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Lahore School of Economics

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Article

Hesitant Probabilistic Fuzzy Linguistic Sets with Applications in Multi-Criteria Group Decision Making Problems

Dheeraj Kumar Joshi ¹, Ismat Beg ^{2,*} and Sanjay Kumar ¹

¹ Department of Mathematics, Statistics and Computer Science, G. B. Pant University of Agriculture and Technology, Pantnagar, Uttarakhand 263145, India; maths.dj44010@gmail.com (D.K.J.); skruhela@hotmail.com (S.K.)

² Centre for Mathematics and Statistical Sciences, Lahore School of Economics, Lahore 53200, Pakistan

* Correspondence: ibeg@lahoreschool.edu.pk

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Abstract: Uncertainties due to randomness and fuzziness comprehensively exist in control and decision support systems. In the present study, we introduce notion of occurring probability of possible values into hesitant fuzzy linguistic element (HFLE) and define hesitant probabilistic fuzzy linguistic set (HPFLS) for ill structured and complex decision making problem. HPFLS provides a single framework where both stochastic and non-stochastic uncertainties can be efficiently handled along with hesitation. We have also proposed expected mean, variance, score and accuracy function and basic operations for HPFLS. Weighted and ordered weighted aggregation operators for HPFLS are also defined in the present study for its applications in multi-criteria group decision making (MCGDM) problems. We propose a MCGDM method with HPFL information which is illustrated by an example. A real case study is also taken in the present study to rank State Bank of India, InfoTech Enterprises, I.T.C., H.D.F.C. Bank, Tata Steel, Tata Motors and Bajaj Finance using real data. Proposed HPFLS-based MCGDM method is also compared with two HFL-based decision making methods.

Keywords: hesitant fuzzy set; hesitant probabilistic fuzzy linguistic set; score and accuracy function; multi-criteria group decision making; aggregation operator

1. Introduction

Uncertainties in decision making problems are due to either randomness or fuzziness, or by both and can be classified into stochastic and non-stochastic uncertainty [1]. Stochastic uncertainties in every system may be well captured by the probabilistic modeling [2,3]. Although several theories have been proposed in the literature to deal with non-stochastic uncertainties but among them fuzzy set theory [4,5] is extensively researched and successfully applied in decision making [6–10]. An extensive literature is due to Mardani et al. [11] on the various fuzzy aggregation operators proposed in last thirty years. Type-2 fuzzy sets [5], interval-valued fuzzy set (IVFS) [4], intuitionistic fuzzy sets (IFS) [12] and interval-valued intuitionistic fuzzy sets (IVIFS) [13], Pythagorean fuzzy set [14] and neutrosophic sets [15] are few other extensions of fuzzy sets practiced in MCGDM problems to include non-stochastic uncertainty and hesitation.

Often decision makers (DMs) in multi-criteria group decision making (MCGDM) problems are not in favor of the same assessment on decision criteria and provide different assessment information on each criterion. Difficulty of agreeing on a common assessment is not because of margin of error or some possible distribution as in case of IFS and type-2 fuzzy sets. To address this issue in MCGDM problems Torra and Narukawa [16] and Torra [17] introduced hesitant fuzzy set

(HFS) and applied in MCGDM problems [18,19]. Various extensions of HFS e.g., triangular hesitant fuzzy set (THFS), generalized hesitant fuzzy set (GHFS), interval valued hesitant fuzzy set (IVHFS), dual hesitant fuzzy set (DHFS), interval valued intuitionistic hesitant fuzzy set (IVIHF) and hesitant pythagorean fuzzy set were used in decision making problems [20–28] considering decision hesitancy and prioritization among decision criteria and developed a fuzzy group decision making method to evaluate complex emergency response in sustainable development. Recently, Garg and Arora [29] proposed distance and similarity measures-based MCDM method using dual hesitant fuzzy soft set.

