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Abstract

I offer an approach linking a welfare criterion to the "sustainable development potential" of the economy. This implies a dependence of a criterion on the information about the current state. I consider the problem for the Dasgupta-Heal-Solow-Stiglitz model with externalities. The economy-linked criterion is constructed on an example of the maximin principle applied to a hybrid level-growth measure. This measure includes as special cases the conventional measures of consumption level and percent change as a measure of growth. The hybrid measure or geometrically weighted percent can be used for measuring sustainable growth as an alternative to percent. The closed form solutions are obtained for the optimal paths including the paths, dynamically consistent with the information updates.

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Keywords: essential nonrenewable resource, modified Hotelling Rule, economy-linked criterion, geometrically weighted percent, normative resource peak

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1 Introduction

Groth et al (2006) argue that the notion of regular growth should be more general than that of exponential growth. In this paper, I obtain the patterns of feasible and optimal sustainable growth for the extended Dasgupta-Heal-Solow-Stiglitz (DHSS) model (Dasgupta and Heal 1974; Solow 1974; Stiglitz 1974) with an essential nonrenewable resource under the standard Hartwick Investment Rule (Hartwick 1977). The extension is that the Hotelling Rule is modified by some phenomena whose total influence can be expressed in terms of an equivalent tax or subsidy.[1] I show that the feasible patterns of growth for this economy are between the constant consumption and the quasiarithmetic growth with parameters depending on the price elasticities of demand for the resource and capital. The approach implies that the final expression for the optimality criterion and therefore the optimal growth paths for the economy are defined via the economy's technological parameters and the initial values for the resource reserve, the rate of the resource extraction, and capital.

Previous results (Bazhanov 2008a & 2008b) have shown that if a criterion is not linked to the initial state of a specific (non-optimal) economy, then the economy guided by this criterion can enter either an inferior or

[1] There is extensive literature discussing a discrepancy between the standard Hotelling Rule and the observed data. The Rule implies that the path of the resource extraction must be decreasing and the resource price must grow at the same rate as the rate of interest. However, this is not the case in the real economy (survey can be found in (Gaudet 2007)). Gaudet (2007) considered different phenomena such as changes in the cost of extraction, durability, peculiarities of the market, and uncertainties. These phenomena can influence both the price dynamics and the pattern of extraction, but they were not considered by Hotelling in his seminal paper (1931). Therefore, the introduction of these effects into the model of Hotelling can reconcile his approach with the observed behavior of the market price and extraction for different kinds of nonrenewable resources including oil.

unsustainable pattern of growth.[2] Both these outcomes can be considered as unacceptable. This implies that a criterion for formulating a long-term sustainable development program in the specific economy should depend on available information about the current state. In other words, the preferences should be adjusted to opportunities. Then, the optimal path should be dynamically corrected with the updates in this information in order to be consistent with the criterion.

The economy-linked criterion is constructed in this paper on an example of the maximin principle applied to a hybrid level-growth utility measure which I call "geometrically weighted percent." The use of the maximin in the problems of intergenerational justice implies that some social welfare measure should be maintained constant over time. Therefore, it is natural to use this convenient property of the maximin for formulating the long-run programs of sustainable development.[3] The hybrid measure, to which the maximin is applied in this paper, includes as particular cases the level of consumption and the rate of growth. In general case it includes all intermediate forms for measuring the level and/or the rate of growth of consumption. This family of measures implies a corresponding family of patterns of optimal growth that can vary from stagnation and quasiarithmetic growth to linear

[2]Koopmans (1965) examined the results of application of various forms of utilitarian criterion to a simple model with a specific technology and a pattern of population growth. The rationale of his research was the idea that "one may wish to choose between principles on the basis of the results of their application. In order to do so, one first needs to know what these results are. This is an economic question logically prior to the ethical or political choice of a criterion."(p. 226) "Ignoring realities in adopting 'principles' may lead one to search for a nonexistent optimum, or to adopt an 'optimum' that is open to unanticipated objections" (p. 229).

[3]Solow (1974) applied the (Rawls 1971) maximin to the level of consumption as a simple social welfare measure that implied the constant-per-capita-consumption criterion. On the other hand, there is a conventional practice to formulate some long-run development goals in terms of constant percent change of GDP (e.g. World 1987, p. 169, p. 173).

and exponential growth. Using this approach, I answer the question: what is the best pattern of growth from this family that a specific economy with the given initial conditions can maintain forever? The approach differs from the conventional methodology in resource economics in that usually the optimal economy is being constructed under the given criterion.

The paper is structured as follows. Sections 2 and 3 introduce the assumptions about the Hotelling Rule and describe the model. The new results presented in the paper include: the methodology of specification of the generalized criterion for the given initial conditions (Section 4); the closed form solutions for the optimal paths of tax, the Hotelling Rule modifier, the rates of extraction, capital, and per capita consumption (Section 4); the condition defining the feasible patterns of sustainable growth (Section 5); the unacceptable consequences of applying the criterion beyond its feasible limits (Section 6); the optimal paths dynamically consistent with the updates in reserve estimates (Section 9). The findings are illustrated with the numerical examples based on the current world oil extraction data (Sections 7-9). The conclusion is in Section 10.

2 The Hotelling Rule assumptions

I divide here the phenomena modifying the Hotelling Rule in two groups:

(a) “natural” processes; for example, technical progress and the worsening quality of resources that influence the cost of extraction;

(b) “externalities”, which are the result of the specific market structure (common property), insecure property rights, or global warming as a result of burning the resource.[4]

[4] Externality connected with global warming is a very interesting special case because it modifies not only the Hotelling Rule but also the production function and/or the utility

I assume that

(1) the effects from the first group are "uncontrollable" essential parts of the process of the resource extraction and they must be included in the modification of the Hotelling Rule as a necessary condition of efficient (in terms of consumption) extraction.[5]

(2) The influence of the phenomena from the second group can be eliminated by institutional changes and environmental policies influencing the resource demand (Caillaud et al. 1988; Pezzey 2002), including compensating tax in such a way that the resulting resource extraction will bring more social welfare to the economy. Hence, I am going to consider the effects of the second group separately from the effects of the first one and call them the "distortions" of the Hotelling Rule or "externalities."

(3) All the effects from the second group ("distortions") can be expressed in terms of equivalent amount of tax/subsidy.

For example, insecure property rights lead to shifting extraction from the future towards the present (Long 1975) or to "overexploitation" (in terms of consumption lost) which is happening also in a common property situation. I assume that the same effect can be obtained by subsidizing production connected with the use of the resource. Thus, I will consider all the "externalities" or "distortions" in the same terms of tax/subsidy including the subsidies themselves.[6]

(Stollery 1998; Hamilton 1994). I think that this case deserves separate consideration.

[5] The necessity of the Hotelling Rule for efficient extraction is shown e.g. in (Dasgupta and Heal 1979).

[6] In fact, subsidies were being applied to stimulate oil use not only in the past but even today "the world fossil fuel industry is still being subsidized by taxpayers at more than $210 billion per year" (Brown 2006).

3 The model

The analysis is based on the DHSS model with zero population growth, zero extraction cost, and with the Cobb-Douglas production function[7]

$$q(t) = f(k(t), r(t)) = k^{\alpha}(t) r^{\beta}(t) \tag{1}$$

where q - output, k - produced capital, r - current resource use, α, $\beta \in (0, 1)$, $\alpha + \beta < 1$, are constants. The assumption about technical change $A(t)$ or TFP (Total Factor Productivity) exactly compensating for capital depreciation δk allows for considering the basic DHSS model with no capital depreciation and no technical progress. At the same time, this assumption makes it possible to examine correctly various patterns of growth in the economy. The pattern of this specific TFP is considered in Section 8.

Without losing generality, assume that population equals to unity and then the lower-case variables are in per capita units. Then $r = -\dot{s}$, s - per capita resource stock ($\dot{s} = ds/dt$). Prices of per capita capital and the resource are $f_k = \alpha q/k$ and $f_r = \beta q/r$ where $f_x = \partial f/\partial x$. Per capita consumption is $c = q - \dot{k}$.

The assumptions imply that in general case the Hotelling Rule can be written as follows:

$$\frac{\dot{f}_r(t)}{f_r(t)} = F\left[f_k(t)\right] + \tau(t) \tag{2}$$

where F - "natural" modification of the Hotelling Rule, τ - distortion by the externalities. In the simplest case, which will be examined below, $F(f_k) \equiv f_k$. Then the Hotelling Rule (2) for $\tau \equiv 0$ with the Hartwick Rule $\dot{k} = r f_r$ implies

[7] There is mixed empirical evidence about the elasticity of factor substitution between capital and resource including the results showing that this value is rather close to unity (Griffin and Gregory 1976; Pindyck 1979), which means that the use of the Cobb-Douglas production function is not implausible in this framework.

for our economy constant consumption over time (Hartwick 1977). In general case, for $\tau \neq 0$, equation (2) follows

$$\frac{d\dot{k}}{dt} = \dot{r} f_r + r \dot{f}_r = \dot{r} f_r + r\left(f_k f_r + \tau f_r\right) \tag{3}$$

and $\dot{c} = f_k \dot{k} + f_r \dot{r} - \ddot{k}$. Substituting (3) for $\ddot{k}$ we have $\dot{c} = f_k \dot{k} + f_r \dot{r} - \dot{r} f_r - r f_k f_r - \tau r f_r = -\tau r f_r$ which goes to zero if $\tau/(r f_r) = \tau/(\beta q)$ goes to zero with $t \to \infty$. Realizing some declining "program" path for modifier τ we can approach the sustainable and efficient path of extraction in a desirable way.

Equation (3) and the saving rule also follow

$$\frac{\dot{f}_r}{f_r} = \beta\left[f_k + \frac{\dot{r}}{r}\left(1 - \frac{1}{\beta}\right)\right] = f_k + \tau$$

or $f_k(\beta - 1) + \frac{\dot{r}}{r}(\beta - 1) = \tau$ that gives

$$\alpha\frac{q}{k} + \frac{\dot{r}}{r} = \frac{\tau}{\beta - 1}. \tag{4}$$

Then

$$\frac{\dot{q}}{q} = \alpha\frac{\dot{k}}{k} + \beta\frac{\dot{r}}{r} = \beta\left(\alpha\frac{q}{k} + \frac{\dot{r}}{r}\right) = \frac{\beta}{\beta - 1}\tau \tag{5}$$

that means that

1) for our economy, growth is associated with negative $\tau(t)$;

2) GDP percent change $\dot{q}/q \to 0$ with any $\tau(t) \to 0$.

According to assumption 3 (Section 2), modifier $\tau(t)$ can be expressed in terms of tax/subsidy. This implies that there exists a Pigovian tax $T(t)$ such that for $F(f_k) \equiv f_k$ equation (2) can take the form[8]

$$\frac{\dot{f}_r + \dot{T}}{f_r + T} = \frac{\dot{f}_r}{f_r} - \tau = f_k \tag{6}$$

[8] This dynamic efficiency condition was used in (Hamilton, 1994) in the form $\dot{n}/n = f_k$ for the net rent per unit of resource $n = f_r - c - T$ with c - marginal cost of extraction.

