Cholesky-ANN models for predicting multivariate realized volatility

Bucci, Andrea

Università Politecnica delle Marche

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Andrea Bucci*

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Abstract

Accurately forecasting multivariate volatility plays a crucial role for the financial industry. The Cholesky-Artificial Neural Networks specification here presented provides a twofold advantage for this topic. On the one hand, the use of the Cholesky decomposition ensures positive definite forecasts. On the other hand, the implementation of artificial neural networks allows to specify nonlinear relations without any particular distributional assumption. Out-of-sample comparisons reveal that Artificial neural networks are not able to strongly outperform the competing models. However, long-memory detecting networks, like Nonlinear Autoregressive model process with eXogenous input and long short-term memory, show improved forecast accuracy respect to existing econometric models.

Keywords: Neural Networks, Machine Learning, Stock market volatility, Realized Volatility
JEL classification: C22, C45, C53, G17

*Department of Economics and Social Sciences, Università Politecnica delle Marche, 60121 Ancona, Italy. email: a.bucci@univpm.it

1 Introduction

Forecasting the volatility of a portfolio has assumed a crucial role in the financial markets, holding the attention of a stream of econometric literature.

This paper draws closely on the contributions made on volatility models and artificial neural networks. Particularly, a strand of literature, e.g. Maheu and McCurdy (2002) and Anderson, Nam, and Vahid (1999), has demonstrated that volatility asymmetrically responds to unexpected news, and that linear methods may not be adequate to model it. A first attempt to model nonlinearities in multivariate conditional volatility has been provided by Kwan, Li, and Ng (2005), which relies on a multivariate threshold GARCH model. Recently, the introduction of a non-parametric measure, like realized volatility, has allowed the implementation of different multivariate models, like the vector smooth transition autoregressive model (Bucci, Palomba, and Rossi (2019)) and the artificial neural networks (ANN) with multiple target variables.

The attention of this paper is focused on the use of neural networks in finance. ANNs have been widely applied in economics and finance, since they are capable of detecting nonlinear dependencies and long-term persistence without any assumption on the distribution of the target variables. However, despite the large amount of papers that refer to the application of ANNs for financial time series forecasting (e.g. White (1988), Khan (2011)), only few works focus specifically on their application on forecasting conditional volatility. The majority of these studies foresees the combination of a GARCH model with a NN architecture, see for example Donaldson and Kamstra (1997) and Hu and Tsoukalas (1999), with few exceptions, e.g. Fernandes, Medeiros, and Scharth (2014), Vortelinos (2017) and Bucci (2019). None of these works is designed for the multivariate context.

Extending the above-mentioned literature, this study seeks to understand whether ANNs provide more accurate forecasts than existing models in a multivariate framework. In doing so, the analysis relies on both feed-forward and recurrent neural networks. Specifically, recurrent neural networks (RNN), which are able to detect long-term dependencies through an architecture replicating the internal feedback of biological neural networks, are analysed. The target variables are modeled through a set of neural networks, including the feed-forward neural network (FNN), the Elman neural network (ENN), the Jordan neural network (JNN), the Nonlinear Autoregressive model process with eXogenous input (NARX) neural network and a long short-term memory (LSTM) neural network.

The approach developed in this paper involves the decomposition of covariance matrices into Cholesky factors. The idea of modelling Cholesky factors follows the works of Halbleib-Chiriac and Voev (2011) and Becker, Clements, and O'Neill (2010). The use of the Cholesky factors ensures positive definite forecast matrices without the imposition of parameter restrictions.

Moreover, in this paper realized covariance matrices assume a crucial role in the implementation of the Cholesky-ANN procedure. As proposed by Barndorff-Nielsen and Shephard (2004) and Andersen, Bollerslev, Diebold, and Labys (2001), high-frequency data may be used to non-parametrically estimate the latent conditional volatility of a portfolio, making the multivariate volatility effectively observable and moldable with existing time series models and neural networks. The realized covariance matrices, computed from the vector of daily returns, have been employed to study the evolution of realized volatility over a monthly horizon, making it possible to analyze the relationship with economic factors that are likely to influence the dynamic behavior of volatility at lower frequencies.

The rest of this paper is structured as follows: Section 2 illustrates Cholesky-ANN procedure; Section 3 discusses the neural network specification selection, while Section 4 provides the empirical results of the out-of-sample forecasting accuracy; finally, Section 5 presents the conclusions.

2 Cholesky-ANN

The estimation of realized covariance matrices is usually not straightforward, since the estimated matrices should be ensured positive semi-definite. Two approaches are commonly used to guarantee positive semidefiniteness: constrained optimization and unconstrained optimization through matrix parametrization, like Cholesky decomposition or matrix logarithmic transformation. Here, according to similar studies (Halbleib-Chiriac and Voev (2011); Becker, Clements, and O'Neill (2010); Bauer and Vorkink (2011)), an unconstrained approach is applied. Specifically, realized covariance matrices are parametrized through Cholesky decomposition, which represents a parsimonious way for obtaining positive definite estimated matrices without special constraints.

The model here presented is used to forecast the stock returns volatility of n stocks. Assuming that $r_{i,t}$ is the $n \times 1$ vector of daily log-returns, the realized covariance matrix can be computed as follows

$$RC_t = \sum_{i=1}^{N_t} r_{i,t} r'_{i,t} \tag{1}$$

where RC_t has been proven to be a consistent estimator for the monthly conditional covariance matrix of log returns by Barndorff-Nielsen and Shephard (2002) and Andersen, Bollerslev, Diebold, and Labys (2003), and N_t is the number of trading days in the month t. The realized covariance matrix is then decomposed applying the Cholesky decomposition, such that

$$RC_t = C_t C_t' \tag{2}$$

where C_t is a unique $n \times n$ lower triangular matrix. The Cholesky factors are obtained by

$$y_t = vech(C_t) \tag{3}$$

where y_t is a vector of $m = n(n+1)/2$ elements. Once obtained the vector of m Cholesky factors, they can be modelled accordingly.

Econometric literature has proved that financial volatility and its underlying processes are often nonlinear in nature, see Martens, De Pooter, and Van Dijk (2004). Some authors, Halbleib-Chiriac and Voev (2011) and Heiden (2015) for example, have shown that covariance matrices, and their Cholesky factors as well, exhibit long-term dependencies. As a result, linear models may not be suited to study the behavior of this phenomena. Extending in the multivariate context a previous paper (see Bucci (2019)), I decided to approximate the relationship between the elements of the Cholesky decomposition and a set of macroeconomic and financial variables through a *universal approximator* such as artificial neural networks (ANN) with multiple outputs.