Qualitative and quantitative analysis of decision criteria with hesitant and uncertain information has always been an important issue for researchers in MCGDM problems. Limited knowledge of decision makers (DMs), nature of considered alternatives and unpredictability of events are main constraints in getting sufficient and accurate information about the decision preferences and decision criteria. Many criteria which are difficult to be analyzed quantitatively can be analyzed using linguistic variables [5]. Linguistic variables improve consistency and flexibility of traditional decision making methods [30] and hence many researchers [31–45] have proposed use of linguistic variable in decision making problems. Kobina et al. [46] proposed few probabilistic linguistic aggregation operators for decision making problem. Garg and Kumar [47], Liu et al. [48] and Garg [49] proposed various aggregation operators, prioritized aggregation operators for linguistic IFS and linguistic neutrosophic set and applied them to MCGDM problems. Lin et al. [50] integrated linguistic sets with HFS to define hesitant fuzzy linguistic set (HFLS) to include hesitancy and inconsistencies among DMs in assessment of an alternative with respect to a certain criterion. Ren et al. [51] and Joshi & Kumar [52] proposed TOPSIS method MCGDM using hesitant fuzzy linguistic and IVIHFL information. Recently few researchers [53–55] have proposed generalized single-valued neutrosophic hesitant fuzzy prioritized aggregation operators and linguistic distribution-based decision making methods using hesitant fuzzy linguistic assessment for decision making methods.

Probabilistic and fuzzy approach-based MCGDM method process only either stochastic or non stochastic uncertainty. One of their major limitations is not to handle both types of uncertainties simultaneously. Comprehensive concurrence of stochastic and non stochastic uncertainty in real life problems attracted researchers to incorporate probability theory with fuzzy logic. Idea of integrating fuzzy set theory with probabilistic theory was initiated by Liang and Song [56] and Meghdadi and Akbarzadeh [1]. In 2005, Liu and Li [57] defined probabilistic fuzzy set (PFS) to handle both stochastic and non stochastic uncertainties in a single framework. To handle simultaneous occurrence of both stochastic and non stochastic uncertainties with hesitation, Xu and Zhou [58] introduced probabilistic hesitant fuzzy set (PHFS). PHFS permits more than one membership degree of an element with different probabilities. Recently many applications of PHFS are found in MCGDM problems [58–65].

Earlier in all HFL-based decision making methods, probabilities of occurrence of elements are assumed to be equal. Assumption of equal probabilities in HFL is too hard to be followed by DMs in real life problems of decision making due to their hesitation. For example, a decision maker provides hesitant fuzzy linguistic element (HFLE) $\{s_2, \langle 0.4, 0.5, 0.6 \rangle\}$ to evaluate the safety level of a vehicle. He or she thinks that the safety level associated with 0.6 and 0.4 are the most and least suitable. However, he or she contradicts with own decision by associating equal probability to each 0.4, 0.5, 0.6. Hence, HFLE $\{s_2, \langle 0.4, 0.5, 0.6 \rangle\}$ with equal probabilities cannot represent DM's accurate assessment of decision criteria. With this limitation in present form of HFLS, we introduce notion of hesitant probabilistic fuzzy linguistic set (HPFLS). This new class of set undertakes both uncertainties caused by randomness and fuzziness in the environment of hesitation in a single framework.

In the present study, we have proposed HPFLS with expected mean, variance, score and accuracy function and a few operations on its elements. We also develop novel hesitant probabilistic fuzzy linguistic weighted averaging (HPFLWA), hesitant probabilistic fuzzy linguistic weighted geometric (HPFLWG), hesitant probabilistic fuzzy linguistic ordered weighted averaging

(HPFLOWA) and hesitant probabilistic fuzzy linguistic ordered weighted geometric (HPFLOWG) aggregation operators to aggregate the HPFL information. A MCGDM method with HPFL information is proposed. Methodology of proposed MCGDM method is illustrated by a numerical example and also applied on a real case study to rank the organizations.

2. Preliminaries

In this section, we briefly review fundamental concepts and definitions of hesitant fuzzy set, linguistic variables, hesitant fuzzy linguistic set and hesitant probabilistic fuzzy set.