This equation can be rewritten as follows:

$$\frac{\dot{f}_r + \dot{T}}{f_r + T} - \frac{\dot{f}_r - \tau f_r}{f_r} = 0$$

or, for $f_r(f_r + T) \neq 0$, we have $\dot{f} f_r + \dot{T} f_r - \dot{f}_r f_r - T\dot{f}_r + \tau f_r(f_r + T) = 0$. This implies (for $f_r \neq 0$ and $\frac{\dot{f}_r}{f_r} - \tau = f_k$) the dynamic condition for the tax

$$\dot{T} - T f_k + \tau f_r = 0 \tag{7}$$

General solution of (7) is

$$T(t) = e^{\int f_k(t)dt} \left[\widehat{T} - \int \tau f_r e^{-\int f_k(t)dt} dt \right] \tag{8}$$

The equation (7) and its solution (8) can be considered with the two types of initial conditions, associated with the two different interpretations of the equation (6).

Initial condition I. If we are looking for the path of tax $T(t)$ corresponding to the "program" decrease in distortion $\tau(t)$ then we will set $T(0) = T_0$. Since we introduce $T(t)$ as a new tax, which will compensate for the distorting phenomena, and which

(a) is continuous,

(b) was not applied before ($T(t) = 0$ for $t \leq 0$),

then we will assume that $T_0 = 0$ which gives us $\dot{T}(0) = -\tau(0) f_r(0)$.

Initial condition II. If we want to estimate the effect of the distorting externalities in terms of tax/subsidy at the current moment $t = 0$, which means that we want to find $T(0)$, then we will assume that the distorting combination of externalities is continuous at $t = 0$, and we can estimate $\dot{T}(0) = \dot{T}_0$, which implies $T(0) = \left[\tau(0) f_r(0) + \dot{T}_0\right] / f_k(0)$.

In problem I (equation (7) with the initial condition I), the observable resource price at $t = 0$ is $f_r(0)$ while in problem II (equation (7) with the initial condition II), the observable price is $f_r(0) + T(0)$ and $f_r(0)$ is the value which price would be without distorting externalities expressed via the tax/subsidy $T(0)$.

4 The economy-linked criterion and the optimal paths

In (Bazhanov 2008a) I have shown that an economy can enter an inferior path if it follows a criterion that is not linked to the "potential" of the economy. The potential of the economy is expressed in the properties of the production function and the initial state. On the other hand, non-linked criterion can imply an unrealizable or unsustainable optimal path for some specific economy. For example, the constant GDP percent change implies exponential growth that cannot be sustained infinitely under the assumptions of the essential nonrenewable resource and a plausible pattern of technical change (Dasgupta and Heal 1979). Another example is Stollery's (1998) combination of the constant-utility criterion ($U(c,T) = c^{1-\gamma}T^{-1}/(1-\gamma) = const$ where $T = T\left[r(t)\right]$ is the atmospheric temperature) with the global temperature rising exponentially with the resource extraction. This combination implies unsustainable behavior of the economy unless the rates of extraction decline very quickly in the initial period (Bazhanov 2008b).

In order to avoid these unacceptable consequences, I construct the economy-linked criterion on an example of the maximin principle applied to a generalized level-growth utility measure.[9] The use of the maximin in the problems of intergenerational justice implies that some social welfare measure must be constant over time. Therefore, it is natural to use this property of the max-

[9] This approach was also considered in (Bazhanov 2007).

imin for the problems connected with formulating the long-run programs of sustainable development.[10] The closed form solutions for the optimal paths of the Hotelling Rule modifier, tax, consumption, capital, and extraction are provided in Lemma 1, Proposition 1, and Corollary 1.

Solow (1974) showed that the maximin applied to the LEVEL of consumption implies constant consumption and no growth in output. I apply the same approach to a more general measure which takes into account not only the LEVEL of consumption but also the rate of its change.[11] I introduce a variant of generalized measure of consumption that includes as the specific cases conventional measures for the LEVEL or for the GROWTH of consumption depending on the values of parameters. Then I examine which values of these parameters correspond to the optimal paths in the Cobb-Douglas economy with a nonrenewable resource, externality, and the tax, internalizing the externality in the optimal way.

The expression $\dot{c}^{\gamma}c^{\mu}$ is considered here as an example of a hybrid level-growth measure. The maximin applied to this expression implies that already this expression, not consumption per se, must be constant over time. Assume for simplicity that $\mu = 1-\gamma$ and then we obtain the constant-utility criterion

[10] One can claim that the overall wealth of an economy could be higher as a result of the alternate ups and downs, however, I will stick here to the evidence that "loss aversion favors social arrangements that provide a steady improvement of rewards or benefits over time, in preference to schedules in which the same total benefit is handed out in equal or diminishing quantities" (Kahneman and Varey, 1991, p. 152).

[11] There are findings supporting the idea that for estimating a consumer's perception of consumption and, consequently, the utility, it is not enough to calculate a vector of measurable static indicators. "We can ask, ... how well a person's life is going and whether that person is...better off than he or she was a year ago" (Scanlon 1991, p. 18). There is also evidence that has "documented the claim that people are relatively insensitive to steady states, but highly sensitive to changes" and that "the main carriers of value are gains and losses rather than overall wealth" (Kahneman and Varey 1991, p. 148). Here I take into account prehistory of consumption in the form of derivative $\dot{c}$.

or the criterion of just growth[12] of consumption in a form[13] of

$$\dot{c}^{\gamma}c^{1-\gamma} = \overline{U} = const \tag{9}$$

which implies quasiarithmetic growth

$$c(t) = c_0(1+\varphi t)^{\gamma} \tag{10}$$

where $\varphi = \left(\overline{U}/c_0\right)^{1/\gamma}/\gamma$.

Hence, the social planner in our case keeps the combination $\dot{c}^{\gamma}c^{1-\gamma}$ constant over time with the conventional restriction on the resource reserve $\int_0^{\infty} r(t)dt = s_0$, production function in a form (1), with the Hotelling Rule modified in a form (2), resource rent investing saving rule $\dot{k} = \beta q$, and non-negative capital, output, and consumption.

Note that criterion (9) includes constant consumption as a specific case for $\gamma = 0$. More general expression $\dot{c}^{\gamma}c^{\mu}$ includes as specific cases

(a) conventional function for measuring the utility of the LEVEL of unlimitedly growing consumption $c^{1-\eta}/(1-\eta)$ for $\gamma = 0$, $\mu = 1-\eta$, and $\overline{U} = \widehat{U}(1-\eta)$;

(b) percent change as a conventional measure of the GROWTH of consumption for $\gamma = 1$ and $\mu = -1$;

(c) a sample value function which relates value to an initial consumption c and to a change of consumption $\dot{c}$ (Kahneman and Varey, 1991, p. 157): $V(\dot{c}, c) = b\dot{c}^{a}/c$ for $\dot{c} > 0$, where $a < 1$ and $b > 0; V(0, c) = 0; V(\dot{c}, c) = -Kb(-\dot{c})^{a}/c$ for $\dot{c} < 0$, where $K > 1$.

The important property of criterion (9) is that it allows for the growth of the economy and that the parameters of the criterion must be specified

[12] For $\gamma > 0$ this version of criterion is applicable only to growth ($\dot{c} > 0$) because at the steady states ($\dot{c} = 0$) this expression is always zero (not sensitive to the LEVEL).

[13] This form can be written as follows $(\dot{c}/c)^{\gamma}c = \overline{U}$ which implies that the decline in the rate of growth in our hybrid utility is compensated by the growing level of consumption.

for the economy's initial conditions. This means that using this criterion, we can consider numerical examples that resemble the behavior of the real economy. The importance of the mechanism of matching of the criterion for just allocation of some scarce resource to the context was emphasized, for example, in (Konow 2003): "the most significant challenge to ... any theory ... is to incorporate the impact of context on justice evaluation, and much work remains in this regard."

The expressions for the optimal paths in the Cobb-Douglas economy with the specific initial conditions are provided in the following Lemma 1, Proposition 1, and Corollary 1.

Lemma 1. *For the economy $q = k^\alpha r^\beta$ with the resource rent investing saving rule $\dot{k} = \beta q$ and the Hotelling Rule modified in a way $\dot{f}_r/f_r = f_k + \tau$, the unique path of the Hotelling Rule modifier*

$$\tau(t) = \frac{\beta - 1}{\beta}\frac{1}{\lambda_1 t + \lambda_0}$$

is socially optimal with respect to criterion (9) with $\gamma = 1/\lambda_1$ and $\overline{U} = (1-\beta)q_0/\lambda_0^{1/\lambda_1}$.

Proof. Condition (9) implies for our economy that $\dot{c}^\gamma c^{1-\gamma} = (1-\beta)^\gamma \dot{q}^\gamma (1-\beta)^{1-\gamma} q^{1-\gamma} = (1-\beta)\dot{q}^\gamma q^{1-\gamma} = \overline{U}$ or

$$\dot{q}^\gamma q^{1-\gamma} = \overline{U}/(1-\beta) \tag{11}$$

The equation (10) gives us $q = c/(1-\beta) = c_0(1+\varphi t)^\gamma/(1-\beta)$ and from the equation (5) we have $\dot{q} = \beta q\tau/(\beta - 1)$. Substituting for $\dot{q}$ we obtain

$$\left(\frac{\beta}{\beta-1}q\tau\right)^\gamma q^{1-\gamma} = \left(\frac{\beta}{\beta-1}\tau\right)^\gamma q = \frac{\overline{U}}{1-\beta}$$

Then substitution for q gives us

$$\left[\frac{\beta}{\beta-1}\tau\left(1+\varphi t\right)\right]^\gamma = \frac{\overline{U}}{c_0}$$

or

$$\tau = \left(\frac{\overline{U}}{c_0}\right)^{\frac{1}{\gamma}} \frac{\beta-1}{\beta}\frac{1}{1+\varphi t} = \frac{\beta-1}{\beta}\frac{\varphi\gamma}{1+\varphi t} = \frac{\beta-1}{\beta}\frac{1}{1/(\varphi\gamma)+t/\gamma}$$

where we have $\lambda_0 = 1/(\varphi\gamma)$ and $\lambda_1 = 1/\gamma$. Substitution for φ into the expression for λ_0 gives us the expression for $\overline{U}$ via λ_0 and λ_1■

Proposition 1. *Let the economy $q = k^\alpha r^\beta$ follow the Hartwick Investment Rule $\dot{k} = \beta q$; the Hotelling Rule is modified in a way $\dot{f}_r/f_r = f_k+\tau$ and the initial conditions are: $\dot{q}_0/q_0$ - the initial rate of growth; $q_0 = q(0) = k_0^\alpha r_0^\beta$ - the initial output where $k_0 = k(0), r_0 = r(0)$, and $s_0 = s(0)$ are the initial values of capital, the resource extraction and the reserve estimate.*

Then the unique path of tax, introduced at $t = 0$ with $T(0) = 0$ in the following way:

$$T(t) = \beta\left\{[k(t)(1+\lambda_1)]^\alpha q_0^{\beta-1}\right\}^{1/\beta}\left[1-(t\lambda_1/\lambda_0+1)^{(\beta-1)/(\beta\lambda_1)}\right]$$

is socially optimal with respect to criterion (9) with $\gamma = 1/\lambda_1$ and $\overline{U} = (1-\beta)q_0/\lambda_0^{1/\lambda_1} = c_0/\lambda_0^{1/\lambda_1}$. The optimal tax implies the following optimal paths of capital and the resource use:

$$\begin{aligned} k(t) &= k_0 + \frac{\beta q_0}{\lambda_0^{1/\lambda_1}(1+\lambda_1)}\left[(\lambda_1 t+\lambda_0)^{(1+1/\lambda_1)} - \lambda_0^{(1+1/\lambda_1)}\right] \\ r(t) &= \left(\frac{q_0}{\lambda_0^{1/\lambda_1}}\right)^{1/\beta}(\lambda_1 t+\lambda_0)^{1/(\beta\lambda_1)}k^{-\alpha/\beta} \end{aligned}$$

where $\lambda_0 = q_0/\dot{q}_0,\ \lambda_1 = \lambda_1(s_0)$.