Neural networks (NN) have attracted much attention as new method for estimating and forecasting time series in many fields of study, including economics and finance. They can be seen as a non-parametric statistical method which replicates the architecture of the human brain by grouping artificial neurons (or nodes) in connected layers. By modifying the number of layers or nodes in each layer, a wide range of econometric models can be approximated. In particular, empirical research demonstrated that ANNs are capable of detecting nonlinearities without a priori information about the relations between the dependent variable(s) and its(their) determinants and without any distribution assumption.

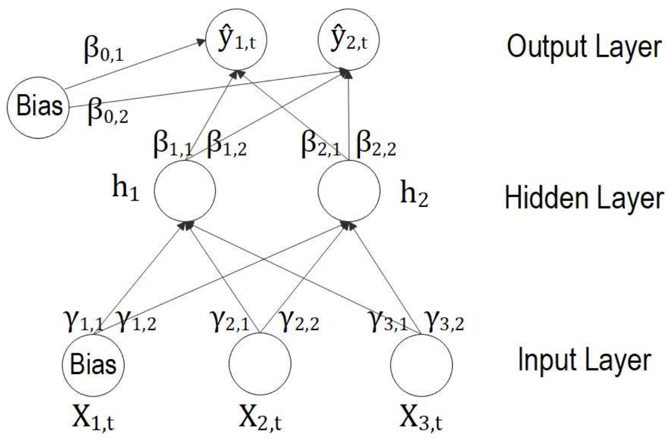
Typically, an ANN consists of three layers: an input layer, which contains the input variables (the equivalent of biological sensors), one or more hidden layers, and an output layer, with one or more output variables. The nodes of a layer connect with nodes of the succeeding layer through weights (parameters) and an activation function. The use of an activation function, like a logistic or hyperbolic tangent function, allows to introduce nonlinearity in the model and to replicate the way biological neurons are activated.

The weights are trained from the data by minimizing the mean squared error, usually through a backpropagation algorithm (BP). Once the input and the output vectors are read by the BP algorithm, the training starts with random weights. After calculating the mean squared error between the observed and the predicted output, the network adjusts the parameters in the direction reducing the error, and so on until there is no improvement.

In econometrics, a significant part of the model specification consists in identifying the explanatory variables and the number of lags leading the most accurate forecasts. When constructing a neural network, the overall task could be much longer, since the process does not only involve the choice of inputs, but also the identification of the network architecture.

Firstly, a neural network researcher should choose the type of neural network to be implemented: feed-forward or recurrent. In feed-forward neural network (FNN), the information moves forward from a layer to the next one. Assuming a single hidden layer architecture, a FNN with multiple outputs can be depicted as in Figure 1.

Figure 1: FNN with two target variables, a single hidden layer with two hidden nodes and three input variables (including the bias)



The output function of the k-th target variable, with $k = 1,\ldots,l$ (where l is the number of target variables, e.g. $l = 2$ in Figure 1), for FNN with a single hidden layer can be specified as follows:

$$\hat{y}_{k,t} = F\big(\beta_{0,k} + \sum_{j=1}^{q} G\big(x_t\gamma_j'\big)\beta_{j,k}\big), \tag{4}$$

where G is the hidden layer activation function, $\beta_{j,k}$ is the weight from the j-th hidden unit to the output unit, $x_t = \left\{1, x_{1,t}, \ldots, x_{s,t}\right\}$ is the $1 \times m$ vector of input variables at time t (with

$m = s+1$), $\beta_{0,k}$ is the bias of the k-th output unit, $\gamma_j = \left\{\gamma_{1,j}, \ldots, \gamma_{m,j}\right\}$ is the vector of weights of dimension $1 \times m$ connecting the input variables and the j-th hidden neuron and q is the number of hidden units.

Generally, F is an identity activation function, such that $F(a) = a$. Equation (4) takes then the following form:

$$\hat{y}_{k,t} = \beta_{0,k} + \sum_{j=1}^{q} G\left(x_t \gamma_j'\right) \beta_{j,k}. \tag{5}$$

In order to replicate the activation state of a biological neuron, all neural networks use non-linear activation functions at some point, usually through G. An ideal activation function should be continuous and differentiable to implement the backpropagation algorithm. The most common choices are a logistic and a hyperbolic tangent function.

The logistic cumulative function is bounded between 0 and 1, where a value closer to 0 (1) equals to a low (high) activation level, and can be specified as

$$G(x) = \frac{1}{1+e^{-x}}.$$

The hyperbolic tangent function is better suited for variables that assume negative values and is bounded between -1 and 1. It can be written as

$$G(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Both the functions have been used in the present work.

An interesting feature of biological neural networks is the presence of internal feedback. Such feedback has been implemented in recurrent neural networks (RNN), where the information is propagated from a layer to the succeeding layer but also back to earlier layers. In this way, RNNs allow to preserve long-memory of time series and capture long-term dependence with a restrained number of parameters. Therefore, in practical applications RNNs are more appropriate than FNN to forecast nonlinear time series.

This analysis relies on four different types of recurrent neural networks: Elman, Jordan, long short-term memory (LSTM) network and nonlinear autoregressive exogenous (NARX) neural network.

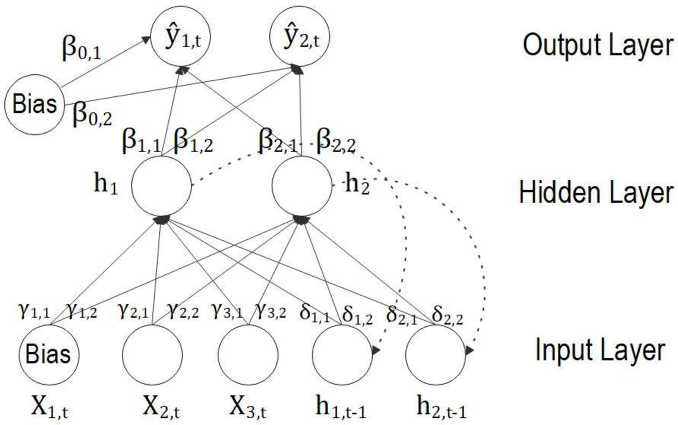
In an Elman neural network, introduced by Elman (1990), the hidden nodes with a time delay are used as additional input neurons. Assuming an identity output function and a single lag for the hidden nodes, the output of the Elman network with multiple outputs can be represented as in Figure 2 and can be written as follows

$$\hat{y}_{k,t} = \beta_{0,k} + \sum_{j=1}^{q} h_{tj} \beta_{j,k} \tag{6}$$

$$h_{tj} = G\left(x_t\gamma_j' + h_{t-1}\delta_j'\right) \qquad j = 1,\ldots,q$$

where h_{t-1} is the vector of lagged hidden units and δ_j is the vector of connection weights between the j-th hidden node and the lagged hidden nodes.