Definition 1. ([16,17]) Let X be a reference set. An HFS A on X is defined using a function $I_A(X)$ that returns a subset of $[0, 1]$. Mathematically, it is symbolized using following expression:

$$A = \{ \langle x, I_A(x) \rangle \mid x \in X \} \tag{1}$$

where $I_A(X)$ is hesitant fuzzy element (HFE) having a set of different values lies between $[0, 1]$.

Definition 2. ([32]) Let $S = \{s_i \mid i = 1, 2, \dots, t\}$ be a finite discrete LTS. Here s_i represents a possible value for a linguistic variable and satisfies the following characteristics:

1. The set is ordered: $s_i > s_j$ if $i > j$
2. $\text{Max} \{s_i, s_j\} = s_i$ if $i \geq j$
3. $\text{Min} \{s_i, s_j\} = s_j$ if $i \geq j$

Xu [66] extended finite discrete LTS $S = \{s_i \mid i = 1, 2, \dots, t\}$ to continuous LTS $S = \{s_\theta \mid s_0 \leq s_\theta \leq s_t, \theta \in [0, t]\}$ to conserve all the provided information. An LTS is original if $s_\theta \in S$, otherwise it is called virtual.

Definition 3. ([50]) Let X be the reference set and $s_\theta \in S$. A hesitant fuzzy linguistic set A in X is a mathematical object of following form:

$$A = \{ \langle x, s_\theta(x), h_A(x) \rangle \mid x \in X \} \tag{2}$$

Here $h_A(x)$ is a set of possible finite number of values belonging to $[0,1]$ and denotes the possible membership degrees that x belongs to $s_\theta(x)$.

Definition 4. ([58]) Let X be the reference set. An HPFS H_f on X is a mathematical object of following form:

$$H_f = \{ \langle \gamma_i, h(\gamma_i \mid p_i) \rangle \mid \gamma_i \in X \} \tag{3}$$

Here $h(\gamma_i \mid p_i)$ is set of elements $\gamma_i \mid p_i$ expressing the hesitant fuzzy information with probabilities to the set $H_f, 0 \leq \gamma_i \leq 1$ ($i = 1, 2, \dots, \#h$) (number of possible elements in $h(\gamma_i \mid p_i)$), $p_i \in [0,1]$ are corresponding probabilities with condition $\sum_{i=1}^{\#h} p_i = 1$ s.

3. Hesitant Probabilistic Fuzzy Linguistic Set (HPFLS) and Hesitant Probabilistic Fuzzy Linguistic Element (HPFLE)

Qualitative and quantitative analysis of decision criteria with hesitant is always been an important issue for researchers in MCGDM problems. Earlier classification of fuzzy sets (hesitant fuzzy set [16,17], hesitant fuzzy linguistic set [50] and probabilistic hesitant fuzzy [58]) are not capable to deal with fuzziness, hesitancy and uncertainty both qualitatively and quantitatively. Keeping in mind the limitations of HFLEs and to fully describe precious information provided by DMs; our aim is to propose a new class of set called HPFLS. This set can easily describe stochastic

and non-stochastic uncertainties with hesitant information using both qualitative and quantitative terms. In this section, we also develop expected mean, variance, score and accuracy function of HPFLEs, along with a comparison method. Some basic operations of HPFLEs are also defined in this section.

Definition 5. Let X and S be the reference set and linguistic term set. An HPFLS H_{PL} on X is a mathematical object of following form:

$$H_{PL} = \{ \langle x, h_{PL}(p_x) \rangle \mid x \in X \} \tag{4}$$

Here $h_{PL}(p_x) = s_\theta(x \mid p_k)$, $h(\gamma_i \mid p_i) \mid \gamma_i, p_i$, $s_\theta(x) \in S$ and $h_{PL}(p_x)$ is set of some elements x denoting the hesitant fuzzy linguistic information with probabilities to the set H_{PL} , $0 \leq \gamma_i \leq 1$, $i = 1, 2, \dots, \# h_{PL}$. Here $\# h_{PL}$ is the number of possible elements in $h_{PL}(p_x)$, $p_i \in [0, 1]$ is the hesitant probability of γ_i and $\sum_{i=1}^{\# h_{PL}} p_i = 1$. We call $h_{PL}(p_x)$ HPFLE and H_{PL} is set of all HPFLEs.

