economic agent refers to a person or legal entity that plays an active role in an economic process". There are a lot of studies of agent-based economic and financial models (Tesfatsion and Judd, 2005; Gaffard and Napoletano, 2012). Our approach has nearly nothing with them. We regard agent as economic unit that has a lot of economic or financial variables like asset and debts, investment and credits, supply and demand and etc. Economic and financial variables can be additive or non-additive. Additive variables are investment, credits, volume and cost of commodities and etc. Non-additive variables are prices, bank rates, inflation, indexes and etc. Non-additive variables can be presented as ratio of two additive variables or ratio of non-additive variables. For example price of commodity equals ratio of cost and volume of commodities purchased by particular transaction. Inflation index during time term *[0,T]* equals ratio of prices at moment *T* and at moment *0*. All additive macroeconomic or financial variables like GDP, investment, credits, supply and demand and etc., are composed as sum of agents variables. For example macroeconomic investment equals sum of investment (without doubling) of all agents of the entire economics. Non-additive macroeconomic variables like inflation, economic growth are determined as ratios of macroeconomic additive variables. Thus description of agents additive economic and financial variables determine evolution of all macroeconomic and financial variables. Let's introduce economic space notion and explain how macroeconomic additive variables can be described by additive variables of economic agents.

To define economic space let's use well-known economic tool – risk ratings. Risk management and risk assessment (Horcher, 2005; Skoglund and Chen, 2015) during at least 50 years establish well-developed sector of economics. Risk assessment is a core tool for banking and corporate management and is necessary issue for any investment and financial markets operations. Top international rating agencies provide risk assessments for major banks, financial securities and etc. Risk ratings of particular agent like bank or corporation or ratings of their securities impact on decisions of financial markets traders. There are many risks that affect macroeconomics and finance like credit, inflation, market risks and etc. We don't argue particular risks but treat any risks as factors that may affect economic and financial properties of agents and hence entire economics.

Let's treat assessments of risk ratings of agents as coordinates of agents alike to coordinates of physical particles. Let's note space that imbed agents by their risk coordinates as economic space (Olkhov, 2016a-b; 2017a,b). Current risk methodologies measure risk ratings by risk grades (Wilier, 1901; McNeil, Frey and Embrechts, 2005; Metz and Cantor, 2007; SEC, 2012; S&P, 2014) that have notations as *AAA, AA, BB, C* etc. Let's take current risk grades as points $x_1,...x_n$ of economic space. Such economic space imbed economic agents by their risk ratings $\boldsymbol{x}$. Risk grades of single risk establish *1*-dimensional economic space. Grades of two or three risks establish 2 or 3 dimensional economic space. Number of risk grades like *AAA, AA, BB, C* etc. depends on risk assessment methodology. Let's assume that one may extend risk methodology so that it adopts continuous risk grades. Then *n*-dimensional economic space that describe action of *n* risks can be treated as R^n. Let's propose that economic statistics provide sufficient data for risk assessments of all economic agents of the macroeconomics. Let's state that risk ratings take continuous values between most secure grade equals *0* and most risky grade equals *1*. Partition of agents by their risk ratings for *n* risks define economic domain (1.1) on economic space R^n. All agents have their risk coordinates inside economic domain (1.1). Partition of agents on economic domain (1.1) establishes distribution of agents economic and financial variables over economic domain. Change of agents risk ratings due to their economic activity, market dynamics, other endogenous or exogenous shocks induce evolution of agents variables and thus change macroeconomic variables. In the next section we show how usage of risk ratings as coordinates of economic agents describes evolution of macroeconomic variables.

4. Economic variables

Description of numerous separate agents and their economic and financial variables is too complex problem. Uncertainty of agents risk coordinates and of their economic and financial variables makes such description too ambiguity. To simplify macroeconomic model and develop more sustainable and reasonable model let's rougher our description. The main idea is simple: let's rougher description of separate agents and their variables and describe same variables as aggregates of variables of agents with coordinates at point $\boldsymbol{x}$ of economic space. Let's regard macroeconomics as system of numerous agents on *n*-dimensional economic domain (1.1). Let's state that agents at moment *t* have risk ratings coordinates $\boldsymbol{x}=(x_1,...x_n)$ and velocities $\boldsymbol{v}=(v_1,...v_n)$. Velocities $\boldsymbol{v}=(v_1,...v_n)$ describe change of agents risk coordinates. Let's assume that a unit volume $dV(\boldsymbol{x})$ at point $\boldsymbol{x}$ on economic space contains many agents but scales *d* (2) of a unit volume $dV(\boldsymbol{x})$ are small to compare with scales of domain (1.1)

$$d \ll 1 \quad ; \quad dV = d^n \tag{2}$$

Let's regard only additive variables of agents and assume that economic statistics able select "independent" agents. Let's call agents as "independent" if sum of their additive variables equals same variable of the entire group. For example sum of Credits of *k* agents equals Credits of the group of these *k* agents. Additive variables are Credits, Investment, Asset and etc. There are a lot of non-additive variables as bank rates, inflation, prices and etc. Non-additive variables are defined as ratio of additive variables or ratio of non-additive variables. For example non-additive variable - price of transaction equals the ratio of cost and volume of this deal. Hence all economic variables are determined by additive variables only. Let's show how description of additive variables models evolution of macroeconomic variables.

Let's define additive economic variable $A(t,\boldsymbol{x})$ at point $\boldsymbol{x}$ as sum of variables $A_i(t,\boldsymbol{x})$ of agents i with coordinates in a unit volume $dV(\boldsymbol{x})$ (2) and then average it during term Δ as:

$$A(t,\boldsymbol{x}) = \sum_{i\in dV(\boldsymbol{x});\,\Delta} A_i(t,\boldsymbol{x}) \tag{3}$$

$$\sum_{i\in dV(\boldsymbol{x});\,\Delta} A_i(t,\boldsymbol{x}) = \frac{1}{\Delta}\int_t^{t+\Delta} d\tau \ \sum_{i\in dV(\boldsymbol{x})} A_i(\tau,\boldsymbol{x}) \tag{4}$$

We use $i \in dV(\boldsymbol{x})$ to denote that risk coordinates $\boldsymbol{x}$ of agent i belong to unit volume $dV(\boldsymbol{x})$. For brevity we use left hand sum (4) to denote averaging during time Δ in a unit volume $dV(\boldsymbol{x})$. We repeat meaning of space scales d and time scale Δ given in Sec.2. Scales d<<1 of volume $dV(\boldsymbol{x})$ are small to compare with scales of economic domain (1.1) but volume $dV(\boldsymbol{x})$ contains a lot of economic agents. Scale Δ is small to compare with time scales of the problem under consideration but a lot of economic and financial transactions between agents are performed during time Δ. Time averaging smooth changes of variables under numerous transactions during time Δ. We aggregate values of variables of numerous agents with risk coordinates inside volume $dV(\boldsymbol{x})$, smooth their changes during time Δ and denote result as density function of variable at point $\boldsymbol{x}$. Density function $A(t,\boldsymbol{x})$ describes economic variable at point $\boldsymbol{x}$ on economic domain (1.1). For example let's take $A_i(t,\boldsymbol{x})$ as Credits of agent i. Then density of Credits $A(t,\boldsymbol{x})$ describe sum of Credits issued by all agents with coordinates $\boldsymbol{x}$ inside unit volume $dV(\boldsymbol{x})$ and averaged during time Δ. Total value of macroeconomic variable $A(t)$ is determined by integral (5) over economic domain (1.1):

$$A(t) = \int d\boldsymbol{x}\ A(t,\boldsymbol{x}) \tag{5}$$

Function $A(t)$ equals sum (without doubling) of variables $A_i(t,\boldsymbol{x})$ of all agents i of entire economics averaged during time Δ. For example Credits $A(t)$ issued in macroeconomics equal integral of Credits $A(t,\boldsymbol{x})$ by $d\boldsymbol{x}$ over economic domain (1.1). Thus function $A(t,\boldsymbol{x})$ (3) can be treated as economic density of variable $A(t)$ (5) on economic space. Now let's describe evolution of economic densities $A(t,\boldsymbol{x})$ defined by relations (3). Economic density $A(t,\boldsymbol{x})$ (3) is composed by variables $A_i(t,\boldsymbol{x})$ of agents i . Risk ratings of each agent can change during time Δ. Such time Δ can be equal a day, a week, a quarter etc. Let's describe change of agent's i risk coordinates on economic space during time Δ by velocity $\boldsymbol{v}_i=(v_1,\dots v_n)$. Thus each agent i with economic variable $A_i(t,\boldsymbol{x})$ carries flow of this economic variable with velocity $\boldsymbol{v}_i=(v_1,\dots v_n)$. Flow $\boldsymbol{p}_{iA}(t,\boldsymbol{x})$ of economic variable $A_i(t,\boldsymbol{x})$ of agent i with velocity $\boldsymbol{v}_i=(v_1,\dots v_n)$ equals:

$$\boldsymbol{p}_{iA}(t,\boldsymbol{x}) = A_i(t,\boldsymbol{x})\boldsymbol{v}_i(t,\boldsymbol{x}) \tag{6}$$

Different agents induce different flows of economic variable A in different directions with different velocities. Let's aggregate flows of variable $A_i(t,\boldsymbol{x})$ in the direction of velocity $\boldsymbol{v}_i$ of agents i with coordinates in a unit volume $dV(\boldsymbol{x})$ (2) and then average this flow during time Δ similar to relations (3, 4). Let's define flow $\boldsymbol{P}_A(t,\boldsymbol{x})$ of variable $A(t,\boldsymbol{x})$ as:

$$\boldsymbol{P}_A(t,\boldsymbol{x}) = \sum_{i\in dV(\boldsymbol{x});\,\Delta} A_i(t,\boldsymbol{x})\boldsymbol{v}_i(t,\boldsymbol{x}) \tag{7}$$

Similar to (5) integral of (7) by $d\boldsymbol{x}$ over economic domain (1.1) define macro flow $\boldsymbol{P}_A(t)$ of variable $A(t)$ as:

$$\boldsymbol{P}_{\boldsymbol{A}}(t) = \int d\boldsymbol{x}\ \boldsymbol{P}_{\boldsymbol{A}}(t,\boldsymbol{x}) \tag{8}$$

Flow $\boldsymbol{P}_A(t,\boldsymbol{x})$ (7) of variable $A(t,\boldsymbol{x})$ (3) defines aggregated velocity $\boldsymbol{v}_A(t,\boldsymbol{x})$ of economic variable $A(t,\boldsymbol{x})$ that adjust the flow (7) as:

$$\boldsymbol{P}_A(t,\boldsymbol{x}) = A(t,\boldsymbol{x})\boldsymbol{v}_A(t,\boldsymbol{x}) \tag{9}$$

Thus (9) describes flow $\boldsymbol{P}_A(t,\boldsymbol{x})$ of economic variable $A(t,\boldsymbol{x})$ with velocity $\boldsymbol{v}_A(t,\boldsymbol{x})$. Relations (5) and (8) define macro velocity $\boldsymbol{v}_A(t)$ on domain (1.1) of macro variable $A(t)$ as:

$$\boldsymbol{P}_A(t) = A(t)\boldsymbol{v}_A(t) \tag{10}$$

Let's mention that due to (3; 5; 7-9 and 10) velocity $\boldsymbol{v}_A(t)$ is not equal to integral of velocity $\boldsymbol{v}_A(t,\boldsymbol{x})$ over economic domain (1.1). Aggregation of agents economic variables defines corresponding economic densities and velocities. Due to relations (3-10) different economic variables A define different economic flows $\boldsymbol{P}_A(t,\boldsymbol{x})$ and different velocities $\boldsymbol{v}_A(t,\boldsymbol{x})$. In other words – motion of different economic variables $A(t,\boldsymbol{x})$ on economic space has different velocities $\boldsymbol{v}_A(t,\boldsymbol{x})$. For example flow $\boldsymbol{P}_C(t,\boldsymbol{x})$ of Credits $C(t,\boldsymbol{x})$ has velocity $\boldsymbol{v}_C(t,\boldsymbol{x})$ different from velocity $\boldsymbol{v}_L(t,\boldsymbol{x})$ that describe flow $\boldsymbol{P}_L(t,\boldsymbol{x})$ of Loans $L(t,\boldsymbol{x})$ or velocity $\boldsymbol{v}_I(t,\boldsymbol{x})$ that describe flow $\boldsymbol{P}_I(t,\boldsymbol{x})$ of Investment $I(t,\boldsymbol{x})$ on economic space. Macroeconomic models should describe dynamics and mutual interactions between numerous economic and financial variables and their flows. Properties of economic flows are completely different from properties of any physical flows.

To model dynamics of economic variables $A(t,\boldsymbol{x})$ and flows $\boldsymbol{P}_A(t,\boldsymbol{x})$ let's describe their change in small unit volume dV. There are two factors that change $A(t,\boldsymbol{x})$ in a unit volume dV. The first factor describes change of $A(t,\boldsymbol{x})$ on a unit volume dV in time and can be presented by time derivative as:

$$\int d\boldsymbol{x}\ \frac{\partial}{\partial t}A(t,\boldsymbol{x}) \tag{11}$$

The second factor describes change of variable $A(t,\boldsymbol{x})$ due to flows $\boldsymbol{P}_A(t,\boldsymbol{x})$. Indeed, amount of economic density $A(t,\boldsymbol{x})$ in a unit volume dV during time dt can grow up or decrease due to in- or out- flows $\boldsymbol{P}_A(t,\boldsymbol{x})$. If there are more in-flows $\boldsymbol{P}_A(t,\boldsymbol{x})$ then out-flows then amount of $A(t,\boldsymbol{x})$ will increase in a volume dV. To calculate balance of in- and out-flows let's take integral of flow $\boldsymbol{P}_A(t,\boldsymbol{x})$ over the surface of a unit volume dV:

$$\oint ds\, \boldsymbol{P}_A(t,\boldsymbol{x}) = \oint ds\ A(t,\boldsymbol{x})\, \boldsymbol{v}_A(t,\boldsymbol{x}) \tag{12}$$

Due to divergence theorem (Strauss 2008, p.179) surface integral of flux $A(t,\boldsymbol{x})\boldsymbol{v}_A(t,\boldsymbol{x})$ through surface equals volume integral of divergence of flow $A(t,\boldsymbol{x})\boldsymbol{v}_A(t,\boldsymbol{x})$

$$\oint ds\, A(t,\boldsymbol{x})\, \boldsymbol{v}_A(t,\boldsymbol{x}) = \int d\boldsymbol{x}\ \nabla\cdot(A(t,\boldsymbol{x})\, \boldsymbol{v}_A(t,\boldsymbol{x})) \tag{13}$$

Relations (11,13) give total change of variable $A(t,\boldsymbol{x})$ in a unit volume dV :

$$\int dx\ \left[\frac{\partial}{\partial t}A(t,\boldsymbol{x}) + \nabla\cdot\big(A(t,\boldsymbol{x})\, \boldsymbol{v}_A(t,\boldsymbol{x})\big)\right]$$

As unit volume dV is arbitrary one can take equations on economic density $A(t,\boldsymbol{x})$ as

$$D_A A(t,\boldsymbol{x}) = \frac{\partial}{\partial t}A(t,\boldsymbol{x}) + \nabla\cdot\big(A(t,\boldsymbol{x})\, \boldsymbol{v}_A(t,\boldsymbol{x})\big) = F_A(t,\boldsymbol{x}) \tag{14}$$

Function $F_A(t,\boldsymbol{x})$ in right side (14) describe factors that impact change of economic density $A(t,\boldsymbol{x})$ as: other variables, transactions, expectations and etc. Equations like (14) are reproduced in any treatise on physics of fluids (Batchelor, 1967; Resibois and De Leener, 1977; Landau and Lifshitz, 1987) and are valid for any additive economic or financial variables defined as aggregates of agents variables on economic space similar to (3). Due to (13) integral of divergence of flow $\nabla\cdot\big(A(t,\boldsymbol{x})\, \boldsymbol{v}_A(t,\boldsymbol{x})\big)$ over economic domain (1.1) equals integral over surface of economic domain (1.1) and hence equals zero as no economic or financial flows exist outside of (1.1):

$$\int d\boldsymbol{x}\ \nabla\cdot\big(A(t,\boldsymbol{x})\, \boldsymbol{v}_A(t,\boldsymbol{x})\big) = \oint ds\, A(t,\boldsymbol{x})\, \boldsymbol{v}_A(t,\boldsymbol{x}) = 0$$

Hence integral over economic domain (1.1) for (14) due to (5) equals:

$$\int d\boldsymbol{x}\left[\frac{\partial}{\partial t}A(t,\boldsymbol{x}) + \nabla\cdot\big(A(t,\boldsymbol{x})\, \boldsymbol{v}_A(t,\boldsymbol{x})\big)\right] = \frac{d}{dt}\int d\boldsymbol{x}\ A(t,\boldsymbol{x}) = \frac{d}{dt}A(t) \tag{15}$$

Thus operator D_A (14) on economic space for economic or financial variable $A(t,\boldsymbol{x})$ (3) plays the same role as usual ordinary derivation by time d/dt for macro variable $A(t)$ (5). Let's underline that different variables $A(t,\boldsymbol{x})$ and $B(t,\boldsymbol{x})$ follow different operators (14) due to different velocities $\boldsymbol{v}_A(t,\boldsymbol{x})$ and $\boldsymbol{v}_B(t,\boldsymbol{x})$. So, economic variable $B(t,\boldsymbol{x})$ follows equations:

$$D_B B(t,\boldsymbol{x}) = \frac{\partial}{\partial t}B(t,\boldsymbol{x}) + \nabla\cdot\big(B(t,\boldsymbol{x})\, \boldsymbol{v}_B(t,\boldsymbol{x})\big) = F_B(t,\boldsymbol{x}) \tag{16}$$

Equations (14; 16) are valid for additive variables. Flow $\boldsymbol{P}_A(t,\boldsymbol{x})$ follows the same operator D_A (14) as variable $A(t,\boldsymbol{x})$ and

$$D_A\boldsymbol{P}_A(t,\boldsymbol{x}) \equiv \frac{\partial}{\partial t}\boldsymbol{P}_A(t,\boldsymbol{x}) + \nabla\cdot\big(\boldsymbol{P}_A(t,\boldsymbol{x})\,\boldsymbol{v}_A(t,\boldsymbol{x})\big) = \boldsymbol{G}_A(t,\boldsymbol{x}) \tag{17}$$

$$\nabla\cdot\big(\boldsymbol{P}_A(t,\boldsymbol{x})\,\boldsymbol{v}_A(t,\boldsymbol{x})\big) = \sum_{j=1,..n}\frac{\partial}{\partial x_j}\Big(\boldsymbol{P}_A(t,\boldsymbol{x})\,v_{Aj}(t,\boldsymbol{x})\Big)$$

Function $\boldsymbol{G}_A(t,\boldsymbol{x})$ in right side (17) describe factors that impact change of economic flow $\boldsymbol{P}_A(t,\boldsymbol{x})$ as: other variables, transactions, expectations and etc.

Equations (14, 17) describe evolution of $A(t,\boldsymbol{x})$ (3) and $\boldsymbol{P}_A(t,\boldsymbol{x})$ (9) under action of factors $F_A(t,\boldsymbol{x})$ and $\boldsymbol{G}_A(t,\boldsymbol{x})$. Integrals of (14; 17) by $d\boldsymbol{x}$ over domain (1.1) give ordinary differential equations as no economic or financial flows exist outside of (1.1) (Strauss 2008, p.179):

$$\int d\boldsymbol{x}\left[\frac{\partial}{\partial t}A(t,\boldsymbol{x}) + \nabla\cdot\big(A(t,\boldsymbol{x})\,\boldsymbol{v}_A(t,\boldsymbol{x})\big)\right] = \frac{d}{dt}A(t) = F_A(t) \tag{18.1}$$

$$\int d\boldsymbol{x}\left[\frac{\partial}{\partial t}\boldsymbol{P}_A(t,\boldsymbol{x}) + \nabla\cdot\big(\boldsymbol{P}_A(t,\boldsymbol{x})\,\boldsymbol{v}_A(t,\boldsymbol{x})\big)\right] = \frac{d}{dt}\boldsymbol{P}_A(t) = \boldsymbol{G}_A(t) \tag{18.2}$$

Ordinary differential equations (18.1, 18.2) describe time evolution of macroeconomic and financial variables of entire economics. It is clear that all complexity of economic dynamics is described by right hand side factors $F_A(t,\boldsymbol{x})$ and $\boldsymbol{G}_A(t,\boldsymbol{x})$ in (14, 17). Equations (14, 17) permit model self-consistent interactions between two macro variables. The simplest case of mutual dependence between two macro variables can be presented as

$$\frac{\partial}{\partial t}A(t,\boldsymbol{x}) + \nabla\cdot\big(A(t,\boldsymbol{x})\,\boldsymbol{v}_A(t,\boldsymbol{x})\big) = F_A(t,\boldsymbol{x}) \tag{19.1}$$

$$\frac{\partial}{\partial t}B(t,\boldsymbol{x}) + \nabla\cdot\big(B(t,\boldsymbol{x})\,\boldsymbol{u}_B(t,\boldsymbol{x})\big) = F_B(t,\boldsymbol{x}) \tag{19.2}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_A(t,\boldsymbol{x}) + \nabla\cdot\big(\boldsymbol{P}_A(t,\boldsymbol{x})\,\boldsymbol{v}_A(t,\boldsymbol{x})\big) = \boldsymbol{G}_A(t,\boldsymbol{x}) \tag{19.3}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_B(t,\boldsymbol{x}) + \nabla\cdot\big(\boldsymbol{P}_B(t,\boldsymbol{x})\,\boldsymbol{u}_B(t,\boldsymbol{x})\big) = \boldsymbol{G}_B(t,\boldsymbol{x}) \tag{19.4}$$

$$F_A(t,\boldsymbol{x}) = F_A(t,\boldsymbol{x}\,;B,\boldsymbol{P}_B) \;\; ; \;\; F_B(t,\boldsymbol{x}) = F_B(t,\boldsymbol{x}\,;A,\boldsymbol{P}_A) \tag{19.5}$$

$$\boldsymbol{G}_A(t,\boldsymbol{x}) = \boldsymbol{G}_A(t,\boldsymbol{x}\,;B,\boldsymbol{P}_B) \;\; ; \;\; \boldsymbol{G}_B(t,\boldsymbol{x}) = F_B(t,\boldsymbol{x}\,;A,\boldsymbol{P}_A) \tag{19.6}$$

Relations (19.5, 19.6) may describe dependence of $F_A(t,\boldsymbol{x})$ and $\boldsymbol{G}_A(t,\boldsymbol{x})$ on variable $B(t,\boldsymbol{x})$ and flow $\boldsymbol{P}_B(t,\boldsymbol{x})$ and $F_B(t,\boldsymbol{x})$ and $\boldsymbol{G}_B(t,\boldsymbol{x})$ on variable $A(t,\boldsymbol{x})$ and flow $\boldsymbol{P}_A(t,\boldsymbol{x})$. $F_A(t,\boldsymbol{x})$ and $\boldsymbol{G}_A(t,\boldsymbol{x})$ may depend on operators like divergence, gradient, rotor and etc. on functions $B(t,\boldsymbol{x})$ and $\boldsymbol{P}_B(t,\boldsymbol{x})$. It is obvious that in real economics macro variables depend on numerous economic and financial factors but (19.1-19.4) permit study simple approximations of mutual relations between two – three or four macroeconomic variables and their flows.

In Part II we describe economic transactions and expectations as density functions on economic space. We derive equations on transactions, expectations and their flows. We show how our approach helps describe asset pricing on economic space and derive equations on price evolution. In Part III of our paper we apply our model equations to description of particular economic problems.

Part II. Economic Transactions, Expectations and Asset Pricing

5. Economic Transactions on Economic Space

In this Section we model economic and financial transactions between agents. In Part I. we show that risk assessments of economic agents permit distribute them by their risk rating as coordinates on economic space. Now let's model economic transactions in a similar way. Let's study additive economic and financial variables that are subject of transactions between agents. For example let's propose that agent i sell some amount of variable E to agent j. As E one may regard any goods, capital, service or commodities as Oil, Steel, Energy and etc. For example let's assume that agent i provide credits C to agent j. Such transactions between agents i and j change amount of credits C provided by i and change amount of loans L received by j. Each transactions take certain time dt and we consider transactions as rate or speed of change of corresponding variable E of agents involved into transaction. For example all transactions of agent i at moment t during time $[0, t]$ define change of variable E (Steel, Energy, Shares, Credits, Assets and etc.) owned by agent i during period $[0, t]$.

To avoid excess specification of transactions between numerous separate agents let's replace description of transactions between separate agents by rougher description of transactions between points of economic space and average it during time Δ alike to (I.3; 4). Let's neglect granularity of separate agents and transactions between them and replace it by density functions of transactions on economic space. Let's take that agents on economic space R^n at moment t have coordinates $\boldsymbol{x}=(x_1,...x_n)$ and risk velocities $\boldsymbol{\upsilon}=(\upsilon_1,...\upsilon_n)$. Risk velocities describe change of agents risk coordinates during time dt. Let's remind that all agents have coordinates inside n-dimensional economic domain (I.1.1). Transactions between agents with risk coordinates $\boldsymbol{x}$ and agents with risk coordinates $\boldsymbol{y}$ are determined on $2n$-dimensional economic domain, $z=(\boldsymbol{x},\boldsymbol{y})$:

$$\boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \ ; \ \boldsymbol{x} = (x_1 \dots x_n) \ ; \ \boldsymbol{y} = (y_1 \dots y_n) \tag{1.1}$$

$$0 \le x_i \le 1 \ ; \ 0 \le y_i \le 1 \, , \quad i = 1, \dots n \tag{1.2}$$

Relations (1.1; 1.2) define economic domain that is filled by pairs of agents with coordinates $z=(\boldsymbol{x},\boldsymbol{y})$ on $2n$-dimensional economic space R^{2n}. Let's rougher description of transactions between agents and replace it by description of transactions between all agents at points $\boldsymbol{x}$ and $\boldsymbol{y}$. Let's take a unit volume $dV(z)$

$$dV(\boldsymbol{z}) = dV(\boldsymbol{x})dV(\boldsymbol{y}) \ ; \ \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{1.3}$$

and assume that *dV(x)* and *dV(y)* follow relations (I.2) and their scales are small to compare with scales of economic domain (I.1.1) for $\boldsymbol{x}$ and $\boldsymbol{y}$. Let's propose:

$$dV_i \ll 1\,, i = 1, \dots n\,;\; dV(\boldsymbol{x}) = \prod_{i=1,..n} dV_i \tag{1.4}$$

$$dV_j \ll 1\,, j = 1, \dots n\,;\; dV(\boldsymbol{y}) = \prod_{j=1,..n} dV_j \tag{1.5}$$

Let's assume that each unit volume *dV(x)* and *dV(y)* contain a lot of agents with risk coordinates inside *dV(x)* and *dV(y)*. Let's take time Δ small to compare with time scales of macroeconomic problem under consideration but assume that during time Δ agents inside *dV(x)* and *dV(y)* perform a lot of transactions. Let's rougher space description of transactions on (1.1; 1.2) by scales dV_i and rougher time description by scale Δ. As we keep space scales dV_i small to compare with scales of economic domain (1.1; 1.2) and time scale Δ small to compare with time scales of the macroeconomic problem hence we still use continuous approximation, but with rougher scales.

Let's denote $bs_{1,2}(t,\boldsymbol{x},\boldsymbol{y})$ as buy-sell transactions by variable *E* from agent *1* at point $\boldsymbol{x}$ to agent *2* at point $\boldsymbol{y}$. Economic variable *E* may be Oil, Steel, Shares, Credits, Assets and etc. that are supplied from agent *1* as seller at point $\boldsymbol{x}$ to agent 2 as buyer at point $\boldsymbol{y}$ at moment *t*. Let's aggregate all transactions by variable *E* performed by all agents inside *dV(x)* and agents inside *dV(y)*. Similar to (I. 3;4) let's define transaction *BS(t,z)* at point $\boldsymbol{z}=(\boldsymbol{x},\boldsymbol{y})$ as sum of transactions $bs_{i,j}(t,\boldsymbol{x},\boldsymbol{y})$ between all agents *i* in a unit volume *dV(x)* at point $\boldsymbol{x}$ and agents *j* in a unit volume *dV(y)* at point $\boldsymbol{y}$ and average this sum during time Δ:

$$BS(t,\boldsymbol{x},\boldsymbol{y}) = \sum_{i\in dV(\boldsymbol{x});\, j\in dV(\boldsymbol{y});\, \Delta} bs_{i,j}(t,\boldsymbol{x},\boldsymbol{y}) \tag{2.1}$$

$$\sum_{i\in dV(\boldsymbol{x});\, \Delta} bs_{i,j}(t,\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\Delta}\int_t^{t+\Delta} d\tau \sum_{i\in dV(\boldsymbol{x});\, j\in dV(\boldsymbol{y})} bs_{i,j}(t,\boldsymbol{x},\boldsymbol{y}) \tag{2.2}$$

Integral of *BS(t,z)* by variable $d\boldsymbol{y}$ over economic domain (I.1.1) defines all sells *BS(t,x)* of variable *E* performed by agents inside a unit volume *dV(x)* at point $\boldsymbol{x}$

$$BS(t,\boldsymbol{x}) = \int d\boldsymbol{y}\; BS(t,\boldsymbol{x},\boldsymbol{y}) \tag{3}$$

and integral of *BS(t,x)* by variable $d\boldsymbol{x}$ over economic domain define all sells *BS(t)* of variable *E* performed by all agents of macroeconomics at moment *t*.

$$BS(t) = \int d\boldsymbol{x}\; BS(t,\boldsymbol{x}) = \int d\boldsymbol{x}d\boldsymbol{y}\; BS(t,\boldsymbol{x},\boldsymbol{y}) \tag{4.1}$$

For example, if *CI(t)* (4.2) defines cumulative investment made in economy during term [0,t] and *BS(t,x,y)* – investment transactions made from $\boldsymbol{x}$ to $\boldsymbol{y}$ during time term *dt* then

$$\frac{d}{dt}CI(t) = BS(t) = \int d\boldsymbol{x}d\boldsymbol{y}\; BS(t,\boldsymbol{x},\boldsymbol{y}) \tag{4.2}$$

Hence transactions define time derivative of cumulative macroeconomic and financial variables like investment, credits and etc., made during time term. Macros transactions *BS(t,z)* on economic domain (1.1; 1.2) describe evolution of macroeconomic and financial variables. Relations (4.1-4.2) define macroeconomic variables as integrals of transactions *BS(t,z)* over economic domain. Let's call *BS(t,z)* as transactions density functions on *2n*-dimensional economic domain similar to economic density function $A(t,\boldsymbol{x})$ (I.3).