Figure 2: ENN with a single hidden layer

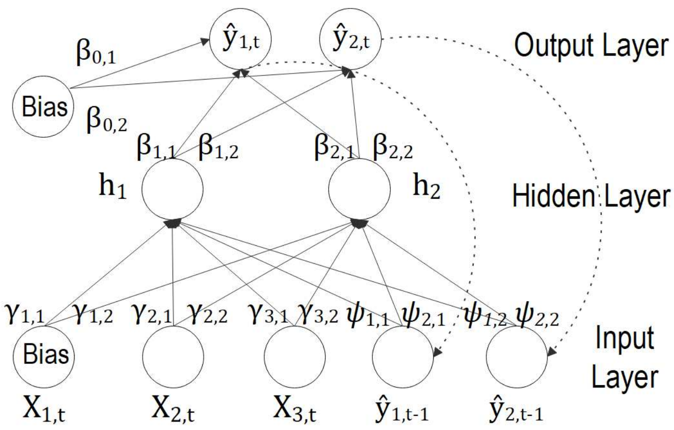


An alternative way to implement internal feedback is specifying a Jordan neural network (JNN), proposed by Jordan (1986). As shown in Figure 3, a JNN exhibits a feedback from the output to the input layer. The lagged output units are then used as additional neurons. The way the output units are fed back has similarities with GARCH models, making JNN an appetible way to forecast conditional volatility. Accordingly, the output of the Jordan neural network (JNN) with multiple target variables can be computed as

$$\hat{y}_{k,t} = \beta_{0,k} + \sum_{j=1}^{q} G\left(x_t\gamma_j' + \sum_{k=1}^{l} \hat{y}_{k,t-1}\psi_{j,k}\right)\beta_{j,k} \tag{7}$$

where $\psi_{j,k}$ is the weight between the lagged output and the j-th hidden unit for the target variable k.

Figure 3: JNN with a single hidden layer



Elman and Jordan neural networks, because of the presence of internal feedbacks, are capable of detecting long-range relations. For this reason, such models have many potential applications in economics and finance, see Kuan and Liu (1995) and Tino, Schittenkopf, and Dorffner (2001).

Although traditional RNNs are able to model and forecast nonlinear time series with great accuracy, they present several pitfalls. On one side, time lags of internal feedbacks should be pre-determined by the researcher. On the other side, RNNs still suffer from the presence of the vanishing gradient problem[1]. To overcome these issues, two further ways to define nonlinear and persistent relationships through neural networks can be specified, namely the NARX neural network and the long short-term memory (LSTM) neural network.

NARX neural network can be seen as an augmented version of FNN or JNN. In fact, it allows to directly include lagged observed target variables or predicted output in the input layer. In doing so, direct connections from the past mitigate the vanishing gradient problem and capture long-term dependence.

There exist two ways to specify a NARX network:

- NARX-P, through a parallel (P) specification the predicted output units at time t are used as additional inputs in a FNN architecture at time $t+1$, similarly to a JNN;

[1]The vanishing gradient problem occurs when neural networks are trained through a gradient based algorithm. When backpropagation is applied, the gradient tends to get smaller and smaller, almost vanishing, as the iteration process goes forward. As a consequence, the learning process may be particularly slow and a local minimum may be reached.

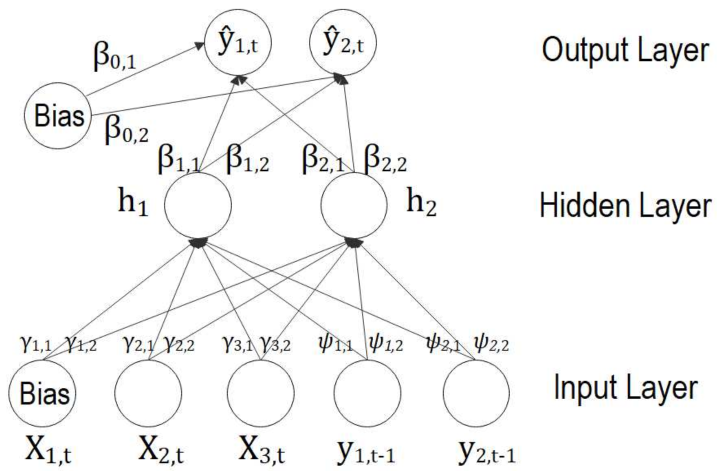
- NARX-SP, through a semi-parallel (SP) specification d lags of the real target variables are included in the set of explanatory variables in the input layer. This architecture behaves like a FNN where the input units are enriched by lagged observed outputs.

Throughout this paper, I will consider only a NARX-SP architecture with zero input order and a multi-dimensional output, as depicted as in Figure 4. The output of this network can be specified as follows:

$$y_{k,t} = \beta_{0,k} + \sum_{j=1}^{q} G\big(x_t \gamma_j' + \sum_{k}^{l} \sum_{d}^{n_d} y_{k,t-d} \psi_{k,d,j}\big) \beta_{j,k} \tag{8}$$

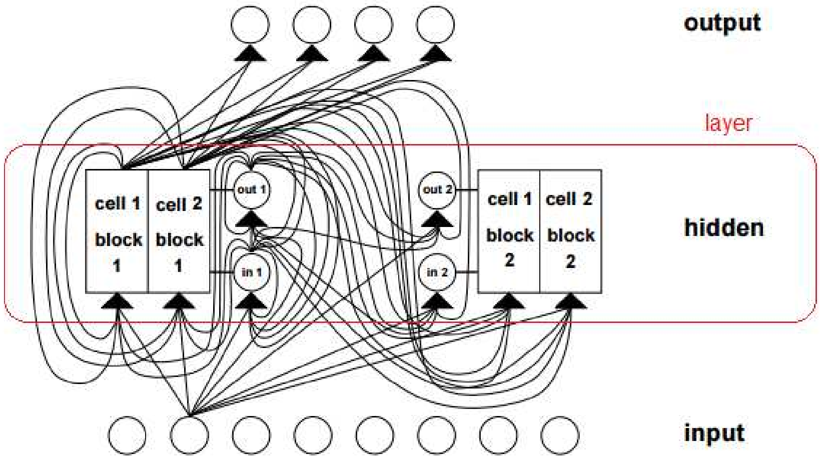
where $\psi_{k,d,j}$ is the connection weight of the d-th delay of output unit k.