Proof: Appendix 1.

Corollary 1. *In conditions of Proposition 1, the optimal path of consumption implied by (9) is*

$$c(t) = c_0\left(1+\frac{\dot{q}_0}{q_0}t\right)^\gamma$$

i.e. the optimal slope of the consumption path at each moment of time is defined by the initial GDP percent change; the rate of growth is defined by $\gamma = 1/\lambda_1(s_0)$;

the expression for the Hotelling Rule is

$$\dot{f}_r(t)/f_r(t) = f_k(t) + \frac{\beta - 1}{\beta} \frac{1}{\lambda_1(s_0)t + q_0/\dot{q}_0}$$

where $\lambda_1(s_0)$ *is uniquely defined from the equation*

$$\begin{aligned} s_0 \;&=\; \frac{\lambda_1 + 1}{(\lambda_1 + 1)(\alpha - \beta) - 1 + \beta} \cdot \frac{k_0 r_0}{q_0} \\ &\times \left\{ 1 + (1 - \beta) \left(k_0 \beta \dot{q}_0 (\lambda_1 + 1) - \beta^2 q_0^2 \right) \right. \\ &\times \left. {}_2F_0 \left(\left[1, \frac{\beta(\lambda_1 + 2) - 1}{\beta(\lambda_1 + 1)} \right], [\,], \left(\beta q_0^2 - k_0 \dot{q}_0 (\lambda_1 + 1) \right) (\lambda_1 + 1) \beta^2 \right) \right\} \end{aligned} \tag{12}$$

where ${}_2F_0(\cdot)$ *is the hypergeometric function with 2 upper parameters and an empty list of lower parameters.*

Proof is the result of straightforward substitution of the expressions for $\overline{U}$, λ_0, and $\lambda_1(s_0)$ obtained in Lemma 1, Proposition 1, and Appendix 2.

5 Compatibility of the criterion with the initial conditions

Before considering the numerical examples, I will examine possible limitations of criterion (9) that can prevent us from calibrating the model on the data from the real economy. It is known, that in the particular case of this criterion, for $\gamma = 0$ (constant consumption), we cannot use in our numerical examples the data from a growing economy with the growing extraction $r(t)$ because in this case the initial value of change of rate of extraction $\dot{r}(0)$ must be negative and it is strictly defined by the initial values of the rate of extraction $r(0)$, reserve $s(0)$, and technological parameters α and β. That is why the economy pursuing this specific type of intergenerational justice must adjust its extraction and saving during some transition period in order to switch to the optimal path in finite time (Bazhanov 2008a).

In general case ($\gamma > 0$), the economy is already allowed to have different patterns of sustainable growth, and the specific type of growth corresponds to the specific set of initial data. This implies that the economy's initial conditions are already not strictly fixed by the criterion but they can belong to some range or satisfy some restricting relationship. In Appendix 1, I have shown that for the ratio $\dot{r}/r$ to be negative (declining extraction) in the long run, the value of λ_1 must be greater than $1/\alpha - 1$, which implies $\gamma < 1/(1/\alpha - 1) = \alpha/(1-\alpha)$ (for $\alpha = 0.3$ we have $\gamma < 0.43$). Now we will examine how the value of λ_1 is restricted by the requirement of convergence of the integral $\int_0^\infty r(t)dt$. We can express $r(t)$ as follows:

$$\begin{aligned} r(t) &= q^{1/\beta} k^{-\alpha/\beta} \\ &= \widehat{q}^{1/\beta} \left[\widehat{k}(\lambda_1 t + \lambda_0)^{-1/(\alpha\lambda_1)} + \frac{\beta\widehat{q}}{1+\lambda_1}(\lambda_1 t + \lambda_0)^{((\lambda_1+1)/\lambda_1 - 1/(\alpha\lambda_1))} \right]^{-\alpha/\beta} \end{aligned}$$

Convergence of this integral is defined by the behavior of the second term in bracket, since $\lim_{t\to\infty}(\lambda_1 t + \lambda_0)^{-1/(\alpha\lambda_1)} = 0$. This gives us the convergence condition $\left[\alpha^2(\lambda_1+1) - \alpha\right]/(\alpha\beta\lambda_1) > 1$ or

$$\lambda_1 > (1-\alpha)/(\alpha-\beta) \tag{13}$$

For example, it requires $\lambda_1 > 14$ ($\gamma < 0.0714$) for $\alpha = 0.3$ and $\beta = 0.25$ while the requirement of negative ratio $\dot{r}/r$ implies only $\lambda_1 > (1-\alpha)/\alpha = 2.333$. Note that the combination of condition (13) with the requirement of declining extraction ($\lambda_1 > (1-\alpha)/\alpha > 0$) implies $\alpha > \beta$ (Solow, 1974).

Inequality (13) shows that in our model the value of γ must be less than $(\alpha-\beta)/(1-\alpha)$ regardless of the values of initial conditions. This restriction

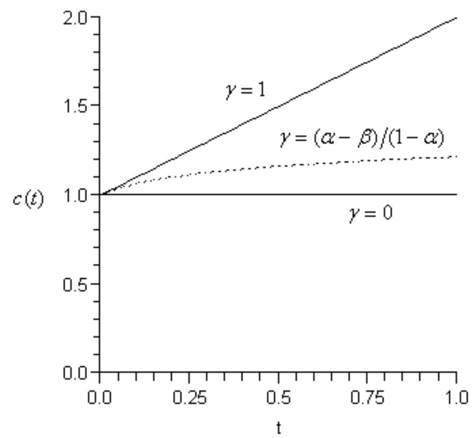

Figure 1: Patterns of feasible growth for the Cobb-Douglas economy with $\alpha = 0.3$ are between the constant ($\gamma = 0$) and the path with $\gamma = (\alpha - \beta)/(1 - \alpha)$

prevents our model with the Cobb-Douglass technology from the patterns of growth that are close to linear if $\alpha < 0.5$. The economy can realize only some variants of quasiarithmetic growth including stagnation ($\gamma = 0$). The set of these feasible sustainable paths is located in Figure 1 between the constant ($\gamma = 0$) and the path for $\gamma = (\alpha - \beta)/(1 - \alpha)$.

Condition (13) gives us only the lower bound for finding λ_1. The exact value of λ_1 must be defined as the solution of equation $\int_0^\infty r(t, \lambda_1)dt = s_0$. Therefore, the question of existence of this solution is in our case the main source of possible incompatibility of criterion (9) with some sets of the initial conditions. Hence, I will define the applicability of a criterion for formulating a long-run (sustainable) development program for the specific economy in the following way.

Definition 1 *We will say that a criterion is applicable for a long-run devel-*

opment program[14] *in an economy* $q = f(k, r)$ *with the given initial state if there exists at least one optimal with respect to this criterion program* $\langle q^*, k^*, r^* \rangle$ *that satisfies the economy's initial conditions.*

The answer to the question about the applicability of criterion (9) for a long-run program in the DHSS economy is formulated in the following

Proposition 2. *Criterion* $\dot{c}^\gamma c^{1-\gamma} = const$ *is applicable for a long-run development program in the economy* $q = k^\alpha r^\beta$ *with* $\dot{k} = \beta q$ *if the initial reserve* s_0 *satisfies the condition*

$$s_0 \geq \frac{k_0 r_0}{q_0(\alpha - \beta)} \tag{14}$$

where q_0, k_0, *and* r_0 *are the initial values of output, capital, and the rate of extraction.*[15]

Proof. In Appendix 2, I have shown that the following formula can be used for defining λ_1 as a good approximation to the solution of the equation $\int_0^\infty r(t, \lambda_1) dt = s_0$ with respect to λ_1 :

$$\lambda_1 = \frac{(1-\alpha) s_0 q_0 + k_0 r_0}{(\alpha - \beta) s_0 q_0 - k_0 r_0} \tag{15}$$

This formula captures the main peculiarities of behavior of the exact solution. In particular, it shows that the denominator can be zero for some sets of parameters, which follows the value of λ_1 going to infinity. This implies that denominator must be positive or $s_0 > k_0 r_0 / \left[q_0(\alpha - \beta)\right]$, which coincides with the condition (14). This means that the value of $\lambda_1(s_0)$ is a decreasing function from infinity at the minimal value for $s_0 = k_0 r_0 / \left[q_0(\alpha - \beta)\right]$ to the minimal value $\lambda_{1\min} = (1-\alpha)/(\alpha - \beta)$ for s_0 going to infinity (Fig. 2).