Let's remind that transactions densities *BS(t,z)*, $z=(\boldsymbol{x},\boldsymbol{y})$ are determined as aggregates of transactions between agents at points $\boldsymbol{x}$ and $\boldsymbol{y}$. As we argue in Part I each agent *i* on economic domain is described by its risk coordinates $\boldsymbol{x}_i$ and its velocity $\boldsymbol{v}_i$. Thus similar to (I.6) let's define flows $\boldsymbol{p}_{ij}(t,z)$ (5.1;5.2) of transactions $bs_{ij}(t,z)$ between agents:

$$\boldsymbol{p}_{ij}(t,\boldsymbol{z}) = \left(\boldsymbol{p}_{xij}(t,\boldsymbol{z}), \boldsymbol{p}_{yij}(t,\boldsymbol{z})\right) \tag{5.1}$$

$$\boldsymbol{p}_{xij}(t,\boldsymbol{z}) = bs_{i,j}(t,\boldsymbol{z})\boldsymbol{v}_i(t,\boldsymbol{x}) \quad ; \quad \boldsymbol{p}_{yij}(t,\boldsymbol{z}) = bs_{i,j}(t,\boldsymbol{z})\boldsymbol{v}_j(t,\boldsymbol{y}) \tag{5.2}$$

Flows $\boldsymbol{p}_{ij}(t,\boldsymbol{x},\boldsymbol{y})$ describe amounts of transactions $bs_{ij}(t,\boldsymbol{x},\boldsymbol{y})$ carried by agents *i* as sellers and carried by agents *j* as buyers of variable *E*. Flows $\boldsymbol{p}_{xij}(t,\boldsymbol{x},\boldsymbol{y})$ describe motion of sellers along axis $\boldsymbol{X}$ and flows $\boldsymbol{p}_{yij}(t,\boldsymbol{x},\boldsymbol{y})$ describe motion of buyers along axis $\boldsymbol{Y}$. Aggregates of flows $\boldsymbol{p}_{ij}(t,\boldsymbol{x},\boldsymbol{y})$ over all agents *i* at point $\boldsymbol{x}$ with coordinates inside $dV(\boldsymbol{x})$ and all agents *j* at point $\boldsymbol{y}$ with coordinates inside $dV(\boldsymbol{y})$ define transactions flows $\boldsymbol{P}(t,\boldsymbol{x},\boldsymbol{y})$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$ similar to (I.7) and (2.1; 2.2) as:

$$\boldsymbol{P}(t,\boldsymbol{z}) = \left(\boldsymbol{P}_{\boldsymbol{x}}(t,\boldsymbol{z}), \boldsymbol{P}_{\boldsymbol{y}}(t,\boldsymbol{z})\right) \ ; \ \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{5.3}$$

$$\boldsymbol{P}_{\boldsymbol{x}}(t,\boldsymbol{z}) = \sum_{i\in dV(x);j\in dV(y)\,\Delta} \boldsymbol{p}_{xij}(t,\boldsymbol{z}) = \sum_{i\in dV(x);j\in dV(y)\,\Delta} bs_{i,j}(t,\boldsymbol{z})\boldsymbol{v}_i(t,\boldsymbol{x}) \tag{5.4}$$

$$\boldsymbol{P}_{\boldsymbol{x}}(t,\boldsymbol{z}) = \sum_{i\in dV(x);j\in dV(y)\,\Delta} \boldsymbol{p}_{yij}(t,\boldsymbol{z}) = \sum_{i\in dV(x);j\in dV(y)\,\Delta} bs_{i,j}(t,\boldsymbol{z})\boldsymbol{v}_j(t,\boldsymbol{y}) \tag{5.5}$$

Transactions flows $\boldsymbol{P}(t,z)$ (5.3-5.5) between points $\boldsymbol{x}$ and $\boldsymbol{y}$ describe amounts of transactions *BS(t,z)* carried by transactions velocities $\boldsymbol{v}(t,z)$ through *2n*-dimensional economic domain (1.1;1.2). Similar to (I.9) let's define transactions velocities $\boldsymbol{v}(t,z)$ as:

$$\boldsymbol{P}(t,\boldsymbol{z}) = BS(t,\boldsymbol{z})\boldsymbol{v}(t,\boldsymbol{z}) \ ; \ \boldsymbol{v}(t,\boldsymbol{z}) = (\boldsymbol{v}_{\boldsymbol{x}}(t,\boldsymbol{z}); \boldsymbol{v}_{\boldsymbol{y}}(t,\boldsymbol{z})) \tag{5.6}$$

$$\boldsymbol{P}_{\boldsymbol{x}}(t,\boldsymbol{z}) = BS(t,\boldsymbol{z})\,\boldsymbol{v}_{\boldsymbol{x}}(t,\boldsymbol{z}) \ ; \ \boldsymbol{P}_{\boldsymbol{y}}(t,\boldsymbol{z}) = BS(t,\boldsymbol{z})\boldsymbol{v}_{\boldsymbol{y}}(t,\boldsymbol{z}) \tag{5.7}$$

Similar to (I.8;9) integrals over economic domain (1.1;1.2) by $d\boldsymbol{x}$ and $d\boldsymbol{y}$ define macroeconomic flows of transactions *BS(t)* (4.1) with velocity $\boldsymbol{v}(t)$ as:

$$\boldsymbol{P}(t) = \int d\boldsymbol{x}d\boldsymbol{y}\,\boldsymbol{P}(t,\boldsymbol{x},\boldsymbol{y}) = BS(t)\boldsymbol{v}(t) \quad ; \quad \boldsymbol{v}(t) = (\boldsymbol{v}_x(t); \boldsymbol{v}_y(t)\,) \tag{5.8}$$

For example let's take *BS(t)* as investments made in macroeconomics during time *dt*. Then relations (5.8) describe flow of investment transactions with velocity $\boldsymbol{v}(t)$ on economic space. Components $\boldsymbol{v}_x(t)$ and $\boldsymbol{v}_y(t)$ describe motion of aggregated investors and aggregated recipients of investments. Positive or negative values of components of velocity $\boldsymbol{v}_{xi}(t)$ along axis x_i of economic space describe motion of investors in risky of safer directions. Positive values of components of velocity $\boldsymbol{v}_{yj}(t)$ along axis y_j of economic space describe risk growth of recipients of investments and negative $\boldsymbol{v}_{yj}(t)$ describes decline of risks of recipients of investments along axis y_j. Aggregated investors and recipients of investments have coordinates inside economic domain (1.1;1.2). Thus velocities (5.8) can't be constant and must change signature and fluctuate as borders of economic domain (1.1; 1.2) reduce motion along each risk axes. Fluctuations of macroeconomic velocities (5.8) of investment transactions describe motion of investors and recipients of investments from safer to risky areas of economic domain (1.1; 1.2) and back from risky to safer areas. Such fluctuations of investors and recipients of investments due to oscillations of velocity $\boldsymbol{v}(t)$ (5.8) describe Investment business cycles. Credit transactions, buy-sell transactions and etc., induce similar macroeconomic transactions flows (5.8) and describe corresponding credit cycles, buy-sell cycles and etc., (Olkhov, 2017d; 2018).

Relations (2.1-2.2; 5.3-5.5) allow derive equations on transactions density $BS(t,\boldsymbol{z})$ and transactions flows $\boldsymbol{P}(t,\boldsymbol{z})$, $\boldsymbol{z}=(\boldsymbol{x},\boldsymbol{y})$ on $2n$-dimensional economic domain similar to equations (I.14; 17). To derive equations on transactions density $BS(t,\boldsymbol{z})$ (2.1; 2.2) and flows $\boldsymbol{P}(t,\boldsymbol{z})$ (5.6) let's describe their change in a small unit volume $dV(\boldsymbol{z})$ (1.4; 1.5). Two factors change $BS(t,\boldsymbol{z})$ in a unit volume $dV(\boldsymbol{z})$. The first change $BS(t,\boldsymbol{z})$ in time as:

$$\int d\boldsymbol{z}\ \frac{\partial}{\partial t} BS(t,\boldsymbol{z}) \tag{5.9}$$

The second factor describes change of $BS(t,\boldsymbol{z})$ due to flows $\boldsymbol{P}(t,\boldsymbol{z})$: amount of $BS(t,\boldsymbol{z})$ in a unit volume $dV(\boldsymbol{z})$ (1.4; 1.5) can grow up or decrease due to in- or out- flows $\boldsymbol{P}(t,\boldsymbol{z})$ during time dt. If in-flows $\boldsymbol{P}(t,\boldsymbol{z})$ are exceed out-flows then $BS(t,\boldsymbol{z})$ will increase in a volume $dV(\boldsymbol{z})$. To calculate balance of in- and out-flows let's take integral of flow $\boldsymbol{P}(t,\boldsymbol{z})$ over the surface of $dV(\boldsymbol{z})$:

$$\oint ds\, \boldsymbol{P}(t,\boldsymbol{z}) = \oint ds\ \ BS(t,\boldsymbol{z})\boldsymbol{v}(t,\boldsymbol{z}) \tag{5.10}$$

Due to divergence theorem (Strauss 2008, p.179) surface integral (5.10) of the flow $\boldsymbol{P}(t,\boldsymbol{z})=BS(t,\boldsymbol{z})\boldsymbol{v}(t,\boldsymbol{z})$ equals its volume integral by divergence of the flow:

$$\oint ds\, BS(t,\boldsymbol{z})\boldsymbol{v}(t,\boldsymbol{z}) = \int d\boldsymbol{z}\ \nabla\cdot(BS(t,\boldsymbol{z})\boldsymbol{v}(t,\boldsymbol{z})) \tag{5.11}$$

Relations (5.9; 5.11) give total change of variable $BS(t,\boldsymbol{z})$ in a unit volume $dV(\boldsymbol{z})$:

$$\int d\boldsymbol{z}\ \left[\frac{\partial}{\partial t} BS(t,\boldsymbol{z}) + \nabla\cdot\big(BS(t,\boldsymbol{z})\ \boldsymbol{v}(t,\boldsymbol{z})\big)\right]$$

As a unit volume $dV(\boldsymbol{z})$ is arbitrary one can take equations on economic density $BS(t,\boldsymbol{z})$ as

$$\frac{\partial}{\partial t} BS(t,\boldsymbol{z}) + \nabla\cdot\big(BS(t,\boldsymbol{z})\ \boldsymbol{v}(t,\boldsymbol{z})\big) = F(t,\boldsymbol{z}) \tag{5.12}$$

Same considerations are valid for the flow $\boldsymbol{P}(t,\boldsymbol{z})$:

$$\frac{\partial}{\partial t}\boldsymbol{P}(t,\boldsymbol{z}) + \nabla\cdot\big(\boldsymbol{P}(t,\boldsymbol{z})\ \boldsymbol{v}(t,\boldsymbol{z})\big) = \boldsymbol{G}(t,\boldsymbol{z}) \tag{5.13}$$

Similar to (I.18.1; 18.2) integrals of (5.12; 5.13) by $dz=(\boldsymbol{dx},\boldsymbol{dy})$ over economic domain (1.1; 1.2) give for (4.1) ordinary time derivation equations:

$$\int d\boldsymbol{z}\left[\frac{\partial}{\partial t}BS(t,\boldsymbol{z})+\nabla\cdot\left(BS(t,\boldsymbol{z})\,\boldsymbol{v}(t,\boldsymbol{z})\right)\right]=\frac{d}{dt}BS(t)=F(t)=\int d\boldsymbol{z}\,F(t,\boldsymbol{z}) \qquad (6.1)$$

$$\int d\boldsymbol{z}\left[\frac{\partial}{\partial t}\boldsymbol{P}(t,\boldsymbol{z})+\nabla\cdot\left(\boldsymbol{P}(t,\boldsymbol{z})\,\boldsymbol{v}(t,\boldsymbol{z})\right)\right]=\frac{d}{dt}\boldsymbol{P}(t)=\boldsymbol{G}(t)=\int d\boldsymbol{z}\,\boldsymbol{G}(t,\boldsymbol{z}) \qquad (6.2)$$

Relations (6.1; 6.2) illustrate that operators in the left hand of (5.12; 5.13) for $BS(t,z)$ and flows $\boldsymbol{P}(t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ on $2n$-dimensional economic space play role alike to ordinary time derivative for macro transactions $BS(t)$ (4.1) and flows $\boldsymbol{P}(t)$ (5.8). Different transactions have different densities, flows and velocities and thus are described by different operators (5.12; 5.13) with different functions $F(t,z)$ and $\boldsymbol{G}(t,z)$. It is assumed that agents are engaged into transactions $BS(t,z)$ with other agents under various expectations. Thus we propose that functions $F(t,z)$ in (5.12) may describe action of expectations of agents involved into transactions $BS(t,z)$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$. In the next section we introduce definitions of expectations between points $\boldsymbol{x}$ and $\boldsymbol{y}$. Functions $\boldsymbol{G}(t,z)$ in (5.13) describe action of factors that impact evolution of transactions flows $\boldsymbol{P}(t,z)$. Thus functions $F(t,z)$ and $\boldsymbol{G}(t,z)$ in (5.12; 5.13) define particular evolution model of transactions $BS(t,z)$ and flows $\boldsymbol{P}(t,z)$. Various economic reasons those define dependence of functions $F(t,z)$ and $\boldsymbol{G}(t,z)$ on other transactions, economic variables or expectations permit study different economic models of evolution of transactions $BS(t,z)$ and flows $\boldsymbol{P}(t,z)$.

The simplest case describes mutual dependence between two transactions $BS_E(t,z)$ and $BS_Q(t,z)$. Let's study exchange by economic variables E and Q in the assumption that functions $F_E(t,z)$ and $\boldsymbol{G}_E(t,z)$ depend on transactions $BS_Q(t,z)$ and its flows $\boldsymbol{P}_Q(t,z)$ and functions $F_Q(t,z)$ and $\boldsymbol{G}_Q(t,z)$ depend on transactions $BS_E(t,z)$ and flows $\boldsymbol{P}_E(t,z)$. This approximation models self-consistent dynamics of two transactions and their flows and describes evolution of corresponding variables E and Q. One can study equations (5.12; 5.13) with functions $F(t,z)$ and $\boldsymbol{G}(t,z)$ that depend on several transactions, expectations or economic variables. Such models describe approximations of economic evolution of transactions and macro variables for different functions $F(t,z)$ and $\boldsymbol{G}(t,z)$.

To describe possible impact of expectations on functions $F(t,z)$ and $\boldsymbol{G}(t,z)$ for equations (5.12; 5.13) let's introduce definitions of expectations densities similar to above models of economic variables and economic transactions.

6. Expectations on Economic Space

Expectations are the most "etheric" economic substance. In this Section we consider expectation as economic substance that determine performance of transactions and thus have substantial impact on evolution of macroeconomic variables.

Expectations are treated as factors that govern economic and financial transactions, price and return at least by Keynes (1936) and actively studied since Muth (1961) and Lucas (1972) and in numerous further publications (Sargent and Wallace, 1976; Hansen and Sargent, 1979; Kydland and Prescott, 1980; Blume and Easley, 1984; Brock and Hommes, 1998; Manski, 2004; Brunnermeier and Parker, 2005; Dominitz and Manski, 2005; Klaauw et al, 2008; Janžek and Ziherl, 2013; Greenwood and Shleifer, 2014; Lof, 2014; Manski, 2017; Thaler, 2018).

Expectations concern all economic and financial variables as inflation and demand, exchange rates, bank rates, price trends and etc. There are a lot of studies on expectations measurements (Manski, 2004; Dominitz and Manski, 2005; Klaauw et al, 2008; Stangl, 2009; Janžek and Ziherl, 2013; Manski, 2017; Tanaka et al, 2018). Manski (2004) indicates that "It would be better to measure expectations as - subjective probabilities". Dominitz and Manski (2005 "analyze probabilistic expectations of equity returns". Stangl (2009) suggests that "Visual Analog Scale (VAS) enables scores between categories, and the respondent can express not only the direction of his attitude but also its magnitude on a 1-to-100 point scale, which comes close to an interval scale measurement". Measurement of such "etheric" economic substance as expectations is a really tough problem. Our approach to expectations modeling as important factor that impact macroeconomic evolution requires that all expectations under consideration should have similar measure. Let's omit here discussion on expectations measure and assume that all expectations are measured as index. It is clear, that scale of index is not important. Expectations may take any values between 0 and 100 or 0 and 1. The only requirement – all expectations are measured by same measure with same scale. For certainty let's take that measure of expectations is an interval [0,1].

Each economic agent can have a lot of different expectations and different expectations force agents accomplish transactions. Let's assume that in economy there are $j=1,..K$ expectations those may impact transactions between agents. Let's transfer description of expectations that define transactions between separate agents to aggregate expectations that describe transactions between points on economic space. To aggregate value and economic importance of agents expectations let's state that economic value of particular expectation of agent should be proportional to value of transactions made under this particular expectation. Indeed, if particular transactions amount 90% of all deals and are made under expectation 1 then this particular expectation 1 is ninety times more important then expectation 2 that is responsible for only 1% of same deals. Thus aggregation of expectations and description of most valuable expectations should be done for expectations weighted by value of transactions made under these expectations.

Let's take buy-sell transactions $bs_{ij}(t,\boldsymbol{x},\boldsymbol{y})$ that describe transfer of economic variable E - assets, shares, commodities, service, credits and etc., from agent i as seller at point $\boldsymbol{x}$ to agent j as buyer at point $\boldsymbol{y}$. Let's denote $ex_i(k;t,\boldsymbol{x})$ as expectations of type $k=1,..K$ of agent i as seller at point $\boldsymbol{x}$. Let's assume that expectations $ex_i(k;t,\boldsymbol{x})$ approve $bs_{ij}(k;t,\boldsymbol{x},\boldsymbol{y})$ - part of transactions $bs_{ij}(t,\boldsymbol{x},\boldsymbol{y})$ with economic variable E made under sellers expectations of type k from agent i as seller at point $\boldsymbol{x}$ to agent j as buyer at point $\boldsymbol{y}$. Further let's denote expectations of buyer $ex_j(t,\boldsymbol{y};l)$ of type $l=1,..K$ that approve part $bs_{ij}(t,\boldsymbol{x},\boldsymbol{y};l)$ of transactions $bs_{ij}(t,\boldsymbol{x},\boldsymbol{y})$ made under buyers expectations of type l by the agent j as buyer at point $\boldsymbol{y}$.

Economic value of sellers expectations $ex_i(k;t,\boldsymbol{x})$ is proportional to amount of transactions $bs_{ij}(k;t,\boldsymbol{x},\boldsymbol{y})$ with variable E made under this type of expectations. For k, $l=1,..K$ let's introduce expected transactions $et_{ij}(k;t,\boldsymbol{x},\boldsymbol{y};l)$ as follows:

$$\boldsymbol{et}_{ij}(k;t,\boldsymbol{z};l) = \left(et_{ij}(k;t,\boldsymbol{z})\,;\; et_{ij}(t,\boldsymbol{z};l)\right) \;;\; \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{7.1}$$

$$et_{ij}(k;t,\boldsymbol{z}) = ex_i(k;t,\boldsymbol{x})bs_{ij}(k;t,\boldsymbol{z}) \;;\; et_{ij}(t,\boldsymbol{z};l) = ex_j(t,\boldsymbol{y};l)bs_{ij}(t,\boldsymbol{z};l)$$

Expected transactions $et_{ij}(k;t,z)$ (7.1) describe sellers expectations $ex_i(k;t,\boldsymbol{x})$ at point $\boldsymbol{x}$ weighted by transactions $bs_{ij}(k;t,z)$ performed between agents i as sellers at $\boldsymbol{x}$ and agents j as buyers at $\boldsymbol{y}$ under expectations of type k. Expected transactions $et_{ij}(t,z;l)$ (7.1) describe buyers expectations $ex_j(t,\boldsymbol{y};l)$ at $\boldsymbol{y}$ weighted by transactions $bs_{ij}(t,z;l)$ performed under buyers expectations $ex_j(t,\boldsymbol{y};l)$ between agents i as sellers at $\boldsymbol{x}$ and agents j as buyers at $\boldsymbol{y}$. Transactions $bs_{ij}(k;t,z)$ between agents i and j are made with variable E under sellers expectations k and transactions $bs_{ij}(t,z;l)$ are made under buyers expectations l and are additive functions.

Let's rougher description of transactions $bs_{ij}(k;t,z)$ and $bs_{ij}(t,z;l)$ and define transactions $BS(k;t,z)$ and transactions $BS(t,z;l)$ with variable E performed by sellers at $\boldsymbol{x}$ under expectations of type k and by buyers at $\boldsymbol{y}$ under expectations of type l as:

$$BS(k;t,\boldsymbol{z}) = \sum\nolimits_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} bs_{i,j}(k;t,\boldsymbol{z}) \;;\; \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{7.2}$$

$$BS(t,\boldsymbol{z};l) = \sum\nolimits_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} bs_{i,j}(k;t,\boldsymbol{z}) \tag{7.3}$$

Functions $BS(k;t,\boldsymbol{x},\boldsymbol{y})$ (7.2) describe part of transactions $BS(t,\boldsymbol{x},\boldsymbol{y})$ (4.2) performed by sellers at $\boldsymbol{x}$ under expectations of type k of with agents at $\boldsymbol{y}$ and all types of buyers expectations. Functions $BS(t,\boldsymbol{x},\boldsymbol{y};l)$ (7.3) describe part of transactions $BS(t,\boldsymbol{x},\boldsymbol{y})$ (4.2) performed by buyers at $\boldsymbol{y}$ under expectations of type l with agents at $\boldsymbol{x}$ and all types of sellers expectations.

$$BS(t,\boldsymbol{z}) = \sum\nolimits_k BS(k;t,\boldsymbol{z}) = \sum\nolimits_l BS(t,\boldsymbol{z};l) \tag{7.4}$$

Sum by k of transactions $BS(k;t,z)$ (7.2) equals sum by l of transactions $BS(t,z;l)$ (7.3) and that equals transactions $BS(t,z)$ (2.1;2.2) performed under all expectations a $z=(\boldsymbol{x}, \boldsymbol{y})$.

Now let's define expected transactions $\boldsymbol{Et}(k;t,\boldsymbol{x},\boldsymbol{y};l)$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$ made under sellers expectations of type k and buyers expectations of type l. Let's aggregate (7.1) in unit volumes (1.3) and average alike to (2.1;2.2) as:

$$\boldsymbol{Et}(k;t,\boldsymbol{z};l) = \big(Et(k;t,\boldsymbol{z}) \;;\; Et(t,\boldsymbol{z};l)\big) \;;\; \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{7.5}$$

$$Et(k;t,\boldsymbol{z}) = \textstyle\sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\Delta} ex_i(k;t,\boldsymbol{x})bs_{ij}(k;t,\boldsymbol{z}) \tag{7.6}$$

$$Et(t,\boldsymbol{z};l) = \textstyle\sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\Delta} ex_j(t,\boldsymbol{y};l)bs_{ij}(t,\boldsymbol{z};l) \tag{7.7}$$

Definitions of $BS(k;t,z)$ (7.2) and $BS(t,z;l)$ (7.3) permit use expected transactions $Et(k;t,z)$ and $Et(t,z;l)$ (7.5-7.7) and introduce expectations densities $Ex(k;t,z)$, $z=(\boldsymbol{x}, \boldsymbol{y})$ of type k of sellers at $\boldsymbol{x}$ and expectations densities $Ex(t,z;l)$ of type l of buyers at $\boldsymbol{y}$ as:

$$Et(k;t,\boldsymbol{z}) = Ex(k;t,\boldsymbol{z})BS(k;t,\boldsymbol{z}) \tag{7.8}$$

$$Et(t,\boldsymbol{z};l) = Ex(t,\boldsymbol{z};l)BS(t,\boldsymbol{z};l) \tag{7.9}$$

Let's underline that expected transactions $Et(k;t,z)$ and expectations $Ex(t,z;l)$ (7.8; 7.9) are determined with respect to transactions with selected economic variable E. Transactions with different variables E – commodities, service, assets and etc., - define different expected transactions and expectations densities. Functions $Ex(k;t,\boldsymbol{x},\boldsymbol{y})$ (7.8) $z=(\boldsymbol{x}, \boldsymbol{y})$, describe sellers expectations of type k at point $\boldsymbol{x}$ for transactions $BS(k;t,\boldsymbol{x},\boldsymbol{y})$ (7.2) with economic variable E under sellers expectations of type k and for all expectations of buyers at $\boldsymbol{y}$. Functions $Ex(t,\boldsymbol{x},\boldsymbol{y};l)$ (7.9) describe buyers expectations of type l at point $\boldsymbol{y}$ for transactions $BS(t,\boldsymbol{x},\boldsymbol{y};l)$ (7.3) made under all expectations of Sellers at $\boldsymbol{x}$. To define expectations of sellers $Ex(k;t)$ and expectations of buyers $Ex(t;l)$ let's take integrals over economic domain (1.1; 1.2):

$$BS(k;t,\boldsymbol{x}) = \int d\boldsymbol{y}\, BS(k;t,\boldsymbol{x};\boldsymbol{y}) \;;\; BS(t,\boldsymbol{y};l) = \int d\boldsymbol{x}\, BS(t,\boldsymbol{x};\boldsymbol{y};l) \tag{8.1}$$

$$Et(k;t,\boldsymbol{x}) = \int d\boldsymbol{y}\, Et(k;t,\boldsymbol{x},\boldsymbol{y}) = Ex(k;t,\boldsymbol{x})BS(k;t,\boldsymbol{x}) \tag{8.2}$$

$$Et(t,\boldsymbol{y};l) = \int d\boldsymbol{x}\, Et(t,\boldsymbol{x},\boldsymbol{y};l) = Ex(t,\boldsymbol{y};l)BS(t,\boldsymbol{y};l) \tag{8.3}$$

$$BS(k;t) = \int d\boldsymbol{x}d\boldsymbol{y}\, BS(k;t,\boldsymbol{x};\boldsymbol{y}) \;;\; BS(t;l) = \int d\boldsymbol{x}d\boldsymbol{y}\, BS(t,\boldsymbol{x};\boldsymbol{y};l) \tag{8.4}$$

$$Et(k;t) = \int d\boldsymbol{x}d\boldsymbol{y}\, Et(k;t,\boldsymbol{x},\boldsymbol{y}) = Ex(k;t)BS(k;t) \tag{8.5}$$

$$Et(t;l) = \int d\boldsymbol{x}d\boldsymbol{y}\, Et(t,\boldsymbol{x},\boldsymbol{y};l) = Ex(k;t)BS(t;l) \tag{8.6}$$

Relations (8.1) define transactions $BS(k;t,\boldsymbol{x})$ with economic variable E performed by sellers at $\boldsymbol{x}$ under their expectations of type k with all buyers of entire economics. Buyers at $\boldsymbol{y}$ under their expectations of type l perform transactions $BS(t,\boldsymbol{y};l)$ (8.1) with all sellers of the entire economics. Relations (8.2) define expected transactions $Et(k;t,\boldsymbol{x})$ made by sellers at $\boldsymbol{x}$ under sellers expectations $Ex(k;t,\boldsymbol{x})$ of type k with all buyers of entire economics. Relations (8.3) define expected transactions $Et(t,\boldsymbol{y};l)$ made by buyers at $\boldsymbol{y}$ under buyers expectations $Ex(t,\boldsymbol{y};l)$ of type l with all sellers. Relations (8.4) define all transactions $BS(k;t)$ with economic variable E made in economics under sellers expectations of type k. Functions $BS(t;l)$ (8.4) define all transactions with economic variable E made in economics under buyers expectations of type l. Relations (8.5) define macroeconomic sellers expectations $Ex(k;t)$ of type k for the transactions $BS(k;t)$ with economic variable E. Relations (8.6) define macroeconomic buyers expectations $Ex(t;l)$ of type l for the transactions $BS(t;l)$ with economic variable E. Thus starting with definitions of expected transactions (7.1) and definitions of partial transactions $BS(k;t,\boldsymbol{x},\boldsymbol{y})$ (7.2) and $BS(t,\boldsymbol{x},\boldsymbol{y};l)$ (7.3) we derive reasonable definitions of macroeconomic expectations of sellers (8.5) and buyers (8.6) for transactions with economic variable E. Let's outline that expectations of type k play different role for transactions with different economic variables E and that makes observations and measurements of expectations a really complex problem.