Figure 4: Architecture of a NARX network



The aforementioned recurrent neural networks are not the only architectures able to replicate the way biological neurons remember through the internal feedback. In fact, the most successful RNN to capture long-term dependencies is the long short-term memory network (LSTM), proposed by Hochreiter and Schmidhuber (1997). The structure of a LSTM is similar to other RNNs with an input layer, a recurrent hidden layer and an output layer. However, in order to retain relevant information and to discard unnecessary parts, the hidden nodes are replaced by a memory block (see Figure 5), where a mechanism of gating allows to control the information flow in the block.

Figure 5: LSTM neural network



As depicted in Figure 6, each memory block is a self-connected system of three gates: input, forget and output. Compared to traditional RNNs, a forget gate is added to the memory block to prevent the gradient from vanishing or exploding, see also Hsu (2017). The gating mechanism allows the network to learn which information should be maintained and which not. The structure of a LSTM network with multiple target variables can be formalized as below:

$$f_{k,t} = G\left(W_{k,f} h_{k,t-1} + U_{k,f} x_t + b_{k,f}\right) \tag{9}$$

$$i_{k,t} = G\left(W_{k,i} h_{k,t-1} + U_{k,i} x_t + b_{k,i}\right) \tag{10}$$

$$\tilde{c}_{k,t} = C\left(W_{k,c} h_{k,t-1} + U_{k,c} x_t + b_{k,c}\right) \tag{11}$$

$$c_{k,t} = f_{k,t} \odot c_{k,t-1} + i_{k,t} \odot \tilde{c}_{k,t} \tag{12}$$

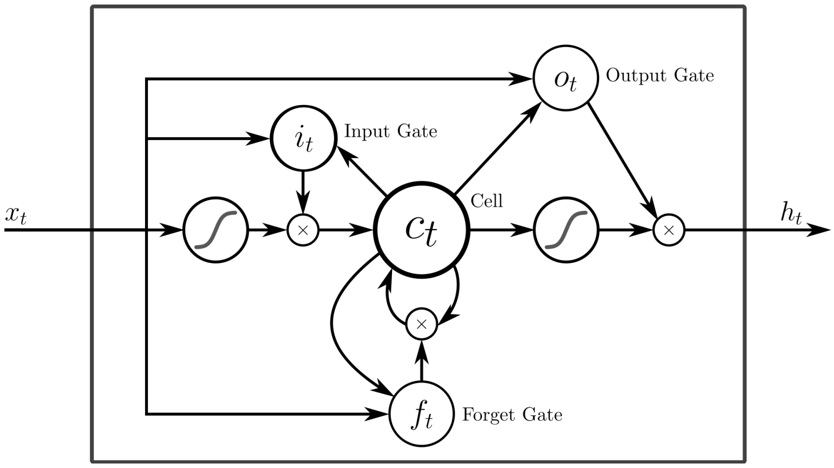
$$o_{k,t} = G\left(W_{k,o} h_{k,t-1} + U_{k,o} x_t + V_{k,o} c_{k,t} + b_{k,o}\right) \tag{13}$$

$$h_{k,t} = o_{k,t} \odot C\left(c_{k,t}\right) \tag{14}$$

$$\hat{y}_{k,t} = h_{k,t} \tag{15}$$

where W_i, W_f, W_c, W_o, U_i, U_f, U_c and U_o are the weight matrices of input, forget, memory cell state, and output gates respectively, V_c is the cell state weight matrix, $\hat{y}_{k,t}$ is the k-th output of the neural network, b_i, b_f, b_c and b_o are the biases of each gate, G is a sigmoid or logistic function and C is a hyperbolic tangent function.

Figure 6: Basic LSTM memory block



LSTM is particularly useful, among others, in situations where the time series exhibits a highly persistent behavior. For such reasons, it has been broadly applied in financial studies, like Xiong, Nichols, and Shen (2016), Bao, Yue, and Rao (2017) and Kim and Won (2018).

All the aforementioned types of neural networks are used to produce forecasts of the Cholesky elements of C_t. Once the forecast of the lower triangular matrix, $\hat{C}_{t+k}$, has been obtained, the forecast realized covariance matrix can be re-constructed as

$$RC_{t+k} = \hat{C}_{t+k}\hat{C}'_{t+k},$$

ensuring a positive definite forecast of the covariance matrix, for $n < N_t$.

3 Neural Networks Architecture

When specifying a neural network architecture, the researcher should identify the set of explanatory variables used as inputs, define the number of hidden layers and hidden nodes, and choose the activation function and the training algorithm for the definition of the connection weights.

The set of determinants is usually gathered from the related literature. From an empirical perspective, several studies studies found that return volatility is countercyclical, see Schwert (1989), Mele (2007), Paye (2012) and Christiansen, Schmeling, and Schrimpf (2012). Here, the most part of the candidates as explanatory variables is based on excess return predictability determinants. Following Mele (2007) and Welch and Goyal (2008), I include in the analysis the dividend-price (DP) and the price-earnings ratio (EP), the equity market return (MKT) and the Fama and French (1993) factors (HML, SMB and STR), which typically influence risk premium. Since the set of independent variables include the returns of two bond Futures, also the 1-month T-Bill rate (TB) and the term spread difference (TS) between long-term and short-term bond yields are used as additional inputs. As in Schwert (1989) and Engle, Ghysels, and Sohn (2009), the set of predictors is also enriched by macroeconomic variables, as inflation rate (INF) and industrial production growth (IP). Finally, the default spread has been included as a proxy for the credit risk. All the exogenous variables have been lagged one time to alleviate the endogeneity problem.