[14] A criterion can be *applicable* for selecting the best path among the feasible paths in an economy, but it can be *not applicable* for a long-run development program because the

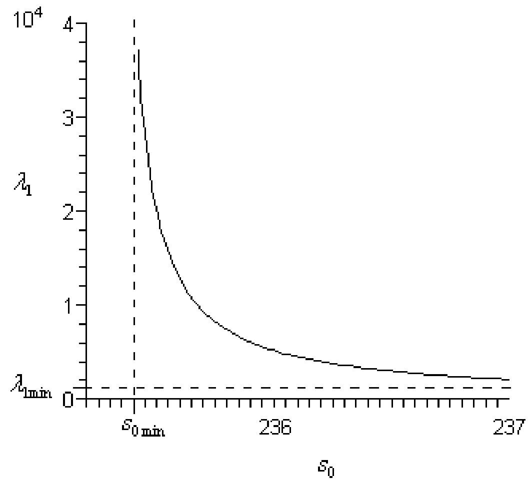

Figure 2: λ_1 as a function of the initial reserve s_0

Indeed, considering the limiting case for the path of extraction with λ_1 going to infinity (corresponds to the smallest possible s_0), we obtain

$$\begin{aligned} r_\infty(t) &\triangleq \lim_{\lambda_1\to\infty} r(t,\lambda_1) = \lim_{\lambda_1\to\infty}\left(\frac{q_0}{\lambda_0^{1/\lambda_1}}\right)^{1/\beta}(\lambda_1 t+\lambda_0)^{1/(\beta\lambda_1)}k^{-\alpha/\beta} \\ &= q_0^{1/\beta}\left[k_0+\beta q_0 t\right]^{-\alpha/\beta} \end{aligned}$$

The total amount of reserve, extracted along this path is

$$\int_0^\infty r_\infty(t)dt = \frac{q_0^{1/\beta}}{\beta q_0\left(1-\frac{\alpha}{\beta}\right)}\left[k_0+\beta q_0 t\right]^{1-\alpha/\beta}\Big|_0^\infty = -\frac{q_0^{1/\beta}k_0^{1-\alpha/\beta}}{q_0\left(\beta-\alpha\right)} = \frac{k_0 r_0}{q_0\left(\alpha-\beta\right)}$$

which is the greatest lower bound for the feasible reserve s_0 ■

optimal path that it implies can be *not realizable* in this economy in the long run.

[15] I do not consider here the initial condition for investment $\dot{k}(0)$ since for simplicity I assume that the economy follows the Hartwick Saving Rule and so this initial condition is always satisfied.

If the initial conditions in an economy do not satisfy (14), then the economy needs a transition period for adjusting its patterns of extraction and saving in order to meet the minimum requirements expressed in (14) and then it can enter a sustainable path (Bazhanov 2008a).

It would be interesting to analyze the practical applicability of the hybrid measure in general form $\dot{c}^{\gamma}c^{\mu}$ if we had had some opportunities for γ to be close to unity for the plausible values of α. However, our analysis for the simple case with $\mu = 1 - \gamma$ and with the conventional value of $\alpha = 0.3$ (see e.g. Nordhaus and Boyer 2000) shows that the DHSS economy in our framework can exhibit only the patterns of quasiarithmetic growth that are closer to constant than to linear function ($\gamma \ll 1$).

Moreover, these patterns of sustainable growth, including constant consumption as a specific case, are affordable not for all initial conditions in the economy. If the economy overuses the resource having relatively small amount of its reserve, then it needs some transition period to adjust the extraction and saving in order to have an opportunity to enter a sustainable path in finite time. This result implies the impossibility of exponential growth for the DHSS model and therefore the inconvenience of the percent change as a measure for sustainable growth.

This follows an important practical application of the hybrid measure. This expression can be called *geometrically weighted percent,* and it can be used as a measure for sustainable growth of some economic indicators e.g. social welfare function or NNP (Hartwick 1990) instead of regular percent change. The rate of regular percent change declines for sustainable growth if this growth is not exponential. Indefiniteness of the rate of this decline makes regular percent an inconvenient and even a misleading measure for sustainable development. For example, this inherently unsustainable indicator was used as a necessary condition for sustainability even in such a seminal document for sustainable development as the Brundtland Report (World 1987): "The key elements of sustainability are: a minimum of 3 percent per capita income growth in developing countries" (p. 169) and "annual global per capita GDP growth rates of around 3 percent can be achieved. This growth is at least as great as that regarded in this report as a minimum for reasonable development" (p. 173). Besides contradictions with the environmental goals, which were noticed e.g. in (Hueting 1990), measuring growth in GDP percent change conflicts with theoretical possibility of realization of this program. In this sense, geometrically weighted percent in the form of (9) is more convenient for formulating the long-run economic goals because maintaining this expression constant implies feasible and sustainable growth.

6 An economy with declining output and/or small reserve s_0

In order to complete the analysis of applicability of the economy-linked criterion in the form of (9) to formulating long-run development programs, I will show that this criterion leads to unacceptable implications for the cases when an economy has declining output ($\dot{q}_0/q_0 < 0$) at the initial moment and/or $\gamma < 0$.[16] The optimal paths of consumption for these cases can be obtained by plotting the formula for consumption in Corollary 1.

For a growing economy ($\dot{q}_0/q_0 > 0$) with $\gamma < 0$ criterion (9) implies optimal consumption asymptotically approaching zero (Fig. 3a). If the economy's output is declining at the initial moment and $\gamma > 0$, then we obtain that the optimal paths of consumption must be decreasing to zero in finite

[16] Negative γ for the Cobb-Douglas technology is equivalent to the initial conditions not satisfying (14).

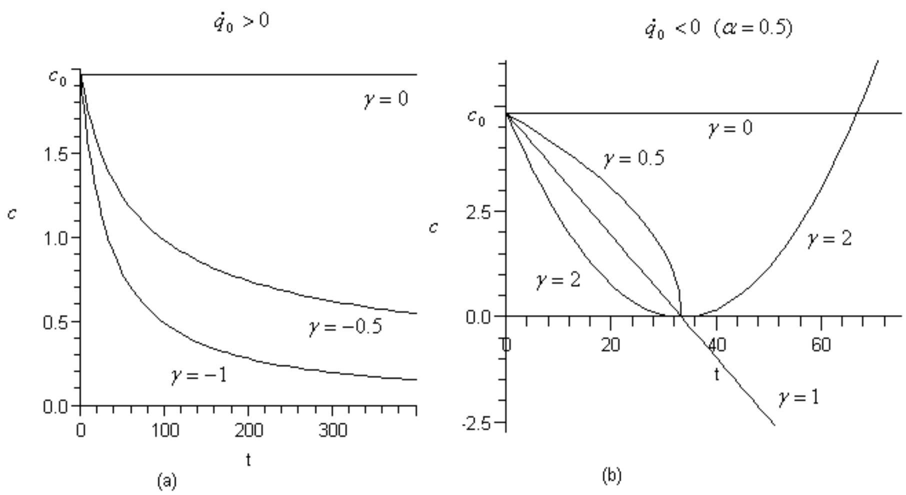

Figure 3: Unacceptable paths of consumption, optimal with respect to criterion (9): (a) growing economy and $\gamma < 0$; (b) declining economy and $\gamma > 0$

time for all positive γ. However, for the even integer values of $\gamma > 1$, the optimal path after hitting zero must have unbounded polynomial growth (Fig. 3b). Note again that $\gamma > 1$ cannot be obtained in the DHSS model for the conventional values of α. In the last, presumably the most pessimistic case where the economy has declining output and can rely only on negative γ, we obtain that criterion (9) requires the consumption to be growing to infinity in a finite period (Fig. 4). This scenario can be realized only in the short run because growing consumption with decreasing output implies negative investment and subsequent collapse.

Hence, the only case when criterion (9) leads to ethically acceptable paths of consumption is growing output at the initial moment and $\gamma > 0$ (or satisfaction of condition (14)). The paths of consumption for this case are depicted in Figure 1.

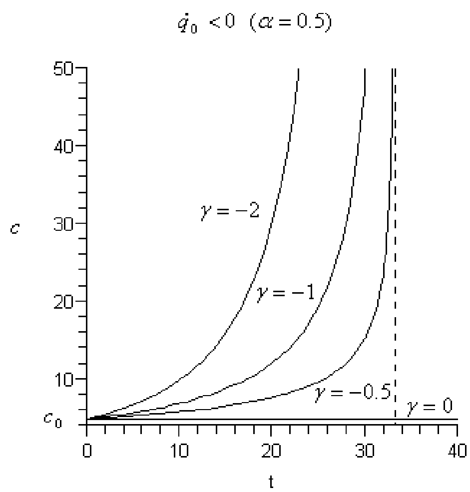

Figure 4: Paths of consumption, assumed by criterion (9) for declining economy and $\gamma < 0$

7 Numerical example

I will start with problem II, which estimates the effect of the distorting externalities in terms of tax/subsidy at the current moment $t = 0$. Assume that the distorting combination of externalities is continuous at $t = 0$, and that it is constant or $\dot{T}_0 = 0$, which implies $T(0) = \tau(0) f_r(0)/f_k(0)$.

The primary initial values are: $\alpha = 0.3$, $\beta = 0.25$, GDP percent change $\dot{q}_0/q_0 = 0.03$, the initial rate of extraction $r_0 = 3.6243$, the initial reserve $s_0 = 2 \cdot 180.4722 = 360.9444$,[17] and the rate of extraction is growing with $\dot{r}_0 = 0.1$. This gives us the other initial values (see (Bazhanov, 2006b) for

[17] I use the world oil extraction on January 1, 2007 as r_0 and the world reserves as s_0 (Radler, 2006): $r_0 = 72{,}486.5$ [1,000 bbl/day] $\times 365 = 26{,}457{,}572$ [1,000 bbl/year] (or 3.6243 bln t/year); $s_0 = 1{,}317{,}447{,}415$ [1,000 bbl] (or 180.4722 bln t). I use coefficient 1 ton of crude oil = 7.3 barrel. According to the report of Cambridge Energy Research Associates (CERA, 2006), actual world reserves (3.74 trillion barrels) are about three times more than the conventional estimate being published in December issues of *Oil & Gas Journal*. I use in the example the "average" of the two estimates.

the estimation of k_0)

$$k_0 = \left\{ \left[\left(\frac{\dot{q}}{q} \right)_0 \frac{1}{\beta} - \frac{\dot{r}_0}{r_0} \right] / \left(\alpha r_0^{\beta} \right) \right\}^{\frac{1}{\alpha - 1}} = 8.5174$$

and $\lambda_0 = q_0/\dot{q}_0 = 33.3333$. This follows $q_0 = k_0^{\alpha} r_0^{\beta} = 2.6236$, $c_0 = (1-\beta) q_0 = 1.9677$, $\dot{q}_0 = (\dot{q}_0/q_0)\, q_0 = 0.0787$, $\tau(0) = (\dot{q}_0/q_0)\,(\beta - 1)/\beta = -0.09$. For these values, condition (14) is satisfied (for our example $s_{0\,\min} = 235.3$)[18] and we have $q_0/k_0 = 0.308$, the rate of interest $f_k(0) = \alpha q_0/k_0 = 0.092$ and the resource price (which would be in problem II without distortions) $f_r(0) = \beta q_0/r_0 = 0.18097$. Note that $\dot{f}_r(0) = f_r(0)\,(\dot{q}_0/q_0 - \dot{r}_0/r_0) = 0.0004$ (price is growing but very slowly). The assumption $\dot{T}_0 = 0$ implies $T(0) = -.1763$. This means that for our simplified economy

1) the distortions are equivalent to the influence of subsidy rather than tax;

2) the observable price $f_r(0) + T(0) = 0.0047$ is about 38.4 times less than it would be without externalities.