Now let's describe how expected transactions and expectations can flow on economic space alike to flows of economic variables (I.6-10) and transactions flows (5.1-5.5). Motion of agents i and j at points $\boldsymbol{x}$ and $\boldsymbol{y}$ with velocities $\boldsymbol{v}_i(t,\boldsymbol{x})$ and $\boldsymbol{v}_j(t,\boldsymbol{y})$ on e-space due to change of their risk ratings induce flows $\boldsymbol{p}_{ij}(k;t,z)$ and $\boldsymbol{p}_{ij}(t,z;l)$ of expected transactions $et_{ij}(k;t,z)$ and $et_{ij}(t,z;l)$ (7.1) similar to flows $\boldsymbol{p}_{xij}(t,z)$ of transactions $bs_{ij}(t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$, as:

$$\boldsymbol{p}_{ij}(k;t,\boldsymbol{z};l) = \left(\boldsymbol{p}_{ij}(k;t,\boldsymbol{z}), \boldsymbol{p}_{ij}(t,\boldsymbol{z};l)\right) \quad ; \quad \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{9.1}$$

$$\boldsymbol{p}_{ij}(k;t,\boldsymbol{z}) = et_{ij}(k;t,\boldsymbol{z})\boldsymbol{v}_i(\boldsymbol{x}) = ex_i(k;t,\boldsymbol{x})bs_{i,j}(k;t,\boldsymbol{z})\boldsymbol{v}_i(\boldsymbol{x}) \tag{9.2}$$

$$\boldsymbol{p}_{ij}(t,\boldsymbol{z};l) = et_{ij}(t,\boldsymbol{z};l)\boldsymbol{v}_j(\boldsymbol{y}) = ex_j(t,\boldsymbol{y};l)bs_{i,j}(t,\boldsymbol{z};l)\boldsymbol{v}_j(\boldsymbol{y}) \tag{9.3}$$

Flows $\boldsymbol{p}_{ij}(k;t,z)$ describe amounts of expected transactions $et_{ij}(k;t,z)$ of type k carried by agent i in the direction of velocity $\boldsymbol{v}_i$. To define aggregate flows of expected transactions at points $\boldsymbol{x}$ and $\boldsymbol{y}$ let's collect flows $\boldsymbol{p}_{ij}(k;t,z)$ of expected transactions $et_{ij}(k;t,z)$ (9.2) of agents i in a unit $dV(t,\boldsymbol{x})$ (1.3-1.5) and flows $\boldsymbol{p}_{ij}(t,z;l)$ of expected transactions $et_{ij}(t,z;l)$ (9.3) of agents j in a unit volume $dV(t,\boldsymbol{y})$ and then average the sum during time Δ similar to (2.1;2.2; 5.4; 5.5) as:

$$\boldsymbol{P}(k;t,\boldsymbol{z};l) = \left(\boldsymbol{P}_{\boldsymbol{x}}(k;t,\boldsymbol{z}), \boldsymbol{P}_{\boldsymbol{y}}(t,\boldsymbol{z};l)\right) \quad ; \quad \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{9.4}$$

$$\boldsymbol{P}_x(k;t,\boldsymbol{z}) = \textstyle\sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} et_{ij}(k;t,\boldsymbol{z})\boldsymbol{v}_i(\boldsymbol{x}) \tag{9.5}$$

$$\boldsymbol{P}_y(t,\boldsymbol{z};l) = \textstyle\sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} et_{ij}(t,\boldsymbol{z};l)\boldsymbol{v}_j(\boldsymbol{y}) \tag{9.6}$$

$$\boldsymbol{P}_x(k;t,\boldsymbol{z}) = Et_E(k;t,\boldsymbol{z})\,\boldsymbol{v}_x(k;t,\boldsymbol{z}) = Ex(k;t,\boldsymbol{z})BS(k;t,\boldsymbol{z})\boldsymbol{v}_x(k;t,\boldsymbol{z}) \tag{9.7}$$

$$\boldsymbol{P}_y(t,\boldsymbol{z};l) = Et_E(t,\boldsymbol{z};l)\boldsymbol{v}_{\boldsymbol{y}}(t,\boldsymbol{z};l) = Ex(t,\boldsymbol{z};l)BS(t,\boldsymbol{z};l)\boldsymbol{v}_y(t,\boldsymbol{z};l) \tag{9.8}$$

$$\boldsymbol{v}(k;t,\boldsymbol{z};l) = \left(\boldsymbol{v}_x(k;t,\boldsymbol{z});\boldsymbol{v}_y(t,\boldsymbol{z};l)\right) \tag{9.9}$$

For transactions $BS(t,\boldsymbol{x},\boldsymbol{y})$ that describe deals with variable E (shares, commodities, service and etc.) from sellers at point $\boldsymbol{x}$ to buyers at point $\boldsymbol{y}$ relations (9.5) define aggregated flows $\boldsymbol{P}_x(k;t,z)$ of expected transactions of type k of sellers at point $\boldsymbol{x}$. Relations (9.6) define aggregated flows $\boldsymbol{P}_y(t,z;l)$ of expected transactions of type l of buyers at point $\boldsymbol{y}$. Relations (9.7-9.9) and expected transactions $Et(k;t,z)$ and $Et(t,z;l)$ (7.5-7.9) define velocities $\boldsymbol{v}_x(k;t,z)$ (9.7) of sellers at point $\boldsymbol{x}$ of expected transaction of type k and velocities $\boldsymbol{v}_y(t,z;l)$ (9.8) of buyers at point $\boldsymbol{y}$ of expected transaction of type l as function of $z=(\boldsymbol{x},\boldsymbol{y})$. Similar to definitions of macroeconomic flows of variables (I. 6-9) and macro flows of transactions (5.3-5.8) integrals by $dz=d\boldsymbol{x}d\boldsymbol{y}$ over economic domain (1.1; 1.2) of relations (9.4-9.9) define macroeconomic flows $\boldsymbol{P}_x(k;t)$, $\boldsymbol{P}_y(t;l)$ and macroeconomic velocities $\boldsymbol{v}_x(k;t)$ and $\boldsymbol{v}_y(t;l)$ of expected transactions $Et_x(k;t)$, $Et_y(t;l)$ and macroeconomic expectations $Ex_x(k;t)$, $Ex_y(t;l)$ as:

$$\boldsymbol{P}_x(k;t) = \int d\boldsymbol{z}\, Et_x(k;t,\boldsymbol{z})\boldsymbol{v}_x(k;t,\boldsymbol{z}) \tag{10.1}$$

$$\boldsymbol{P}_x(k;t) = Et_x(k;t)\,\boldsymbol{v}_x(k;t) = Ex_x(k;t)BS(k;t)\boldsymbol{v}_x(k;t) \tag{10.2}$$

$$\boldsymbol{P}_{\boldsymbol{y}}(\boldsymbol{t};l) = \int d\boldsymbol{z}\, Et_{\boldsymbol{y}}(t,\boldsymbol{z};l)\boldsymbol{v}_{\boldsymbol{y}}(\boldsymbol{t},\boldsymbol{z};l) \tag{10.3}$$

$$\boldsymbol{P}_y(t;l) = Et_y(t;l)\,\boldsymbol{v}_y(t;l) = Ex_y(t;l)BS(t;l)\boldsymbol{v}_y(t;l) \tag{10.4}$$

Relations (10.1-2) define macroeconomic flows of $\boldsymbol{P}_x(k;t)$ and macroeconomic velocities $\boldsymbol{v}_x(k;t)$ of expected transaction $Et_x(k;t)$ of type k for sellers of variable E. As well flows $\boldsymbol{P}_x(k;t)$ (10.2) describe motion of macroeconomic expectations $Ex_x(k;t)$ of type k for sellers of variable E. Relations (10.3-4) define macroeconomic flows $\boldsymbol{P}_y(t;l)$ and velocities $\boldsymbol{v}_y(t;l)$ of buyers expected transaction $Et_y(t;l)$ of type l and motion of buyers macroeconomic expectations $Ex_y(t;l)$ of type l. In other words, sellers expectations $Ex_x(k;t)$ of type k change in time due to motion on economic domain with velocity $\boldsymbol{v}_x(k;t)$. Borders of economic domain (1.1;1.2) reduce motion along risk axes and hence values and direction of sellers flows $\boldsymbol{P}_x(k;t)$ and velocities $\boldsymbol{v}_x(k;t)$ should fluctuate. That induce time oscillations of macroeconomic expectations $Ex_x(k;t)$ and transactions $BS(k;t)$ and should correlate with the business cycles induced by oscillations of flows $\boldsymbol{P}(t)$ and velocities $\boldsymbol{v}(t)$ (5.8).

Let's underline that velocities of $\boldsymbol{v}_x(t)$ of sellers and velocities $\boldsymbol{v}_y(t)$ of buyers (5.8) differs from velocities $\boldsymbol{v}_x(k;t)$ of sellers expectations $Ex_x(k;t)$ of type k and velocities $\boldsymbol{v}_y(t;l)$ of buyers expectations $Ex_y(t;l)$ of type l. Flows of different variables E, transactions and expectations have different velocities and their mutual interaction on economic domain reflect high complexity of real economic processes.

Definitions (7.5-7.7) of expected transactions $Et(k;t,z)$ and $Et(t,z;l)$ and definitions (9.4-9.6) of their flows $\boldsymbol{P}_x(k;t,z)$ and $\boldsymbol{P}_y(t,z;l)$ and definitions (9.7; 9.8) of their velocities $\boldsymbol{v}_x(k;t,z)$ and

$\boldsymbol{v}_y(t,z;l)$ allow take equations on expected transactions and their flows similar to equations on transactions and their flows (5.12; 5.13) as:

$$\frac{\partial}{\partial t}Et(k;t,\boldsymbol{z}) + \nabla\cdot\left(Et(k;t,\boldsymbol{z})\,\boldsymbol{v}_x(k;t,\boldsymbol{z})\right) = W_x(k;t,\boldsymbol{z}) \tag{10.5}$$

$$\frac{\partial}{\partial t}Et(t,\boldsymbol{z};l) + \nabla\cdot\left(Et(t,\boldsymbol{z};l)\,\boldsymbol{v}_{yl}(t,\boldsymbol{z};l)\right) = W_y(t,\boldsymbol{z};l) \tag{10.6}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_x(k;t,\boldsymbol{z}) + \nabla\cdot\left(\boldsymbol{P}_x(k;t,\boldsymbol{z})\,\boldsymbol{v}_x(k;t,\boldsymbol{z})\right) = \boldsymbol{R}_x(k;t,\boldsymbol{z}) \tag{10.7}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_y(t,\boldsymbol{z};l) + \nabla\cdot\left(\boldsymbol{P}_y(t,\boldsymbol{z};l)\,\boldsymbol{v}_y(t,\boldsymbol{z};l)\right) = \boldsymbol{R}_y(t,\boldsymbol{z};l) \tag{10.8}$$

Functions W_x, W_y and $\boldsymbol{R}_x$, $\boldsymbol{R}_y$ in equations (10.5-10.8) describe action of economic and financial variables, transactions and different expectations, technology, political and other factors that may impact change of expected transactions $Et(k;t,z)$ and $Et(t,z;l)$ and their flows $\boldsymbol{P}_x(k;t,z)$ and $\boldsymbol{P}_y(t,z;l)$ and hence impact change of expectations $Ex(k;t,z)$ and $Ex(t,z;l)$. That makes economic modeling a really exciting problem.

Equations (I.14; 17) on macroeconomic and financial variables $A(t,\boldsymbol{x})$ and their flows $\boldsymbol{P}_A(t,\boldsymbol{x})$, equations (5.12; 5.13) on transactions $BS(t,z)$ and transactions flows $\boldsymbol{P}(t,z)$ and equations (10.5-10.8) on expected transaction $Et(k;t,z)$ and $Et(t,z;l)$ and their flows $\boldsymbol{P}_x(k;t,z)$ and $\boldsymbol{P}_y(t,z;l)$ complete our approximation of macroeconomic evolution based on description of relations between macroeconomic and financial variables, transactions and expectations on economic space. It is obvious that description of any particular macroeconomic problem requires definition of right hand side of equations (I.14; 17), (5.12; 5.13), (10.5-10.8). All specifics and details of macroeconomic processes are hidden in and are determined by function $F_A(t,\boldsymbol{x})$ and $\boldsymbol{G}_A(t,\boldsymbol{x})$, $F(t,z)$ and $\boldsymbol{G}(t,z)$, W_x and W_x, W_y and $\boldsymbol{R}_x$, $\boldsymbol{R}_y$. We describe some particular economic problems in Part III.

7. Asset Pricing

Asset pricing is one of the most important problems of economics and finance. We refer (Cochrane and Hansen, 1992; Cochrane and Culp, 2003; Hansen, 2013; Campbell, 2014; Fama, 2014; Cochrane, 2017) as only small part of asset pricing studies.

Let's mention that in this paper we don't argue ***why*** asset prices should take certain values, but study ***how*** economic equations on variables, transactions expectations and their flows determine equations on asset prices. Below we show that expectations and economic flows induce equations on asset pricing and argue different definitions of transactions prices.

Above in Sec.5 and 6 we derive equations (5.12; 5.13) and (10.5-10.8) on transactions $BS(t,z)$ with economic variable E and expected transactions $Et(k;t,z)$ and $Et(t,z;l)$. As variable E one can take assets, investment, credits, commodities and etc. Meanwhile any economic transactions from agent i to agent j with particular asset or commodities implies payments for assets or commodities from agent j to agent i. Thus transactions with variable E between agents i and j should describe trading volume Q_{ij} from i to j and trading value or cost C_{ij} from j to i. For example let's assume that agent i sell $Q_{ij} = 100\ bbl.$ of Brent crude oil to agent j for $C_{ij} = 6000\ \$$. Thus Brent oil price p_{ij} of this particular transaction equals $p_{ij} = C_{ij}/Q_{ij} = 60\ \$/bbl.$ Let's treat transactions as two component functions and describe prices of separate deals between two agents. That helps describe prices of aggregate transactions between points $\boldsymbol{x}$ and $\boldsymbol{y}$ and prices aggregated over entire economics.

In Appendix A we give notion (A.1) of transaction as two component function. Transactions $\boldsymbol{BS}$ with variable E as two components function define trading volume Q and cost C of variable E:

$$\boldsymbol{BS}(\boldsymbol{k};t,\boldsymbol{z}) = \big(Q(k_1;t,\boldsymbol{z});C(k_2;t,\boldsymbol{z})\big) \ ; \quad \boldsymbol{k} = (k_1,k_2) \tag{11.1}$$

Relations (11.1) double the number of equations that describe transactions and expectations. Each transaction should be approved by sellers expectations. Sellers expectations of type k_1 approve trading volume $Q(k_1;t,z)$ and expectations of type k_2 approve trading value or cost $C(k_2;t,z)$ of transactions. Thus sellers expectations $\boldsymbol{k}=(k_1,k_2)$ approve price $p(\boldsymbol{k};t,z)$ (A.12.7) or (11.2) of variable E for the transaction $\boldsymbol{BS}$

$$C(k_2;t,\boldsymbol{z}) = p(\boldsymbol{k};t,\boldsymbol{z})Q(k_1;t,\boldsymbol{z}) \ ; \quad \boldsymbol{k} = (k_1,k_2) \tag{11.2}$$

All transactions transaction $\boldsymbol{BS}$ with variable E performed in economics at moment t define (A.12.14) price $p(t)$ as:

$$C(t) = p(t)Q(t) \tag{11.3}$$

In Appendix we derive equations that describe sellers transactions $\boldsymbol{BS}(\boldsymbol{k};t,z)$ (A.12.1) of type $\boldsymbol{k}=(k_1;k_2)$ made under sellers expectations $Ex_Q(k_1;t,z)$ of type k_1 (A.13.7) on trading volume $Q(k_1;t,z)$ (A.12.2) and sellers expectations $Ex_C(k_2;t,z)$ of type k_2 (A.13.8) on cost $C(k_2;t,z)$ of transaction (A.12.3). In other words – sellers expectations $Ex_Q(k_1;t,z)$ of type k_1 approve trading volumes $Q(k_1;t,z)$ (A.12.2) of variable E for transactions $\boldsymbol{BS}(\boldsymbol{k};t,z)$ (A.12.1). Sellers expectations $Ex_C(k_2;t,z)$ of type k_2 approve trading values or costs $C(k_2;t,z)$ (A.12.3) of transactions with variable E. We derive similar equations on buyers transactions of type $\boldsymbol{l}=(l_1;l_2)$ $\boldsymbol{BS}(t,z;\boldsymbol{l})$ (A.12.4) made under buyers expectations $Ex_Q(t,z;l_1)$ (A.13.9) of type l_1 on trading volumes $Q(t,z;l_1)$ (A.12.5) of variable E and buyers expectations $Ex_C(t,z;l_2)$ (A.13.10) on costs $C(t,z;l_2)$ (A.12.6) of type k_2.

Let's state that notion of price should always be treated in regard to definite transactions only. For example, sellers price $p(\boldsymbol{k};t,z)$ (A.12.7) or (11.2) correspond to all transactions made under sellers expectations of type $k=(k_1,k_2)$ at moment t between points $\boldsymbol{x}$ and $\boldsymbol{y}$; $z=(\boldsymbol{x},\boldsymbol{y})$. Definition of price $p(t,z)$ (A.12.9) corresponds to all transactions performed between points $\boldsymbol{x}$ and $\boldsymbol{y}$; $z=(\boldsymbol{x},\boldsymbol{y})$ under all expectations of sellers and buyers. Price $p(t)$ (A.12.14) or (11.3) corresponds to all transactions in economy made at moment t with variable E. Different definitions of price describe different states of prices due to different aggregations of transactions and cause different equations.

Economic equations on transactions $\boldsymbol{BS}(\boldsymbol{k};t,z)$ (A.18.1-4) made under sellers expectations and equations on transactions made under buyers expectations (A.19.1-4) describe evolution of transactions as two component functions and their flows. Further we derive equations on sellers expected transactions and their flows (A.20.1-4) and buyers expected transactions and their flows (A.21.1-4). Equations (A.18.1–21.4) complete system of equations on transactions and expected transactions and their flows under expectations of type $\boldsymbol{k}=(k_1;k_2)$ and $\boldsymbol{l}=(l_1;l_2)$.

Equations on transactions and their flows define equations on prices (A.12.7-16). For example, (A.22.3-4) define equations on sellers price $p(k_1,k_2;t)$ (A.12.7) for transactions (A.12.15) follows equations made in economics under expectations of type $\boldsymbol{k}=(k_1;k_2)$. Relations (A.23.1-6) define equations on price $p(t)$ (A.12.14) of all transactions made in economy at moment t with variable E.

$$\frac{d}{dt}Q(t) = F_Q(t) \quad ; \quad Q(t)\frac{d}{dt}p(t) + p(t)F_Q(t) = F_C(t) \tag{11.4}$$

$$Q(t)\frac{d}{dt}\boldsymbol{v}_Q(t) + F_Q(t)\boldsymbol{v}_Q(t) = \boldsymbol{G}_Q(t) \;; \quad Q(t)p(t)\frac{d}{dt}\boldsymbol{v}_C(t) + \boldsymbol{v}_C(t)F_C(t) = \boldsymbol{G}_C(t) \tag{11.5}$$

It is clear that representations (11.4, 11.5) allow model some cases and arise a lot of new problems. Asset pricing is too complex problem to be described by (11.4, 11.5) or by any definite equations only. In Appendix A we argue hidden complexity of (11.4; 11.5) and problems of equations on economic variables (Part I, 18.1, 18.2), on transactions (5.12; 5.13), on expected transactions (10.5-10.8). As well in Part III we apply equations (11.4, 11.5) to model some simple cases of price fluctuations.

Part III. Economic Waves, Business Cycles, Asset and Option Pricing

8. Economic waves

Wave propagation of small disturbances is one of most general properties of any complex systems. In this Sec. we describe wave propagation of small disturbances of density functions of economic variables and transactions on economic domain (1.1) of economic space (Olkhov, 2016a-2017c).

8.1. Waves of economic variables

Any model of economic phenomena implies definite approximation. In this Sec we assume that equations (I.14; 17) on density functions of economic variables and their flows depend on other economic variables only. To simplify the problem we study mutual interactions between two economic variables and their flows. Such approximation permits describe self-consistent model of mutual dependence between two variables and describe wave propagation of small disturbances of economic variables. Let's study wave propagation of disturbances of economic variables on economic space (Olkhov, 2016a-2017a). As example let's take familiar demand-price relations that propose price growth with rise of demand and demand decline as price increases. Let's derive equations that describe wave propagation of perturbations of price and demand. Demand $A(t,\boldsymbol{x})$ is additive variable and price $p(t,\boldsymbol{x})$ is non-additive. Supply $S(t,\boldsymbol{x})$ of assets, commodities, service can be measured in physical units as cars, shares, tons et., and in currency units. For simplicity let's assume that supply $S(t,\boldsymbol{x})$ measured in physical units is constant $S(t,\boldsymbol{x})=S$ - const., and supply $B(t,\boldsymbol{x})$ measured in currency units equals product of $S(t,\boldsymbol{x})$ and price $p(t,\boldsymbol{x})$

$$B(t,\boldsymbol{x}) = S\,p(t,\boldsymbol{x}) \;\; ; \;\; S - const \tag{1.2}$$

For such simplified assumptions demand $A(t,\boldsymbol{x})$ and supply $B(t,\boldsymbol{x})$ are additive variables and follow equations (I.14;17). We define flows of variables $A(t,\boldsymbol{x})$ and $B(t,\boldsymbol{x})$ in (I.6-10). Let's take equations (I.14; 17) on economic variables $A(t,\boldsymbol{x})$ and $B(t,\boldsymbol{x})$ and their flows $\boldsymbol{P_A}(t,\boldsymbol{x})$ and $\boldsymbol{P_B}(t,\boldsymbol{x})$:

$$\frac{\partial}{\partial t}A(t,\boldsymbol{x}) + \nabla\cdot\big(A(t,\boldsymbol{x})\,\boldsymbol{v}(t,\boldsymbol{x})\big) = F_A(t,\boldsymbol{x}) \tag{2.1}$$

$$\frac{\partial}{\partial t}B(t,\boldsymbol{x}) + \nabla\cdot\big(B(t,\boldsymbol{x})\,\boldsymbol{u}(t,\boldsymbol{x})\big) = F_B(t,\boldsymbol{x}) \tag{2.2}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_A(t,\boldsymbol{x}) + \nabla\cdot\big(\boldsymbol{P}_A(t,\boldsymbol{x})\,\boldsymbol{v}(t,\boldsymbol{x})\big) = \boldsymbol{G}_A(t,\boldsymbol{x}) \tag{2.3}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_B(t,\boldsymbol{x}) + \nabla\cdot\big(\boldsymbol{P}_B(t,\boldsymbol{x})\,\boldsymbol{u}(t,\boldsymbol{x})\big) = \boldsymbol{G}_B(t,\boldsymbol{x}) \tag{2.4}$$

$$\boldsymbol{P}_A(t,\boldsymbol{x}) = A(t,\boldsymbol{x})\,\boldsymbol{v}(t,\boldsymbol{x}) \;\; ; \;\; \boldsymbol{P}_B(t,\boldsymbol{x}) = B(t,\boldsymbol{x})\,\boldsymbol{u}(t,\boldsymbol{x}) \tag{2.5}$$

To describe Demand-Price model (2.1-2.5) let's define functions $F_A(t,\boldsymbol{x})$ and $F_B(t,\boldsymbol{x})$. Let's remind that ∇ - indicates gradient and $\nabla\cdot$ - indicates divergence. Let's assume that function $F_A(t,\boldsymbol{x})$ is proportional to time derivative of supply $B(t,\boldsymbol{x})$:

$$F_A(t,\boldsymbol{x}) = \alpha_1 \frac{\partial}{\partial t}B(t,\boldsymbol{x}) \;\; ; \;\; F_B(t,\boldsymbol{x}) = \alpha_2 \frac{\partial}{\partial t}A(t,\boldsymbol{x}) \;\; ; \;\; \alpha_1 < 0\,; \;\; \alpha_2 > 0 \tag{3.1}$$

and function $F_B(t,\boldsymbol{x})$ is proportional to time derivative of demand $A(t,\boldsymbol{x})$. These assumptions for $\alpha_1 < 0$ give simple model of demand decline with price growth and price growth with demand increase for $\alpha_2 > 0$. Indeed, due to assumption (1.2) supply $B(t,\boldsymbol{x})$ measured in currency units is proportional to price $p(t,\boldsymbol{x})$ and hence time derivative of supply $B(t,\boldsymbol{x})$ equals time derivative of price $p(t,\boldsymbol{x})$. To define functions $\boldsymbol{G}_A(t,\boldsymbol{x})$ and $\boldsymbol{G}_B(t,\boldsymbol{x})$ in equations (2.3; 2.4) let's take

$$\boldsymbol{G}_A(t,\boldsymbol{x}) = \beta_1 \nabla B(t,\boldsymbol{x}) \;;\; \boldsymbol{G}_B(t,\boldsymbol{x}) = \beta_2 \nabla A(t,\boldsymbol{x}) \;;\; \beta_1 < 0 \;;\; \beta_2 > 0 \tag{3.2}$$

Relations (3.2) propose that demand velocity $\boldsymbol{v}(t,\boldsymbol{x})$ decrease in the direction of economic domain with high supply prices (3.3) with

$$\nabla B(t,\boldsymbol{x}) > 0 \tag{3.3}$$

and (3.2) represents that supply velocity $\boldsymbol{u}(t,\boldsymbol{x})$ grows up in the direction of economic domain with high demand (3.4):

$$\nabla A(t,\boldsymbol{x}) > 0 \tag{3.4}$$

Thus equations (2.1-2.4) take form:

$$\frac{\partial}{\partial t} A(t,\boldsymbol{x}) + \nabla \cdot \big(A(t,\boldsymbol{x})\, \boldsymbol{v}(t,\boldsymbol{x})\big) = \alpha_1 \frac{\partial}{\partial t} B(t,\boldsymbol{x}) \tag{4.1}$$

$$\frac{\partial}{\partial t} B(t,\boldsymbol{x}) + \nabla \cdot \big(B(t,\boldsymbol{x})\, \boldsymbol{u}(t,\boldsymbol{x})\big) = \alpha_2 \frac{\partial}{\partial t} A(t,\boldsymbol{x}) \tag{4.2}$$

$$\frac{\partial}{\partial t} \boldsymbol{P}_A(t,\boldsymbol{x}) + \nabla \cdot \big(\boldsymbol{P}_A(t,\boldsymbol{x})\, \boldsymbol{v}(t,\boldsymbol{x})\big) = \beta_1 \nabla B(t,\boldsymbol{x}) \tag{4.3}$$

$$\frac{\partial}{\partial t} \boldsymbol{P}_B(t,\boldsymbol{x}) + \nabla \cdot \big(\boldsymbol{P}_B(t,\boldsymbol{x})\, \boldsymbol{u}(t,\boldsymbol{x})\big) = \beta_2 \nabla A(t,\boldsymbol{x}) \tag{4.4}$$

$$\alpha_1 < 0 \;;\; \alpha_2 > 0 \;;\; \beta_1 < 0 \;;\; \beta_2 > 0 \tag{4.5}$$

To derive equations that describe wave propagation of disturbances of demand and price let's take linear approximation for equations (4.1-4.4) for disturbances of demand $A(t,\boldsymbol{x})$ and price $p(t,\boldsymbol{x})$. Let's take disturbances as follows:

$$A(t,\boldsymbol{x}) = A_0\big(1 + \varphi(t,\boldsymbol{x})\big);\; B(t,\boldsymbol{x}) = Sp_0\big(1 + \pi(t,\boldsymbol{x})\big) \tag{5.1}$$

Relations (5.1) define dimensionless disturbances of demand $\varphi(t,\boldsymbol{x})$ and price $\pi(t,\boldsymbol{x})$. Let's take that velocities $\boldsymbol{v}(t,\boldsymbol{x})$ and $\boldsymbol{u}(t,\boldsymbol{x})$ are small and in linear approximation equations (4.1-4.4) take form:

$$\frac{\partial}{\partial t} \varphi(t,\boldsymbol{x}) + \nabla \cdot \boldsymbol{v}(t,\boldsymbol{x}) = \alpha_1 C \frac{\partial}{\partial t} \pi(t,\boldsymbol{x}) \;;\; C = \frac{Sp_0}{A_0} \tag{5.2}$$

$$C\left(\frac{\partial}{\partial t} \pi(t,\boldsymbol{x}) + \nabla \cdot \boldsymbol{u}(t,\boldsymbol{x})\right) = \alpha_2 \frac{\partial}{\partial t} \varphi(t,\boldsymbol{x}) \tag{5.3}$$

$$\frac{\partial}{\partial t} \boldsymbol{v}(t,\boldsymbol{x}) = \beta_1 C \nabla \pi(t,\boldsymbol{x}) \;;\; C \frac{\partial}{\partial t} \boldsymbol{u}(t,\boldsymbol{x}) = \beta_2 \nabla \varphi(t,\boldsymbol{x}) \tag{5.4}$$

In Appendix B we show that equations (5.2-5.4) can take form of equations (5.5) on disturbances of demand $\varphi(t,\boldsymbol{x})$ and price $\pi(t,\boldsymbol{x})$:

$$\left[(1 - \alpha_1\alpha_2)\frac{\partial^4}{\partial t^4} + (\alpha_1\beta_2 + \beta_1\alpha_2)\Delta \frac{\partial^2}{\partial t^2} - \beta_1\beta_2\Delta^2\right] \varphi(t,\boldsymbol{x}) = 0 \tag{5.5}$$

As we show in Appendix B for $\alpha_1\alpha_2<0$ for any negative $\beta_1<0$ there exist domain with positive $\beta_2>0$ for which equations on disturbances of demand $\varphi(t,\boldsymbol{x})$ and price $\pi(t,\boldsymbol{x})$ take form of bi-wave equation (5.6):

$$\left(\frac{\partial^2}{\partial t^2} - c_1^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - c_2^2\Delta\right)\varphi(t,\boldsymbol{x}) = 0 \tag{5.6}$$

with different values of wave speed c_1 and c_2 determined by α_1, α_2, β_1, β_2 (B.5; 6). Bi-wave equations (5.6) describe more complex wave propagation than common second order wave equations. In Appendix B we show that equations (5.6) allow wave propagation of price disturbances $\pi(t,\boldsymbol{x})$ (B.8) with exponential growth of amplitude as $exp(\gamma t)$. Thus exponential growth of small price disturbances $\pi(t,\boldsymbol{x})$ may disturb sustainable economic evolution.