As an illustration of Definition 5, we assume two HPFLEs $h_{PL}(p_x) = [(s_1 \mid 1), \{0.2 \mid 0.3, 0.4 \mid 0.2, 0.5 \mid 0.5\}]$, $h_{PL}(p_y) = [(s_3 \mid 1), \{0.4 \mid 0.2, 0.5 \mid 0.4, 0.2 \mid 0.4\}]$ on reference set $X = \{x, y\}$.

An object $H_{PL} = [\langle x, (s_1 \mid 1), \{0.2 \mid 0.3, 0.4 \mid 0.2, 0.5 \mid 0.5\} \rangle, \langle y, (s_3 \mid 1), \{0.4 \mid 0.2, 0.5 \mid 0.4, 0.2 \mid 0.4\} \rangle]$ represents an HPFLS.

It is important to note that if the probabilities of the possible values in HPFLEs are equal, i.e., $p_1 = p_2 = \dots = p_{\#h}$, then HPFLE reduced to HFLE.

3.1. Some Basic Operations on Hesitant Probabilistic Fuzzy Linguistic Element (HPFLEs)

Based on operational rules of hesitant fuzzy linguistic set [50] and hesitant probabilistic set [61], we propose following operational laws for $h_{PL}^1(p_x) = \langle s_\theta((x) \mid p), h(\gamma_j \mid p_j) \mid \gamma_j, p_j \rangle$ and $h_{PL}^2(p_y) = \langle s_\theta((y) \mid p), h(\gamma_k \mid p_k) \mid \gamma_k, p_k \rangle$ then

- (1) $(h_{PL}^1)^\lambda = \langle s_{\theta^\lambda}((x) \mid p), h(\gamma_j^\lambda \mid p_j) \mid \gamma_j, p_j \rangle$ for some $\lambda > 0$
- (2) $\lambda(h_{PL}^1) = \langle s_{\lambda\theta}((x) \mid p), h(1 - (1 - \gamma_j)^\lambda \mid p_j) \mid \gamma_j, p_j \rangle$ for some $\lambda > 0$
- (3) $h_{PL}^1 \oplus h_{PL}^2 = \langle s_{\theta((x) \oplus \theta((y)))} \mid p, h((\gamma_j + \gamma_k - \gamma_j \gamma_k) \mid p_j p_k) \rangle$
- (4) $h_{PL}^1 \otimes h_{PL}^2 = \langle s_{\theta((x) \otimes \theta((y)))} \mid p, h((\gamma_j \gamma_k) \mid p_j p_k) \rangle$
- (5) $h_{PL}^1 \cup h_{PL}^2 = \langle s_{\theta((x) \cup \theta((y)))} \mid p, h((\gamma_j \vee \gamma_k) \mid (p_j \vee p_k) \sum (Max(p_j, p_k))) \rangle$
- (6) $h_{PL}^1 \cap h_{PL}^2 = \langle s_{\theta((x) \cap \theta((y)))} \mid p, h((\gamma_j \wedge \gamma_k) \mid (p_j \wedge p_k) \sum (Min(p_j, p_k))) \rangle$