We turn to solving problem I where we will estimate the optimal tax $T(t)$ and the paths of capital and extraction. This problem implies that there is no tax at the initial moment ($T_0 = 0$) which gives us $\dot{T}_0 = 0.016$ (growing optimal tax). Then we estimate $\lambda_1 = 60.11$[19] using the feasibility condition $\int_0^{\infty} r(t)dt = s_0$ (Appendix 2). This gives us the optimal path of capital that is very close to linear (solid line in Fig. 8), $k(t) = 8.16 + 0.0101 \cdot (60.11t + 33.33)^{1.0166}$, and the paths of the resource extraction (solid line in Fig. 9) and

[18] If we take s_0 equal to 180.4722 bln t (*Oil & Gas Journal* estimate) then condition (14) will be not satisfied or our model of the world economy will be not compatible with the sustainable growth in the sense of criterion (9) and it will need a transition period in order to adjust the initial state.

[19] Numerical calculation of the integral gives $\lambda_1 = 60.11$; the expression via the hypergeometric function (Appendix 2) implies $\lambda_1 = 72.33$, and the approximate formula (15) gives $\lambda_1 = 42.1$.

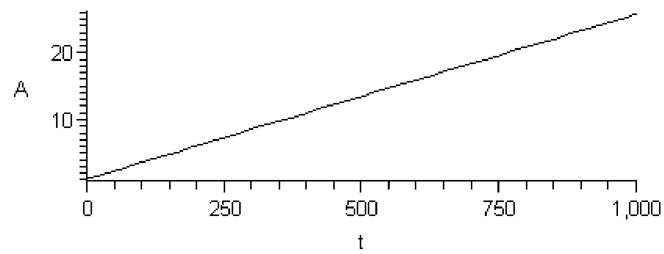

Figure 5: Technical progress compensating for capital depreciation

tax (solid line in Fig. 7). Quasiarithmetic growth of consumption is depicted in solid line in Fig. 10.

8 Technical progress compensating for capital depreciation

The assumption about no capital depreciation and no technical progress can be interpreted as an equivalent assumption about the specific TFP that exactly compensates for the decay of capital. Then the path of this technical progress can be constructed in order to estimate its plausibility. In other words, our assumption implies that

$$q(t) = A(t)k^{\alpha}r^{\beta} - \delta k$$

and technical progress $A(t)$ is such that $A(t)k^{\alpha}r^{\beta} - \delta k = k^{\alpha}r^{\beta}$. This follows

$$A(t) = 1 + \delta k^{1-\alpha}r^{-\beta}$$

Substituting for $r = \widehat{r}\,(\lambda_1 t + \lambda_0)^{1/(\beta\lambda_1)}\,k^{-\alpha/\beta}$ where $\widehat{r} = \widehat{q}^{1/\beta}$ and $\widehat{q} = q_0/\lambda_0^{1/\lambda_1}$ we have

$$A(t) = 1 + \frac{\delta}{\widehat{q}}\left[\frac{\widehat{k}}{(\lambda_1 t + \lambda_0)^{1/\lambda_1}} + \frac{\beta\widehat{q}}{1+\lambda_1}\,(\lambda_1 t + \lambda_0)\right]$$

which is asymptotically linear with the slope $\delta\beta/\left(1+\lambda_1\right)$. For our example, given $\delta = 0.1$, the slope is $0.1 \cdot 0.25/(1 + 60.11) = 0.000409$ (Fig. 5).

9 Variable reserves and dynamic corrections

The amount of reserve s_0 was considered so far as a constant, though in practice the value of the proven recoverable reserve is being updated annually. This value decreases because of the extraction and it can increase due to the discovery of new oil fields and due to the changes in oil prices and in extracting technologies. Nevertheless, in many theoretical problems we can consider s_0 as all the amount of the reserve including proven, unproven, and as yet not discovered so we can assume correctly that s_0 is a constant in these problems. However, if we are going to estimate numerically the path of tax which depends on s_0 and which controls the economy in the optimal way, we should estimate s_0 as accurately as possible. Otherwise, the economy will follow an inferior path in the case of underestimation of s_0 or it will overconsume if s_0 is overestimated.

In this section, I will examine a procedure of dynamic policy correction that will depend on the information about the changes in the resource reserves. The paths of our economy are defined by the value of s_0 (via $\lambda_1(s_0)$) at the initial moment $t = 0$. With time, we obtain additional information about s_0 that was not available at the initial moment. Using this information at each moment $t > 0$ we will reestimate s_0 which will imply the dynamic correction of the tax and of all the paths in the economy.

Assume that with time our revaluation of s_0 is growing and asymptotically approaches a constant $\widehat{s}_0$, for example, in the following way (Fig. 6):

$$s_0(t) = \widehat{s}_0 - e^{-wt}(\widehat{s}_0 - \overline{s}_0) \tag{16}$$

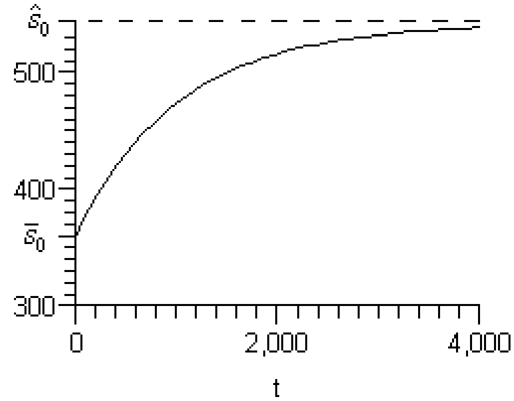

Figure 6: Information updates about the reserve estimate

I will take for the numerical example $s_0(0) = \overline{s}_0 = 2 \cdot 180.4722 = 360.94$ [bln t] and $\widehat{s}_0 = \lim_{t\to\infty} s_0(t) = 3 \cdot 180.4722 = 541.41$ (CERA's reserve estimate). The parameter w here is $w = 0.001$. Then we can make use of the explicit expression (15) for $\lambda_1(s_0)$. Substituting (16) for s_0 in (15) and then substituting it into (9) we obtain the measure of the optimal sustainable growth which is dynamically responding to the new information about the reserves. Substitution of the dynamically changing $\lambda_1(s_0(t))$ implies corresponding updates in paths of tax, capital, extraction, and consumption (Figs. 7 - 10, time in years). The paths corresponding to the precommitment policy with $s_0(t) \equiv \overline{s}_0$ are depicted as a solid line, precommitment paths with $s_0(t) \equiv \widehat{s}_0$ (assuming that we know everything about reserves at the initial moment) are in crosses, and the dynamically updated paths are in circles.

We can see that the reaction of the economy on the larger amount of the initial reserve ($s_0(t) \equiv \widehat{s}_0$, paths in crosses) is rather plausible. The level of tax is lower, the levels of capital and rates of extraction are higher and, as a result, the level of the optimal per capita consumption is also higher.

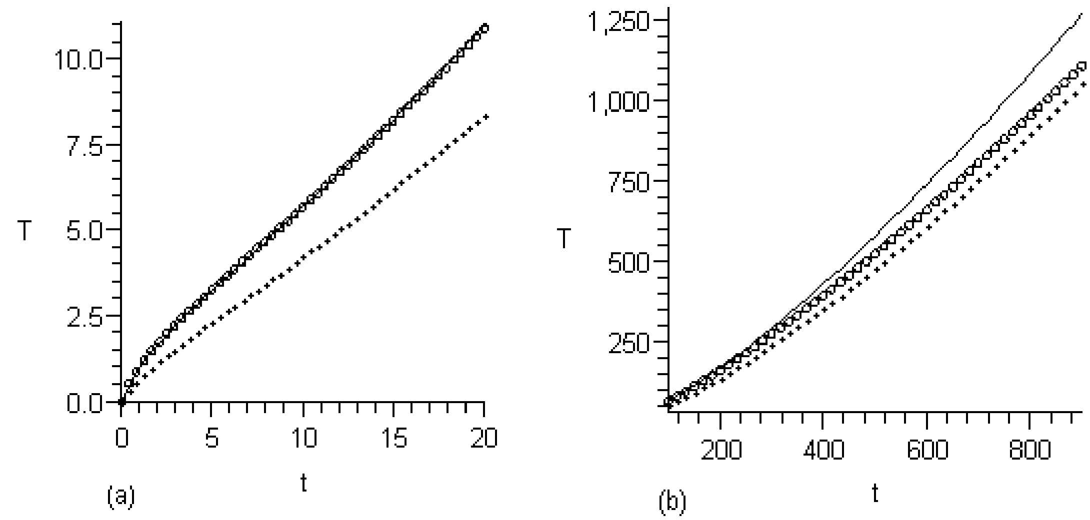

Figure 7: The optimal paths of tax: (a) in the short run; (b) in the long run. For fixed reserve $\overline{s}_0$ - as a solid line; for fixed reserve $\widehat{s}_0 = 1.5\overline{s}_0$ - in crosses; dynamically changing path - in circles

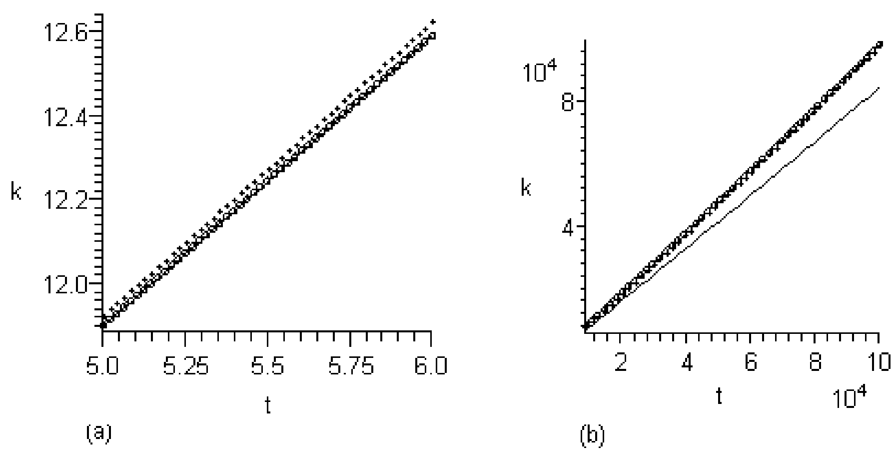

Figure 8: The optimal paths of capital: (a) in the short run; (b) in the long run. For fixed reserve $\overline{s}_0$ - as a solid line; for fixed reserve $\widehat{s}_0$ - in crosses; dynamically changing path - in circles