8.2 Waves of transactions

Transactions and their flows are determined on economic domain (II.1.1; 1.2):

$$\boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \;;\; \boldsymbol{x} = (x_1 \dots x_n) \;;\; \boldsymbol{y} = (y_1 \dots y_n) \tag{6.1}$$

$$0 \le x_i \le 1 \,, i = 1, \dots n \;;\; 0 \le y_j \le 1 \,, j = 1, \dots n \tag{6.2}$$

and are described by (II.5.9; 5.10). Let's take transactions $S(t,z)$ at $z=(\boldsymbol{x},\boldsymbol{y})$ that describe supply of goods, commodities or assets from point $\boldsymbol{x}$ to $\boldsymbol{y}$ and may depend on macroeconomic variables, other transactions and expectations (Olkhov, 2017b). Self-consistent description of transactions, expectation, variables and other transaction is a too complex problem. Let's study simple self-consistent model of mutual interaction between two transactions and their flows. Let's assume that transaction $S(t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ supply goods or commodities from point $\boldsymbol{x}$ to point $\boldsymbol{y}$ as respond to demand $D(t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ for these commodities from point $\boldsymbol{y}$ to point $\boldsymbol{x}$. Let's assume that interactions between transactions $S(t,z)$ and $D(t,z)$ and their flows $\boldsymbol{P}(t,z)$ and $\boldsymbol{Q}(t,z)$ are described by functions $F_1(t,z)$, $F_2(t,z)$ and $\boldsymbol{G}_1(t,z)$, $\boldsymbol{G}_2(t,z)$ and depend only on each other and their flows. Both transactions follow equations alike to (II.5.9; 5.10). Let's define functions $F_1(t,z)$, $F_2(t,z)$ and $\boldsymbol{G}_1(t,z)$, $\boldsymbol{G}_2(t,z)$ for equations on $S(t,z)$ and $D(t,z)$ and flows $\boldsymbol{P}(t,z)$ and $\boldsymbol{Q}(t,z)$ respectively as (see 2.5):

$$F_1(t,\boldsymbol{z}) = \alpha_1 \nabla\cdot \boldsymbol{Q}(t,\boldsymbol{z}) \;;\; F_2(t,\boldsymbol{z}) = \alpha_2 \nabla\cdot \boldsymbol{P}(t,\boldsymbol{z}) \tag{6.3}$$

$$\boldsymbol{G}_1(t,\boldsymbol{z}) = \beta_1 \nabla D(t,\boldsymbol{z}) \;;\; \boldsymbol{G}_2(t,\boldsymbol{z}) = \beta_2 \nabla S(t,\boldsymbol{z}) \tag{6.4}$$

Economic meaning of (6.3; 6.4) is follows. Due to (II.5.6) flows $\boldsymbol{P}(t,z)$ and $\boldsymbol{Q}(t,z)$ looks as:

$$\boldsymbol{P}(t,\boldsymbol{z}) = S(t,\boldsymbol{z})\boldsymbol{v}(t,\boldsymbol{z}) \;;\; \boldsymbol{v}(t,\boldsymbol{z}) = (\boldsymbol{v}_x(t,\boldsymbol{z}); \boldsymbol{v}_y(t,\boldsymbol{z})) \tag{6.5}$$

$$\boldsymbol{Q}(t,\boldsymbol{z}) = D(t,\boldsymbol{z})\boldsymbol{u}(t,\boldsymbol{z}) \;;\; \boldsymbol{u}(t,\boldsymbol{z}) = (\boldsymbol{u}_x(t,\boldsymbol{z}); \boldsymbol{u}_y(t,\boldsymbol{z})) \tag{6.6}$$

Velocity $\boldsymbol{v}_x$ of supply flow $\boldsymbol{P}(t,z)$ describes motion of suppliers at and velocity $\boldsymbol{v}_y$ describe motion of consumers on economic domain. Divergence in (6.3) describes sources and run-off of flows in a unit volume

$$dV = dV_x dV_y$$

Volume dV_x describes a unit volume of variable $\boldsymbol{x}$ and dV_y describes a unit volume near variable $\boldsymbol{y}$**.** Transactions *S(t,z)*, $z=(\boldsymbol{x},\boldsymbol{y})$ supply goods from a unit volume dV_x near point $\boldsymbol{x}$ to a unit volume dV_y near $\boldsymbol{y}$. Transactions *D(t,z)* describe demand of goods from a unit volume dV_y near $\boldsymbol{y}$ to a unit volume dV_x near $\boldsymbol{x}$. Divergence in (6.3) equals:

$$\nabla\cdot\ \boldsymbol{Q}(t,\boldsymbol{z}) = \nabla_x\cdot\ \boldsymbol{Q}(t,\boldsymbol{x},\boldsymbol{y}) + \nabla_y\cdot\ \boldsymbol{Q}(t,\boldsymbol{x},\boldsymbol{y}) \tag{6.7}$$

Here $\boldsymbol{x}$-divergence $\nabla_x\cdot\ Q(t,\boldsymbol{x},\boldsymbol{y})$ describes sources and sinks of demand flow ***Q****(t,z)* of suppliers at point $\boldsymbol{x}$ in a unit volume dV_x . Divergence $\nabla_y\cdot\ Q(t,\boldsymbol{x},\boldsymbol{y})$ – describes sources and sinks of demand flow ***Q****(t,z)* of consumers of goods, those who generate demand at point $\boldsymbol{y}$ in a unit volume dV_y. Let's treat

$$\nabla_x\cdot\ \boldsymbol{Q}(t,x,y) < 0 \tag{6.8}$$

as sinks of demand flow into point $\boldsymbol{x}$ that is met by supply *S(t,z)* from point $\boldsymbol{x}$. Let's present divergence of supply flow ***P****(t,z)* (6.9) similar to (6.7):

$$\nabla\cdot\ \boldsymbol{P}(t,\boldsymbol{z}) = \nabla_x\cdot\ \boldsymbol{P}(t,\boldsymbol{x},\boldsymbol{y}) + \nabla_y\cdot\ \boldsymbol{P}(t,\boldsymbol{x},\boldsymbol{y}) \tag{6.9}$$

Here $\boldsymbol{x}$-divergence $\nabla_x\cdot P(t,\boldsymbol{x},\boldsymbol{y})$ describes sources and sinks of supply flow ***P****(t,z)* of from $\boldsymbol{x}$ in a unit volume dV_x. Relations (6.10)

$$\nabla_x\cdot\ \boldsymbol{P}(t,x,y) > 0 \tag{6.10}$$

describe sources of supply flow ***P****(t,z)* from point $\boldsymbol{x}$ to $\boldsymbol{y}$. Due to (6.3; 6.4) equations on transactions *S(t,z)* and *D(t,z)* take form similar to (II.5.9):

$$\frac{\partial}{\partial t}S + \nabla\cdot(S\ \boldsymbol{v}) = \alpha_1\ \nabla\cdot\ \boldsymbol{Q}(t,\boldsymbol{z}) \tag{7.1}$$

$$\frac{\partial}{\partial t}D + \nabla\cdot(D\ \boldsymbol{u}) = \alpha_2\ \nabla\cdot\ \boldsymbol{P}(t,\boldsymbol{z}) \tag{7.2}$$

and equations on flows ***P****(t,z)* and ***Q****(t,z)*

$$\boldsymbol{P}(t,\boldsymbol{z}) = S(t,\boldsymbol{z})\boldsymbol{v}(t,\boldsymbol{z})\ \ ;\ \ \boldsymbol{Q}(t,\boldsymbol{z}) = D(t,\boldsymbol{z})\boldsymbol{u}(t,\boldsymbol{z}) \tag{7.3}$$

on *2n*-dimensional economic domain $z=(\boldsymbol{x},\boldsymbol{y})$ take form similar to (II.5.10):

$$\frac{\partial}{\partial t}\boldsymbol{P}(t,\boldsymbol{z}) + \nabla\cdot\big(\boldsymbol{P}(t,\boldsymbol{z})\ \boldsymbol{v}(t,\boldsymbol{z})\big) = \beta_1\ \nabla D(t,\boldsymbol{z}) \tag{7.4}$$

$$\frac{\partial}{\partial t}\boldsymbol{Q}(t,\boldsymbol{z}) + \nabla\cdot\big(\boldsymbol{Q}(t,\boldsymbol{z})\ \boldsymbol{u}(t,\boldsymbol{z})\big) = \beta_2\ \nabla S(t,\boldsymbol{z}) \tag{7.5}$$

Equations (7.1; 7.2; 7.3; 7.4) cause equations on macroeconomic supply *S(t)* and demand *D(t)* (II.4.1). Functions *S(t)* and *D(t)* (7.6) describe macroeconomic supply and demand of selected goods, commodities etc.

$$S(t) = \int d\boldsymbol{x}d\boldsymbol{y}\ S(t,\boldsymbol{x},\boldsymbol{y})\ \ ;\ \ D(t) = \int d\boldsymbol{x}d\boldsymbol{y}\,D(t,\boldsymbol{x},\boldsymbol{y}) \tag{7.6}$$

$$\frac{d}{dt}S(t) = 0\ \ ;\ \ \frac{d}{dt}D(t) = 0\ ;\ \frac{d}{dt}\boldsymbol{P}(t) = 0\ \ ;\ \ \frac{d}{dt}\boldsymbol{Q}(t) = 0 \tag{7.7}$$

Relations (7.7) valid as integral of divergence over economic space equals zero due to divergence theorem (Gauss' Theorem) (Strauss, 2008, p.179) because no flows exist outside of economic domain and because transactions are equal zero outside of economic domain. Thus model interactions (6.3; 6.4) and equations (7.1-7.5) describe constant or slow-changing macroeconomic supply and demand, but allow model wave propagation of small disturbances of supply and demand. To derive wave equations let's study small perturbations of transactions *S(t,z)* and *D(t,z)* and assume that velocities $\boldsymbol{v}(t,z)$ and $\boldsymbol{u}(t,z)$ of supply and demand flows are small. Let's take:

$$S(t,\boldsymbol{z}) = S_0(1 + s(t,\boldsymbol{z}))\ ;\ D(t,\boldsymbol{z}) = D_0(1 + d(t,\boldsymbol{z})) \tag{7.8}$$

$$\boldsymbol{P}(t,\boldsymbol{z}) = S_0\boldsymbol{v}(t,\boldsymbol{z})\ ;\ \boldsymbol{Q}(t,\boldsymbol{z}) = D_0\boldsymbol{u}(t,\boldsymbol{z}) \tag{7.9}$$

and let's assume that velocities $\boldsymbol{v}(t,z)$ and $\boldsymbol{u}(t,z)$ in (7.9) are small. Relations (7.7) model S_0 and D_0 that are constant or slow-changing to compare with small disturbances *s(t,z)* and *d(t,z)*. Let's take equations (7.1; 7.2; 7.4; 7.5) in linear approximation by perturbations *s(t,z), d(t,z)* (7.8) and $\boldsymbol{v}(t,z)$ and $\boldsymbol{u}(t,z)$.

$$S_0\frac{\partial}{\partial t}s(t,\boldsymbol{z}) + S_0\nabla\cdot\boldsymbol{v} = \alpha_1 D_0\nabla\cdot\boldsymbol{u}\ \ ;\ D_0\frac{\partial}{\partial t}d(t,\boldsymbol{z}) + D_0\nabla\cdot\boldsymbol{u} = \alpha_2 S_0\nabla\cdot\boldsymbol{v} \tag{8.1}$$

$$S_0\frac{\partial}{\partial t}\boldsymbol{v}(t,\boldsymbol{z}) = \beta_1 D_0\nabla\, d(t,\boldsymbol{z})\ ;\ \ D_0\frac{\partial}{\partial t}\boldsymbol{u}(t,\boldsymbol{z}) = \beta_2 S_0\nabla\, s(t,\boldsymbol{z}) \tag{8.2}$$

Equations (8.1; 8.2) cause (see Appendix C, C.5) equations on *s(t,z), d(t,z)* (8.3):

$$\left[\frac{\partial^4}{\partial t^4} - a\Delta\,\frac{\partial^2}{\partial t^2} + b\Delta^2\,\right]s(t,\boldsymbol{z}) = 0 \tag{8.3}$$

Equations (8.3) may take form of bi-wave equation (C.7):

$$\left(\frac{\partial^2}{\partial t^2} - c_1^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - c_2^2\Delta\right)s(t,\boldsymbol{z}) = 0 \tag{8.4}$$

Wave propagation of small disturbances of supply *s(t,z)* and demand *d(t,z)* transactions induces wave propagation of disturbances of economic variables (C.14.1-C.16.5) determined by transactions $S(t,\boldsymbol{x},\boldsymbol{y})$ and $D(t,\boldsymbol{x},\boldsymbol{y})$. Bi-wave equations describe wave propagation of disturbances of economic variables induced by transactions and take form (C.17.3) similar to (8.4). Wave propagation of small disturbances of transactions induces fluctuations (C.18.1; 18.2) of macroeconomic variables *S(t)* and *D(t)* (7.6). As we show in Appendix B disturbances *s(t)* of macroeconomic supply *S(t)* at moment *t* may grow up as $exp(\gamma t)$ for $\gamma>0$ or dissipate to constant rate S_0 for $\gamma<0$ and fluctuate with frequency ω.

8.3 Economic surface-like waves

In Sections 8.1 and 8.2 we study wave propagation of small disturbances of densities functions of economic variables and transactions. These waves have parallels to sound waves in continuous media. Now let's show that disturbances of velocities of transactions flows may be origin of waves alike to surface waves in fluids (Olkhov, 2017c). Let's study simple model of economics under action of a single risk on *1*-dimensional economic space. Hence economic transactions are determined on *2*-dimensional economic domain (6.1; 6.2). Borders of economic domain establish bound lines for economic transactions. Disturbances of transactions near these bound lines may disturb bound lines and induce surface-like waves of along borders of economic domain. On other hand disturbances of transactions at bound lines may induce surface-like waves of perturbations that propagate inside economic domain and cause disturbances of transactions and economic variables far from borders of economic domain. Such surface-like waves may propagate along with growth of wave amplitude and thus impact of such waves of small perturbations may grow up in time. Thus description of economic surface-like waves may explain propagation and amplification of transactions disturbances near borders of economic domain. Let's remind that borders of economic domain are areas with maximum or minimum risk ratings. Thus, for example, perturbations of transactions near maximum risk ratings may propagate inside economic domain to areas with low risk ratings and growth of amplitudes of such perturbation may hardly disturb economic processes with low risk ratings.

For simplicity let's consider same example as in Sec. 8.2 and Appendix C. Let's take model relations between supply transactions $S(t,z)$ and Demand transactions $D(t,z)$ on economic domain (6.1; 6.2), $z=(x,y)$ and study small disturbances of transactions and flows similar to (7.8; 7.9) and equations (8.1; 8.2). Velocities of transactions on *2*-dimensinal economic domain take form:

$$\boldsymbol{v}(t,x,y) = \left(v_x(t,x,y); v_y(t,x,y)\right); \boldsymbol{u}(t,x,y) = \left(u_x(t,x,y); u_y(t,x,y)\right) \quad (9.1)$$

Let's take that transactions $D(t,z)$, $z=(x,y)$ transfer demand request from consumes at y to suppliers at x. Hence velocities υ_x and u_x along axis X describe motion of suppliers and velocities υ_y and u_y along Y describe motion of consumers of goods and services provided by suppliers. Let's study possible waves that can be generated by disturbances (7.8; 7.9) near border $y=1$ of economic domain (6.1; 6.2). Border $y=1$ describes consumers with maximum risks. Let's define perturbations of the border as $y=\xi(t,x)$. Interactions between transactions $S(t,z)$ and $D(t,z)$ require that border $y=\xi(t,x)$ should be common for both. Otherwise interaction between them will be violated. Time derivations of function $y=\xi(t,x)$ define y-velocities υ_y and u_y at $y=\xi(t,x)$ as:

$$\frac{\partial}{\partial t}\xi(t,x) = v_y(t,x,y = \xi(t,x)) = u_y(t,x,y = \xi(t,x)) \quad (9.2)$$

Time derivation (9.2) describes velocities υ_y of consumers with maximum risks and velocities

u_y of demanders of goods. Let's modify equations (8.2) and assume that near border $y=1$

$$S_0 \frac{\partial}{\partial t} \boldsymbol{v}(t, \boldsymbol{z}) = D_0(\beta_1 \nabla\, d(t, \boldsymbol{z}) + \boldsymbol{g})\, ; \;\; D_0 \frac{\partial}{\partial t} \boldsymbol{u}(t, \boldsymbol{z}) = S_0(\beta_2 \nabla\, s(t, \boldsymbol{z}) + \boldsymbol{h}) \qquad (9.3)$$

As $\boldsymbol{g}$ and $\boldsymbol{h}$ we introduce constant economic or financial *"accelerations"* $\boldsymbol{h}=(h_x,h_y)$ and $\boldsymbol{g}=(g_x, g_y)$ that act on economic agents, supply *S(t,z)* and demand *D(t,z)* transactions along axes *X* and *Y* and prevent agents from taking excess risk. Let's introduce functions *G* and *H*:

$$G(x, y) = g_x\, x + g_y y \;\; ; \; H(x, y) = h_x\, x + h_y y\, ; \; g_x\, , g_y, h_x, h_y - const \qquad (9.4)$$

Let's assume that potentials φ and ψ determine velocities $\boldsymbol{v}$ and $\boldsymbol{u}$ as:

$$\boldsymbol{v} = \nabla\, \varphi \;\; ; \;\; \boldsymbol{u} = \nabla\, \psi \qquad (9.5)$$

Thus equations (8.2) on velocities take form:

$$S_0 \frac{\partial}{\partial t} v_x = D_0(\beta_1 \frac{\partial}{\partial x} d - g_x)\, ; \;\; S_0 \frac{\partial}{\partial t} v_y = D_0(\beta_1 \frac{\partial}{\partial y} d - g_y) \qquad (9.6)$$

$$D_0 \frac{\partial}{\partial t} u_x = S_0 \left(\beta_2 \frac{\partial}{\partial x} s - h_x\right); \; B_0 \;\frac{\partial}{\partial t} u_y = S_0(\, \beta_2 \frac{\partial}{\partial y}\; s - h_y) \qquad (9.7)$$

Relations (9.5) allow present (9.6; 9.7) as

$$S_0 \frac{\partial}{\partial t} \frac{\partial}{\partial x} \varphi = D_0(\beta_1 \frac{\partial}{\partial x} d - g_x) \quad ; \;\; S_0 \frac{\partial}{\partial t} \frac{\partial}{\partial y} \varphi = D_0(\beta_1 \frac{\partial}{\partial y} d - g_y) \qquad (9.8)$$

$$D_0 \frac{\partial}{\partial t} \frac{\partial}{\partial x} \psi = S_0 \left(\beta_2 \frac{\partial}{\partial x} s - h_x\right) \quad ; \; D_0 \;\frac{\partial}{\partial t} \frac{\partial}{\partial y} \psi = S_0(\, \beta_2 \frac{\partial}{\partial y}\; s - h_y) \qquad (9.9)$$

Then (9.4) supply *s(t,x,y)* and demand *d(t,x,y)* transactions can be written as:

$$\beta_2 S_0 s(t, x, y) = S_0\big[h_x(x-1) + h_y(y-1)\big] + D_0 \frac{\partial}{\partial t}\; \psi(t, x, y) \qquad (10.1)$$

$$\beta_1 D_0 d(t, x, y) = D_0\big[g_x(x-1) + g_y(y-1)\big] + S_0 \frac{\partial}{\partial t}\; \varphi(t, x, y) \qquad (10.2)$$

For $\varphi=\psi=0$ (10.1; 10.2) describe steady state of supply *s(t,x,y)* and demand *d(t,x,y)* perturbations and on border $y=1$ *s(t,x,y)* and *d(t,x,y)* take form (10.3):

$$\beta_2 s(t, x, 1) = h_x(x-1)\, ; \;\; \beta_1 d(t, x, 1) = g_x(x-1) \qquad (10.3)$$

On surface $y= \xi(t,x)$ disturbances *s(t,x,y)* and *d(t,x,y)* take form:

$$\beta_2 S_0 s(t, x, y)|_{y=\,\xi(t,x)} = S_0\big[h_x(x-1) + h_y(\xi(t, x) - 1)\big] + \; D_0 \frac{\partial}{\partial t}\; \psi\big(t, x, \xi(t, x)\big) \qquad (10.4)$$

$$\beta_1 D_0 d(t, x, y)|_{y=\,\xi(t,x)} = D_0\big[g_x(x-1) + g_y(\xi(t, x) - 1)\big] + S_0 \frac{\partial}{\partial t}\; \varphi\big(t, x, \xi(t, x)\big) \qquad (10.5)$$

Let's propose that perturbations $y= \xi(t,x)$ near $y=1$ are small and assume that *s(t,x,y)* and *d(t,x,y)* take values *s(t,x,1)* and *d(t,x,1)* in a steady state for $\varphi=\psi=0$ on $y=1$ (10.3). Hence from (10.4; 10.5) obtain:

$$S_0 h_y(\xi(t, x) - 1) = -\, D_0 \frac{\partial}{\partial t}\; \psi\big(t, x, \xi(t, x)\big) \qquad (10.6)$$

$$D_0 g_y(\xi(t, x) - 1) = -S_0 \frac{\partial}{\partial t}\; \varphi\big(t, x, \xi(t, x)\big) \qquad (10.7)$$

Hence obtain:

$$\xi(t,x) - 1 = -\frac{D_0}{S_0 h_y}\frac{\partial}{\partial t}\psi\big(t,x,\xi(t,x)\big) = -\frac{S_0}{D_0 g_y}\frac{\partial}{\partial t}\varphi\big(t,x,\xi(t,x)\big) \tag{10.8}$$

Equations (10.8) determine relations between h_y and g_y

$$S_0^{\,2} h_y = D_0^{\,2} g_y$$

$$\frac{\partial}{\partial t}\xi(t,x) = \frac{\partial}{\partial y}\psi = \frac{\partial}{\partial y}\varphi = -\frac{S_0}{D_0 g_y}\frac{\partial^2}{\partial t^2}\varphi\big(t,x,y=\xi(t,x)\big) \tag{10.9}$$

Equation (10.9) describes constraints on potentials φ and ψ at $y=\xi(t,x)$. To derive equations on potentials φ and ψ let's substitute (10.1; 10.2) into (8.1) and neglect all non-linear terms with potentials and financial "*accelerations*". Equations on φ and ψ take form:

$$S_0\left(\frac{\partial^2}{\partial t^2} - \alpha_2\beta_1\Delta\right)\varphi = -\beta_1 D_0\Delta\psi \;;\; D_0\left(\frac{\partial^2}{\partial t^2} - \alpha_1\beta_2\Delta\right)\psi = -\beta_2 S_0\Delta\varphi \;;\; \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{11.1}$$

From (11.1) obtain:

$$\left[\left(\frac{\partial^2}{\partial t^2} - \alpha_2\beta_1\,\Delta\right)\left(\frac{\partial^2}{\partial t^2} - \alpha_1\beta_2\,\Delta\right) - \beta_1\beta_2\Delta^2\right]\varphi = 0 \tag{11.2}$$

Let's take functions φ and ψ *as*:

$$\varphi = \psi = cos(kx - \omega t)\, f(y-1) \;;\; f(0) = 1 \tag{11.3}$$

Let's take into account that perturbations $\xi(t,x)$ near steady boundary $y=X$ are small and hence relations (10.9) for (11.3) at $y=1$ give:

$$\frac{\partial}{\partial y} f(0) = \frac{S_0\omega^2}{D_0 g_y} > 0 \tag{11.4}$$

and substitute (11.3) into (11.2). Then (C.17.2) obtain equation on function $f(y)$ as ordinary differential equation of forth order :

$$\left(q_4\frac{\partial^4}{\partial y^4} + q_2\frac{\partial^2}{\partial y^2} + q_0\right) f(y) = 0 \tag{11.5}$$

$$q_4 = b \;;\; q_2 = a\omega^2 - 2bk^2 \;;\; q_0 = \omega^4 - a\omega^2 k^2 + bk^4 \tag{11.6}$$

Characteristic equation (11.7) of equation (11.5)

$$q_4\gamma^4 + q_2\gamma^2 + q_0 = 0 \tag{11.7}$$

defines roots γ^2:

$$\gamma_{1,2}^2 = \frac{-q_2 +/- \sqrt{q_2^2 - 4q_0 q_4}}{2q_4} = \frac{-q_2 +/- \omega^2\sqrt{a^2 - 4b}}{2b} \tag{11.8}$$

For single positive root $\gamma>0$ obtain simplest potentials φ and ψ as:

$$\varphi = \psi = cos(kx - \omega t)\, exp(\gamma(y-1)) \;;\; \gamma = \frac{S_0\omega^2}{D_0 g_y} > 0 \tag{12.1}$$

Function $y=\xi(t,x)$ (10.8) takes form:

$$\xi(t,x) = 1 - \frac{S_0\omega}{D_0 g_y} sin(kx - \omega t) = 1 - \sqrt{\frac{S_0\gamma}{D_0 g_y}}\, sin(kx - \omega t) \tag{12.2}$$

Border *y=1* define position of consumers for supply transactions *s(t,x,y)* and consumers as origin of demand for demand transactions *d(t,x,y)*. Supply *s(t,x,y)* and demand *d(t,x,y)* waves at stationary border *y=1* take form:

$$\beta_2 S_0 s(t,x,1) = S_0 h_x (x-1) + D_0 \omega\, sin(kx - \omega t) \tag{12.3}$$

$$\beta_1 D_0 d(t,x,1) = D_0 g_x (x-1) + S_0 \omega\, sin(kx - \omega t) \tag{12.4}$$

Surface-like waves of supply transactions *s(t,x,1)* (12.3) reflect change of supply for consumers at *y=1* from suppliers at *x*. Relations (12.4) describe change of demand from consumers at *y=1* to suppliers at *x*. Integral of supply transactions *s(t,x,1)* by *dx* (12.3) along border *y=1* over *(0,1)* define supply *s(t,1)* at risk border *y=1* as function of time:

$$\beta_2 S_0 s(t,1) = S_0[1 - \frac{h_x}{2}] + 2\frac{D_0\omega}{k} sin\left(\omega t - \frac{k}{2}\right) sin\left(\frac{k}{2}\right) \tag{12.5}$$

Function *s(t,1)* (12.5) describes fluctuations of supply to consumers at *y=1* with frequency ω from all suppliers of the economy. Simplest solution (12.1) with $\gamma>0$ describe exponential dissipation of disturbances induced by surface-like waves inside macro domain *y<1*.

Actually there might be surface-like waves that describe amplification of disturbances at *y=1* inside economic domain along axis *Y* for *y<<1*. For root $\gamma^2>0$ (11.8) let's take two roots:

$$\gamma_{1,2} = +/- \sqrt{\gamma^2}$$

Then from (11.3; 11.4) obtain:

$$f(0) = \lambda_1 + \lambda_2 = 1 \;\; ; \;\; \frac{\partial}{\partial y} f(0) = \gamma(\lambda_1 - \lambda_2) = \frac{S_0\omega^2}{D_0 g_y} > 0$$

$$\lambda_1 = \frac{1}{2} + \frac{S_0\omega^2}{2\gamma D_0 g_y} \;\; ; \;\; \lambda_2 = \frac{1}{2} - \frac{S_0\omega^2}{2\gamma D_0 g_y}$$

$$\varphi = \psi = cos(kx - \omega t)\,[\lambda_1 \exp(\gamma(y-1)) + \lambda_2 \exp(-\gamma(y-1))]$$

$$\beta_2 S_0 s(t,x,y) = S_0\big[h_x(x-1) + h_y(y-1)\big] + \omega D_0\; sin(kx - \omega t)\,[\lambda_1\, exp(\gamma(y-1)) + \lambda_2\, exp(-\gamma(y-1))]$$

$$\beta_1 D_0 d(t,x,y) = D_0\big[g_x(x-1) + g_y(y-1)\big] + \omega S_0\, sin(kx - \omega t)\,[\lambda_1\, exp(\gamma(y-1)) + \lambda_2\, exp(-\gamma(y-1))]$$

and supply *s(t,x,y)* and demand *d(t,x,y)* transactions grow up as exponent for *(y-1)<0*

$$s(t,x,y) \sim d(t,x,y) \sim \lambda_2\, exp(-\gamma(y-1)) \tag{12.6}$$

This example shows that small disturbances of supply to consumers at *y=1* may induce exponentially growing (12.6) disturbances of supply and demand at *y<1* far from risk border. Suppliers at *x* may stop provide goods to consumers at *y* with high risks at border *y=1* and redirect their supply to more secure consumers with *y<1*.

9 Business cycles

In Sec 8 we show that waves of small disturbances of economic variables or transactions on economic domain (6.1; 6.2) induce time fluctuations of small perturbations of macroeconomic variables. Velocities of these waves define time scales of such fluctuations. Let's call these economic fluctuations as "fast" contrary to "slow" fluctuations of economic variables noted as business cycles. In this section we show that "slow" fluctuations of flows of variables and transactions can cause oscillations of credits, investment, demand and economic growth noted as business cycles. Business cycles as slow fluctuations of macroeconomic and financial variables as GDP, investment, demand and etc., for decades are under permanent research (Tinbergen, 1935, Schumpeter, 1939, Lucas, 1980, Kydland & Prescott, 1991, Zarnowitz, 1992, Diebold & Rudebusch, 1999; Kiyotaki, 2011; Jorda, Schularick & Taylor, 2016). Below we present approximation of the business cycles induced by flows of economic transactions (Olkhov, 2017b; 2019a). For simplicity let's take same supply *S(t,z)* and demand *D(t,z)* transactions as in Sec.8 and let's describe business cycles of supply and demand. Let's take equations on *S(t,z)* and *D(t,z)* similar to (II. 5.9; 5.10) as:

$$\frac{\partial}{\partial t}S + \nabla\cdot(S\boldsymbol{v}) = F_S(t,\boldsymbol{z})\ ;\ \ \frac{\partial}{\partial t}D + \nabla\cdot(D\boldsymbol{u}) = F_D(t,\boldsymbol{z}) \qquad (13.1)$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_S + \nabla\cdot(\boldsymbol{P}_S\,\boldsymbol{v}) = \boldsymbol{G}_S(t,\boldsymbol{z})\ \ ;\ \ \frac{\partial}{\partial t}\boldsymbol{P}_D + \nabla\cdot(\boldsymbol{P}_D\,\boldsymbol{u}) = \boldsymbol{G}_D(t,\boldsymbol{z}) \qquad (13.2)$$

For simplicity let's study economic evolution under action of a single risk similar to Sec.8.3 and study business cycles on 2-dimensional economic domain (6.1; 6.2). Thus coordinates *x* describe evolution of suppliers with economic variable *E* and *y* evolution of consumers of variable *E, z=(x,y)*. As variable *E* one may study any goods, commodities, credits, service, shares, assets and etc. To simplify model calculations let's assume that supply transactions *S(t,z)* and their flows $\boldsymbol{P}_S(t,z)$ depend on demand *D(t,z)* transactions and their flows $\boldsymbol{P}_D(t,z)$ only. We propose that demand transactions *D(t,z)* describe demand from consumers of variable *E* at *y* to suppliers at *x*. Let's take F_S and F_D for (13.1) as (*a* and *b* – const):

$$F_S(t,\boldsymbol{z}) = a\,\boldsymbol{z}\cdot\boldsymbol{P}_D(t,\boldsymbol{z}) = a(\,x\cdot P_{Dx}(t,\boldsymbol{z}) + y\cdot P_{Dy}(t,\boldsymbol{z})) \qquad (13.3)$$

$$F_D(t,\boldsymbol{z}) = b\,\boldsymbol{z}\cdot\boldsymbol{P}_S(t,\boldsymbol{z}) = b(\,x\cdot P_{Sx}(t,\boldsymbol{z}) + y\cdot P_{Sy}(t,\boldsymbol{z})) \qquad (13.4)$$

Relations (13.3-13.4) describe model with supply *S(t,z)* growth up if F_S is positive and hence (13.3) for $a>0$ is positive if at least one component of demand velocities

$$\boldsymbol{u}(t,\boldsymbol{z}) = \left(u_x(t,\boldsymbol{z});\ u_y(t,\boldsymbol{z})\right) \qquad (13.5)$$

direct from safer to risky direction. In other words: if demand transactions *D(t,z)* flew into risky direction that can increase supply *S(t,z).* As well negative value of (13.3) models demand flows from risky to secure domain and cause decrease supply *S(t,z)* as suppliers may prefer more secure consumers. Such assumptions simplify relations between suppliers and consumers and neglect time gaps between providing supply from *x* to consumers at *y* and receiving demand from consumers at *y* to suppliers at *x* and neglect other factors that impact supply. Actually we neglect direct dependence of economic variables and transactions on risk coordinates of economic domain. This assumption simplifies the model and allows outline impact of mutual interactions between transactions *S(t,z)* and *D(t,z)* and their flows on the business cycle fluctuations of variable *E*. Let's take $\boldsymbol{G}_S(t,z)$ and $\boldsymbol{G}_D(t,z)$ for (13.2) as:

$$\boldsymbol{G}_{Sx}(t,\boldsymbol{z}) = c_x P_{Dx}(t,\boldsymbol{z}) \;\; ; \;\; \boldsymbol{G}_{Sy}(t,\boldsymbol{z}) = c_y P_{Dy}(t,\boldsymbol{z}) \quad (13.6)$$

$$\boldsymbol{G}_{Dx}(t,\boldsymbol{z}) = d_x P_{Sx}(t,\boldsymbol{z}) \;\; ; \;\; \boldsymbol{G}_{Dy}(t,\boldsymbol{z}) = d_y P_{Sy}(t,\boldsymbol{z}) \quad (13.7)$$

Equations (13.2; 13.6; 13.7) describe simple linear dependence between transaction flows $\boldsymbol{P}_S(t,z)$ and $\boldsymbol{P}_D(t,z)$. Integrals by *dz* over economic domain (6.1; 6.2) for components of flows due to (II. 4.1; 5.6; 5.7; 5.8) equal:

$$\boldsymbol{P}_S(t) = \int d\boldsymbol{z}\; \boldsymbol{P}_S(t,\boldsymbol{z}) = \int dxdy\;\; S(t,\boldsymbol{z})\boldsymbol{v}(t,\boldsymbol{z}) = S(t)\boldsymbol{v}(t) \;\; ; \;\; \boldsymbol{v} = (v_x; v_y) \quad (13.8)$$

$$\boldsymbol{P}_D(t) = \int d\boldsymbol{z}\; \boldsymbol{P}_D(t,\boldsymbol{z}) = \int dxdy\;\; D(t,\boldsymbol{z})\boldsymbol{u}(t,\boldsymbol{z}) = D(t)\boldsymbol{u}(t) \;\; ; \;\; \boldsymbol{u} = (u_x; u_y) \quad (13.9)$$

$$S(t) = \int dxdy\;\; S(t,x,y) \;\; ; \;\; D(t) = \int dxdy\;\; D(t,x,y) \quad (13.10)$$

As we show in Appendix D, distributions of economic agents by their risk ratings as coordinates on economic domain permit derive mean risk coordinates for each economic variable of transactions (Olkhov, 2017d; 2019a). Relations (D.2.3) define mean risk $X_S(t)$ of suppliers *S(t)* with economic variable *E* and mean risk $Y_C(t)$ of consumers of variable *E*:

$$S(t)X_S(t) = \int dxdy\;\; x\, S(t,x,y) \;\; ; \;\; S(t)Y_C(t) = \int dxdy\;\; y\, S(t,x,y) \quad (14.1)$$

We argue the business cycles of economic variables *E* (credit, investment, assets, commodities and etc.,) as processes induced and correlated with fluctuations of mean risks $X_S(t)$ of suppliers and mean risk $Y_C(t)$ of consumers of variable *E*. Flows of economic transactions of supply $\boldsymbol{P}_S(t)$ and action (13.3, 13.4) of demand flows $\boldsymbol{P}_D(t)$ cause fluctuations of mean risks $X_S(t)$ of suppliers and consumers $Y_C(t)$ as well as mean risks of demanders $Y_D(t)$ and $X_D(t)$ (14.2, 13.10):

$$D(t)X_D(t) = \int dxdy\;\; x\, D(t,x,y) \;\; ; \;\; D(t)Y_D(t) = \int dxdy\;\; y\, D(t,x,y) \quad (14.2)$$

We show in Appendix D (D.2.5-2.7) mean risk $X_S(t)$ (14.1) moves as

$$\frac{d}{dt}X_S(t) = v_x(t) + w_x(t) \quad (14.3)$$

$$w_x(t) = [X_{SF}(t) - X_S(t)]\frac{d}{dt}lnS(t) \quad (14.4)$$

$$F_S(t) = \int dxdy\, F_S(t,x,y) \quad ; \quad X_{SF}(t)F_S(t) = \int dxdy\;\; x\, F_S(t,x,y) \quad (14.5)$$

Borders of economic domain (6.1, 6.2) reduce motion of mean risks (14.1,14.3) and thus velocities $\upsilon_x(t)$ (13.8) and $w_x(t)$ (14.4) should fluctuate and cause oscillations of mean risks. Frequencies of $\upsilon_x(t)$ describe impact of flow fluctuations and frequencies of $w_x(t)$ describe oscillations induced by interactions between supply and demand transactions. In Appendix D we study model equations (D.2.1-2.2) that describe fluctuations of macro supply $S(t)$ (D.1.4) with variable E determined by flows $\boldsymbol{P}_S(t)$, $\boldsymbol{P}_D(t)$ (C.3.4-3.5) and derive relations for $S(t)$ (D.5.6) in simple form as:

$$S(t) = S(0) + a\left[S_x(1)\sin\omega t + S_y(1)\sin\nu t\right] + a\,S_x(3)exp\,\gamma t \qquad (14.6)$$

Relations (14.6) model the business cycles with frequencies ω and ν of macro supply $S(t)$ with variable E accompanied by exponential growth as $exp(\gamma t)$ due to economic growth of $S(t)$. Hence (14.6) may model credit cycles determined by fluctuations of creditors with frequencies ω and borrowers with frequencies ν with exponential growth as $exp(\gamma t)$ of credits provided in economy due to economic growth. The same approach may model investment cycles, consumption cycles and etc.