Using definition of \oplus and \otimes , it can be easily proved that $h_{PL}^1 \oplus h_{PL}^2$ and $h_{PL}^1 \otimes h_{PL}^2$ are commutative. In order to show that $(h_{PL}^1)^\lambda$, $\lambda(h_{PL}^1)$, $h_{PL}^1 \oplus h_{PL}^2$, $h_{PL}^1 \otimes h_{PL}^2$, $h_{PL}^1 \cup h_{PL}^2$ and $h_{PL}^1 \cap h_{PL}^2$ are again HPFLE, we assume that $h_{PL}^1(p_x) = [(s_2 \mid 1), \{0.2 \mid 0.3, 0.4 \mid 0.2, 0.5 \mid 0.5\}]$ and $h_{PL}^2(p_y) = [(s_3 \mid 1), \{0.4 \mid 0.2, 0.5 \mid 0.4, 0.2 \mid 0.4\}]$ are two HPFLEs on reference set $X = \{x, y\}$ and perform the operation laws as follows:

$$\begin{aligned} (h_{PL}^1)^\lambda &= [(s_4 \mid 1), \{0.04 \mid 0.3, 0.16 \mid 0.2, 0.25 \mid 0.5\}] \\ \lambda(h_{PL}^1) &= [(s_4 \mid 1), \{0.36 \mid 0.3, 0.64 \mid 0.2, 0.75 \mid 0.5\}] \\ h_{PL}^1 \oplus h_{PL}^2 &= [(s_5 \mid 1), \{0.54 \mid 0.06, 0.58 \mid 0.12, 0.28 \mid 0.12, 0.76 \mid 0.04, 0.82 \mid 0.08, 0.52 \mid 0.08, 0.8 \mid 0.1, 0.8 \mid 0.2, 0.5 \mid 0.2\}] \end{aligned}$$

$$\begin{aligned}
 h_{PL}^1 \otimes h_{PL}^2 &= [(s_5 | 1), \{0.08 | 0.06, 0.1 | 0.12, 0.04 | 0.12, 0.16 | 0.04, 0.2 | 0.08, 0.08 | 0.08, 0.2 | 0.1, 0.25 | 0.2, 0.1 | 0.2\}] \\
 h_{PL}^1 \cup h_{PL}^2 &= [(s_5 | 1), \{0.4 | 0.08, 0.5 | 0.11, 0.2 | 0.11, 0.4 | 0.06, 0.5 | 0.11, 0.4 | 0.11, 0.5 | 0.14, 0.5 | 0.14, 0.5 | 0.14\}] \\
 h_{PL}^1 \cap h_{PL}^2 &= [(s_5 | 1), \{0.2 | 0.08, 0.2 | 0.13, 0.2 | 0.13, 0.4 | 0.08, 0.4 | 0.08, 0.2 | 0.08, 0.4 | 0.08, 0.5 | 0.17, 0.2 | 0.17\}]
 \end{aligned}$$

3.2. Score and Accuracy Function for Hesitant Probabilistic Fuzzy Linguistic Element (HPFLE)

Comparison is an indispensable and is required if we tend to apply HPFLE in decision making and optimization problems. Hence, we define expected mean, variance, score and accuracy function of HPFLE in this sub section as follows:

Definition 6. Expected mean $E(h_{PL}(p_x))$ and variance $V(h_{PL}(p_x))$ for a HPFLE $h_{PL}(p_x) = \langle s_\theta((x) | p_k), h(\gamma_i | p_i) | \gamma_i, p_i \rangle$ are defined as follows:

$$E(h_{PL}(p_x)) = \frac{\sum_{i=1}^{\#h} (\gamma_i p_i)}{\#h} \tag{5}$$

$$V(h_{PL}(p_x)) = \sum_{i=1}^{\#h} (\gamma_i - E(h_{PL}(p_x)))^2 p_i \tag{6}$$

Definition 7. Score function $S(h_{PL}(p_x))$ and accuracy function $A(h_{PL}(p_x))$ for a HPFLE $h_{PL}(p_x) = \langle s_\theta((x) | p_k), h(\gamma_i | p_i) | \gamma_i, p_i \rangle$ are defined as follows:

$$S(h_{PL}(p_x)) = E(h_{PL}(p_x)) (s_\theta(x)(p_k)) \tag{7}$$

$$A(h_{PL}(p_x)) = V(h_{PL}(p_x)) (s_\theta(x)(p_k)) \tag{8}$$

Using score and accuracy functions two HPFLEs $h_{PL}(p_x), h_{PL}(p_y)$ can be compared as follows:

- (1) If $S(h_{PL}(p_x)) > S(h_{PL}(p_y))$, then $h_{PL}(p_x) > h_{PL}(p_y)$
- (2) If $S(h_{PL}(p_x)) < S(h_{PL}(p_y))$, then $h_{PL}(p_x) < h_{PL}(p_y)$
- (3) If $S(h_{PL}(p_x)) = S(h_{PL}(p_y))$,
 - (a) If $A(h_{PL}(p_x)) > A(h_{PL}(p_y))$, then $h_{PL}(p_x) > h_{PL}(p_y)$
 - (b) If $A(h_{PL}(p_x)) < A(h_{PL}(p_y))$, then $h_{PL}(p_x) < h_{PL}(p_y)$
 - (c) If $A(h_{PL}(p_x)) = A(h_{PL}(p_y))$, then $h_{PL}(p_x) = h_{PL}(p_y)$

As an illustration of Definitions 6 and 7, we compare two HPFLEs $h_{PL}(p_x) = [(s_2 | 1), \{0.2 | 0.3, 0.4 | 0.2, 0.5 | 0.5\}]$ and $h_{PL}(p_y) = [(s_3 | 1), \{0.4 | 0.2, 0.5 | 0.4, 0.2 | 0.4\}]$ using score and accuracy functions as follows:

$$\begin{aligned}
 E(h_{PL}(p_x)) &= (0.2 * 0.3 + 0.4 * 0.2 + 0.5 * 0.5) / 3 = 0.13 \\
 E(h_{PL}(p_y)) &= (0.4 * 0.2 + 0.5 * 0.4 + 0.2 * 0.4) / 3 = 0.12 \\
 V(h_{PL}(p_x)) &= \\
 &(((0.2 - 0.13)^2) * 0.3 + ((0.4 - 0.13)^2) * 0.2 + ((0.5 - 0.13)^2) * 0.5) / 3 = 0.0279 \\
 V(h_{PL}(p_y)) &= \\
 &(((0.4 - 0.12)^2) * 0.2 + ((0.5 - 0.12)^2) * 0.4 + ((0.2 - 0.12)^2) * 0.4) / 3 = 0.0252 \\
 S(h_{PL}(p_x)) &= S(2 * 1) * ((0.2 * 0.3 + 0.4 * 0.2 + 0.5 * 0.5) / 3) = S_{0.26}
 \end{aligned}$$

$$\begin{aligned}
 S(h_{PL}(p_y)) &= S_{(3*1) \vee 0.4*0.2+0.5*0.4+0.2*0.4 \vee 3} = S_{0.36} \\
 A(h_{PL}(p_x)) &= S_{(2*1)((0.2-0.13)^2)*0.3+((0.4-0.13)^2)*0.2+((0.5-0.13)^2)*0.5)/3} = S_{0.0558} \\
 A(h_{PL}(p_y)) &= S_{(3*1)((0.4-0.12)^2)*0.2+((0.5-0.12)^2)*0.4+(0.2-0.12)^2*0.4)/3} = S_{0.0756}
 \end{aligned}$$

Since $S(h_{PL}(p_y)) > S(h_{PL}(p_x))$ therefore $h_{PL}(p_y) < h_{PL}(p_x)$.

Different HPFLEs may have different number of PFNs. To make them equal in numbers we extend HPFLEs until they have same number of PFNs. It can be extended according to DMs risk behavior.

4. Aggregation Operators for Hesitant Probabilistic Fuzzy Linguistic Set (HPFLS)

In group decision making problems, an imperative task is to aggregate the assessment information obtained from DMs about alternatives against each criterion. Various aggregation operators for HFSL [18,50] and HPFS [58–63,67] have been developed in the past few decades. As we propose HPFLS for MCGDM problems, we also develop few aggregations operators to aggregate information in the form of HPFLEs. In this section, we define HPFLW and HPFLOW operators.