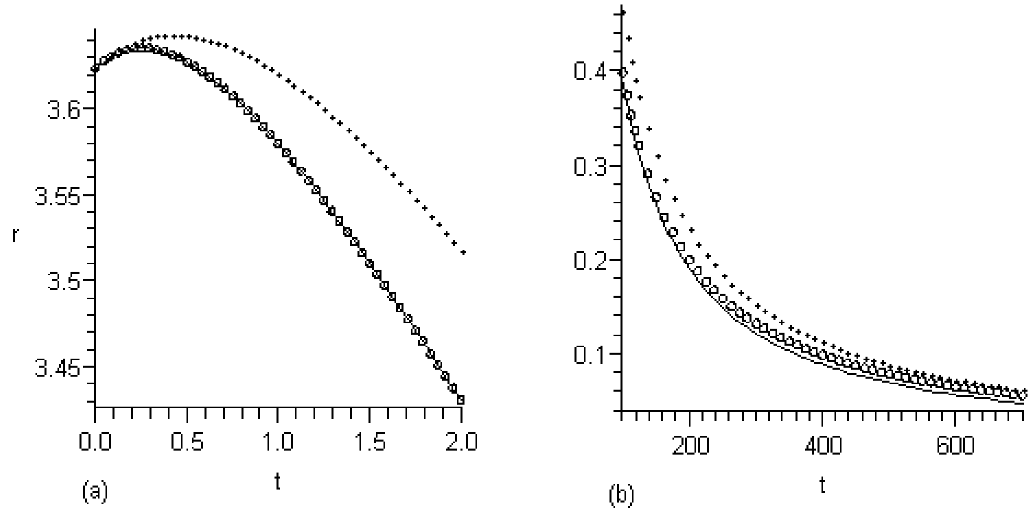

Figure 9: The optimal paths of extraction: (a) in the short run; (b) in the long run. For fixed reserve $\overline{s}_0$ - as a solid line; for fixed reserve $\widehat{s}_0$ - in crosses; dynamically changing path - in circles

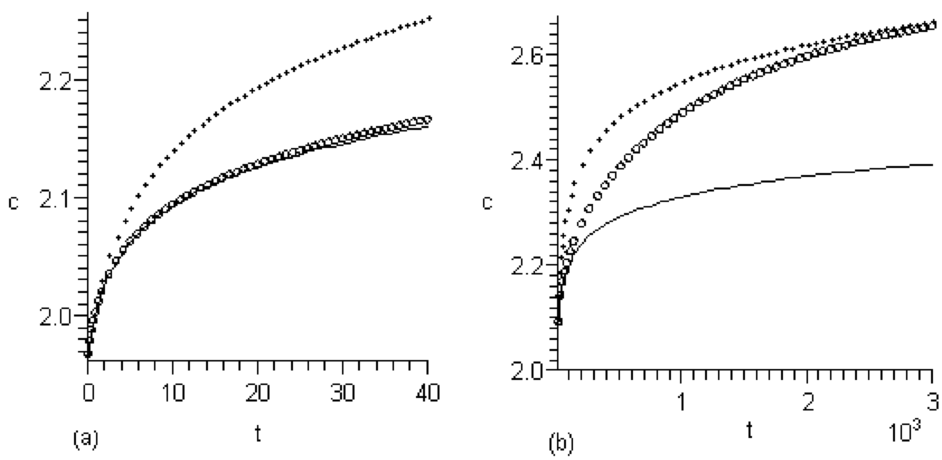

Figure 10: The optimal paths of consumption: (a) in the short run; (b) in the long run. For fixed reserve $\overline{s}_0$ - as a solid line; for fixed reserve $\widehat{s}_0$ - in crosses; dynamically changing path - in circles

Note that the criterion linked to the initial conditions combined with the assumption about modification of the Hotelling Rule in a generalized way can imply hump-shaped optimal paths of extraction. This result implies the notion of *the normative resource peak.* This peak can be compared with the one, being forecasted from the point of view of "physical possibility" of reaching the maximum level of extraction.[20]

One could expect that if an economy chooses an inferior path at the initial point due to the lack of knowledge about the reserve, then the difference in consumption with respect to the optimal "full-knowledge" path (line in crosses, Fig. 10) will only increase with time unless we correct the saving rule. However, the example shows that under the standard Hartwick Rule the consumption in the economy with the dynamically defined parameters (line in circles) is asymptotically "catching-up" to the optimal level of consumption in the process of updating the information about the reserve. The maximum difference in consumption during this process is less than 5%.

Another implication of the dynamically updated parameters is that the level of $\overline{U}$ in criterion (9) becomes variable ($\overline{U}(t) = c_0/\lambda_0^{1/\lambda_1(s_0(t))}$). This could undermine the argument about convenience of the geometrically weighted percent as a measure for sustainable growth. However, in our numerical example with substantially changing information about the reserve, the change in $\overline{U}$ is nothing more then 5% (from $\overline{U}(0) = 1.81$ to $\overline{U}(\infty) = 1.71$), which is negligible in comparison with the mismeasurements in the real economy.

[20] The theories of estimating the "physical" oil peak have been developing since the work of geologist M.K. Hubbert (1956). A methodology different from the Hubbert's oil-peak approach was used in the CERA's report (CERA 2006) according to which the world oil reserves are about three times larger than the conventional estimates and the "physical" oil peak is not expected before 2030. However, the optimal paths of extraction obtained in this paper imply that the normative oil peak must be much closer, namely, in 6 months even for the CERA's reserve estimate.

10 Concluding remarks

This paper has shown that from all patterns of growth offered in (Groth et al 2006) as regular growth, the extended Dasgupta-Heal-Solow-Stiglitz (DHSS) model can realize for the conventional value of $\alpha = 0.3$ only the (sustainable) paths of quasiarithmetic growth that are much closer to constant consumption than to linear function (Fig. 1). The DHSS model is extended here by the assumption that the Hotelling Rule is modified by the phenomena that total influence can be expressed in terms of an equivalent tax or subsidy (Sections 2 and 3). I interpreted the absence in the model of both technical change (TFP) and capital depreciation as presence of the specific TFP exactly compensating for the capital decay (Section 8).

The approach linking the optimality criterion to the economy's "abilities to grow" (Section 4) is described on an example of the maximin applied to a generalized level-growth measure (geometrically weighted percent). The parameter of this measure ($\gamma = 1/\lambda_1$) was calibrated on the economy's technological parameters (the price elasticities of demand for the resource and capital) and the initial conditions. In this framework, I have obtained the closed form solutions for the optimal paths of the Hotelling Rule modifier, tax internalizing the externalities, capital, the resource extraction, and per capita consumption under the standard Hartwick Investment Rule. I have derived the closed-form expression for the dependance of the parameter, specifying the criterion (λ_1), on the reserve estimate. This formula was used to examine the optimal paths dynamically responding to the updates in the information about the reserve estimates (Section 9).

The assumption about the generalized form of the Hotelling Rule modifier made it possible to calibrate the model on the world oil extraction data.

In particular, this modification allowed for nondecreasing extraction in the initial period. This property of the problem introduces the notion of the normative oil (resource) peak. It turned out that in the framework of this paper the optimal oil peak must be in 2-6 months depending on the amount of reserve. In other words, the socially-optimal oil peak is much closer to the initial moment than the various forecasts of the "physical" oil peak which show for how long the rates of extraction can grow (Sections 7 and 9).

It would be interesting to apply

(1) the economy-linked criterion for the problem with the specific externality like Stollery's (1998) and Hamilton's (1994) global warming where the rising temperature affects not only the Hotelling Rule but also the utility and/or the production function;

(2) the methodology of linking a criterion to the specific economy for different hybrid measures and different criteria of justice;

(3) the methodology of linking a criterion to the specific economy with the specific patterns of endogenous technical change.

I think these problems deserve separate consideration.

11 Acknowledgments

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12 Appendix 1 (Proof of Proposition 1)

Lemma 1 gives the optimal pattern of the Hotelling Rule modifier

$$\tau(t) = \frac{\beta - 1}{\beta} \frac{1}{\lambda_1 t + \lambda_0}$$

Indeed, equation (5) implies

$$\frac{\dot{q}}{q} = \frac{\beta}{\beta - 1}\tau = \frac{1}{\lambda_1 t + \lambda_0}$$

which gives us $\lambda_0 = q_0/\dot{q}_0$ (for $\dot{q}_0 \neq 0$) and (solving it for $q(t)$) $q(t) = \widehat{q}\,(\lambda_1 t + \lambda_0)^{1/\lambda_1}$ where the constant of integration $\widehat{q}$ is defined from the initial condition $q(0) = q_0$:

$$\widehat{q} = \frac{q_0}{\lambda_0^{1/\lambda_1}} = (\dot{q}_0/q_0)^{1/\lambda_1}\, q_0$$

Then $\dot{q}(t) = \widehat{q}\,(\lambda_1 t + \lambda_0)^{1/\lambda_1 - 1}$ and expression $\dot{q}^\gamma q^{1-\gamma}$ with $\gamma = 1/\lambda_1$ gives us

$$\begin{aligned} \dot{q}^\gamma q^{1-\gamma} &= \widehat{q}^{1/\lambda_1}\,(\lambda_1 t + \lambda_0)^{(1/\lambda_1 - 1)/\lambda_1}\,\widehat{q}^{1-1/\lambda_1}\,(\lambda_1 t + \lambda_0)^{(1-1/\lambda_1)/\lambda_1} \\ &= \widehat{q} = const = \overline{U}/(1-\beta) \end{aligned}$$

We can rewrite $q(t)$ as follows $q(t) = q_0\,(1 + t\lambda_1/\lambda_0)^{1/\lambda_1}$.

Given expression for q and the saving rule $\dot{k} = \beta\widehat{q}\,(\lambda_1 t + \lambda_0)^{1/\lambda_1}$ we have the path for capital

$$k(t) = \widehat{k} + \frac{\beta\widehat{q}}{1 + \lambda_1}\,(\lambda_1 t + \lambda_0)^{(1+1/\lambda_1)}$$

where the initial condition $k(0) = k_0$ gives us the constant of integration $\widehat{k} = k_0 - \beta\widehat{q}\lambda_0^{(1+1/\lambda_1)}/(1+\lambda_1) = k_0 - \beta q_0\lambda_0/(1+\lambda_1)$. Then we have

$$k(t) = k_0 + \frac{\beta\widehat{q}}{(1 + \lambda_1)}\left[(\lambda_1 t + \lambda_0)^{(1+1/\lambda_1)} - \lambda_0^{(1+1/\lambda_1)}\right]$$