10. Expectations, Assets Price and Return

Assets pricing is the key issue of modern finance. Assets pricing research account thousands studies and we chose (Campbell, 1985; Campbell and Cochrane, 1995; Heaton and Lucas, 2000; Cochrane, 2001; Cochrane and Culp, 2003; Cochrane, 2017) for clear, precise and general treatment of the problem. Expectations as factors that impact assets pricing are studied at least since Muth (1961) and (Fama, 1965; Lucas, 1972; Sargent and Wallace, 1976; Hansen and Sargent, 1979; Blume and Easley, 1984; Brunnermeier and Parker, 2005; Dominitz and Manski, 2005; Greenwood and Shleifer, 2014; Lof, 2014; Manski, 2017). Assets pricing and return are studied by (Keim and Stambaugh, 1986; Mandelbrot, Fisher and Calvet, 1997; Brock and Hommes, 1998; Fama, 1998; Plerou et.al., 1999; Andersen et.al., 2001; Gabaix et.al., 2003; Stanley et.al., 2008; Hansen, 2013; Greenwald, Lettau and Ludvigson, 2014) and present only small part of publications. Below we study a simple case and describe possible impact of expectations on transactions, assets pricing and return (Olkhov, 2018; 2019b).

Let's study transactions with particular assets E at Exchange. Let's assume that agents perform different parts of transactions with assets E at Exchange under different expectations. Each transaction defines quantity Q of assets E (for example number of shares) and cost or value C of the deal. Obvious relations define assets price p of this transaction:

$$C = pQ$$

Transactions performed under different expectations may have different quantity, cost and asset price. Let's assume that agent i at point $\boldsymbol{x}$ have $k,l=1,..K$ different expectations $\boldsymbol{ex}_i(k,l;t,\boldsymbol{x})$ that approve transactions $\boldsymbol{bs}_i(k,l;t,\boldsymbol{x})$ of asset E with Exchange:

$$\boldsymbol{bs}_i(k,l;t,\boldsymbol{x}) = \big(Q_i(k;t,\boldsymbol{x}); C_i(l;t,\boldsymbol{x})\big) \tag{15.1}$$

Here $Q_i(k;t,\boldsymbol{x})$ and $C_i(l;t,\boldsymbol{x})$– quantity and cost of transaction performed by agent i under expectation k,l. We propose that decision on quantity $Q_i(k;t,\boldsymbol{x})$ of transaction is taken under expectation of type k and decision on cost $C_i(l;t,\boldsymbol{x})$ of transaction is taken under expectation of type l. Let's define expectations $\boldsymbol{ex}_i(k,l;t,\boldsymbol{x})$ as:

$$\boldsymbol{ex}_i(k,l;t,\boldsymbol{x}) = \Big(ex_{Qik}(k;t,\boldsymbol{x}), ex_{Ci}(l;t,\boldsymbol{x})\Big);\ k,l = 1,\dots K \tag{15.2}$$

Expectations $ex_{Qi}(k;t,\boldsymbol{x})$ and $ex_{Ci}(l;t,\boldsymbol{x})$ approve quantity Q and cost C of the transaction $\boldsymbol{bs}_i(k,l;t,\boldsymbol{x})$. Relations (II, 2.1, 2.2, 7.2) for define macro transaction $\boldsymbol{BS}(k,l;t,\boldsymbol{x})$ under expectation of type $k,l=1,\dots K$ as

$$\boldsymbol{BS}(k,l;t,\boldsymbol{x}) = \big(Q(k;t,\boldsymbol{x}); C(l;t,\boldsymbol{x})\big) = \sum_{i\in dV(\boldsymbol{x});\,\Delta} \boldsymbol{bs}_i(k,l;t,\boldsymbol{x}) \tag{15.3}$$

$$Q(k;t,\boldsymbol{x}) = \sum_{i\in dV(\boldsymbol{x});\Delta} Q_i(k;t,\boldsymbol{x}) \quad ; \quad C(t,\boldsymbol{x}) = \sum_{i\in dV(\boldsymbol{x});\Delta} C_i(l;t,\boldsymbol{x})$$

Similar to (II, 7.5-7.7) let's introduce expected transactions $\boldsymbol{Et}(k,l;t,\boldsymbol{x})$at point $\boldsymbol{x}$ as

$$\boldsymbol{Et}(k,l;t,\boldsymbol{x}) = (Et_Q(k;t,\boldsymbol{x}); Et_C(l;t,\boldsymbol{x})) \tag{15.4}$$

$$Et_Q(t,\boldsymbol{x}) = \sum_{i\in dV(\boldsymbol{x});\Delta} ex_{Qi}(k;t,\boldsymbol{x})Q_i(k;t,\boldsymbol{x})$$

$$Et_C(l;t,\boldsymbol{x}) = \sum_{i\in dV(\boldsymbol{x});\Delta} ex_{Ci}(l;t,\boldsymbol{x})C_i(l;t,\boldsymbol{x})$$

Let's study relations between transactions $\boldsymbol{BS}(k,l;t)$ (15.3) and expected transactions $\boldsymbol{Et}(k,l;t)$ (15.4) of entire economics as functions of time t only:

$$\boldsymbol{BS}(k,l;t) = \int d\boldsymbol{x}\ \ \boldsymbol{BS}(k,l;t,\boldsymbol{x}) \ ; \ \ \boldsymbol{Et}(k,l;t) = \int d\boldsymbol{x}\ \ \boldsymbol{Et}(k,l;t,\boldsymbol{x})\,; k,l = 1,..K \tag{15.5}$$

Integrals in (15.5) define $\boldsymbol{BS}(k,l;t)$ all transactions with asset E made by all agents of entire economics at Exchange under expected transactions $\boldsymbol{Et}(k,l;t)$. Due to equations (5.1-5.3), (8.1, 8.2) and (9.1, 9.2) equations on (15.5) take form:

$$\frac{d}{dt}Q(k;t) = F_Q(k;t)\ ;\ \frac{d}{dt}C(l;t) = F_C(l;t) \tag{15.6}$$

$$\boldsymbol{F}(k;t) = (F_Q; F_C);\ \ F_Q(k;t) = \int d\boldsymbol{x}\ F_Q(k;t,\boldsymbol{x})\ ;\ F_C(l;t) = \int d\boldsymbol{x}\, F_C(l;t,\boldsymbol{x}) \tag{15.7}$$

$$\frac{d}{dt}Et_Q(k;t) = Fe_Q(k;t)\ ;\ \frac{d}{dt}Et_C(l;t) = Fe_C(l;t) \tag{15.8}$$

$$\boldsymbol{Fe}(k,l;t) = \big(Fe_Q; Fe_C\big)\,; Fe_Q(k;t) = \int d\boldsymbol{x}\ Fe_Q(k;t,\boldsymbol{x}); Fe_C(l;t) = \int d\boldsymbol{x}\, Fe_C(l;t,\boldsymbol{x}) \tag{15.9}$$

Relations (15.1-15.3) define expectations $Ex_{kl}(t)$ of entire economics as:

$$\boldsymbol{Ex}(k,l;t) = \big(Ex_Q; Ex_C\big)$$

$$Et_Q(k;t) = Ex_Q(k;t)Q(k;t) \quad ; \quad Et_C(l;t) = Ex_C(l;t)C(l;t) \qquad (15.10)$$

Equations (15.6-9) describe transactions ***BS****(k,l;t)* (15.5) with assets *E* of the entire economics under expectations ***Ex****(k,l;t)* (15.10). Let's describe a model of mutual action between small disturbances of transactions and expectations in a linear approximation. Let's consider (15.6-9) and assume that mean transactions $\boldsymbol{BS}_0(k,l;t)$ and $\boldsymbol{Et}_0(k,l;t)$ are slow to compare with small dimensionless disturbances ***bs****(k,l;t)* and ***et****(k,l;t)* and let's take $\boldsymbol{BS}_0(k,l)$ and $\boldsymbol{Et}_0(k,l)$ as const. Due to (15.3-5):

$$\boldsymbol{BS}(k,l;t) = (Q;C);\ Q(k;t) = Q_{0k}\big(1+q(k;t)\big);\ C(l;t) = C_{0l}\big(1+c(l;t)\big) \qquad (16.1)$$

$$\boldsymbol{Et}(k,l;t) = \big(Et_Q(k;t); Et_C(l;t)\big) \qquad (16.2)$$

$$Et_Q(k;t) = Et_{Q0k}\Big(1+et_q(k;t)\Big);\ Et_C(l;t) = Et_{C0l}\big(1+et_c(l;t)\big) \qquad (16.3)$$

Equations on small disturbances ***bs****(k,l;t)* and ***et****(k,l;t)* take form:

$$Q_{0k}\frac{d}{dt}q(k;t) = f_q(k;t)\ ;\ C_{0l}\frac{d}{dt}c(l;t) = f_c(l;t) \qquad (16.2)$$

$$Et_{Q0k}\frac{d}{dt}et_q(k;t) = fe_q(k;t)\ ;\ Et_{C0l}\frac{d}{dt}et_c(l;t) = fe_c(l;t) \qquad (16.3)$$

$$Fe_{Qk} = Fe_{Q0k} + fe_Q(k;t)\ ; Fe_{Cl} = Fe_{C0l} + fe_c(l;t\ ) \qquad (16.4)$$

Let's assume that factors $f_q(k;t)$ and $f_c(l;t)$ in (16.2) depend on disturbances of expected transactions $et_q(k;t)$ and $et_c(l;t)$ and $fe_q(k;t)$ and $fe_c(l;t)$ in (16.3) depend on disturbances of *q(k;t)* and *c(l;t)*. For linear approximation by disturbances let's take (16.2-3) as:

$$Q_{0k}\frac{d}{dt}q(k;t) = a_{qk}Et_{Q0k}et_q(k;t)\ ;\ C_{0l}\frac{d}{dt}c(l;t) = a_{cl}Et_{C0l}\,et_c(l;t) \qquad (16.5)$$

$$Et_{Q0k}\frac{d}{dt}et_q(k;t) = be_{qk}Q_{0k}q(k;t)\ ;\ Et_{C0l}\frac{d}{dt}et_c(l;t) = be_{cl}C_{0l}\,c(l;t) \qquad (16.6)$$

$$\omega_{qk}^2 = -a_{qk}be_{qk} > 0\ ;\ \omega_{cl}^2 = -\,a_{cl}be_{cl} > 0 \qquad (16.7)$$

If relations (16.7) are valid, then (16.5-6) are equations for harmonic oscillators:

$$\left(\frac{d^2}{dt^2} + \omega_{qk}^2\right)q(k;t) = 0\ ;\ \left(\frac{d^2}{dt^2} + \omega_{cl}^2\right)c(l;t) = 0 \qquad (16.8)$$

$$\left(\frac{d^2}{dt^2} + \omega_{qk}^2\right)et_q(k;t) = 0\ ;\ \left(\frac{d^2}{dt^2} + \omega_{cl}^2\right)et_c(l;t) = 0\ ; k,l = 1,..K \qquad (16.9)$$

Simple solutions of (16.8) for dimensionless disturbances $q_k(t)$ and $c_l(t)$:

$$q(k;t) = g_{qk}sin\omega_{qk}t +\ d_{qk}cos\omega_{qk}t \qquad (17.1)$$

$$c(l;t) = g_{cl}sin\omega_{cl}t + d_{cl}cos\omega_{cl}t \qquad (17.2)$$

$$g_{qk}, d_{qk}\ , g_{cl}, d_{cl} \ll 1 \qquad (17.3)$$

Relations (17.1-3) describe simple harmonic fluctuations of disturbances of volume *Q(k;t)* and cost *C(l;t)* of transactions ***BS****(k,l;t)* performed under different expectations ***Ex****(k,l;t)* (16.10).

Price fluctuations. Let's note price of transaction made by all agents of entire economics under expectations of type *k, l* as *p(k,l;t)*

$$C(k,l;t) = p(k,l;t)Q(k,l;t) \quad (18.1)$$

Now for convenience let's call *C(k,l;t)* as cost of transaction made under expectation of type *l* for volume *Q(k,l;t)* of transaction made under expectation of type *k*. Thus transaction ***BS****(k,l;t)* has cost *C(k,l;t)* made under expectation of type *l* and volume *Q(k,l;t)* of transaction made under expectation of type *k*. Double indexes *(k,l)* determine transaction with cost under expectation *l* and volume under expectation *k*. Sum of transactions ***BS****(k,l;t)* (16.1) by all expectations *k,l=1,…K* define transactions ***BS****(t)* in the entire economics:

$$\boldsymbol{BS}(t) = (Q(t); C(t)) \;;\; Q(t) = \sum_{kl} Q(k,l;t) ; \quad C(t) = \sum_{k,l} C(k.l;t) \quad (18.2)$$

Price *p(t)* of transactions ***BS****(t)* (18.2) equals:

$$C(t) = p(t)Q(t) \quad (18.3)$$

Let's study disturbances of cost *C(t)*, volume *Q(t)* and price *p(t)* for (18.3) as:

$$Q(t) = \sum_{k,l} Q_{0kl}\big(1 + q(k,l;t)\big) = Q_0 \sum_{k,l} \lambda_{kl}\big(1 + q(k,l;t)\big) \quad (18.4)$$

$$C(t) = \sum_{k,l} C_{0kl}\big(1 + c(k,l;t)\big) = C_0 \sum_{k,l} \mu_{kl}\big(1 + c(k,l;t)\big) \quad (18.5)$$

Relations (18.4) describe impact of dimensionless disturbances *q(k,l;t)* on volume *Q(t)* and (18.5) describe impact of dimensionless disturbances *c(k,l;t)* on cost C(t) of transactions.

$$Q_0 = \sum_{k,l} Q_{0kl} \;;\; \lambda_{kl} = \frac{Q_{0kl}}{Q_0} \;;\; C_0 = \sum_{k,l} C_{0kl} \;;\; \mu_{kl} = \frac{C_{0kl}}{C_0} \;;\; \sum \lambda_{kl} = \sum \mu_{kl} = 1 \quad (18.6)$$

Relations (18.3) define price *p(t)* for *Q(t)* (18.4) and *C(t)* (18.5):

$$p(t) = \frac{C(t)}{Q(t)} = \frac{\sum_{k,l} C(k,l;t)}{\sum_{k,l} Q(k,l;t)} \quad ; \quad p_0 = \frac{C_0}{Q_0} = \frac{\sum_{k,l} C_{0kl}}{\sum_{k,l} Q_{0kl}} \quad (18.7)$$

In linear approximation by disturbances *q(k,l;t)* and *c(k,l;t)* price *p(t)* (18.7) take form:

$$p(t) = \frac{C(t)}{Q(t)} = \frac{C_0 \sum_{k,l} \mu_{kl}\big(1 + c(k,l;t)\big)}{Q_0 \sum_{k,l} \lambda_{kl}\big(1 + q(k,l;t)\big)} = p_0 \left[1 + \sum\nolimits_{k,l} \mu_{kl} c(k,l;t) - \sum\nolimits_{k,l} \lambda_{kl} q(k,l;t)\right]$$

$$p(t) = p_0[1 + \pi(t)] = p_0\big[1 + \sum_{k,l}(\mu_{kl} c(k,l;t) - \lambda_{kl} q(k,l;t))\big] \quad (18.8)$$

Dimensionless fluctuations of price *π(t)* (18.8) equals weighted sum of disturbances *q(k,l;t)* and *c(,lk;t)* as (18.9):

$$\pi(t) = \sum_{k,l} \mu_{kl}\, c(k,l;t) - \lambda_{kl} q(k,l;t) \quad (18.9)$$

Now let's take (18.1) and present *π(t)* in other form:

$$C(k,l;t) = C_{0kl}[1 + c(k,l;t)] = p_{0kl}[1 + \pi(k,l;t)]Q_{0kl}[1 + q(k,l;t)] \quad (19.1)$$

From (18.6-7) and (19.1) in linear approximation by *c(k,l;t), π(k,l;t)* and *q(k,l;t)* obtain:

$$C_{0kl} = p_{0kl} Q_{0kl} \quad ; \quad c(k,l;t) = \pi(k,l;t) + q(k,l;t) \quad (19.2)$$

Let's substitute (19.2) into (18.9):

$$\pi(t) = \sum_{k,l} \mu_{kl} \pi(k,l;t) + \sum_{k,l} (\mu_{kl} - \lambda_{kl})\, q(k,l;t) \tag{19.3}$$

Relations (19.3) describe price perturbations $\pi(t)$ as weighted sum of partial price disturbances $\pi(k,l;t)$ and volume disturbances $q(k,l;t)$. Thus statistics of price disturbances $\pi(t)$ is defined by statistics of partial price disturbances $\pi(k,l;t)$ and statistics of volume disturbances $q_k(k,l;t)$.

Return perturbations. Price disturbances (19.3) cause perturbations of return $r(t,d)$:

$$r(t,d) = \frac{p(t)}{p(t-d)} - 1 \tag{20.1}$$

Let's introduce partial returns $r(k,l;t,d)$ for price $p(k,l;t)$ (18.1) and "returns" $w(k,l;t,d)$ for volumes $Q(k,l;t)$ (18.2):

$$r(k,l;t,d) = \frac{p(k,l;t)}{p(k,l;t-d)} - 1 \quad ; \quad w(k,l;t,d) = \frac{Q(k,l;t)}{Q(k,l;t-d)} - 1 \tag{20.2}$$

Let's assume for simplicity that mean price p_{0kl} and trade volumes Q_{0kl} are constant during time term d and (18.7; 19.3) present (20.1, 20.2) as

$$r(t,d) = \frac{\pi(t)-\pi(t-d)}{1+\pi(t-d)} \;\; ; \;\; w(k,l;t,d) = \frac{q(k,l;t)-q(k,l;t-d)}{1+q(k,l;t-d)} \tag{20.3}$$

$$r(t,d) = \sum \mu_{kl} \frac{1+\pi(k,l;t-d)}{1+\pi(t-d)} r(r,l;t,d) + \sum (\mu_{kl} - \lambda_{kl}) \frac{1+q(k;t-d)}{1+\pi(t-d)} w(k,l;t,d) \tag{20.4}$$

Let's define

$$\varepsilon_{kl}(t-d) = \mu_{kl} \frac{1+\pi(k;t-d)}{1+\pi(t-d)} \quad ; \; \eta_{kl}(t-d) = (\mu_{kl} - \lambda_{kl}) \frac{1+q(k;t-d)}{1+\pi(t-d)} \tag{20.5}$$

$$\sum_{k,l} [\varepsilon_{kl}(t-d) + \eta_{kl}(t-d)] = 1 \tag{20.6}$$

$$r(t,d) = \sum_{k,l} \varepsilon_{kl}\,(t-d) r(k,l;t,d) + \sum_{k,l} \eta_{kl}\,(t-d) w(k,l;t,d) \tag{20.7}$$

Relations (20.6-7) describe return (20.1) as sum of partial returns and volume "returns" $w(k,l;t,d)$ (20.2, 20.3). Sum for coefficients μ_{kl} and μ_{kl}-λ_{kl} for price $p(t)$ (18.7; 19.3) and $\varepsilon_{kl}(t)$ and $\eta_{kl}(t)$ for return $r(t,d)$ (20.1) equals unit but (19.3) and (20.7) can't be treated as averaging procedure as some coefficients μ_{kl}-λ_{kl} and $\eta_{kl}(t)$ should be negative. If mean price (19.2) $p_{0kl}=p_0$ for all pairs of expectations (k,l) then from (18.6, 18.7) obtain

$$p_{0kl} = p_0 = const \;\rightarrow\; \lambda_{kl} = \mu_{kl} \;\; ; \;\; \eta_{kl}(t) = 0 \text{ for all } k,l \tag{20.8}$$

and relations (19.3; 20.7) take simple form

$$\pi(t) = \sum_{k,l} \mu_{kl}\, \pi(k,l;t) \tag{20.9}$$

$$r(t,d) = \sum_{k,l} \mu_{kl} \frac{1+\pi(k,l;t-d)}{1+\pi(t-d)} r(k,l;t,d) = \sum_{k,l} \mu_{kl} \frac{\pi(k,l;t)-\pi(k,l;t-d)}{1+\pi(t-d)} \tag{20.10}$$

Thus assumption (20.8) on prices (19.2) for all pairs of expectations (k,l) cause representation (20.9, 20.10) of price disturbances $\pi(t)$ as weighted sum of partial price disturbances $\pi(k,l;t)$ for different pairs of expectations (k,l). Otherwise price disturbances $\pi(t)$ should take (19.3) and depend on volume perturbations $q(k,l;t)$. Assumption (20.8) cause returns as (20.10), otherwise returns take (20.7). Actually expectations are key factors for market competition and different expectations (k,l) should cause different mean partial prices p_{0kl}. That should cause complex representation of price (19.3) and return (20.7) disturbances as well as impact volatility and statistic distributions of price and return disturbances.

11. Option Pricing

Option pricing accounts thousands articles published since classical Black, Scholes (1973) and Merton (1973) (BSM) studies (Hull and White, 1987; Hansen, Heaton, and Luttmer, 1995; Hull, 2009). Current observations of market data show that option pricing don't follow Brownian motion and classical BSM model (Fortune, 1996). Stochastic volatility is only one of factors that cause BSM model violation (Heston, 1993, Bates, 1995). Studies of economic origin of price stochasticity are important for correct modeling asset and option pricing. We propose that economic space modeling may give new look on description of asset stochasticity and option pricing. Indeed, economic space establishes ground for description of density functions of economic variables and transactions. On other hand economic space allows describe price evolution of assets for selected agent in a random economic environment. Random evolution of risk coordinates of selected assets impact assets and option pricing. Nevertheless it is clear that Brownian motion models don't fit real market option pricing, simple Brownian considerations allow argue some hidden complexities of option pricing problem. Below we discuss classical BSM treatment of option pricing based on assumption of price Brownian motion (Hull, 2009). We start with classical BSM approximation and describe model for option price caused by Brownian motion of economic agent on economic space that gives generalization of the classical BSM equations (Olkhov, 2016a-2016c). Further we argue BSM assumptions and restrictions that arise from previous Section and may impact assets and option pricing models.

Let's start with classical derivation of the BSM (Hull, 2009) based on assumption that price p of selected agent's assets obeys Brownian motion $dW(t)$ with volatility σ and linear trend v:

$$dp(t) = p\,v\,dt + p\sigma dW(t)\ ;\quad <dW(t)>= 0;\ <dW(t)dW(t)> = dt \qquad (21.1)$$

Assumptions (21.1) give the classical BSM equation for the option price $V(p;t)$ for risk-free rate r (Hull, 2009):

$$\frac{\partial V}{\partial t} + rp\frac{\partial V}{\partial p} + \frac{1}{2}\sigma^2 p^2 \frac{\partial^2}{\partial p^2}V \ = \ rV \qquad (21.2)$$

In Sec.10 we use coordinates $\boldsymbol{x}$ to define positions of agents those involved in transactions at Exchange with assets of selected agent $\boldsymbol{A}$. Let's note $\boldsymbol{y}$ as coordinates of selected agent $A(t,\boldsymbol{y})$. Let's assume that price p of assets of selected agent $A(t,\boldsymbol{y})$ depends on time t and on risk coordinates $\boldsymbol{y}$ as $p(t,\boldsymbol{y})$. Let's propose that disturbances of risk coordinates $\boldsymbol{y}$ of selected agent $A(t,\boldsymbol{y})$ follow Brownian motion $d\boldsymbol{Y}(t)$ on n-dimensional economic space:

$$d\boldsymbol{y} = \boldsymbol{v}dt + d\boldsymbol{Y}(t) \quad ; \; d\boldsymbol{Y}(t) = (dY_1, .. dY_n) \quad ; \quad < dY_i(t) > = 0 \tag{21.3}$$

$$< dY_i(t) dY_j(t) > = \eta_{ij} \, dt \quad ; \quad < dW(t) \, dY_i(t) > = b_i$$

Factors η_{ii} describe volatility of Brownian motion dY_i along axis i and η_{ij} for $i \neq j$ describe correlations between Brownian motions dY_i along axes i and dY_j along axes j. Factors b_i – describe correlations between Brownian motion dW and dY_i along axes i. Now let's extend assumption (21.1) and let's propose (21.4) that price $p(t,\boldsymbol{y})$ depend on time t and on Brownian motion $d\boldsymbol{Y}(t)$ (21.3) of selected agent $A(t,\boldsymbol{y})$ on economic space:

$$dp(t,\boldsymbol{y}) = p \, v \, dt + p\sigma dW(t) + p \, \boldsymbol{k} \cdot d\boldsymbol{Y} \quad ; \; \boldsymbol{k} = (k_1, \ldots k_n) - \; const \tag{21.4}$$

Similar to (Hall, 2009) for risk-free rate r from (21.4) obtain extension of the classical BSM equation (21.2) for the option price $V(p;t,\boldsymbol{y})$ on n-dimensional economic space (Olkhov, 2016b,c) :

$$\frac{\partial V}{\partial t} + rp\frac{\partial V}{\partial p} + ry_i\frac{\partial V}{\partial y_i} + \frac{1}{2}p^2 q^2 \frac{\partial^2 V}{\partial p^2} + p\left(\sigma b_i + k_j \, \eta_{ji}\right)\frac{\partial^2 V}{\partial p \partial y_i} + \frac{\eta_{ij}}{2}\frac{\partial^2 V}{\partial y_i \partial y_j} = rV \tag{21.5}$$

$$q^2 = \left(\sigma^2 + k_i \, k_j \, \eta_{ij} + 2\sigma k_i \, b_i\right); \; i,j = 1, \ldots n$$

Additional parameters k_i, b_i, η_{ij}, $i,j=1,\ldots n$, define volatility q^2 and coefficients for additional terms of equation (21.5) and impact option price $V(p;t,\boldsymbol{y})$. Extension (21.5) of the classical BSM equations (21.2) may uncover hidden complexities of option pricing that have origin in the random motion of agents $A(t,\boldsymbol{y})$ on economic space. As special case for (21.5) one can study equation on option price $V(p;t,\boldsymbol{y})$ on 1-dimensional economic space for $\sigma=0$ without classical BSM assumptions (21.1):

$$\frac{\partial V}{\partial t} + rp\frac{\partial V}{\partial p} + ry\frac{\partial V}{\partial y} + \frac{1}{2}p^2 k^2 \eta \frac{\partial^2 V}{\partial p^2} + pk\eta\frac{\partial^2 V}{\partial p \partial y} + \frac{\eta}{2}\frac{\partial^2 V}{\partial y^2} = rV \tag{21.6}$$

Equations (21.6) describe option price $V(p;t,y)$ of assets which price $p(t,y)$ depends only on Brownian motion $dY(t)$ (21.3) of agents coordinates y on 1-dimensional economic space. Let's mention that assumptions (21.3, 21.4) simplify assets pricing model that we argue in Sec.4. Indeed, in Sec.4 we discuss that asset price and its disturbances should depend on relations between transactions and expectations. Thus assumptions on Brownian motion (21.3) of coordinates of selected agent $A(t,\boldsymbol{y})$ on economic space should impact transactions with assets of particular agent $A(t,\boldsymbol{y})$ and corresponding expectations. Let's take relations (19.3) for price disturbances $\pi(t,\boldsymbol{y})$ of assets of selected agent $A(t,\boldsymbol{y})$ with coordinates $\boldsymbol{y}$

$$\pi(t,\boldsymbol{y}) = \sum_{k,l} \mu_{kl} \, \pi(k,l;t,\boldsymbol{y}) + \sum_{k,l} (\mu_{kl} - \lambda_{kl}) \, q(k,l;t,\boldsymbol{y}) \tag{22.1}$$

Let's remind that $\pi(k,l;t,\boldsymbol{y})$ describe partial price disturbances of assets of agent $A(t,\boldsymbol{y})$ for transactions of all economic agents with Exchange made under expectations of type k for decisions on trading volume $Q(k,l;t,\mathbf{y})$ and expectations of type l for decisions on cost $C(k,l;t,\mathbf{y})$ of transaction. As we mention in Sec.10, if partial price p_{0kl} (19.2) is constant for all type of expectations k,l then price disturbances $\pi(t,\mathbf{y})$ take form (20.9) and equal weighted sum of partial prices $\pi(k,l;t,\mathbf{y})$. Otherwise price disturbances $\pi(t,\mathbf{y})$ should depend on disturbances of partial prices $\pi(k,l;t,\mathbf{y})$ and on perturbations of trading volumes $q(k,l;t,\mathbf{y})$. Let's mention that statistic distribution of price disturbances $\pi(t,\mathbf{y})$ (22.1) may depend also on coefficients λ_{kl} and μ_{kl} (18.6) that can fluctuate due to random change of coordinates of selected agent $A(t,\mathbf{y})$. Possible impact of these numerous factors on option pricing should be studied further.