The definition of the number of hidden nodes and layers is more complicated. In fact, there is no theoretical basis to determine the appropriate number of hidden layers or neurons in a network. According to other studies and to the guidelines provided by Xiang, Ding, and Lee (2005), a single-hidden layer architecture is sufficient to approximate a wide range of nonlinear functions. Thus, in this paper a single hidden layer has been implemented. Conversely, a poor number of hidden units does not allow to detect complex nonlinear patterns in the data. However, when too many hidden units are included in the architecture, the network may lead to overfitted out-of-sample forecasts and the number of connection weights may significantly increase. In order to select the optimum number of weights, I compare the sum of the RMSE for different architecture where the number of hidden nodes varies from 1 to 8, as suggested by Tang and Fishwick (1993). Similarly to nonlinear estimation techniques, the risk of a local minimum is high also for neural networks. In fact, the training procedure is sensitive to the initial values of the weights randomly selected. To assess the performance of the network, the parameters have been trained with 300 different starting values. Since the architecture of a LSTM is significantly different from the rest of the neural networks, a number of 100 memory blocks is chosen in line with similar works.

As shown in Table 1, a different number of hidden nodes provides very different results in terms of cumulated RMSE. For FNN, the measure of performance is lower for a number of hidden neurons equal to 5. ENN, ENN with lagged target variables and NARX network exhibit the best performance when 3 hidden nodes are used, while JNN and JNN with lags seem to forecast better with 6 and 2 hidden nodes, respectively.

When implementing a neural network with multiple target variables, the risk of a local minimum could be significant. Accordingly, the connection weights have been trained with a gradient descent with momentum and adaptive learning rate (*gdx*) algorithm for all the networks, except for the NARX neural network. In fact, the NARX network has been trained with a Bayesian Regularization (BR) algorithm, which allows to obtain robust forecasts with a NARX network, as suggested by Guzman, Paz, and Tagert (2017).

Finally, all the activation functions, G, in Equation (4), (6), (7) and (8), are logistic functions.

Table 1: RMSE for increasing number of hidden nodes

The number of hidden neurons, the cumulated RMSE of the m series, the number of connection weights and the delays of the dependent variables used as inputs are included in the table. The maximum number of iterations is equal to 1000. ENN_{lags} and JNN_{lags} are ENN and JNN with lagged target variables as additional inputs. The training sample is approximately equal to the 70% of the whole sample (from May 2007 to December 2017), while the testing sample is equal to 128 observations.

Model	N. Hidden	Performance	N. weights	Lags	Model	N. Hidden	Performance	N. weights	Lags
FNN	1	0.0762	24	-	NARX	1	0.0640	30	1
	2	0.0771	42	-		2	0.0564	54	1
	3	0.0732	60	-		3	0.0521*	78	1
	4	0.0727	78	-		4	0.0561	102	1
	5	0.0674*	96	-		5	0.0560	126	1
	6	0.0686	114	-		6	0.0550	150	1
	7	0.0698	132	-		7	0.0575	174	1
	8	0.0709	150	-		8	0.0574	198	1
ENN	1	0.0743	25	-	ENN_{lags}	1	0.0739	31	1
	2	0.0706	46	-		2	0.0729	58	1
	3	0.0652*	69	-		3	0.0666*	87	1
	4	0.0708	94	-		4	0.0718	118	1
	5	0.0743	121	-		5	0.0776	151	1
	6	0.0751	150	-		6	0.0866	186	1
	7	0.0759	181	-		7	0.0845	223	1
	8	0.0911	214	-		8	0.0865	262	1
JNN	1	0.0763	30	-	JNN_{lags}	1	0.0735	36	1
	2	0.0769	54	-		2	0.0648*	66	1
	3	0.0720	78	-		3	0.0667	96	1
	4	0.0665	102	-		4	0.0657	126	1
	5	0.0690	126	-		5	0.0668	156	1
	6	0.0646*	150	-		6	0.0693	186	1
	7	0.0682	174	-		7	0.0672	216	1
	8	0.0696	198	-		8	0.0725	246	1

* denotes the lowest MSE

4 Empirical Results

The realized covariance matrices are computed according to Equation (1) and are based on the daily returns on the S&P 500, on the 10-year Treasury note future and on the 1-month Treasury bond future, both traded on the Chicago Board of Trade (CBOT)[2]. Thus, there are $T = 428$ monthly realized covariance matrices.

Table 2 shows summary statistics for the realized volatility and co-volatility series, and for the Cholesky factors of the covariance matrices. The volatility of stock market is much larger than average volatility of treasury futures. As is well known, realized variances and covariances are highly persistent for stock market, as shown by the autocorrelation coefficients.

The series of the Cholesky factors preserve the persistence exhibited by the original covariance series, as suggested by Andersen and Bollerslev (1997) and Andersen, Bollerslev, Diebold, and Ebens (2001), while unit root tests provide mixed results on the hypothesis of stationary series. As proved by the skewness and the excess kurtosis of the time series, the distribution of the Cholesky factors is far from being normal, for this reason a method free from distribution assumptions, such as neural networks, could provide improved forecasts.

[2]Future contracts provide two major advantages: they are highly liquid and they permits to define bond returns without approximating them by yields.

Table 2: Descriptive statistics Realized Covariance and Cholesky factors series

The table includes the mean, the standard deviation, skewness, excess kurtosis and the sample autocorrelation. The last three columns report the p-value of the Dickey-Fuller (ADF) with constant (c) and with constant and trend (c-t) and the KPSS test statistics.

Realized Variances									
							ADF		
Symbol	Mean	St.Dev	Skew.	Ex. Kurt.	ρ_1	ρ_2	c	c-t	KPSS
SP	25.526	50.284	8.54	95.36	0.44	0.31	0.0000	0.0000	0.1099
10Y	4.135	6.927	13.46	229.6	0.11	0.10	0.0000	0.0000	0.5813**
1m	10.190	17.739	14.16	242.9	0.09	0.07	0.0000	0.0000	0.1201
Realized Covariances									
							ADF		
Symbol	Mean	St.Dev.	Skew.	Ex. Kurt.	ρ_1	ρ_2	c	c-t	KPSS
SP-10Y	-0.090	5.616	-1.164	12.22	0.55	0.41	<0.001	<0.001	1.9393***
SP-1m	-0.284	9.218	-0.829	8.500	0.57	0.42	<0.001	<0.001	2.0829***
10Y-1m	6.023	10.667	15.22	275.9	0.08	0.08	<0.001	<0.001	0.4254*
Cholesky Factors									
							ADF		
Symbol	Mean	St.Dev.	Skew.	Ex. Kurt.	ρ_1	ρ_2	c	c-t	KPSS
Cholesky 1	4.338	2.592	3.46	19.68	0.63	0.52	<0.001	<0.001	0.1433
Cholesky 2	0.124	0.899	0.13	0.246	0.70	0.67	0.075	<0.001	2.5495***
Cholesky 3	1.633	0.805	4.81	48.06	0.38	0.24	0.031	<0.001	0.8056***
Cholesky 4	0.149	1.457	-0.03	-0.09	0.72	0.71	0.206	<0.001	2.6791***
Cholesky 5	2.380	1.149	6.48	81.03	0.31	0.27	<0.001	<0.001	0.3891*
Cholesky 6	0.823	0.626	9.87	149.5	0.26	0.24	<0.001	<0.001	0.5334**