The modified Hotelling Rule in form of (4) gives an equation for $r(t)$

$$\begin{aligned}\frac{\dot{r}}{r} &= \frac{\tau}{\beta-1}-\alpha q/k=\frac{1}{\beta\left(\lambda_1 t+\lambda_0\right)}-\alpha\frac{\widehat{q}\left(\lambda_1 t+\lambda_0\right)^{1/\lambda_1}}{\widehat{k}+\frac{\beta\widehat{q}}{1+\lambda_1}\left(\lambda_1 t+\lambda_0\right)^{(1+1/\lambda_1)}} \\ &= \frac{\widehat{k}+\frac{\beta\widehat{q}}{1+\lambda_1}\left(\lambda_1 t+\lambda_0\right)^{(1+1/\lambda_1)}-\alpha\beta\widehat{q}\left(\lambda_1 t+\lambda_0\right)^{(1+1/\lambda_1)}}{\beta\widehat{k}\left(\lambda_1 t+\lambda_0\right)+\frac{\beta^2\widehat{q}}{1+\lambda_1}\left(\lambda_1 t+\lambda_0\right)^{(2+1/\lambda_1)}}\end{aligned}$$

or

$$\frac{\dot{r}}{r}=\frac{\widehat{k}+\beta\widehat{q}\left(\frac{1}{1+\lambda_1}-\alpha\right)\left(\lambda_1 t+\lambda_0\right)^{(1+1/\lambda_1)}}{\beta\widehat{k}\left(\lambda_1 t+\lambda_0\right)+\frac{\beta^2\widehat{q}}{1+\lambda_1}\left(\lambda_1 t+\lambda_0\right)^{(2+1/\lambda_1)}} \tag{17}$$

which implies[21]

$$\begin{aligned}r(t) &= \widehat{r}_1(\lambda_1 t+\lambda_0)^{1/\beta\lambda_1}\left[\widehat{k}(1+\lambda_1)+\beta\widehat{q}\left(\lambda_1 t+\lambda_0\right)^{(1+1/\lambda_1)}\right]^{-\alpha/\beta} \\ &= \widehat{r}(\lambda_1 t+\lambda_0)^{1/\beta\lambda_1}k^{-\alpha/\beta}\end{aligned}$$

where the constant of integration $\widehat{r}$ can be defined via the initial value of extraction $r_0 : \widehat{r}=r_0\lambda_0^{-1/\beta\lambda_1}\left[\widehat{k}+\beta\widehat{q}\lambda_0^{(1+1/\lambda_1)}/(1+\lambda_1)\right]^{\alpha/\beta}$. The more simple expression for $\widehat{r}$ can be obtained using the production function $q = k^\alpha r^\beta$ which gives us $\widehat{r}=\widehat{q}^{1/\beta}$. Given the expression for $r(t)$ we can adjust parameter λ_1 using the feasibility and efficiency condition $s_0=\int_0^\infty r(t)dt$ (Appendix 2).

Note that equation (17) implies that $\dot{r}/r\to 0$ with $t\to\infty$ and in order to obtain feasible behavior of $r(t)$ it is necessary that the ratio $\dot{r}/r$ is negative for t big enough. Assuming $\lambda_1>0$ we can see that for t big enough the denominator in (17) is positive and the nominator is negative if and only if $\alpha>1/(1+\lambda_1)$ or $\lambda_1>1/\alpha-1$ which justifies our assumption about the sign of λ_1 for $\alpha\in(0,1)$. This condition for $\lambda_1=\lambda_1(s_0)$ can be interpreted as a possibility condition for realization of the economy-linked optimal (in a

[21] Actually in our problem we can obtain $r(t)$ in more simple way just expressing it from $q(t)$, since we know already expressions for $q(t)$ and $k(t)$. The expression for $r(t)$ in this case is the same.

sense of criterion (9)) paths for the economy with technological parameter α and reserve s_0.

In order to express explicitly the path of tax from formula (8): $T(t) = \exp\left[\int f_k(t)dt\right]\left\{\widehat{T} - \int \tau f_r \exp\left[-\int f_k(\xi)d\xi\right]dt\right\}$ I will consider the following integral which for the case of the Hartwick Rule is

$$\int f_k(t)dt = \alpha \int \frac{q}{k}dt = \frac{\alpha}{\beta}\int \frac{\dot{k}dt}{k} = \frac{\alpha}{\beta}\ln k + C_1$$

It implies that $\exp\left[\int f_k(t)dt\right] = C_2 k(t)^{\alpha/\beta}$ and

$$\int \tau f_r \exp\left[-\int f_k(\xi)d\xi\right]dt = \frac{1}{C_2}\left[\frac{\beta\widehat{q}(1+\lambda_1)^{\alpha/\beta}}{\widehat{r}}(\lambda_1 t + \lambda_0)^{\frac{\beta-1}{\beta\lambda_1}}\right] + C_3$$

which gives us

$$T(t) = k(t)^{\alpha/\beta}\left[\widehat{T} - \frac{\beta\widehat{q}(1+\lambda_1)^{\alpha/\beta}}{\widehat{r}}(\lambda_1 t + \lambda_0)^{\frac{\beta-1}{\beta\lambda_1}}\right] \tag{18}$$

where $\widehat{T} = \widehat{T}(C_2, C_3)$. Since $\widehat{q} = q_0/\lambda_0^{1/\lambda_1}$ and $\widehat{r} = \widehat{q}^{1/\beta}$, and given $T_0 = T(0)$ we have $\widehat{T} = T_0 k_0^{-\alpha/\beta} + \beta\widehat{q}^{1-1/\beta}(1+\lambda_1)^{\alpha/\beta}\lambda_0^{(\beta-1)/(\beta\lambda_1)}$ or

$$\widehat{T} = T_0 k_0^{-\alpha/\beta} + \beta q_0^{(\beta-1)/\beta}(1+\lambda_1)^{\alpha/\beta}$$

Substituting it into (18) we obtain

$$T(t) = k(t)^{\alpha/\beta}\left\{T_0 k_0^{-\alpha/\beta} + \beta(1+\lambda_1)^{\alpha/\beta} q_0^{(\beta-1)/\beta}\left[1 - (\frac{\lambda_1}{\lambda_0}t + 1)^{(\beta-1)/(\beta\lambda_1)}\right]\right\}$$

which for $T_0 = 0$ gives us the expression formulated in the proposition ■

13 Appendix 2 (Estimation of $\lambda_1(s_0)$)

The value of λ_1 can be expressed via reserve estimate s_0 using the feasibility-efficiency condition $\int_0^\infty r(t,\lambda_1)dt = s_0$. The first approach to find $\lambda_1(s_0)$ is to calculate the integral $\int_0^\infty r[t,\lambda_1^i(s_0)]dt$ numerically[22] inside of an iterative procedure which in general case can be described as follows.

(1) Set $\lambda_1^0 = (1-\alpha)/(\alpha-\beta)+\overline{\lambda}$, where $\overline{\lambda}>0$ (arbitrary).

(2) Adjust $\lambda_1^i = \lambda_1^i\left(\lambda_1^{i-1}, \int_0^\infty r[t,\lambda_1^{i-1}(s_0)]dt - s_0\right)$ starting with $i=1$ in such a way that $\left|\int_0^\infty r[t,\lambda_1^i(s_0)]dt - s_0\right| \to 0$ (with desirable accuracy) with $i\to\infty$ and then we will have $\lambda_1^i \to \lambda_1^*$ assuming that λ_1^* exists.

In order to check the accuracy of numerical integration with the infinite upper limit we can transform this operation to integration with a finite limit. For this the integral should be rewritten as follows

$$\int_0^\infty rdt = \widehat{q}^{1/\beta}\left\{\int_0^\infty \left[\frac{\beta\widehat{q}}{1+\lambda_1}(\lambda_1 t+\lambda_0)^{\left(\frac{\lambda_1+1}{\lambda_1}-\frac{1}{\alpha\lambda_1}\right)}\right]^{-\alpha/\beta} dt - \int_0^\infty \Delta r dt\right\} \tag{19}$$

where

$$\begin{aligned}\Delta r \quad = \quad & \left[\frac{\beta\widehat{q}}{1+\lambda_1}(\lambda_1 t+\lambda_0)^{\left(\frac{\lambda_1+1}{\lambda_1}-\frac{1}{\alpha\lambda_1}\right)}\right]^{-\alpha/\beta} \\ & -\left[\widehat{k}\,(\lambda_1 t+\lambda_0)^{-1/(\alpha\lambda_1)} + \frac{\beta\widehat{q}}{1+\lambda_1}\,(\lambda_1 t+\lambda_0)^{(1+1/\lambda_1-1/(\alpha\lambda_1))}\right]^{-\alpha/\beta}\end{aligned}$$

The first integral in (19) can be expressed in elementary functions and integral $\int_0^\infty \Delta r(t,\lambda_1)dt$ converges much faster than the original one and so it can be calculated with finite upper limit with the desirable accuracy. It implies

$$\int_0^\infty r(t,\lambda_1)dt = (1+\lambda_1)^{\alpha/\beta} q_0^{(1-\alpha)/\beta}\frac{(\beta\lambda_0)^{(\beta-\alpha)/\beta}}{(1+\lambda_1)(\alpha-\beta)+\beta-1} - \widehat{q}^{1/\beta}\int_0^\infty \Delta r dt$$

[22] I used procedure _d01amc in Maple.

where $\int_0^\infty \Delta r dt = \lim_{A\to\infty} \int_0^A \Delta r dt$. For our numerical example it is enough to take $A = 10^3$ in order to have the error of integration about 1%.

As the second approach to find $\lambda_1(s_0)$ I will use sequential integration of $r(t, \lambda_1)$ by parts which will follow representation of s_0 as a series. For this I will express r in the following way $r = q^{1/\beta} k^{-\alpha/\beta} = (1/\beta)^{1/\beta} \dot{k}^{1/\beta-1} \dot{k} k^{-\alpha/\beta}$. Denote $u = \dot{k}^{1/\beta-1}$ and $dv = k^{-\alpha/\beta} \dot{k} dt$. Then

$$\begin{aligned}
\int_0^\infty r dt &= (1/\beta)^{1/\beta} \int_0^\infty u dv = (1/\beta)^{1/\beta} \left[uv - \int_0^\infty v du \right] \\
&= (1/\beta)^{1/\beta} \left[-\frac{\dot{k}_0^{1/\beta-1} k_0^{1-\alpha/\beta}}{1-\alpha/\beta} - \frac{1-\beta}{\beta-\alpha} I_2 \right]
\end{aligned}$$

where $I_2 = \int_0^\infty k^{1-\alpha/\beta} \dot{k}^{1/\beta-2} \ddot{k}\, dt$. Substituting for $\ddot{k} = \beta\widehat{q} (\lambda_1 t + \lambda_0)^{1/\lambda_1 - 1} = (\beta\widehat{q})^{\lambda_1} \dot{k}^{1-\lambda_1}$ we have $I_2 = (\beta\widehat{q})^{\lambda_1} I_3$ where $I_3 = \int_0^\infty k^{1-\alpha/\beta} \dot{k}^{1/\beta-1-\lambda_1} dt$. Since $k/\dot{k}^{(1+\lambda_1)} = \widehat{k} \dot{k}^{-1-\lambda_1} + (\beta\widehat{q})^{-\lambda_1}/(1+\lambda_1)$ then $k^{1-\alpha/\beta} \dot{k}^{1/\beta-1-\lambda_1} = k^{-\alpha/\beta} \dot{k}^{1/\beta} k/\dot{k}^{(1+\lambda_1)} = k^{-\alpha/\beta} \dot{k}^{1/\beta} \left[\widehat{k} \dot{k}^{-1-\lambda_1} + (\beta\widehat{q})^{-\lambda_1}/(1+\lambda_1) \right]$. It implies $I_3 = \widehat{k} \int_0^\infty k^{-\alpha/\beta} \dot{k}^{1/\beta-1-\lambda_1} dt + (\beta\widehat{q})^{-\lambda_1}/(1+\lambda_1) \int_0^\infty k^{-\alpha/\beta} \dot{k}^{1/\beta} dt$. The second integral, expressed via the original one, equals to $\beta^{1/\beta} \int_0^\infty r dt$. Then the original integral is

$$\begin{aligned}
\int_0^\infty r dt &= (1/\beta)^{1/\beta} \left\{ -\frac{k_0^{1-\alpha/\beta} \dot{k}_0^{1/\beta-1}}{1-\alpha/\beta} - \frac{1-\beta}{\beta-\alpha} (\beta\widehat{q})^{\lambda_1} \right. \\
&\quad \left. \times \left[\frac{(\beta\widehat{q})^{-\lambda_1}}{(1+\lambda_1)} \beta^{1/\beta} \int_0^\infty r dt + \widehat{k} I_4 \right] \right\}
\end{aligned} \tag{20}$$

where $I_4 = \int_0^\infty k^{-\alpha/\beta} \dot{k}^{1/\beta-(\lambda_1+1)} dt$. Expressing $\int_0^\infty r dt$ from (20) we obtain

$$\begin{aligned}
\int_0^\infty r dt &= \frac{\lambda_1 + 1}{(\lambda_1+1)(-\alpha/\beta+1) - 1 + 1/\beta} (1/\beta)^{1/\beta} \\
&\quad \times \left\{ -k_0^{1-\alpha/\beta} \dot{k}_0^{1/\beta-1} - (1/\beta - 1) (\beta\widehat{q})^{\lambda_1} \widehat{k} I_4 \right\}
\end{aligned} \tag{21}$$