12. Conclusions

Economic theory is an endless problem. We present only beginnings, essentials of economic theory framework, tools and approximations and argue some outcomes. We model economy by three elements – economic variables, transactions and expectations of economic agents. Starting with these properties of economic agents we model macroeconomic variables, transactions and expectations. We show that change of risk ratings of agents due to their economic activity or any factors induce economic flows of variables, transactions and expectations and these flows make significant contribution to macroeconomic evolution. Flows of variables, transactions and expectations double number of properties that define state and evolution of economy. We regard risks as main drivers of macroeconomic evolution and development. Any economic activity is related with risks. No risk-free financial success is possible and risk-free models have nothing common with economic reality.

Our economic model has no assumptions on market equilibrium, utility functions, rational expectations and etc., those ground general equilibrium (Arrow and Debreu, 1954; Tobin, 1969; Arrow, 1974; Smale, 1976; Kydland and Prescott, 1990; Starr, 2011). We show that these assumptions are not necessary for economic modelling. Economic statistics as source for agents risk assessments, alike to measurements of coordinates in physics can provide sufficient data for economic theory. Hence excessive assumptions can be put aside of economic modeling or may be applied for description of few specific cases only.

Our approach uncovers a lot of economic problems that should be studied further to clarify elements of the economic model. Let's argue some those concern economic space. Dimension of economic space is determined by choice of n risks those impact macroeconomic evolution. To develop reasonable economic model one should reduce number of risks and chose major two-three risks to define economic space of 2 or 3 dimensions. Hence one should develop methods to compare and forecast impact of risks on macroeconomic dynamics and procedure for selection most important risks. Choice of definite risks defines distribution of agents, form of density functions and economic dynamics on selected economic space. Different sets of risks cause different economic dynamics. Random nature of economic risks means that impact of some current risks may decline in time and influence of some new risk may unexpectedly grow up. Such collision underlines internal random properties of macroeconomic evolution and modeling. We state that economic development can occurs only under action of risks and different risks may set different directions for economic dynamics. Thus change of major risks results in change of dynamics determined by economic equations on density functions and flows of variables, transactions and expectations. In this paper we study economic evolution in the assumption that major risks and economic space don't change. The problems of random change of major risks should be studied further.

Risk assessments play central role for our model. It is impossible to provide exact risks assessments of all agents in the entire economics. We propose the roughening procedure that transfers description of numerous separate particles to description of aggregated agents and density functions on economic space. Such roughening procedure has some parallels to transition from description of separate physical particles to description of continuous media or physics of fluids in hydrodynamic approximation. Such transition in physics significantly reduce amount of data required for model description. We seek the same effect in economic modeling. Roughening of risk ratings of separate agents and transition to description of density functions and flows of economic variables, transactions and expectations reduce amount of econometric statistics required for such approximation. Our approximation becomes intermediate between extra precise description based on modeling macroeconomics as system of numerous separate agents and description based on modeling macroeconomics as aggregated functions of time only. We propose that achievements of econometrics (Fox, et.al, 2014) and efforts in developing risks assessments methodologies should solve that complex problem for sure.

Any economic flows are accompanied by generation of small perturbations of economic variables, transactions and expectations. Description of propagation of small economic and financial disturbances on economic space reflect most general problem of evolution of any complex system. Wave propagation of small perturbations on economic space may explain interactions between different markets, industries, countries and describe transfer of economic and financial influence over macroeconomics. Total distinction of economic processes from physical problems cause room for amplification of small economic and financial perturbations during wave propagation over economic domain. Growth of wave amplitudes of economic disturbances during propagation on economic space may impact huge perturbations and shocks of entire macroeconomics. Economic wave propagation has analogy in hydrodynamics but nature and properties of economic waves are completely different. Borders of economic domain reduce economic flows of variables and transactions and cause business cycles. Fluctuations of credit mean risks reflect credit cycles, fluctuations of investment mean risks reflect investment cycles and so on. Interactions between major economic and financial variables cause correlations of corresponding cycles. Description of these fluctuations requires relatively complex economic equations.

Many problems should be studied further. Econometric problems and observation of economic and financial variables, transactions and expectations of agents and agents risk assessment are among the central. Up now there are no sufficient econometric data required to establish distribution of economic agents by their risk ratings as coordinates on economic space. Nevertheless we hope that our model may be useful for better understanding and description of economic and financial processes.

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Appendix A. Transactions and expectations as two component functions and assets pricing equations

To describe trading volume Q_{ij} and cost C_{ij} of transaction $\boldsymbol{bs}_{i,j}(t,z)$ with economic variable E let's define transaction as two component function:

$$\boldsymbol{bs}_{i,j}(t,\boldsymbol{z}) = \left(Q_{ij}(t,\boldsymbol{z}); C_{ij}(t,\boldsymbol{z})\right) \quad ; \quad \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{A.1}$$

Each component Q_{ij} and C_{ij} (A.1) of transaction $\boldsymbol{bs}_{i,j}(t,z)$ should be approved by expectations of agent i as seller and expectations of agent j as buyer. Let's define transaction $\boldsymbol{bs}_{i,j}(k;t,z;l)$ performed under sellers expectations of type $\boldsymbol{k}=(k_1;k_2)$ and buyers expectations of type $\boldsymbol{l}=(l_1;l_2)$, $k1,k2, l_1,l_2 =1,...K$ as:

$$\boldsymbol{bs}_{i,j}(\boldsymbol{k};t,\boldsymbol{z};\boldsymbol{l}) = \left(Q_{ij}(k_1;t,\boldsymbol{z};l_1); C_{ij}(k_2;t,\boldsymbol{z};l_2)\right) \tag{A.2}$$

$$\boldsymbol{k} = (k_1,k_2) \quad ; \quad \boldsymbol{l} = (l_1,l_2)$$

Relation (A.2) define transactions $\boldsymbol{bs}_{i,j}(\boldsymbol{k};t,z;\boldsymbol{l})$ determined by trading volume Q_{ij} and cost C_{ij}. Relations (A.2) define price $p_{i,j}(\boldsymbol{k};t,z;\boldsymbol{l})$ of variable E for transaction $\boldsymbol{bs}_{i,j}(\boldsymbol{k};t,z;\boldsymbol{l})$ between agents i and j as:

$$C_{ij}(k_2;t,\boldsymbol{z};l_2) = p_{ij}(\boldsymbol{k};t,\boldsymbol{z};\boldsymbol{l})Q_{ij}(k_1;t,\boldsymbol{z};l_1) \tag{A.2.1}$$

Sum over all buyers expectations of $\boldsymbol{l}=(l_1;l_2)$ define sellers price $p_{i,j}(\boldsymbol{k};t,z)$

$$C_{ij}(k_2;t,\boldsymbol{z}) = p_{ij}(\boldsymbol{k};t,\boldsymbol{z})Q_{ij}(k_1;t,\boldsymbol{z}) \tag{A.2.2}$$

$Q_{i,j}(k_1;t,z)$ and $C_{i,j}(k_2;t,z)$ are defined by (A.7). Sum over all sellers expectations of $\boldsymbol{k}=(k_1;k_2)$ define buyers price $p_{i,j}(t,z;\boldsymbol{l})$

$$C_{ij}(t,\boldsymbol{z};l_2) = p_{ij}(t,\boldsymbol{z};\boldsymbol{l})Q_{ij}(t,\boldsymbol{z};l_1) \tag{A2.3}$$

$Q_{i,j}(t,z;l_1)$ and $C_{i,j}(t,z;l_2)$ are defined by (A.11). And sum over sellers and buyers expectations define price $p_{i,j}(t,z)$ of transactions between agents i and j at $\boldsymbol{x}$ and $\boldsymbol{y}$, $z=(\boldsymbol{x},\boldsymbol{y})$ as:

$$C_{ij}(t,\boldsymbol{z}) = p_{ij}(t,\boldsymbol{z})Q_{ij}(t,\boldsymbol{z}) \tag{A.2.4}$$

$$Q_{ij}(t,\boldsymbol{z}) = \sum\nolimits_{k_1;l_1} Q_{ij}(k_1;t,\boldsymbol{z};l_1) \;; C_{ij}(t,\boldsymbol{z}) = \sum\nolimits_{k_2;l_2} C_{ij}(k_2;t,\boldsymbol{z};l_2)$$

Trading volumes Q_{ij} are approved by sellers expectations of type k_1 and buyers expectations of type l_1. The trading values or costs C_{ij} of transaction are approved by sellers expectations of type k_2 and buyers expectations of type l_2. Let's introduce seller's expectations $\boldsymbol{ex}_i(\boldsymbol{k};t,\boldsymbol{x})$ of type $\boldsymbol{k}=(k_1;k_2)$ of agent i at $\boldsymbol{x}$ as

$$\boldsymbol{ex}_i(\boldsymbol{k};t,\boldsymbol{x}) = \left(ex_{iQ}(k_1;t,\boldsymbol{x});\ ex_{iC}(k_2;t,\boldsymbol{x})\right) \tag{A.3}$$

and buyer's expectations $\boldsymbol{ex}_j(t,\boldsymbol{y};\boldsymbol{l})$ of type $\boldsymbol{l}=(l_1;l_2)$ of agent j at $\boldsymbol{y}$ as

$$\boldsymbol{ex}_j(t,\boldsymbol{y};\boldsymbol{l}) = \left(ex_{jQ}(t,\boldsymbol{y};l_1);\ ex_{jC}(t,\boldsymbol{y};l_2)\right) \tag{A.4}$$

that approve Q_{ij} and C_{ij} (A.2) of transaction ***bs****$_{i,j}$(k;t,z;**l**)* respectively. Similar to (II.7.1) let's define sellers and buyers expected transactions of as:

$$\boldsymbol{et}_{ij}(\boldsymbol{k};t,\boldsymbol{z}) = \left(et_{ijQ}(k_1;t,\boldsymbol{z})\ ;\ et_{ijC}(k_2;t,\boldsymbol{z})\right) \tag{A.4}$$

$$et_{ijQ}(k_1;t,\boldsymbol{z}) = ex_{iQ}(k_1;t,\boldsymbol{x})Q_{ij}(k_1;t,\boldsymbol{z}) \tag{A.5}$$

$$et_{ijC}(k_2;t,\boldsymbol{z}) = ex_{iC}(k_2;t,\boldsymbol{x})C_{ij}(k_2;t,\boldsymbol{z}) \tag{A.6}$$

$$Q_{ij}(k_1;t,\boldsymbol{z}) = \sum\nolimits_{l_1} Q_{ij}(k_1;t,\boldsymbol{z};l_1)\ \ ;\ \ C_{ij}(k_2;t,\boldsymbol{z}) = \sum\nolimits_{l_2} C_{ij}(k_2;t,\boldsymbol{z};l_2) \tag{A.7}$$

$$\boldsymbol{et}_{ij}(t,\boldsymbol{z};\boldsymbol{l}) = \left(et_{ijQ}(t,\boldsymbol{z};l_1)\ ;\ et_{ijC}(t,\boldsymbol{z};l_2)\right) \tag{A.8}$$

$$et_{ijQ}(t,\boldsymbol{z};l_1) = ex_{jQ}(t,\boldsymbol{y};l_1)Q_{ij}(t,\boldsymbol{z};l_1) \tag{A.9}$$

$$et_{ijC}(t,\boldsymbol{z};l_2) = ex_{jC}(t,\boldsymbol{y};l_2)C_{ij}(t,\boldsymbol{z};l_2) \tag{A.10}$$

$$Q_{ij}(t,\boldsymbol{z};l_1) = \sum\nolimits_{k_1} Q_{ij}(k_1;t,\boldsymbol{z};l_1)\ \ ;\ \ C_{ij}(t,\boldsymbol{z};l_2) = \sum\nolimits_{k_2} C_{ij}(k_2;t,\boldsymbol{z};l_2) \tag{A.11}$$

Relations (A.4) define sellers expected transactions of type ***k****=(k$_1$,k$_2$)*. Relations (A.5) define sellers expected transactions for trading volume Q_{ij} and (A.6) define sellers expected transactions for cost C_{ij} of the transaction. Relations (A.7-A.9) define expected transactions for buyers of type ***l****=(l$_1$,l$_2$)*. Relations (II.11.2) for transaction ***bs****$_{i,j}$(**k**;t,z;**l**)* and (A.4-A.11) for expected transactions ***et****$_{i,j}$(**k**;t,z)* and ***et****$_{i,j}$(t,z;**l**)* derive sellers aggregated transactions ***BS(k****;t,z)* and buyers aggregated transactions ***BS****(t,z;**l**)* and expected transactions ***Et(k****;t,z)* and ***Et****(t,z;**l**)* similar to (II. 2.1; 2.2) as:

$$\boldsymbol{BS}(\boldsymbol{k};t,\boldsymbol{z}) = \left(Q(k_1;t,\boldsymbol{z});C(k_2;t,\boldsymbol{z})\right);\ \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{A.12.1}$$

$$Q(k_1;t,\boldsymbol{z}) = \sum\nolimits_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta}\sum\nolimits_{l_1} Q_{ij}(k_1;t,\boldsymbol{z};l_1) \tag{A.12.2}$$

$$C(k_2;t,\boldsymbol{z}) = \sum\nolimits_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta}\sum\nolimits_{l_2} C_{ij}(k_2;t,\boldsymbol{z};l_2) \tag{A.12.3}$$

$$\boldsymbol{BS}(t,\boldsymbol{z};\boldsymbol{l}) = \left(Q(t,\boldsymbol{z};l_1);C(t,\boldsymbol{z};l_2)\right) \tag{A.12.4}$$

$$Q(t,\boldsymbol{z};l_1) = \sum\nolimits_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta}\sum\nolimits_{k_1} Q_{ij}(k_1;t,\boldsymbol{z};l_1) \tag{A.12.5}$$

$$C(t,\boldsymbol{z};l_2) = \sum\nolimits_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta}\sum\nolimits_{k_2} C_{ij}(k_2;t,\boldsymbol{z};l_2) \tag{A.12.6}$$

Relations (A.12.2; 12.3) define sellers aggregated price *p(**k**;t,z)* of variable *E* for the transaction ***BS(k****;t,z)* (A.12.1) under expectations of type ***k****=(k$_1$;k$_2$)* as:

$$C(k_2;t,\boldsymbol{z}) = p(\boldsymbol{k};t,\boldsymbol{z})Q(k_1;t,\boldsymbol{z}) \tag{A.12.7}$$

Relations (A.12.5; 12.6) define buyers aggregated price *p(t,z;**l**)* for expectations of type ***l****=(l$_1$;l$_2$)* as:

$$C(t,\boldsymbol{z};l_2) = p(t,\boldsymbol{z};\boldsymbol{l})Q(t,\boldsymbol{z};l_1) \tag{A.12.8}$$

Sum by all sellers expectations (A.12.10) or all buyers expectations (A.12.11) define aggregate price $p(t,z)$ of transactions between agents at $z=(\boldsymbol{x}, \boldsymbol{y})$:

$$C(t,\boldsymbol{z}) = p(t,\boldsymbol{z})Q(t,\boldsymbol{z}) \tag{A.12.9}$$

$$Q(t,\boldsymbol{z}) = \sum_{k_1} Q_{ij}(k_1;t,\boldsymbol{z}) = \sum_{l_1} Q_{ij}(t,\boldsymbol{z};l_1) \tag{A.12.10}$$

$$C(t,\boldsymbol{z}) = \sum_{k_2} C_{ij}(k_2;t,\boldsymbol{z}) = \sum_{l_2} C_{ij}(t,\boldsymbol{z};l_2) \tag{A.12.11}$$

Integral of $C(t,\boldsymbol{x},\boldsymbol{y})$ and $Q(t,\boldsymbol{x},\boldsymbol{y})$ by $d\boldsymbol{y}$ over economic domain (II. 1.1; 1.2) defines mean price $p_S(t,\boldsymbol{x})$ of sellers for transactions with variable E from point $\boldsymbol{x}$:

$$C_S(t,\boldsymbol{x}) = \int d\boldsymbol{y}\, C(t,\boldsymbol{x},\boldsymbol{y}) = p_S(t,\boldsymbol{x})Q_S(t,\boldsymbol{x}) \ ; \ Q_S(t,\boldsymbol{x}) = \int d\boldsymbol{y}\, Q(t,\boldsymbol{x},\boldsymbol{y}) \tag{A.12.12}$$

Relations (A.12.12) define sellers trading volume $Q_S(t,\boldsymbol{x})$ and cost $C_S(t,\boldsymbol{x})$ of all transactions from $\boldsymbol{x}$ and thus define sellers price $p_S(t,\boldsymbol{x})$ from point $\boldsymbol{x}$. Integral of $C(t,\boldsymbol{x},\boldsymbol{y})$ and $Q(t,\boldsymbol{x},\boldsymbol{y})$ by $d\boldsymbol{x}$ over economic domain (II. 1.1; 1.2) defines mean price $p_B(t,\boldsymbol{y})$ of buyers at $\boldsymbol{y}$:

$$C_B(t,\boldsymbol{y}) = \int d\boldsymbol{x}\, C(t,\boldsymbol{x},\boldsymbol{y}) = p_B(t,\boldsymbol{y})Q_B(t,\boldsymbol{y}) \ ; \ Q_B(t,\boldsymbol{y}) = \int d\boldsymbol{x}\, Q(t,\boldsymbol{x},\boldsymbol{y}) \tag{A.12.13}$$

Relations (A.12.13) define buyers trading volume $Q_B(t,\boldsymbol{y})$ and cost $C_B(t,\boldsymbol{y})$ of all transactions to $\boldsymbol{y}$ and thus define buyers price $p_B(t,\boldsymbol{y})$ at point $\boldsymbol{y}$.

$$C(t) = \int d\boldsymbol{x}d\boldsymbol{y}\, C(t,\boldsymbol{x},\boldsymbol{y}) = p(t)Q(t) \ ; \ Q(t) = \int d\boldsymbol{x}d\boldsymbol{y}\, Q(t,\boldsymbol{x},\boldsymbol{y}) \tag{A.12.14}$$

Relations (A.12.14) define trading volume $Q(t)$ and cost $C(t)$ of all transactions with variable E in economy thus define price $p(t)$ of variable E in macroeconomics at time t. Relations (A.12.15) define sellers price $p(\boldsymbol{k};t)=p(k_1,k_2;t)$

$$C(k_2;t) = \int d\boldsymbol{z}\, C(k_2;t,\boldsymbol{z}) = p(\boldsymbol{k};t)Q(k_1;t) \ ; \ Q(k_1;t) = \int d\boldsymbol{z}\, Q(k_1;t,\boldsymbol{z}) \tag{A.12.15}$$

for transactions with trading volume $Q(k_1;t)$ and cost $C(k_2;t)$ of economic variable E under sellers expectations of type $\boldsymbol{k}=(k_1,k_2)$.

$$C(t;l_2) = \int d\boldsymbol{z}\, C(t,\boldsymbol{z};l_2) = p(t;\boldsymbol{l})Q(t;l_1) \ ; \ Q(t;l_1) = \int d\boldsymbol{z}\, Q(t,\boldsymbol{z};l_1) \tag{A.12.16}$$

Relations (A.12.16) define buyers price $p(t;\boldsymbol{l})= p(t;l_1,l_2)$ of variable E for transactions with trading volume $Q(t;l_1)$ and cost $C(t;l_2)$ under buyers expectations of type $\boldsymbol{l}=(l_1,l_2)$. Definitions (A.2.1-2.4) and (A.12.7-12.16) define different sellers and buyers states of price p of economic variable E under transactions and different expectations. We show below that relations (A.12.7-12.16) define equations on price evolution of economic variable E.

Relations (A.12.1-12.6) define transactions $\boldsymbol{BS}(\boldsymbol{k};t,z)$ made under sellers expectations of type $\boldsymbol{k}=(k_1;k_2)$ and transactions $\boldsymbol{BS}(t,z;\boldsymbol{l})$ made under buyers expectations of type $\boldsymbol{l}=(l_1;l_2)$.

$$\boldsymbol{Et}(\boldsymbol{k};t,\boldsymbol{z}) = \left(Et_Q(k_1;t,\boldsymbol{z})\ ;\ Et_C(k_2;t,\boldsymbol{z})\right) \ ; \ \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{A.13.1}$$

$$Et_Q(k_1;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} \sum_{l_1} ex_{iQ}(k_1;t,\boldsymbol{x})\, Q_{ij}(k_1;t,\boldsymbol{z};l_1) \tag{A.13.2}$$

$$Et_C(k_2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} \sum_{l_2} ex_{iC}(k_2;t,\boldsymbol{x})\, C_{ij}(k_2;t,\boldsymbol{z};l_2) \tag{A.13.3}$$

$$\boldsymbol{Et}(t,\boldsymbol{z};\boldsymbol{l}) = \left(Et_Q(t,\boldsymbol{z};l_1)\ ;\ Et_C(t,\boldsymbol{z};l_2)\right) \tag{A.13.4}$$

$$Et_Q(t,\boldsymbol{z};l_1) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\Delta}\sum_{k_1} ex_{jQ}\,(t,\boldsymbol{y};l_1)Q_{ij}(k_1;t,\boldsymbol{z};l_1) \quad (A.13.5)$$

$$Et_C(t,\boldsymbol{z};l_2) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\Delta}\sum_{k_2} ex_{jC}\,(t,\boldsymbol{y};l_2)C_{ij}(k_2;t,\boldsymbol{z};l_2) \quad (A.13.6)$$

$$Et_Q(k_1;t,\boldsymbol{z}) = Ex_Q(k_1;t,\boldsymbol{z})Q(k_1;t,\boldsymbol{z}) \quad (A.13.7)$$

$$Et_C(k_2;t,\boldsymbol{z}) = Ex_C(k_2;t,\boldsymbol{z})C(k_2;t,\boldsymbol{z}) \quad (A.13.8)$$

$$Et_Q(t,\boldsymbol{z};l_1) = Ex_Q(t,\boldsymbol{y};l_1)Q(t,\boldsymbol{z};l_1) \quad (A.13.9)$$

$$Et_C(t,\boldsymbol{z};l_2) = Ex_C(t,\boldsymbol{y};l_2)C(t,\boldsymbol{z};l_2) \quad (A.13.10)$$

Relations (A.13.1-13.6) define expected transactions $\boldsymbol{Et(k;t,z)}$ of sellers made under expectations of type $\boldsymbol{k}=(k_1;k_2)$ and expected transactions $\boldsymbol{Et(t,z;l)}$ of buyers made under buyers expectations of type $\boldsymbol{l}=(l_1;l_2)$. Relations (A.13.7) for variable *E* define sellers aggregate expectations $Ex_Q(k_1;t,z)$ of type k_1 on trading volume $Q(k_1;t,z)$ (A.12.2) and (A.13.8) sellers aggregate expectations $Ex_C(k_2;t,z)$ of type k_2 on cost $C(k_2;t,z)$ of transaction (A.12.3) with variable *E*. Relations (A.13.9) define buyers aggregate expectations $Ex_Q(t,z;l_1)$ of type l_1 on trading volume $Q(t,z;l_1)$ (A.12.5) and (A.13.10) define buyers expectations $Ex_C(t,z;l_2)$ of type k_2 on cost $C(t,z;l_2)$ of transaction (A.12.6) with variable *E*. Now similar to (II. 2.1; 2.2; 5.1; 5.2) and (7.1) let's introduce flows $\boldsymbol{p}_{ij}(\boldsymbol{k};t,z)$ and $\boldsymbol{p}_{ij}(t,z;\boldsymbol{l})$, $z=(\boldsymbol{x},\boldsymbol{y})$ of transactions (A.2; A.4; A.8):

$$\boldsymbol{p}_{i,j}(\boldsymbol{k};t,\boldsymbol{z}) = \left(\boldsymbol{p}_{Qij}(k_1;t,\boldsymbol{z});\boldsymbol{p}_{Cij}(k_2;t,\boldsymbol{z})\right)\;;\;\boldsymbol{z}=(\boldsymbol{x},\boldsymbol{y}) \quad (A.14.1)$$

$$\boldsymbol{p}_{Qij}(k_1;t,\boldsymbol{z}) = Q_{ij}(k_1;t,\boldsymbol{z})\boldsymbol{v_i}(t,\boldsymbol{x}) \quad (A.14.2)$$

$$\boldsymbol{p}_{Cij}(k_2;t,\boldsymbol{z}) = C_{ij}(k_2;t,\boldsymbol{z})\boldsymbol{v_i}(t,\boldsymbol{x}) \quad (A.14.3)$$

$$\boldsymbol{p}_{i,j}(t,\boldsymbol{z};\boldsymbol{l}) = \left(\boldsymbol{p}_{Qij}(t,\boldsymbol{z};l_1);\boldsymbol{p}_{Cij}(t,\boldsymbol{x},\boldsymbol{y};l_2)\right) \quad (A.14.4)$$

$$\boldsymbol{p}_{Qij}(t,\boldsymbol{z};l_1) = Q_{ij}(t,\boldsymbol{z};l_1)\boldsymbol{v}_j(t,\boldsymbol{y}) \quad (A.14.5)$$

$$\boldsymbol{p}_{Cij}(t,\boldsymbol{z};l_2) = C_{ij}(t,\boldsymbol{z};l_2)\boldsymbol{v}_j(t,\boldsymbol{y}) \quad (A.14.6)$$

Flows $\boldsymbol{pe}_{ij}(\boldsymbol{k};t,z)$ and $\boldsymbol{pe}_{ij}(t,z;\boldsymbol{l})$ of expected transactions $\boldsymbol{et}_{i,j}(\boldsymbol{k};t,z)$ (A.4-6) and $\boldsymbol{et}_{i,j}(t,z;\boldsymbol{l})$ (A.8-10) take form:

$$\boldsymbol{pe}_{i,j}(\boldsymbol{k};t,\boldsymbol{z}) = \left(\boldsymbol{pe}_{Qij}(k_1;t,\boldsymbol{z});\boldsymbol{pe}_{Cij}(k_2;t,\boldsymbol{z})\right)\;;\;\boldsymbol{z}=(\boldsymbol{x},\boldsymbol{y}) \quad (A.15.1)$$

$$\boldsymbol{pe}_{Qij}(k_1;t,\boldsymbol{z}) = ex_i(k_1;t,\boldsymbol{x})Q_{ij}(k_1;t,\boldsymbol{z})\boldsymbol{v_i}(t,\boldsymbol{x}) \quad (A.15.2)$$

$$\boldsymbol{pe}_{Cij}(k_2;t,\boldsymbol{z}) = ex_i(k_2;t,\boldsymbol{x})C_{ij}(k_2;t,\boldsymbol{z})\boldsymbol{v_i}(t,\boldsymbol{x}) \quad (A.15.3)$$

$$\boldsymbol{pe}_{i,j}(t,\boldsymbol{z};\boldsymbol{l}) = \left(\boldsymbol{pe}_{Qij}(t,\boldsymbol{z};l_1);\boldsymbol{pe}_{Cij}(t,\boldsymbol{z};l_2)\right) \quad (A.15.4)$$

$$\boldsymbol{pe}_{Qij}(t,\boldsymbol{z};l_1) = ex_j(t,\boldsymbol{y};l_1)Q_{ij}(t,\boldsymbol{z};l_1)\boldsymbol{v}_j(t,\boldsymbol{y}) \quad (A.15.5)$$

$$\boldsymbol{pe}_{Cij}(t,\boldsymbol{z};l_2) = ex_j(t,\boldsymbol{y};l_2)C_{ij}(t,\boldsymbol{z};l_2)\boldsymbol{v}_j(t,\boldsymbol{y}) \quad (A.15.6)$$