Where: *** - p-value < 1% , ** - p-value < 5%, * - p-value < 10%

KPSS critical values: 0.348 (10%), 0.462 (5%) and 0.742 (1%)

The aim of the paper is to produce more accurate forecasts than existing econometric models in a multivariate context. To this end, time series were first divided in an in-sample set, from May 1982 to April 2007, and an out-of-sample set, from May 2007 to December 2017 (128 monthly observations), which equals to one-third of the whole sample. The one-step-ahead out-of-sample forecasts from the neural networks model are compared with the forecasts from a Cholesky-VAR (*Vector Autoregressive*) model with and without predictors, a VAR on the original realized covariance matrices, and a DCC model.

The evaluation of the out-of-sample forecasts is a crucial aspect for the choice of the model best fitting the data. Common problems arising in the comparison of volatility forecasts concern the latent nature of the target variable and the choice of an adequate loss function. The former issue has been addressed by using a volatility proxy, such as realized volatility. Thus, throughout the paper volatility is assumed as observable. In regard to the latter, Patton (2011) provided the necessary and sufficient conditions for a loss function. Based on these conditions, the loss function here used are the MSE loss function for univariate comparison, and Frobenius distance between matrices and Euclidean distance between vectors for multivariate comparison, such that

$$\begin{aligned}
&L_{MSE} = (\hat{\sigma}_t - h_t)^2 \\
&L_F(\hat{\Sigma}_t, H_t) = \| \hat{\Sigma}_t - H_t \|_F^2 \\
&L_E(\hat{\sigma}_t, h_t) = \mathrm{vech}(\hat{\Sigma}_t - H_t)' \mathrm{vech}(\hat{\Sigma}_t - H_t)
\end{aligned}$$

where $\hat{\sigma}_t$ is the single element of the volatility matrix proxy, $\hat{\Sigma}_t$, while h_t is the single element of the variance matrix forecast, H_t.

The performance is evaluated for each element of the covariance matrices through Mean Absolute Error (MAE), the Root Mean Squared Error (RMSE) and through the equal predictive accuracy test by Diebold and Mariano (1995) (DM). The DM test is also performed in the multivariate framework through the multivariate loss functions above mentioned.

In Table 3, the accuracy measures are computed for each covariance series. For several realized covariances, MAE and RMSE are minimum for the NARX and the LSTM neural networks. Not surprisingly, neural networks provide overall better results than linear models for the majority of the series. Moreover, the use of the Cholesky decomposition seems to improve the out-of-sample forecasting accuracy.

The DM tests for the single elements of the covariance matrices have been computed using the forecasts from the VAR model without exogenous variables as benchmark model. It should be remembered that a positive and significant test implies that the benchmark model performs worse than the compared model. In the most of the pairwise comparison in Table 4, the loss difference is positive but almost never significant. Recurrent neural networks are able to outperform VAR model only in forecasting the realized variance of S&P 500, while the test provides not significant loss differences for all the other time series.

Additionally, DM test has been conducted on multivariate loss function differences, as shown in Table 5, together with the error measured with Frobenius and Euclidean distances. VAR, VARExo and DCC exhibit the highest error among the compared models. The best performance is shown by the JNN with lagged target variables and NARX models, underlying the relevance of detecting long-term persistence with an adequate architecture. When considering the DM on multivariate loss functions, the test statistics is positively significant only when Euclidean norm is applied and only for three models: VAR on Cholesky factors with exogenous variables, JNN with lagged realized covariances as inputs and LSTM. In the multivariate context, NARX is able to outperform all the competing models, except for the DM-Euclidean statistics.

When looking at each single element of the covariance matrices,only minor gains are found from using neural networks in forecasting realized covariances. In a multivariate context, the choice of the forecasting method seems to slightly affect the forecasting accuracy if a single target variable is considered. Conversely, there is evidence that the recurrent neural networks, especially LSTM and NARX networks, provide the best realized covariance forecasts when the whole set of forecasts is evaluated. Interestingly, neural networks are shown to be particularly useful to forecast highly persistent series, once again underlying the relevance of an approach suited for detecting long-term dependencies.

A reason why neural networks cannot outperform competing models in this context can be found in the presence of unit roots in several Cholesky factor series, such as *Cholesky 2*, *Cholesky 3* and *Cholesky 4*, on which the networks are trained. Furthermore, the lack of a significant difference in the forecasting accuracy with econometric models can be dictated by the large number of parameters respect to the number of observations, which may lead to poor forecasting performance.