Integrating I_4 by parts with $u = \dot{k}^{1/\beta-1-(\lambda_1+1)}$, $dv = k^{-\alpha/\beta} \dot{k} dt$ and applying the same substitutions we have

$$
\begin{aligned}
I_4 &= \frac{\lambda_1+1}{(\lambda_1+1)(-\alpha/\beta)-1+1/\beta} \\
&\quad \times \left[-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-(\lambda_1+1)} - (1/\beta-1-(\lambda_1+1))(\beta\widehat{q})^{\lambda_1}\widehat{k}I_8\right]
\end{aligned}
$$

where $I_8 = \int_0^\infty k^{-\alpha/\beta}\dot{k}^{1/\beta-2(\lambda_1+1)}dt$. Substituting for I_4 in (21) we obtain

$$
\begin{aligned}
\int_0^\infty r dt &= \frac{\lambda_1+1}{(\lambda_1+1)(-\alpha/\beta+1)-1+1/\beta}(1/\beta)^{1/\beta} \\
&\quad \times \left\{-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1} - \frac{(\lambda_1+1)(1/\beta-1)}{(\lambda_1+1)(-\alpha/\beta)-1+1/\beta}(\beta\widehat{q})^{\lambda_1}\widehat{k}\right. \\
&\quad \left.\times \left[-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-(\lambda_1+1)} - (1/\beta-1-(\lambda_1+1))(\beta\widehat{q})^{\lambda_1}\widehat{k}I_8\right]\right\}
\end{aligned}
$$

Integrating I_8 by parts with $u = \dot{k}^{1/\beta-1-2(\lambda_1+1)}$, $dv = k^{-\alpha/\beta}\dot{k}dt$ we have

$$
\begin{aligned}
I_8 &= \frac{\lambda_1+1}{(\lambda_1+1)(-\alpha/\beta-1)-1+1/\beta} \\
&\quad \times \left[-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-2(\lambda_1+1)} - (1/\beta-1-2(\lambda_1+1))(\beta\widehat{q})^{\lambda_1}\widehat{k}I_{12}\right]
\end{aligned}
$$

which gives us

$$
\begin{aligned}
\int_0^\infty r dt &= \frac{\lambda_1+1}{(\lambda_1+1)(-\alpha/\beta+1)-1+1/\beta}(1/\beta)^{1/\beta} \\
&\quad \times \left\{-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1} - \frac{(\lambda_1+1)(1/\beta-1)}{(\lambda_1+1)(-\alpha/\beta)-1+1/\beta}(\beta\widehat{q})^{\lambda_1}\widehat{k}\right. \\
&\quad \times \left[-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-(\lambda_1+1)} - \frac{(\lambda_1+1)(1/\beta-1-(\lambda_1+1))}{(\lambda_1+1)(-\alpha/\beta-1)-1+1/\beta}(\beta\widehat{q})^{\lambda_1}\widehat{k}\right. \\
&\quad \left.\left.\times \left\{-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-2(\lambda_1+1)} - (1/\beta-1-2(\lambda_1+1))(\beta\widehat{q})^{\lambda_1}\widehat{k}I_{12}\right\}\right]\right\}
\end{aligned}
$$

This makes visible the pattern of expressions for integrals $I_4, I_8, I_{12}, \ldots$ and so (multiplying fractions by $-\beta$) we can show that the original integral is

$$
\int_0^\infty r dt = \frac{\lambda_1+1}{(\lambda_1+1)(\alpha-\beta)-1+\beta} \cdot \beta^{(1-1/\beta)}
$$

$$\times\left\{k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1}+\frac{(\lambda_1+1)(1-\beta)}{(\lambda_1+1)\alpha-1+\beta}(\beta\widehat{q})^{\lambda_1}\widehat{k}\right.$$

$$\times\left[k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-(\lambda_1+1)}+\frac{(\lambda_1+1)(1-\beta[1+(\lambda_1+1)])}{(\lambda_1+1)(\alpha+\beta)-1+\beta}(\beta\widehat{q})^{\lambda_1}\widehat{k}\right.$$

$$\times\left\{k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-2(\lambda_1+1)}+\frac{(\lambda_1+1)(1-\beta[1+2(\lambda_1+1)])}{(\lambda_1+1)(\alpha+2\beta)-1+\beta}(\beta\widehat{q})^{\lambda_1}\widehat{k}\right.$$
$$\left.\left.\left.\times\left[k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-3(\lambda_1+1)}+\ldots\right]\right\}\right]\right\}$$

Substituting for $\widehat{q}$, λ_0, and for $\dot{k}_0=\beta k_0^{\alpha}r_0^{\beta}$ we obtain

$$\int_0^\infty rdt=\frac{\lambda_1+1}{(\lambda_1+1)(\alpha-\beta)-1+\beta}\cdot\frac{k_0r_0}{q_0}$$

$$\times\left\{1+\frac{(\lambda_1+1)(1-\beta)}{(\lambda_1+1)\alpha-1+\beta}\cdot\widehat{k}\cdot\left[\beta\dot{q}_0\right.\right.$$
$$+\frac{(\lambda_1+1)(1-\beta[1+(\lambda_1+1)])}{(\lambda_1+1)(\alpha+\beta)-1+\beta}\cdot\widehat{k}\cdot\left\{(\beta\dot{q}_0)^2\right.$$
$$\left.\left.\left.+\frac{(\lambda_1+1)(1-\beta[1+2(\lambda_1+1)])}{(\lambda_1+1)(\alpha+2\beta)-1+\beta}\cdot\widehat{k}\cdot\left[(\beta\dot{q}_0)^3+\ldots\right]\right\}\right]\right\}$$

This gives us a closed form solution for our integral as a series

$$\begin{aligned}\int_0^\infty rdt &= \frac{\lambda_1+1}{(\lambda_1+1)(\alpha-\beta)-1+\beta}\cdot\frac{k_0r_0}{q_0}\\ &\quad\times\left\{1+\sum_{i=1}^{\infty}\left[\widehat{k}(\lambda_1)\beta\dot{q}_0(\lambda_1+1)\right]^i\cdot\prod_{j=0}^{i-1}\frac{1-\beta[1+j(\lambda_1+1)]}{(\lambda_1+1)(\alpha+j\beta)+\beta-1}\right\}\end{aligned}$$

The series can be expressed via special functions,[23] namely,

$$\prod_{j=0}^{i-1}\frac{1-\beta[1+j(\lambda_1+1)]}{(\lambda_1+1)(\alpha+j\beta)+\beta-1}=[-\beta(\lambda_1+1)]^i\,\Gamma\left(i-\frac{1-\beta}{\beta(\lambda_1+1)}\right)/\Gamma\left(-\frac{1-\beta}{\beta(\lambda_1+1)}\right)$$

[23]The expression of the series via special functions can be obtained in Maple.

and then

$$\int_0^\infty r dt = \frac{\lambda_1 + 1}{(\lambda_1+1)(\alpha-\beta) - 1 + \beta} \cdot \frac{k_0 r_0}{q_0} \\ \times \Big\{ 1 + (1-\beta)\beta \widehat{k}(\lambda_1)\dot{q}_0(\lambda_1+1) \\ \times {}_2F_0\left(\left[1, \frac{\beta(\lambda_1+2)-1}{\beta(\lambda_1+1)} \right], [\,], -\widehat{k}(\lambda_1)\dot{q}_0\beta^2(\lambda_1+1)^2 \right) \Big\} \tag{22}$$

where ${}_2F_0(\cdot)$ is the hypergeometric function with 2 upper parameters and empty list of lower parameters. Substituting for $\widehat{k} = k_0 - \beta q_0^2 / [\dot{q}_0(1+\lambda_1)]$ (Appendix 1) we obtain equation (12) in Corollary 1. For our numerical example the second term in bracket $\{\cdot\}$ equals to 0.247 and so, taking into account the existing uncertainty in reserve estimate, we can consider as a good approximation for the value of reserve the following formula

$$s_0 = \int_0^\infty r dt = \frac{\lambda_1 + 1}{(\lambda_1+1)(\alpha-\beta) - 1 + \beta} \cdot \frac{k_0 r_0}{q_0}$$

which gives an explicit expression for $\lambda_1(s_0)$:

$$\lambda_1 = \frac{(1-\alpha)s_0 q_0 + k_0 r_0}{(\alpha-\beta)s_0 q_0 - k_0 r_0}$$

This formula captures the main peculiarities of behavior of the exact solution. Particularly, it has the same horizontal and vertical asymptotes as the closed form solution (22) (Fig. 11).

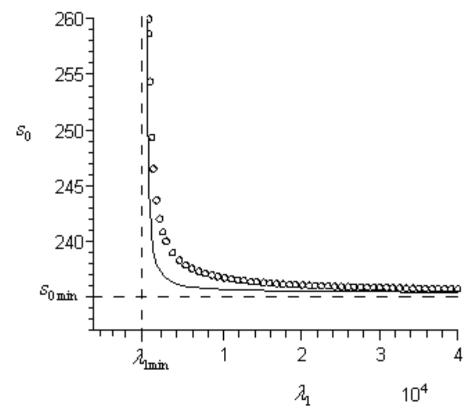

Figure 11: Dependence of reserve s_0 (the value of integral $\int_0^\infty r(t, \lambda_1)dt$) on λ_1: closed form solution (22) - in circles; approximate formula - solid line