Relations (A.14.1-6) are required to define flows $\boldsymbol{P(k;t,z)}$ and $\boldsymbol{P(t,z;l)}$ and velocities $\boldsymbol{v(k;t,z)}$

and $\boldsymbol{v}(t,z;\boldsymbol{l})$ of transactions $\boldsymbol{BS}(\boldsymbol{k};t,z)$ and $\boldsymbol{BS}(t,z;\boldsymbol{l})$ (A.12.1-6). Relations (A.15.1-6) allow define flows $\boldsymbol{Pe}(\boldsymbol{k};t,z)$ and $\boldsymbol{Pe}(t,z;\boldsymbol{l})$ and velocities $\boldsymbol{v}_{et}(\boldsymbol{k};t,z)$ and $\boldsymbol{v}_{et}(t,z;\boldsymbol{l})$ of expected transactions $\boldsymbol{Et}(\boldsymbol{k};t,z)$ and $\boldsymbol{Et}(t,z;\boldsymbol{l})$ (A.13.1-6). Let's define flows $\boldsymbol{P}(\boldsymbol{k};t,z)$ and $\boldsymbol{P}(t,z;\boldsymbol{l})$, $z=(\boldsymbol{x},\boldsymbol{y})$ similar to (9.4-9.9) as:

$$\boldsymbol{P}(\boldsymbol{k};t,\boldsymbol{z}) = \left(\boldsymbol{P}_Q(k_1;t,\boldsymbol{z});\boldsymbol{P}_C(k_2;t,\boldsymbol{z})\right) \quad ; \quad \boldsymbol{z}=(\boldsymbol{x},\boldsymbol{y}) \tag{A.16.1}$$

$$\boldsymbol{P}_Q(k_1;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} Q_{ij}(k_1;t,\boldsymbol{z})\boldsymbol{v}_{\boldsymbol{i}}(t,\boldsymbol{x}) \tag{A.16.2}$$

$$\boldsymbol{P}_C(k_2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} C_{ij}(k_2;t,\boldsymbol{z})\boldsymbol{v}_{\boldsymbol{i}}(t,\boldsymbol{x}) \tag{A.16.3}$$

$$\boldsymbol{P}(t,\boldsymbol{z};\boldsymbol{l}) = \left(\boldsymbol{P}_Q(t,\boldsymbol{z};l_1);\boldsymbol{P}_C(t,\boldsymbol{x},\boldsymbol{y};l_2)\right) \tag{A.16.4}$$

$$\boldsymbol{P}_Q(t,\boldsymbol{z};l_1) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} Q_{ij}(t,\boldsymbol{z};l_1)\boldsymbol{v}_j(t,\boldsymbol{y}) \tag{A.16.5}$$

$$\boldsymbol{P}_C(t,\boldsymbol{z};l_2) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} C_{ij}(t,\boldsymbol{z};l_2)\boldsymbol{v}_j(t,\boldsymbol{y}) \tag{A.16.6}$$

$$\boldsymbol{P}_Q(k_1;t,\boldsymbol{z}) = Q(k_1;t,\boldsymbol{z})\boldsymbol{v}_Q(k_1;t,\boldsymbol{z}) \ ; \ \boldsymbol{P}_C(k_1;t,\boldsymbol{z}) = C(k_2;t,\boldsymbol{z})\boldsymbol{v}_C(k_2;t,\boldsymbol{z}) \tag{A.16.7}$$

$$\boldsymbol{P}_Q(t,\boldsymbol{z};l_1) = Q(t,\boldsymbol{z};l_1)\boldsymbol{v}_Q(t,\boldsymbol{z};l_1) \ ; \ \boldsymbol{P}_C(t,\boldsymbol{z};l_2) = C(t,\boldsymbol{z};l_2)\boldsymbol{v}_C(t,\boldsymbol{z};l_2) \tag{A.16.8}$$

$$\boldsymbol{v}(\boldsymbol{k};t,\boldsymbol{z}) = \left(\boldsymbol{v}_Q(k_1;t,\boldsymbol{z});\boldsymbol{v}_C(k_2;t,\boldsymbol{z})\right) \ ; \ \boldsymbol{k}=(k_1,k_2) \tag{A.16.9}$$

$$\boldsymbol{v}(t,\boldsymbol{z};\boldsymbol{l}) = \left(\boldsymbol{v}_Q(t,\boldsymbol{z};l_1);\boldsymbol{v}_C(t,\boldsymbol{z};l_2)\right) \ ; \ \boldsymbol{l}=(l_1,l_2) \tag{A.16.10}$$

Relations (A.16.7-16.8) define velocities $\boldsymbol{v}(\boldsymbol{k};t,z)$ (II.16.9) and $\boldsymbol{v}(t,z;\boldsymbol{l})$ (II.16.10). These velocities determine equations on transactions $\boldsymbol{BS}(\boldsymbol{k};t,z)$ (A.12.1-12.3) made under sellers expectations of type $\boldsymbol{k}=(k_1;k_2)$ and transactions $\boldsymbol{BS}(t,z;\boldsymbol{l})$ (A.12.4-12.6) made under buyers expectations of type $\boldsymbol{l}=(l_1;l_2)$. Flows $\boldsymbol{Pe}(\boldsymbol{k};t,z)$ and $\boldsymbol{Pe}(t,z;\boldsymbol{l})$, $z=(\boldsymbol{x},\boldsymbol{y})$ of expected transactions $\boldsymbol{Et}(\boldsymbol{k};t,z)$ and $\boldsymbol{Et}(t,z;\boldsymbol{l})$ (A.13.1-10) take form:

$$\boldsymbol{Pe}(\boldsymbol{k};t,\boldsymbol{z}) = \left(\boldsymbol{Pe}_Q(\boldsymbol{k};t,\boldsymbol{z});\boldsymbol{Pe}_C(k_2;t,\boldsymbol{z})\right) \quad ; \quad \boldsymbol{z}=(\boldsymbol{x},\boldsymbol{y}) \tag{A.17.1}$$

$$\boldsymbol{Pe}_Q(k_1;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} ex_{iQ}(k_1;t,\boldsymbol{x})Q_{ij}(k_1;t,\boldsymbol{z})\boldsymbol{v}_{\boldsymbol{i}}(t,\boldsymbol{x}) \tag{A.17.2}$$

$$\boldsymbol{Pe}_C(k_2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} ex_{iC}(k_2;t,\boldsymbol{x})C_{ij}(k_2;t,\boldsymbol{z})\boldsymbol{v}_{\boldsymbol{i}}(t,\boldsymbol{x}) \tag{A.17.3}$$

$$\boldsymbol{Pe}(t,\boldsymbol{z};\boldsymbol{l}) = \left(\boldsymbol{Pe}_Q(t,\boldsymbol{z};l_1);\boldsymbol{Pe}_C(t,\boldsymbol{z};l_2)\right) \tag{A.17.4}$$

$$\boldsymbol{Pe}_Q(t,\boldsymbol{z};l_1) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} ex_{jQ}(t,\boldsymbol{y};l_1)Q_{ij}(t,\boldsymbol{z};l_1)\boldsymbol{v}_j(t,\boldsymbol{y}) \tag{A.17.5}$$

$$\boldsymbol{Pe}_C(t,\boldsymbol{z};l_2) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} ex_{jC}(t,\boldsymbol{y};l_2)C_{ij}(t,\boldsymbol{z};l_2)\boldsymbol{v}_j(t,\boldsymbol{y}) \tag{A.17.6}$$

$$\boldsymbol{Pe}_Q(k_1;t,\boldsymbol{z}) = Et_Q(k_1;t,\boldsymbol{z})\boldsymbol{v}_{eQ}(k_1;t,\boldsymbol{z}) = Ex_Q(k_1;t,\boldsymbol{z})Q(k_1;t,\boldsymbol{z})\boldsymbol{v}_{eQ}(k_1;t,\boldsymbol{z}) \tag{A.17.7}$$

$$\boldsymbol{Pe}_C(k_2;t,\boldsymbol{z}) = Et_C(k_2;t,\boldsymbol{z})\boldsymbol{v}_{eC}(k_2;t,\boldsymbol{z}) = Ex_C(k_2;t,\boldsymbol{z})C(k_2;t,\boldsymbol{z})\boldsymbol{v}_{eC}(k_2;t,\boldsymbol{z}) \tag{A.17.8}$$

$$\boldsymbol{Pe}_Q(t,\boldsymbol{z};l_1) = Ex_Q(t,\boldsymbol{z};l_1)Q(t,\boldsymbol{z};l_1)\boldsymbol{v}_{eQ}(t,\boldsymbol{z};l_1) \tag{A.17.9}$$

$$\boldsymbol{Pe}_C(t,\boldsymbol{z};l_2) = Ex_C(t,\boldsymbol{z};l_2)C(t,\boldsymbol{z};l_2)\boldsymbol{v}_{eC}(t,\boldsymbol{z};l_2) \tag{A.17.10}$$

$$\boldsymbol{v}_e(\boldsymbol{k};t,\boldsymbol{z}) = \left(\boldsymbol{v}_{eQ}(k_1;t,\boldsymbol{z});\ \boldsymbol{v}_{eC}(k_2;t,\boldsymbol{z})\right) \tag{A.17.11}$$

$$\boldsymbol{v}_e(t,\boldsymbol{z};\boldsymbol{l}) = \left(\boldsymbol{v}_{eQ}(t,\boldsymbol{z};l_1);\ \boldsymbol{v}_{eC}(t,\boldsymbol{z};l_2)\right) \tag{A.17.12}$$

Relations (A.17.1-17.3) and (A.17.7-17.8) for $z=(\boldsymbol{x},\boldsymbol{y})$ define expectations $Ex_Q(k_1;t,z)$ and $Ex_C(k_2;t,z)$ of sellers that approve transactions with trading volume $Q(k_1;t,z)$ (A.12.2) and cost $C(k_2;t,z)$ (A.12.3) as well as velocities $\boldsymbol{v}_{eQ}(k_1;t,z)$ and $\boldsymbol{v}_{eC}(k_2;t,z)$ (A.17.11) that describe motion of sellers expectations. Relations (A.17.4-17.6) and (A.17.9-17.10) define expectations $Ex_Q(t,z;l_1)$, $z=(\boldsymbol{x},\boldsymbol{y})$ of buyers that approve transactions with trading volume $Q(t,z;l_1)$ (A.12.5) and expectations $Ex_C(t,z;l_2)$ that approve transactions with trading cost $C(t,z;l_2)$ (A.12.6) as well as velocities $\boldsymbol{v}_{eQ}(t,z;l_1)$ and $\boldsymbol{v}_{eC}(t,z;l_2)$ (A.17.12) that describe motion of buyers expectations.

Equations (A.18.1-18.4) describe transactions $\boldsymbol{BS}(\boldsymbol{k};t,z)$ (A.12.1-12.3) and flows $\boldsymbol{P}(\boldsymbol{k};t,z)$ (A.16.1-16.3) made under sellers expectations of type $\boldsymbol{k}=(k_1;k_2)$

$$\frac{\partial}{\partial t}Q(k_1;t,\boldsymbol{z}) + \nabla\cdot\left(Q(k_1;t,\boldsymbol{z})\boldsymbol{v}_Q(k_1;t,\boldsymbol{z})\right) = F_Q(k_1;t,\boldsymbol{z}) \tag{A.18.1}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_Q(k_1;t,\boldsymbol{z}) + \nabla\cdot\left(\boldsymbol{P}_Q(k_1;t,\boldsymbol{z})\boldsymbol{v}_Q(k_1;t,\boldsymbol{z})\right) = \boldsymbol{G}_Q(k_1;t,\boldsymbol{z}) \tag{A.18.2}$$

$$\frac{\partial}{\partial t}C(k_2;t,\boldsymbol{z}) + \nabla\cdot\left(C(k_2;t,\boldsymbol{z})\boldsymbol{v}_C(k_2;t,\boldsymbol{z})\right) = F_C(k_2;t,\boldsymbol{z}) \tag{A.18.3}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_C(k_2;t,\boldsymbol{z}) + \nabla\cdot\left(\boldsymbol{P}_C(k_2;t,\boldsymbol{z})\boldsymbol{v}_C(k_2;t,\boldsymbol{z})\right) = \boldsymbol{G}_C(k_2;t,\boldsymbol{z}) \tag{A.18.4}$$

Equations (A.19.1-19.4) describe transactions $\boldsymbol{BS}(t,z;\boldsymbol{l})$ (A.12.4-12.6) and flows $\boldsymbol{P}(t,z;\boldsymbol{l})$ (A.16.4-16.7) made under Buyers expectations of type $\boldsymbol{l}=(l_1;l_2)$ are similar to (II.6.1; 6.2):

$$\frac{\partial}{\partial t}Q(t,\boldsymbol{z};l_1) + \nabla\cdot\left(Q(t,\boldsymbol{z};l_1)\boldsymbol{v}_Q(t,\boldsymbol{z};l_1)\right) = F_Q(t,\boldsymbol{z};l_1) \tag{A.19.1}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_Q(t,\boldsymbol{z};l_1) + \nabla\cdot\left(\boldsymbol{P}_Q(t,\boldsymbol{z};l_1)\boldsymbol{v}_Q(t,\boldsymbol{z};l_1)\right) = \boldsymbol{G}_Q(t,\boldsymbol{z};l_1) \tag{A.19.2}$$

$$\frac{\partial}{\partial t}C(t,\boldsymbol{z};l_2) + \nabla\cdot\left(C(t,\boldsymbol{z};l_2)\boldsymbol{v}_C(t,\boldsymbol{z};l_2)\right) = F_C(t,\boldsymbol{z};l_2) \tag{A.19.3}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}_C(t,\boldsymbol{z};l_2) + \nabla\cdot\left(\boldsymbol{P}_C(t,\boldsymbol{z};l_2)\boldsymbol{v}_C(t,\boldsymbol{z};l_2)\right) = \boldsymbol{G}_C(t,\boldsymbol{z};l_2) \tag{A.19.4}$$

Velocities $\boldsymbol{v}_e(\boldsymbol{k};t,z)$ (A.17.11) and $\boldsymbol{v}_e(t,z;\boldsymbol{l})$ (A.17.12) define equations (A.20.1-20.4) on expected transactions $\boldsymbol{Et}(\boldsymbol{k};t,z)$ (II.13.6-13.8) and their flows $\boldsymbol{Pe}(\boldsymbol{k};t,z)$ (A.17.1-17.3):

$$\frac{\partial}{\partial t}Et_Q(k_1;t,\boldsymbol{z}) + \nabla\cdot\left(Et_Q(k_1;t,\boldsymbol{z})\boldsymbol{v}_{eQ}(k_1;t,\boldsymbol{z})\right) = F_{eQ}(k_1;t,\boldsymbol{z}) \tag{A.20.1}$$

$$\frac{\partial}{\partial t}\boldsymbol{Pe}_Q(k_1;t,\boldsymbol{z}) + \nabla\cdot\left(\boldsymbol{Pe}_Q(k_1;t,\boldsymbol{z})\boldsymbol{v}_{eQ}(k_1;t,\boldsymbol{z})\right) = \boldsymbol{G}_{eQ}(k_1;t,\boldsymbol{z}) \tag{A.20.2}$$

$$\frac{\partial}{\partial t}Et_C(k_2;t,\boldsymbol{z}) + \nabla\cdot\left(Et_C(k_2;t,\boldsymbol{z})\boldsymbol{v}_{eC}(k_2;t,\boldsymbol{z})\right) = F_{eC}(k_2;t,\boldsymbol{z}) \tag{A.20.3}$$

$$\frac{\partial}{\partial t}\boldsymbol{Pe}_C(k_2;t,\boldsymbol{z}) + \nabla\cdot\left(\boldsymbol{Pe}_C(k_2;t,\boldsymbol{z})\boldsymbol{v}_{eC}(k_2;t,\boldsymbol{z})\right) = \boldsymbol{G}_{eC}(k_2;t,\boldsymbol{z}) \tag{A.20.4}$$

Equations (A.21.1-21.4) on expected transactions $\boldsymbol{Et}(t,z;\boldsymbol{l})$ (A.13.1-6) and their flows $\boldsymbol{Pe}(t,z;\boldsymbol{l})$ (A.17.4-17.6):

$$\frac{\partial}{\partial t}Et_Q(t,\boldsymbol{z};l_1)+\nabla\cdot\left(Et_Q(t,\boldsymbol{z};l_1)\boldsymbol{v}_{eQ}(t,\boldsymbol{z};l_1)\right)=F_{eQ}(t,\boldsymbol{z};l_1) \quad \text{(A.21.1)}$$

$$\frac{\partial}{\partial t}\boldsymbol{Pe}_Q(t,\boldsymbol{z};l_1)+\nabla\cdot\left(\boldsymbol{Pe}_Q(t,\boldsymbol{z};l_1)\boldsymbol{v}_{eQ}(t,\boldsymbol{z};l_1)\right)=\boldsymbol{G}_{eQ}(t,\boldsymbol{z};l_1) \quad \text{(A.21.2)}$$

$$\frac{\partial}{\partial t}Et_C(t,\boldsymbol{z};l_2)+\nabla\cdot\left(Et_C(t,\boldsymbol{z};l_2)\boldsymbol{v}_{eC}(t,\boldsymbol{z};l_2)\right)=F_{eC}(t,\boldsymbol{z};l_2) \quad \text{(A.21.3)}$$

$$\frac{\partial}{\partial t}\boldsymbol{Pe}_C(t,\boldsymbol{z};l_2)+\nabla\cdot\left(\boldsymbol{Pe}_C(t,\boldsymbol{z};l_2)\boldsymbol{v}_{eC}(t,\boldsymbol{z};l_2)\right)=\boldsymbol{G}_{eC}(t,\boldsymbol{z};l_2) \quad \text{(A.21.4)}$$

Equations (A.18.1 – 21.4) complete system of equations on transactions and expected transactions and their flows made under expectations of type $\boldsymbol{k}=(k_1;k_2)$ and $\boldsymbol{l}=(l_1;l_2)$. Equations (A.18.1 – 21.4) and definitions of price p (A.12.7-12.16) permit derive equations on price of economic variable E due to transactions $\boldsymbol{BS}$ (A.12.1-6). To derive equations on price $p(k_1,k_2;t)$ (A.12.7) for transactions (A.12.15) made under sellers expectations k_1 and k_2 let's take integrals of (A.18.1-18.4) by $dz=d\boldsymbol{x}d\boldsymbol{y}$ over economic domain:

$$C(k_2;t)=p(k_1,k_2;t)Q(k_1;t)$$

$$\frac{d}{dt}Q(k_1;t)=F_Q(k_1;t)\ ;\ \frac{d}{dt}C(k_2;t)=F_C(k_2;t) \quad \text{(A.22.1)}$$

$$Q(k_1;t)\frac{d}{dt}p(k_1,k_2;t)+p(k_1,k_2;t)F_Q(k_1;t)=F_C(k_2;t) \quad \text{(A.22.2)}$$

Transactions made in economy at moment t with variable E under all expectations of sellers and buyers define equations on price $p(t)$ (A.12.14):

$$C(t)=p(t)Q(t) \quad \text{(A.23.1)}$$

$$\frac{d}{dt}Q(t)=F_Q(t)\quad ;\ Q(t)\frac{d}{dt}p(t)+p(t)F_Q(t)=F_C(t) \quad \text{(A.23.2)}$$

Let's underline two issues on equations (A.23.2). First: price $p(t)$ (A.23.2) depends on functions $F_Q(t)$ that determine evolution of quantity $Q(t)$ (A.23.1) and $F_C(t)$ that determine evolution of cost $C(t)$ (A.12.14) of transactions. Second - complexity of price $p(t)$ equation (A.23.2) is hidden by direct form of functions $F_Q(t)$, $F_C(t)$ that define dependence of transactions (A.18.1) and (A.18.3) on $F_Q(k_1;t,z)$, $F_C(k_2;t,z)$ under sellers expectations of type $\boldsymbol{k}=(k_1;k_2)$ or (A.19.1) and (A.19.3) on $F_Q(t,\boldsymbol{z};l_1)$, $F_C(t,\boldsymbol{z};l_2)$ under buyers expectations of type $\boldsymbol{l}=(l_1;l_2)$. These functions describe dependence of transactions on expectations and their flows. Expectations may depend on economic variables, transactions, other expectations and their flows. Thus expectations that should define functions $F_Q(k_1;t,z)$, $F_C(k_2;t,z)$ for (A.18.1-18.4) or $F_Q(t)$, $F_C(t)$ for (A.23.2) in play core role for transmitting impact of different economic variables, transactions and their flows on price $p(t)$ (A.23.2) of variable E. That makes description of price $p(t)$ a really tough problem. Let's repeat that dependence of expectations on flows of variables, transactions and other expectations may cause dependence of price $p(t)$ on flows and velocities $\boldsymbol{v}_Q(t)$ and $\boldsymbol{v}_C(t)$ or velocities of transactions and etc. Analysis of price evolution and fluctuations requires development of econometrics data that can verify model dependence of expectations on economic variables, transactions and their flows.

Equations (A.22.1-4) describe sellers price $p(k_1,k_2;t)$ (A.12.15) that model price of variable E in entire economics due to sellers expectations of type $\boldsymbol{k}=(k_1, k_2)$. Let's mention that sellers price $p(k_1,k_2;t)$ (A.12.15) can differs from buyers price $p(t;l_1,l_2)$ (A.12.16) but nevertheless they both define same price $p(t)$ (A.12.14) determined by all transactions with variable E in the entire economics. Fluctuations of sellers $p(k_1,k_2;t)$ (A.12.15) can differs from statistics of buyers price $p(t;l_1,l_2)$ (A.12.16). This and many other problems concern modeling price dynamics and fluctuations should be studied further.

Moreover, equations on economic variables (I.18.1, 18.2), on transactions (II. 5.12; 5.13), expected transactions (II.10.5-10.8) and their flows should model direct dependence of variables, transactions and expectations on risk coordinates of economic space. Indeed, growth or decline of risk ratings should directly impact the value of economic variables, transactions and expectations. Economic modeling should take into account evolution of value of economic variables, transactions and expectations during motion on economic domain. Such dependence can be modeled in two ways. First approximation may model dependence of economic variables on transaction and dependence of transactions on numerous expectations. That describes numerous kinds of mutual economic and financial interactions between economic variables, transactions and expectations. The second approximation should describe direct dependence of economic variables, transactions and expectations on value of risk coordinates on economic domain. That requires introduction into economic equations (I.18.1, 18.2), (II.5.12; 5.13), (II.10.5-10.8) economic "risk potentials" that model direct impact of risk coordinates on variables, transactions, expectations and their flows. In Part III we present simple model of such direct dependence of risk coordinates to model economic surface-like waves on economic domain. The problem of economic "risk potentials" that models dependence of density functions for economic variables, transactions and expectations straightly relates to the problem of risk ratings assessments. Actually risk assessments methodologies and assessment procedures directly impact economic theory and vice versa. These problems are very interesting and we shall study mutual impact of economic theory and risk assessment in our future research.

Appendix B. Wave equations for economic variables

Let's start with equations (III.5.2) and take time derivative. We obtain with help of (III.5.4):

$$\frac{\partial^2}{\partial t^2}\varphi(t,\boldsymbol{x}) = \alpha_1 C\frac{\partial^2}{\partial t^2}\pi(t,\boldsymbol{x}) - \beta_1 C\Delta\pi(t,\boldsymbol{x}) \tag{B.1}$$

We have the similar equation from (III.5.3; 5.4):

$$C\frac{\partial^2}{\partial t^2}\pi(t,\boldsymbol{x}) = \alpha_2\frac{\partial^2}{\partial t^2}\varphi(t,\boldsymbol{x}) - \beta_2\Delta\varphi(t,\boldsymbol{x}) \tag{B.2}$$

Thus for (B.1) and (B.2) obtain:

$$(1-\alpha_1\alpha_2)\frac{\partial^2}{\partial t^2}\varphi(t,\boldsymbol{x}) = -\alpha_1\beta_2\Delta\varphi(t,\boldsymbol{x}) - \beta_1 C\Delta\pi(t,\boldsymbol{x}) \tag{B.3}$$

Let's take second time derivative from (B.3) and with (B.1; B.2) obtain for $\varphi(t,\boldsymbol{x})$ and $\pi(t,\boldsymbol{x})$:

$$\left[(1-\alpha_1\alpha_2)\frac{\partial^4}{\partial t^4} + (\alpha_1\beta_2+\beta_1\alpha_2)\Delta\frac{\partial^2}{\partial t^2} - \beta_1\beta_2\Delta^2\right]\varphi(t,\boldsymbol{x}) = 0 \tag{B.4}$$

To derive wave equations let's take Fourier transform by time and coordinates or let's substitute the wave type solution $\varphi(t,\boldsymbol{x}) = \varphi(\boldsymbol{x}\text{-}\boldsymbol{c}t)$. Than (A.4) takes form

$$(1-\alpha_1\alpha_2)c^4 + (\alpha_1\beta_2+\alpha_2\beta_1)c^2 - \beta_1\beta_2 = 0 \tag{B.5}$$

$$a = 1-\alpha_1\alpha_2 > 1\ ;\ b = \alpha_1\beta_2 + \alpha_2\beta_1 < 0\ ;\ d = \beta_1\beta_2 < 0$$

For positive roots c^2

$$c_{1,2}^2 = \frac{-b+/-\sqrt{b^4+4ad}}{2a} \tag{B.6}$$

equation (B.4) takes form of bi-wave equation (B.7) for $\varphi(t,\boldsymbol{x})$ and $\pi(t,\boldsymbol{x})$:

$$\left(\frac{\partial^2}{\partial t^2} - c_1^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - c_2^2\Delta\right)\varphi(t,\boldsymbol{x}) = 0 \tag{B.7}$$

Bi-wave equations (B.7) describe propagation of waves with two different speeds c_1 and c_2. If α_1 and α_2 equals zero, there are no wave equations and (B.4) take form

$$\left[\frac{\partial^4}{\partial t^4} - d\Delta^2\right]\varphi(t,\boldsymbol{x}) = 0;\ \ d < 0$$

Due to (III.1) supply $B(t,\boldsymbol{x})$ is proportional to price $p(t,\boldsymbol{x})$ and supply disturbances are proportional to price disturbances $\pi(t,\boldsymbol{x})$ (III.5.1). Let's take $\pi(t,\boldsymbol{x})$ as:

$$\pi(t,\boldsymbol{x}) = \pi_0\cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\exp(\gamma t + \boldsymbol{p}\cdot\boldsymbol{x})\ \ ;\ \ \pi_0 \ll 1 \tag{B.8}$$

Here $\boldsymbol{k}\cdot\boldsymbol{x}$ is scalar product of vectors $\boldsymbol{k}$ and $\boldsymbol{x}$. For price disturbances $\pi(t,\boldsymbol{x})$ (B.8) equation (B.4) becomes a system of two equations:

$$a[(\gamma^2-\omega^2)^2 - 4\gamma^2\omega^2] + b\,[(p^2-k^2)(\gamma^2-\omega^2) + 4\gamma\omega\,\boldsymbol{k}\cdot\boldsymbol{p}] - d[(p^2-k^2)^2 - 4(\,\boldsymbol{k}\cdot\boldsymbol{p}\,)^2] = 0 \tag{B.9}$$

$$4a\omega\gamma(\gamma^2-\omega^2) + b[\,2\omega\gamma\,(p^2-k^2) - 2(\gamma^2-\omega^2)\,\boldsymbol{k}\cdot\boldsymbol{p}\,] + 4d(p^2-k^2)\,\boldsymbol{k}\cdot\boldsymbol{p} = 0$$

Let's study simple case. Let's $\boldsymbol{p}=0$. Then (B.9) takes form:

$$a[(\gamma^2-\omega^2)^2 - 4\gamma^2\omega^2] - bk^2(\gamma^2-\omega^2) - dk^4 = 0$$

$$\gamma^2 - \omega^2 = \frac{bk^2}{2a} \;\; ; \;\; 4ad + b^2 < 0 \qquad \text{(B.10)}$$

Thus due to (B.10) roots $c^2_{1,2}$ (B.6) of equations (B.5) become complex numbers.

$$\gamma^4 - \frac{bk^2}{2a}\gamma^2 + \frac{k^4(b^2 + 4ad)}{16a^2} = 0 \;\; ; \;\; \gamma^2{}_{1,2} = \frac{k^2}{4a}\left(b +/- \sqrt{-4ad}\;\right)$$

Thus $\gamma^2 > 0$ for

$$\gamma^2 = \frac{k^2}{4a}\left(b + \sqrt{-4ad}\;\right) > 0 \quad ; \quad \omega^2 = \frac{k^2}{4a}\left(-b + \sqrt{-4ad}\;\right) > 0$$

For $\gamma > 0$ wave amplitude (B.8) grows up as $exp(\gamma t)$. Thus waves of small price disturbances $\pi(t,\boldsymbol{x})$ can propagate on economic domain with exponential growth of amplitude in time and that may disturb sustainable economic evolution.