Table 3: Out-of-sample Forecasts Evaluation

	SP	SP-10Y	SP-1m	10Y	10Y-1m	1m
			MAE			
VAR	0.2860	0.0361	0.0717	0.0405	0.0558	0.0977
VARExo	0.3119	0.0352	0.0727	0.0393	0.0515	0.0901
CholVAR	0.2136	0.0401	0.1694	0.0235	0.0794	0.0635
CholVARExo	0.1765	0.0323	0.1480	0.0264	0.0740	0.0636
DCC	0.3292	0.0410	0.0734	0.0332	0.0521	0.1068
FNN	0.1924	0.0385	0.1528	0.0276	0.0619	0.0508
ENN	0.2077	0.0306	0.1491	0.0238	0.0966	0.0501
ENN_{lags}	0.2653	0.0321	0.1546	0.0214	0.0784	0.0501
JNN	0.1877	0.0425	0.1632	0.0268	0.0634	0.0517
JNN_{lags}	0.1611	0.0300	0.1682	0.0280	0.0670	0.0556
LSTM	0.1488	0.0385	0.1613	0.0215	0.0513	0.0406
NARX	0.1485	0.0280	0.1529	0.0243	0.0867	0.0456
			RMSE			
	SP	SP-10Y	SP-1m	10Y	10Y-1m	1m
VAR	0.6073	0.0621	0.1241	0.1126	0.1653	0.3190
VARExo	0.6822	0.0770	0.1675	0.0722	0.0729	0.1505
CholVAR	0.4794	0.0633	0.2192	0.0365	0.0997	0.1935
CholVARExo	0.3886	0.0682	0.2105	0.0403	0.0103	0.1773
DCC	0.6896	0.0684	0.1173	0.0467	0.0656	0.1588
FNN	0.4327	0.0604	0.2070	0.0403	0.0845	0.1190
ENN	0.3383	0.0636	0.1834	0.0401	0.1229	0.1099
ENN_{lags}	0.4442	0.0528	0.1962	0.0384	0.0912	0.1182
JNN	0.4017	0.0736	0.2557	0.0366	0.0831	0.1196
JNN_{lags}	0.3291	0.0621	0.2151	0.0400	0.0879	0.1111
LSTM	0.4220	0.0600	0.2192	0.0385	0.0727	0.0909
NARX	0.3089	0.0568	0.2243	0.0375	0.1038	0.1136

Values have been multiplied by 10^2.

Table 4: Diebold-Mariano test - MSE Loss Function

	SP	SP-10Y	SP-1m	10Y	10Y-1m	1m
VARExo	-0.6309	-0.8942	-1.0472	0.6589	0.8982	0.9178
CholVAR	1.4217	-0.3653	-4.8336^{***}	1.0301	0.7108	1.0532
CholVARExo	1.7873^{*}	-0.8061	-3.3350^{***}	1.0123	0.7099	1.0283
DCC	-0.8045	-1.1784	0.3256	0.9517	0.9403	0.8879
FNN	1.3850	0.3362	-3.4702^{***}	1.0028	0.8234	1.0168
ENN	1.6905^{*}	-0.1670	-3.0251^{***}	1.0039	0.4957	1.0410
ENN_{lags}	1.4462	1.9721^{*}	-3.3147^{***}	1.0167	0.7759	1.0230
JNN	1.6830	-1.0598	-3.2517^{***}	1.0288	0.8332	1.0149
JNN_{lags}	1.9081^{*}	0.0013	-3.6584^{***}	1.0052	0.7994	1.0382
LSTM	1.8202^{*}	0.4082	-3.5759^{***}	1.0158	0.8995	1.0853
NARX	1.8647^{*}	1.1590	-3.2338^{***}	1.0239	0.6771	1.0326

Table 5: Statistical Evaluation with Multivariate Loss Functions

	Distance		**Diebold-Mariano**	
	Frobenius	Euclidean	Frobenius	Euclidean
VAR	0.00298	0.0825	-	-
VARExo	0.00297	0.0823	-1.5988	-1.6168
CholVAR	0.00235	0.0651	-1.0629	0.3177
CholVARExo	0.00201	0.0558	0.7739	1.7458^{*}
DCC	0.00296	0.0820	-0.5772	-0.6924
FNN	0.00207	0.0573	0.0154	0.7305
ENN	0.00174	0.0481	-0.7226	-0.1151
ENN_{lags}	0.00209	0.0579	0.3222	1.1033
JNN	0.00206	0.0571	0.5635	1.4634
JNN_{lags}	0.00173	0.0480	0.8781	1.7483^{*}
LSTM	0.00202	0.0559	0.8230	1.8269^{*}
NARX	0.00170	0.0472	1.4543	1.5659

5 Conclusions

This study represents the first attempt to model multivariate volatility through artificial neural networks, in order to detect nonlinear dynamics and long-term dependencies in the realized covariance series. The use of such models allows to detect possible nonlinear relationships without the a priori knowledge of the distribution of the variables at a significantly reduced computational effort. However, this approach is not costless, since with an increasing number of hidden layers and hidden nodes the number of parameters to be trained and the risk of a local minimum increase rapidly.

The main purpose of the paper is to understand whether the use of the Cholesky decomposition combined with the implementation of neural networks, specifically recurrent neural networks, is able to outperform traditional econometric method in forecasting realized covariance matrices. At this end, the out-of-sample forecasting accuracy of a set of neural networks is compared with existing econometric methods.

Several conclusions emerge from the statistical evaluation of the out-of-sample forecasts. First, contrary to the literature on univariate models, NNs show a modest gain in forecasting accuracy when applied to multivariate persistent time series. Second, long-term dependence detecting models, like NARX and LSTM, seem to provide the best forecasting performance, as underlined by the comparison through multivariate loss functions.

Moreover, the use of the Cholesky decomposition seems to improve the forecasting accuracy respect to models which are not based on a parametrization, like the VAR, the VARExo and the DCC model.

Future works could implement different parametrizations of the realized covariance matrices, like the logarithmic transformation used by Bauer and Vorkink (2011). A further analysis could rely on different input variables and different target variables, like the exchange rate volatility or the crude oil volatility. Another interesting application would be using such models with realized covariance matrices sampled at higher frequencies.

References

ANDERSEN, T. G., AND T. BOLLERSLEV (1997): "Heterogeneous information arrivals and return volatility dynamics: Uncovering the long-run in high frequency returns," *Journal of Finance*, 52, 975–1005.

ANDERSEN, T. G., T. BOLLERSLEV, F. X. DIEBOLD, AND H. EBENS (2001): "The distribution of realized stock return volatility," *Journal of Financial Economics*, 61, 43–76.

ANDERSEN, T. G., T. BOLLERSLEV, F. X. DIEBOLD, AND P. LABYS (2001): "The distribution of exchange rate volatility," *Journal of American Statistical Association*, 96, 42–55.

——— (2003): "Modeling and Forecasting Realized Volatility," *Econometrica*, 71, 579–625.

ANDERSON, H. M., K. NAM, AND F. VAHID (1999): *Asymmetric Nonlinear Smooth Transition Garch Models*pp. 191–207. Springer US.

BAO, W., H. YUE, AND Y. RAO (2017): "A deep learning framework for financial time series using stacked autoencoders and long-short term memory," *PLoS One*, 12(7).