Appendix C. Wave equations for perturbations of economic transactions

Let's start with equation on perturbations *s(t,z)* (III.8.1) and take time derivative $\partial/\partial t$:

$$S_0 \frac{\partial^2}{\partial t^2} s(t,\boldsymbol{z}) + S_0 \nabla \cdot \frac{\partial}{\partial t} \boldsymbol{v} = \alpha_1 D_0 \nabla \cdot \frac{\partial}{\partial t} \boldsymbol{u} \quad \text{(C.1)}$$

and substitute equations on velocity $\boldsymbol{v}(t,z)$ and $\boldsymbol{u}(t,z)$ (III.8.2):

$$S_0 \frac{\partial^2}{\partial t^2} s(t,\boldsymbol{z}) - \alpha_1 \beta_2 S_0 \Delta\, s(t,\boldsymbol{z}) = -\beta_1 D_0 \Delta\, d(t,\boldsymbol{z}) \quad \text{(C.2)}$$

The same obtain for equation for perturbations of demand *d(t,z):*

$$D_0 \frac{\partial^2}{\partial t^2} d(t,\boldsymbol{z}) = \alpha_2 \beta_1 D_0 \Delta\, d(t,\boldsymbol{z}) - \beta_2 S_0 \Delta\, s(t,\boldsymbol{z}) \quad \text{(C.3)}$$

Let's take second derivative by time $\partial^2/\partial t^2$ of (C.2):

$$S_0 \frac{\partial^4}{\partial t^4} s(t,\boldsymbol{z}) - S_0 \alpha_1 \beta_2 \Delta \frac{\partial^2}{\partial t^2} s(t,\boldsymbol{z}) = -D_0 \beta_1 \Delta \frac{\partial^2}{\partial t^2} d(t,\boldsymbol{z})$$

and substitute (C.3):

$$S_0 \left[\frac{\partial^4}{\partial t^4} s(t,\boldsymbol{z}) - \alpha_1 \beta_2 \Delta \frac{\partial^2}{\partial t^2} s(t,\boldsymbol{z}) - \beta_1 \beta_2 \Delta^2\, s(t,\boldsymbol{z})\right] = -D_0 \alpha_2 \beta_1 \beta_1 \Delta^2 d(t,\boldsymbol{z}) \quad \text{(C.4)}$$

Now take operator Δ of (C.2) and obtain:

$$S_0 \frac{\partial^2}{\partial t^2} \Delta s(t,\boldsymbol{z}) - S_0 \alpha_1 \beta_2 \Delta^2 s(t,\boldsymbol{z}) = -D_0 \beta_1 \Delta^2\, d(t,\boldsymbol{z})$$

and substitute into (C.4) obtain equations for perturbations of supply *s(t,z)* and demand *d(t,z)*:

$$\left[\frac{\partial^4}{\partial t^4} - (\alpha_1 \beta_2 + \alpha_2 \beta_1) \Delta \frac{\partial^2}{\partial t^2} + \beta_1 \beta_2 (\alpha_1 \alpha_2 - 1) \Delta^2\right] s(t,\boldsymbol{z}) = 0 \quad \text{(C.5)}$$

Let's define

$$a = (\alpha_1 \beta_2 + \alpha_2 \beta_1) \quad ; \quad b = \beta_1 \beta_2 (\alpha_1 \alpha_2 - 1) \quad \text{(C.6)}$$

Let's take

$$s(t,\boldsymbol{z}) = s(\boldsymbol{z} - \boldsymbol{c}t)$$

and (C.5) takes form of bi-wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - c_1^2 \Delta\right)\left(\frac{\partial^2}{\partial t^2} - c_2^2 \Delta\right) s(t,\boldsymbol{z}) = 0\,; \;\; \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \quad \text{(C.7)}$$

$$c_{1,2}^4 - a c_{1,2}^2 + b = 0$$

1. For $a>0$; $b>0$ there are two positive roots for squares of velocities c^2

$$c_{1,2}^2 = \frac{a +/- \sqrt{a^2 - 4b}}{2} > 0 \quad \text{(C.8)}$$

2. For $a>0$; $b<0$ or for $a<0$; $b<0$ there is one positive root for speed square

$$c_1^2 = \frac{a + \sqrt{a^2 - 4b}}{2} > 0 \quad \text{(C.9)}$$

3. For $a<0$; $b>0$ there are no positive roots and thus no wave regime.

For each positive square of speed c^2

$$c^2 = c_x^2 + c_y^2 > 0 \quad \text{(C.10)}$$

Here $c_x{}^2$ – describes wave speed of suppliers along axes $\boldsymbol{x}$ and $c_y{}^2$ – describes wave speed of consumers of goods along axes $\boldsymbol{y}$. Thus single positive value of c^2 means that there can be a lot of different waves of supply perturbations with different wave speed $\boldsymbol{c}_x$ along axes $\boldsymbol{x}$ and speed $\boldsymbol{c}_y$ along axes $\boldsymbol{y}$. The same value c^2 (C.8) or (C.9) may induce waves of supply $s(t,z)$ and demand $d(t,z)$ perturbations with different waves speed $\boldsymbol{c}_s$ of supply and $\boldsymbol{c}_d$ of demand that fulfill the conditions (C.10):

$$\boldsymbol{c_s} = (\boldsymbol{c_{sx}}\, ;\, \boldsymbol{c_{sy}}) \;\; {c_s}^2 = c_{sx}^2 + c_{sy}^2 > 0 \quad \text{(C.11)}$$

$$\boldsymbol{c_d} = (\boldsymbol{c_{dx}}\, ;\, \boldsymbol{c_{dy}}) \;\; {c_d}^2 = c_{dx}^2 + c_{dy}^2 > 0 \quad \text{(C.12)}$$

$$\boldsymbol{c_s} = (\boldsymbol{c_{sx}}\, ;\, \boldsymbol{c_{sy}}) \neq \boldsymbol{c_d} = (\boldsymbol{c_{dx}}\, ;\, \boldsymbol{c_{dy}}) \;\; but \;\; {c_s}^2 = c_d^2 > 0$$

Let show that equations (C.5) allow propagation of supply disturbances waves with amplitudes growing as exponent. Let take $s(t,z)$ as:

$$s(t,\boldsymbol{z}) = \cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{z})\exp(\gamma t) \;\; ; \;\; \boldsymbol{k} = (\boldsymbol{k}_x, \boldsymbol{k}_x) \quad \text{(C.13)}$$

Function (C.13) satisfies equations (C.5) if:

$$\omega^2 = \gamma^2 + \frac{ak^2}{2} \qquad 4\gamma^2\omega^2 = k^4\left(b - \frac{a^2}{4}\right) > 0 \;\; ; \; 4b > a^2$$

$$\gamma^2 = k^2\frac{\sqrt{4b+3a^2}-2a}{8} > 0 \qquad \omega^2 = k^2\frac{\sqrt{4b+3a^2}+2a}{8} > 0$$

For $\gamma > 0$ wave amplitude grows up as $exp(\gamma t)$. Let's show that equations (III.8.1; 8.2) on disturbances of supply transactions from $\boldsymbol{x}$ to $\boldsymbol{y}$ and demand transactions from $\boldsymbol{y}$ to $\boldsymbol{x}$ induce equations on perturbations of economic variables – densities of supply $S_{out}(t,\boldsymbol{x})$ from point $\boldsymbol{x}$, supply $S_{in}(t,\boldsymbol{y})$ to point $\boldsymbol{y}$, demand $D_{out}(t,\boldsymbol{y})$ from point $\boldsymbol{y}$ and demand $D_{in}(t,\boldsymbol{x})$ at point $\boldsymbol{x}$ and their flows. To do that let's take integral by $\boldsymbol{dy}$ over economic domain (II.1.1; 1.2). Due to (II.3) supply $S_{out}(t,\boldsymbol{x})$ from point $\boldsymbol{x}$ and supply $S_{in}(t,\boldsymbol{y})$ to point $\boldsymbol{y}$ are defined as:

$$S_{out}(t,\boldsymbol{x}) = \int d\boldsymbol{y}\; S(t,\boldsymbol{x},\boldsymbol{y}) \;\; ; \;\; S_{in}(t,\boldsymbol{y}) = \int d\boldsymbol{x}\; S(t,\boldsymbol{x},\boldsymbol{y}) \quad \text{(C.14.1)}$$

and use (III.7.3) to define their flows $\boldsymbol{P}_{out}(t,\boldsymbol{x})$ and $\boldsymbol{P}_{in}(t,\boldsymbol{y})$:

$$\boldsymbol{P}_{out}(t,\boldsymbol{x}) = \int d\boldsymbol{y}\; \boldsymbol{P}(t,\boldsymbol{x},\boldsymbol{y}) \;\; ; \;\; \boldsymbol{P}_{in}(t,\boldsymbol{y}) = \int d\boldsymbol{x}\; \boldsymbol{P}(t,\boldsymbol{x},\boldsymbol{y}) \quad \text{(C.14.2)}$$

The similar relations define demand $D_{out}(t,\boldsymbol{y})$ from point $\boldsymbol{y}$ and demand $D_{in}(t,\boldsymbol{x})$ at point $\boldsymbol{x}$ and their flows:

$$D_{out}(t,\boldsymbol{y}) = \int d\boldsymbol{x}\; D(t,\boldsymbol{x},\boldsymbol{y}) \;\; ; \;\; D_{in}(t,\boldsymbol{x}) = \int d\boldsymbol{y}\; D(t,\boldsymbol{x},\boldsymbol{y}) \quad \text{(C.14.3)}$$

$$\boldsymbol{Q}_{out}(t,\boldsymbol{y}) = \int d\boldsymbol{x}\; \boldsymbol{Q}(t,\boldsymbol{x},\boldsymbol{y}) \;\; ; \;\; \boldsymbol{Q}_{in}(t,\boldsymbol{x}) = \int d\boldsymbol{y}\; \boldsymbol{Q}(t,\boldsymbol{x},\boldsymbol{y}) \quad \text{(C.14.4)}$$

Economic meaning of supply $S_{out}(t,\boldsymbol{x})$ - it is total supply of selected goods, commodities etc., from point $\boldsymbol{x}$. Function $S_{in}(t,\boldsymbol{y})$ describes total supply of selected goods to point $\boldsymbol{y}$. Economic density function $D_{out}(t,\boldsymbol{y})$ describes total demand from point $\boldsymbol{y}$ and $D_{in}(t,\boldsymbol{x})$ – total demand at point $\boldsymbol{x}$ from entire economy. Equations on density functions $S_{out}(t,\boldsymbol{x})$, $S_{in}(t,\boldsymbol{y})$, $D_{in}(t,\boldsymbol{x})$, $D_{out}(t,\boldsymbol{y})$ and their flows can be derived from (III.7.1; 7.2; 7.4; 7.5). Let's take integrals by $d\boldsymbol{x}$ or $d\boldsymbol{y}$ over economic space:

$$\frac{\partial}{\partial t} S_{out}(t,\boldsymbol{x}) + \nabla\cdot(S_{out}\,\boldsymbol{v_{out}}) = \alpha_1\,\nabla\cdot\,\boldsymbol{Q}_{in}(t,\boldsymbol{x}) \quad (C.15.1)$$

$$\frac{\partial}{\partial t} D_{in}(t,\boldsymbol{x}) + \nabla\cdot(D_{in}\,\boldsymbol{u_{in}}) = \alpha_2\,\nabla\cdot\,\boldsymbol{P}_{out}(t,\boldsymbol{x}) \quad (C.15.2)$$

$$\frac{\partial}{\partial t} \boldsymbol{P}_{out}(t,\boldsymbol{x}) + \nabla\cdot(\boldsymbol{P}_{out}\,\boldsymbol{v}_{out}) = \beta_1\,\nabla D_{in}(t,\boldsymbol{x}) \quad (C.15.3)$$

$$\frac{\partial}{\partial t} \boldsymbol{Q}_{in}(t,\boldsymbol{x}) + \nabla\cdot(\boldsymbol{Q}_{in}\,\boldsymbol{u}_{in}) = \beta_2\,\nabla S_{out}(t,\boldsymbol{x}) \quad (C.15.4)$$

$$\boldsymbol{P}_{out}(t,\boldsymbol{x}) = S_{out}(t,\boldsymbol{x})\boldsymbol{v}_{out}(t,\boldsymbol{x}) \;;\; \boldsymbol{Q}_{in}(t,\boldsymbol{x}) = D_{in}(t,\boldsymbol{x})\boldsymbol{u}_{in}(t,\boldsymbol{x}) \quad (C.15.5)$$

Similar equations are valid for $S_{in}(t,\boldsymbol{y})$, $D_{out}(t,\boldsymbol{y})$ and their flows $\boldsymbol{P}_{in}(t,\boldsymbol{y})$, $\boldsymbol{Q}_{out}(t,\boldsymbol{y})$. To derive wave equations on disturbances of $S_{out}(t,\boldsymbol{x})$, $D_{in}(t,\boldsymbol{x})$ and their flows let's take integrals by $d\boldsymbol{y}$ of (III.7.8; 7.9):

$$S_{out}(t,\boldsymbol{x}) = S_{0out}(1+s_{out}(t,\boldsymbol{x}))\;;\; D_{in}(t,\boldsymbol{x}) = D_{0in}(1+d_{in}(t,\boldsymbol{x})) \quad (C.16.4)$$

$$\boldsymbol{P}_{out}(t,\boldsymbol{x}) = S_{0out}\boldsymbol{v}_{out}(t,\boldsymbol{x}) \;;\; \boldsymbol{Q}_{in}(t,\boldsymbol{x}) = D_{0in}\boldsymbol{u}_{in}(t,\boldsymbol{x}) \quad (C.16.5)$$

Equations on disturbances $s_{out}(t,\boldsymbol{x})$, $d_{in}(t,\boldsymbol{x})$ and their flows are similar to (III.8.1; 8.2) but perturbations depend on $\boldsymbol{x}$ only:

$$\frac{\partial}{\partial t} s_{out}(t,\boldsymbol{x}) + S_0\nabla\cdot\boldsymbol{v}_{out} = \alpha_1 D_0\nabla\cdot\boldsymbol{u}_{in}(t,\boldsymbol{x}) \quad (C.16.6)$$

$$\frac{\partial}{\partial t} d_{in}(t,\boldsymbol{x}) + D_0\nabla\cdot\boldsymbol{u}_{in} = \alpha_2 S_0\nabla\cdot\boldsymbol{v}_{out}(t,\boldsymbol{x}) \quad (C.16.7)$$

$$S_0\frac{\partial}{\partial t}\boldsymbol{v}_{out}(t,\boldsymbol{z}) = \beta_1\nabla\, d(t,\boldsymbol{x})\;;\; D_0\frac{\partial}{\partial t}\boldsymbol{u}_{in}(t,\boldsymbol{x}) = \beta_2\nabla\, s(t,\boldsymbol{x}) \quad (C.16.8)$$

Equations on disturbances $s_{out}(t,\boldsymbol{x})$ and $d_{in}(t,\boldsymbol{x})$ as well on $s_{in}(t,\boldsymbol{x})$ and $d_{out}(t,\boldsymbol{x})$ take form similar to (C.5; C.6):

$$\left[\frac{\partial^4}{\partial t^4} - a\Delta\,\frac{\partial^2}{\partial t^2} + b\Delta^2\right]\, s_{out}(t,\boldsymbol{x}) = 0 \quad (C.17.1)$$

Let's argue signs of α_1, α_2, β_1, β_2. Positive divergence $D_0\nabla\cdot\boldsymbol{u}_{in}(t,\boldsymbol{x}) > 0$ for disturbances of demand flow means that demand flows out of a unit volume dV at point $\boldsymbol{x}$ and thus reduce amount of demand at $\boldsymbol{x}$. Decline of demand may decline supply $s_{out}(t,\boldsymbol{x})$ and hence we take $\alpha_1<0$. As well positive divergence $S_0\nabla\cdot\boldsymbol{v}_{out}(t,\boldsymbol{x}) > 0$ for disturbances of supply flow means that supply flows out of a unit volume dV at point $\boldsymbol{x}$ and hence decline supply at $\boldsymbol{x}$. Reduction of supply at $\boldsymbol{x}$ may increase demand at this point and we take $\alpha_2>0$. Equations (C.16.8) model relations between supply flows $S_0v(t,\boldsymbol{x})$ and gradient of demand perturbations. We propose that supply flows $S_0\boldsymbol{v}(t,\boldsymbol{x})$ grow up in the direction of higher demand determined by gradient of demand perturbations $\nabla d(t,\boldsymbol{x})$ and thus take $\beta_1>0$. As well demand flows $D_0\boldsymbol{u}(t,\boldsymbol{x})$ decline in the direction of higher supply determined by gradient of supply perturbations $\nabla s(t, \boldsymbol{x})$ and thus take $\beta_2<0$. Hence we obtain:

$$\alpha_1 < 0 \ ; \ \alpha_2 > 0 \ ; \beta_1 > 0 \ ; \ \ \beta_2 < 0 \qquad \text{(C.17.2)}$$

$$a = (\alpha_1\beta_2 + \alpha_2\beta_1) > 0 \ ; \ b = \beta_1\beta_2(\alpha_1\alpha_2 - 1) > 0$$

and due to (C.8) there are two positive roots for c^2 of (B.7). Same considerations are valid for equations on $s_{in}(t,\boldsymbol{x})$ and $d_{out}(t,\boldsymbol{x})$. Thus disturbances of economic variables $s_{out}(t,\boldsymbol{x})$ and $d_{in}(t,\boldsymbol{x})$ follow bi-wave equations

$$(\frac{\partial^2}{\partial t^2} - c_1^2\Delta)(\frac{\partial^2}{\partial t^2} - c_2^2\Delta)s(t, \boldsymbol{x}) = 0 \qquad \text{(C.17.3)}$$

Wave equations (C.7) on transactions disturbances induce similar wave equations on disturbances of *–in* and *–out* economic variables that are determined by transactions. Let's show that these waves induce small fluctuations of macroeconomic variables. Let's study economics under action of a single risk. Due to (II.1.1; 1.2) transactions are defined on *2*-dimensional economic domain. For (III.7.8) and (C.13) macroeconomic supply *S(t)* at moment *t* (II.4.1; 4.2)

$$S(t) = S_0\big(1 + s(t)\big) \ ; \quad s(t) = \int_0^1 dxdy \ s(t, x, y) \qquad \text{(C.18.1)}$$

$$s(t) = \frac{4\exp(\gamma t)}{k_x k_y}\cos\left(\frac{k_x + k_y}{2} - \omega t\right) sin\frac{k_x}{2} \ sin\frac{k_y}{2} \qquad \text{(C.18.2)}$$

Hence disturbances *s(t)* of macroeconomic supply *S(t)* at moment *t* may grow up as $exp(\gamma t)$ for $\gamma>0$ or dissipate to constant rate S_0 for $\gamma<0$ and fluctuate with frequency ω.

Appendix D. The business cycle equations

Let's show that macroeconomic supply *S(t)* and demand *D(t)* follow fluctuations that can be treated as business cycles. To derive equations on *S(t)* and *D(t)* as (II.4.1) let's take integral by *dz=dxdy* of (III.13.1; 13.3):

$$\frac{d}{dt}S(t) = \frac{d}{dt}\int d\boldsymbol{z}\, S(t,\boldsymbol{z}) = -\int d\boldsymbol{z}\ \nabla\cdot\big(\boldsymbol{v}(t,\boldsymbol{z})S(t,\boldsymbol{z})\big) + a\int d\boldsymbol{z}\ \boldsymbol{z}\cdot\boldsymbol{P}_D(t,\boldsymbol{z}) \qquad \text{(D.1.1)}$$

First integral in the right side (D.1.1) is integral of divergence over 2-dimensional economic domain (III.6.1; 6.2) and due to divergence theorem (Strauss 2008, p.179) it equals integral of flux through surface of economic domain and hence equals zero as no economic fluxes exist outside of economic domain (III.6.1; 6.2). Let's define *Pz(t)* and *Dz(t)* as:

$$P_Sz(t) = \int dxdy\, xP_{Sx}(t,x,y) + yP_{Sy}(t,x,y) = P_Sx(t) + P_Sy(t) \qquad \text{(D.1.2)}$$

$$P_Dz(t) = \int dxdy\, xP_{Dx}(t,x,y) + yP_{Dy}(t,x,y) = P_Dx(t) + P_Dy(t) \qquad \text{(D.1.3)}$$

Due to (D.1.1-1.3) equations on *S(t)* and *D(t)* take form:

$$\frac{d}{dt}S(t) = a\,[P_Dx(t) + P_Dy(t)\,] \quad ; \quad \frac{d}{dt}D(t) = b\,[P_Sx(t) + P_Sy(t)] \qquad \text{(D.1.4)}$$

To derive equations on *Pz(t)* and *Dz(t)* let's use equations (III.13.2; 13.4) on flows $\boldsymbol{P}_S(t)$, $\boldsymbol{P}_S(t)$ and matrix operators as (III.13.6; 13.7).

$$P_{Sx}(t) = \int dxdy\, P_{Sx}(t,x,y) = S(t)v_x(t) \qquad \text{(D.1.5)}$$

$$P_{Sy}(t) = \int dxdy\, P_{Sy}(t,x,y) = S(t)v_y(t) \qquad \text{(D.1.6)}$$

$$P_{Dx}(t) = \int dxdy\, P_{Dx}(t,x,y) = D(t)u_x(t) \qquad \text{(D.1.7)}$$

$$P_{Dy}(t) = \int dxdy\, P_{Dy}(t,x,y) = D(t)u_y(t) \qquad \text{(D.1.8)}$$

Similar to (D.1.1) from (III.13.2; 13.6; 13.7) for (D.1.5- D.1.8) obtain:

$$\frac{d}{dt}P_{Sx}(t) = c_1P_{Dx}(t) \quad ; \quad \frac{d}{dt}P_{Dx}(t) = d_1P_{Sx}(t) \qquad \text{(D.2.1)}$$

$$\frac{d}{dt}P_{Sy}(t) = c_2P_{Sy}(t) \quad ; \quad \frac{d}{dt}P_{Dy}(t) = d_2P_{Sy}(t) \qquad \text{(D.2.2)}$$

As we mentioned before, flows (D.1.5-1.8) can't have constant sign of velocities (D.1.5-1.8). Indeed, let's define mean risk $X_S(t)$ of suppliers with variable *E* and mean risk $Y_C(t)$ of consumers of variable *E* as:

$$S(t)X_S(t) = \int dxdy\, x\, S(t,x,y) \ ; \ S(t)Y_C(t) = \int dxdy\, y\, S(t,x,y) \qquad \text{(D.2.3)}$$

It is easy to show that for $F_S(t,x,y)=0$ one derive from (III.13.1; 13.8):

$$\frac{d}{dt}S(t) = 0\,;\ S(t) = S_0 = const;\ \frac{d}{dt}X_S(t) = v_x(t)\,;\ \frac{d}{dt}Y_C(t) = v_y(t) \qquad \text{(D.2.4)}$$

Thus in the absence of interaction $F_S(t,x,y)=0$ mean risk $X_S(t)$ of suppliers of variable *E* moves along axis *X* with velocity $\upsilon_x(t)$ (D.2.4) and mean risk $Y_C(t)$ of consumers of variable *E* moves along axis *Y* with velocity $\upsilon_y(t)$ (D.2.4). Borders of economic domain reduce motion of mean risks. Hence velocities $\upsilon_x(t)$ and $\upsilon_y(t)$ must change sign and should fluctuate. Let's underline that relations (D.2.3, 2.4) simplify real economic processes as we neglect interactions between transactions $F_S(t,x,y)$ and neglect direct dependence of economic variables and transactions on risk coordinates $z=(x,y)$ on economic domain. Indeed, risks impact on economic performance and activity of economic agents. Thus change of risk coordinates should change value of density functions of economic variables and transactions. Starting with (13.1) it is easy to show that in the presence of interactions between supply $S(t,x,y)$ and demand $D(t,x,y)$ transactions mean risks $X_S(t)$ of suppliers of variable E change due to two factors as:

$$\frac{d}{dt} X_S(t) = \upsilon_x(t) + w_x(t) \tag{D2.5}$$

$$w_x(t) = [X_{SF}(t) - X_S(t)] \frac{d}{dt} lnS(t) \tag{D.2.6}$$

$$F_S(t) = \int dxdy\, F_S(t,x,y) \;\; ; \;\; X_{SF}(t) F_S(t) = \int dxdy\;\, x\, F_S(t,x,y) \tag{D.2.7}$$

Here $\upsilon_x(t)$ is determined by (III.13.8) and velocity $w_x(t)$ (D.2.6, 2.7) describes motion (D.2.5) of mean risk $X_S(t)$ (D.2.3) of suppliers along axis X due to interaction $F_S(t,x,y)$ (III.13.1) of supply and demand transactions. Mean risk $X_S(t)$ of suppliers and mean risk $Y_C(t)$ of consumers (D.2.3) of variable E on economic domain (III.6.1; 6.2) are reduced by borders of economic domain (D.2.8):

$$0 \le X_S(t) \le 1 \;\; ; \;\; 0 \le Y_C(t) \le 1 \tag{D.2.8}$$

Hence velocities $\upsilon_x(t)$ (D.1.5-1.8) and $w_x(t)$ (D.2.6-7) should fluctuate as (D.2.8) reduce motion of mean risks (D.2.3, 2.5). Thus (D.2.5) describes two sources of fluctuations caused by velocities $\upsilon_x(t)$ (D.1.5-1.8) and $w_x(t)$ (D.2.6-7). Let's model fluctuations of flows $\boldsymbol{P}_S(t)$ and $\boldsymbol{P}_D(t)$ by equations (D.2.1-2) that describe harmonique oscillations with frequencies ω, ν:

$$\omega^2 = -c_1 d_1 > 0 \;\; ; \;\; \nu^2 = -c_2 d_2 > 0 \tag{D.3.1}$$

$$\left[\frac{d^2}{dt^2} + \omega^2\right] P_{Sx}(t) = 0 \;\; ; \;\; \left[\frac{d^2}{dt^2} + \omega^2\right] P_{Dx}(t) = 0 \tag{D.3.2}$$

$$\left[\frac{d^2}{dt^2} + \nu^2\right] P_{Sy}(t) = 0 \;\; ; \;\; \left[\frac{d^2}{dt^2} + \nu^2\right] P_{Dy}(t) = 0 \tag{D.3.3}$$

Frequencies ω describe oscillations of mean risk $X_S(t)$ (D.2.3-2.4) of suppliers along axis X and ν describe oscillations of consumers mean risk $Y_C(t)$ along axis Y. Solutions (D.3.1-3.3):

$$P_{Sx}(t) = P_{Sx}(1) \sin \omega t + P_{Sx}(2) \cos \omega t \,; P_{Sy}(t) = P_{Sy}(1) \sin \nu t + P_{Sy}(2) \cos \nu t \tag{D.3.4}$$

$$P_{Dx}(t) = P_{Dx}(1) \sin \omega t + P_{Dx}(2) \cos \omega t \,; P_{Dy}(t) = P_{Dy}(1) \sin \nu t + P_{Dy}(2) \cos \nu t \tag{D.3.5}$$

To derive equations on $Pz(t)$ and $Dz(t)$ let's derive equations on their components $P_Sx(t)$, $P_Sy(t)$, $P_Dx(t)$, $P_Dy(t)$ (D.1.2;1.3) and use equations (III.13.2; 13.6). Let's multiply equations (III.13.2) by $z=(x,0)$ and take integral by $dxdy$

$$\frac{d}{dt}P_S x(t) = \frac{d}{dt}\int dxdy\, xP_{Sx}(t,x,y) = \int dxdy\,[-x\frac{\partial}{\partial x}(v_x\, P_{Sx}) + c_1 xP_{Dx}(t,x,y)]$$

$$-\int dxdy\, x\frac{\partial}{\partial x}(v_x\, P_{Sx}) = \int dxdy\, v_x^2(t,x,y)S(t,x,y)$$

For $P_Sx(t)$, $P_Sy(t)$, $P_Dx(t)$, $P_Dy(t)$ (D.1.2;1.3) obtain equations:

$$\frac{d}{dt}P_S x(t) = ESx(t) + c_1 P_D x(t) \;\; ; \;\; \frac{d}{dt}P_D x(t) = EDx(t) + d_1 P_S x(t)$$

$$\frac{d}{dt}P_S y(t) = ESy(t) + c_2 P_D x(t) \;\; ; \;\; \frac{d}{dt}P_D y(t) = EDy(t) + d_2 P_S y(t)$$

Let's use (III.13.10) and denote *ESx(t,x,y), ESy(t,x,y), EDx(t,x,y) EDy(t,x,y)* and *ESx(t), ESy(t), EDx(t) EDy(t)* as:

$$ESx(t) = \int dxdy\; ESx(t,x,y) = \int dxdy\, v_x^2(t,x,y)S(t,x,y) = S(t)v_x^2(t) \tag{D.4.1}$$

$$ESy(t) = \int dxdy\; ESy(t,x,y) = \int dxdy\, v_y^2(t,x,y)S(t,x,y) = S(t)v_y^2(t) \tag{D.4.2}$$

$$EDx(t) = \int dxdy\; EDx(t,x,y) = \int dxdy\, u_x^2(t,x,y)D(t,x,y) = D(t)u_x^2(t) \tag{D.4.3}$$

$$EDy(t) = \int dxdy\; EDy(t,x,y) = \int dxdy\, u_y^2(t,x,y)D(t,x,y) = D(t)u_y^2(t) \tag{D.4.4}$$

Equations on $P_Sx(t)$, $P_Sy(t)$, $P_Dx(t)$, $P_Dy(t)$ take form:

$$\left[\frac{d^2}{dt^2} + \omega^2\right]P_S x(t) = \frac{d}{dt}ESx(t) + c_1 EDx(t)\, ; \left[\frac{d^2}{dt^2} + \omega^2\right]P_D x(t) = \frac{d}{dt}EDx(t) + d_1 ESx(t) \tag{D.4.5}$$

$$\left[\frac{d^2}{dt^2} + v^2\right]P_S y(t) = \frac{d}{dt}ESy(t) + c_2 EDy(t)\, ; \left[\frac{d^2}{dt^2} + v^2\right]P_D y(t) = \frac{d}{dt}EDy(t) + d_2 ESy(t) \tag{D.4.6}$$

Equations (D.4.5-4.6) describe fluctuations of $P_Sx(t)$, $P_Sy(t)$, $P_Dx(t)$, $P_Dy(t)$ with frequencies ω and v under action of *ESx, ESy, EDx, EDy* (D.4.1-4.4). To close system of ordinary differential equations (D.4.5-4.6) let's define equations on *ESx, ESy, EDx, EDy*. Let's outline that relations (D.4.1-4.4) are proportional to product of supply *S(t)* and velocity square $v^2(t)$ and *looks alike to* energy of a particle with mass *S(t)* and velocity square velocity $v^2(t)$. We underline that this is only *similarity* between (D.4.1-4.5) and energy of a particle and have no further analogies. To define equations on (D.4.1-4.5) let's propose that:

$$\frac{\partial}{\partial t}ESx(t,x,y) + \frac{\partial}{\partial x}(v_x ESx) = \mu_1 EDx \;\; ; \;\; \frac{\partial}{\partial t}EDx(t,x,y) + \frac{\partial}{\partial x}(u_x EDx) = \eta_1 ESx \tag{D.5.1}$$

$$\frac{\partial}{\partial t}ESy(t,x,y) + \frac{\partial}{\partial y}\left(v_y ESy\right) = \mu_2 EDy \;\; ; \;\; \frac{\partial}{\partial t}EDy(t,x,y) + \frac{\partial}{\partial y}\left(u_y EDy\right) = \eta_2 ESx \tag{D.5.2}$$

$$\gamma_1^2 = \mu_1\eta_1 > 0 \;\; ; \;\; \gamma_2^2 = \mu_2\eta_2 > 0 \tag{D.5.3}$$

Equations (D.5.1-3) give equations on *ESx(t), ESy(t), EDx(t), EDy(t)*

$$\left[\frac{d^2}{dt^2} - \gamma_1^2\right]ESx(t) = 0 \;\; ; \;\; \left[\frac{d^2}{dt^2} - \gamma_1^2\right]EDx(t) = 0 \tag{D.5.4}$$

$$\left[\frac{d^2}{dt^2} - \gamma_2^2\right]ESy(t) = 0 \;\; ; \;\; \left[\frac{d^2}{dt^2} - \gamma_2^2\right]EDy(t) = 0 \tag{C.5.5}$$

Let's explain economic meaning of (D.5.1-5.5): "energies" *ESx(t), ESy(t), EDx(t), EDy(t)* grow up or decay in time by exponent $exp(\gamma_1 t)$ and $exp(\gamma_2 t)$ that can be different for each risk axis. Here γ_1 define exponential growth or decay in time of *ESx(t)* induced by motion of suppliers along axis *X* and γ_2 describe exponential growth or decrease in time of *ESy(t),* induced by motion of consumers along axis *Y*. The same valid for *EDx(t)* and *EDy(t)* respectively. Solutions of (D.5.4-5.5; D.4.5-4.6) with exponential growth have form:

$$ESx(t) = ESx(1)\, exp\, \gamma_1 t \;\; ; \; ESy(t) = ESy(1)\, exp\, \gamma_2 t$$

$$EDx(t) = EDx(1)\, exp\, \gamma_1 t \; ; \;\; EDy(t) = EDy(1)\, exp\, \gamma_2 t$$

$$P_S x(t) = P_S x(1) \sin \omega t + P_S x(2) \cos \omega t + P_S x(3)\, exp\, \gamma_1 t$$

$$P_S y(t) = P_S y(1) \sin \nu_i t + P_S y(2) \cos \nu_i t + P_S y(3)\, exp\, \gamma_2 t$$

$$P_D x(t) = P_D x(1) \sin \omega t + P_D x(2) \cos \omega t + P_D x(3)\, exp\, \gamma_1 t$$

$$P_D y(t) = P_D y(1) \sin \nu_i t + P_D y(2) \cos \nu_i t + P_D y(3)\, exp\, \gamma_2 t$$

Macroeconomic supply *S(t)* of variable *E* as solution of (D.1.4) takes form:

$$S(t) = S(0) + a\big[S_x(1) \sin \omega t + S_x(2) \cos \omega t + S_y(1) \sin \nu t + S_y(2) \cos \nu t\big] + a\big[S_x(3) exp\, \gamma_1 t + S_y(3)\, exp\, \gamma_2 t\big] \quad \text{(D.5.6)}$$

Initial values and equations (D.1.4-D.5.5) define simple but long relations on constants $S_x(j)$, $S_y(j)$, $j=0,..3$ and we omit them here. Similar relations valid for demand *D(t).*