BARNDORFF-NIELSEN, O. E., AND N. SHEPHARD (2002): "Estimating Quadratic Variation Using Realised Variance," *Journal of Applied Econometrics*, 17, 457–477.

——— (2004): "Power and Bipower Variation with Stochastic Volatility and Jumps," *Journal of Financial Econometrics*, 2, 1–37.

BAUER, G. H., AND K. VORKINK (2011): "Forecasting multivariate realized stock market volatility," *Journal of Econometrics*, 160, 93–101.

BECKER, R., A. CLEMENTS, AND R. O'NEILL (2010): "A Cholesky-MIDAS model for predicting stock portfolio volatility," Working paper, Centre for Growth and Business Cycle Research Discussion Paper Series.

BUCCI, A. (2019): "Realized Volatility Forecasting with Neural Networks," Working paper.

BUCCI, A., G. PALOMBA, AND E. ROSSI (2019): "Does macroeconomics help in predicting stock market volatility comovements? A nonlinear approach," Working paper.

CHRISTIANSEN, C., M. SCHMELING, AND A. SCHRIMPF (2012): "A comprehensive look at financial volatility prediction by economic variables," *Journal of Applied Econometrics*, 27, 956–977.

DIEBOLD, F. X., AND R. S. MARIANO (1995): "Comparing predictive accuracy," *Journal of Business and Economic Statistics*, 13, 253–263.

DONALDSON, G. R., AND M. KAMSTRA (1997): "An artificial neural network-GARCH model for international stock return volatility," *Journal of Empirical Finance*, 4, 17–46.

ELMAN, J. L. (1990): "Finding structure in time," *Cognitive Science*, 14(2), 179–211.

ENGLE, R. F., E. GHYSELS, AND B. SOHN (2009): "On the Economic Sources of Stock Market Volatility," Working paper.

FAMA, E., AND K. FRENCH (1993): "Common Risk Factors in the Returns on Stocks and Bonds," *Journal of Financial Economics*, 33, 3–56.

FERNANDES, M., M. C. MEDEIROS, AND M. SCHARTH (2014): "Modeling and predicting the CBOE market volatility index," *Journal of Banking & Finance*, 40, 1–10.

GUZMAN, S. M., J. O. PAZ, AND M. L. M. TAGERT (2017): "The Use of NARX Neural Networks to Forecast Daily Groundwater Levels," *Water Resour Manage*, 31, 1591–1603.

HALBLEIB-CHIRIAC, R., AND V. VOEV (2011): "Modelling and Forecasting Multivariate Realized Volatility," *Journal of Applied Econometrics*, 26, 922–947.

HEIDEN, M. D. (2015): "Pitfalls of the Cholesky decomposition for forecasting multivariate volatility," Working paper.

HOCHREITER, S., AND J. SCHMIDHUBER (1997): "Long short-term memory," *Neural Computation*, 9(8), 1735–1780.

HSU, D. (2017): "Time Series Forecasting Based on Augmented Long Short-Term Memory," Working paper.

HU, Y. M., AND C. TSOUKALAS (1999): "Combining conditional volatility forecasts using neural networks: an application to the EMS exchange rates," *Journal of International Financial Markets, Institutions & Money*, 9, 407–422.

JORDAN, M. I. (1986): "Serial Order: A Parallel Distributed Processing Approach," Discussion paper, Institute for Cognitive Science Report 8604, UC San Diego.

KHAN, A. I. (2011): “Financial Volatility Forecasting by Nonlinear Support Vector Machine Heterogeneous Autoregressive Model: Evidence from Nikkei 225 Stock Index,” *International Journal of Economics and Finance*, 3(4), 138–150.

KIM, H. Y., AND C. H. WON (2018): “Forecasting the volatility of stock price index: A hybrid model integrating LSTM with multiple GARCH-type models,” *Expert Systems with Applications*, 103, 25–37.

KUAN, C., AND T. LIU (1995): “Forecasting exchange rates using feedforward and recurrent neural networks,” *Journal of Applied Econometrics*, 10(4), 347–364.

KWAN, C., W. LI, AND K. NG (2005): “A Multivariate Threshold GARCH Model with Time-varying Correlations,” Working Paper.

MAHEU, J. M., AND T. H. MCCURDY (2002): “Nonlinear features of realized volatility,” *Review of Economics and Statistics*, 84, 668–681.

MARTENS, M., M. DE POOTER, AND D. J. VAN DIJK (2004): “Modeling and Forecasting S&P 500 Volatility: Long Memory, Structural Breaks and Nonlinearity,” Tinbergen Institute Discussion Paper No. 04-067/4.

MELE, A. (2007): “Asymmetric stock market volatility and the cyclical behavior of expected returns,” *Journal of Financial Economics*, 86, 446–478.

PATTON, A. J. (2011): “Volatility forecast comparison using imperfect volatility proxies,” *Journal of Econometrics*, 160(1), 246 – 256.

PAYE, B. S. (2012): “Dèjà vol: Predictive regressions for aggregate stock market volatility using macroeconomic variables,” *Journal of Financial Economics*, 106, 527–546.

SCHWERT, W. G. (1989): “Why does stock market volatility change over time,” *Journal of Finance*, 44, 1115–1153.

TANG, Z., AND P. A. FISHWICK (1993): “Feed-forward Neural Nets as Models for Time Series Forecasting,” *ORSA Journal on Computing*, 5, 374–385.

TINO, P., C. SCHITTENKOPF, AND G. DORFFNER (2001): “Financial volatility trading using recurrent neural networks,” *IEEE Transactions on Neural Networks*, 12(4), 865–874.

VORTELINOS, D. I. (2017): "Forecasting realized volatility: HAR against Principal Components Combining, neural networks and GARCH," *Research in International Business and Finance*, 39, 824–839.

WELCH, I., AND A. GOYAL (2008): "A comprehensive look at the empirical performance of equity premium prediction," *Review of Financial Studies*, 21(4), 1455–1508.

WHITE, H. (1988): *Economic prediction using neural networks: the case of IBM daily stock returns*. IEEE 1988 International Conference on Neural Networks.

XIANG, C., S. DING, AND T. H. LEE (2005): "Geometrical interpretation and architecture selection of MLP," *IEEE Transactions on Neural Networks*, 16(1), 84–96.

XIONG, R., E. P. NICHOLS, AND Y. SHEN (2016): "Deep Learning Stock Volatility with Google Domestic Trends," *CoRR*, abs/1512.04916.