4.1. Hesitant Probabilistic Linguistic Fuzzy Weighted Aggregation Operators

Let $H_{PL}^i = h_{PL}(p_x) = \langle s_{\theta_i}((x) | p), h(\gamma_i | p_i) | \gamma_i, p_i, (i = 1, 2, \dots, n) \rangle$ be collection of HPFLEs. Hesitant probabilistic fuzzy linguistic weighted averaging (HPFLWA) operator and hesitant probabilistic fuzzy linguistic weighted geometric (HPFLWG) operator are defined as follows:

Definition 8. HPFLWA is a mapping $H_{PL}^n \rightarrow H_{PL}$ such that

$$\begin{aligned}
 \text{HPFLWA } (H_1, H_2, \dots, H_n) &= \bigoplus_{i=1}^n (\omega_i H_i) \\
 &= \left\langle \left[\sum_{i=1}^n \omega_i s_{\theta_i}((x) | p) \right], \left[\bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, \dots, \gamma_n \in H_n, \left\{ 1 - \prod_{i=1}^n (1 - \gamma_i)^{\omega_i} \mid p_1 p_2 \dots p_n \right\}} \right] \right\rangle \tag{9}
 \end{aligned}$$

Definition 9. HPFLWG operator is a mapping $H_{PL}^n \rightarrow H_{PL}$ such that

$$\begin{aligned}
 \text{HPFLWG } (H_1, H_2, \dots, H_n) &= \bigotimes_{i=1}^n (H_i)^{\omega_i} \\
 &= \left\langle \left[\sum_{i=1}^n s_{\theta_i}^{\omega_i}((x) | p) \right], \left[\bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, \dots, \gamma_n \in H_n, \left\{ \prod_{i=1}^n (\gamma_i)^{\omega_i} \mid p_1 p_2 \dots p_n \right\}} \right] \right\rangle \tag{10}
 \end{aligned}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is weight vector of $H_i (i = 1, 2, \dots, n)$ with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, p_n is

the probability of γ_i in the HPFLEs $H_i (i = 1, 2, \dots, n)$. In particular if $\omega = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n} \right)^T$ then HPFLWA and HPFLWG operator are reduced to following hesitant probabilistic fuzzy linguistic averaging (HPFLA) operator and hesitant probabilistic fuzzy linguistic geometric (HPFLG) operator respectively:

$$\text{HPFLA } (H_1, H_2, \dots, H_n) = \bigoplus_{i=1}^n \left(\frac{1}{n} H_i \right) \tag{11}$$

$$\begin{aligned}
 &= \left\langle \left[\sum_{i=1}^n \frac{1}{n} s_{\theta_i}((x) | p) \right], \left[\bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, \dots, \gamma_n \in H_n} \left\{ 1 - \prod_{i=1}^n (1 - \gamma_i)^{\frac{1}{n}} | p_1 p_2 \dots p_n \right\} \right] \right\rangle \\
 &\quad \text{HPFLG} (H_1, H_2, \dots, H_n) = \bigotimes_{i=1}^n (H_i)^{\frac{1}{n}} \\
 &= \left\langle \left[\sum_{i=1}^n s_{\theta_i}^{\frac{1}{n}}((x) | p) \right], \left[\bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, \dots, \gamma_n \in H_n} \left\{ \prod_{i=1}^n (\gamma_i)^{\frac{1}{n}} | p_1 p_2 \dots p_n \right\} \right] \right\rangle \tag{12}
 \end{aligned}$$

Lemma 1. ([17] Let $\alpha_i > 0, \omega_i > 0, i = 1, 2, \dots, n$ and $\sum_{i=1}^n \omega_i = 1$, then $\prod_{i=1}^n \alpha_i^{\omega_i} \leq \sum_{i=1}^n \alpha_i \omega_i$ and equality holds if and only if $\alpha_1 = \alpha_2 = \dots = \alpha_n$.

Theorem 1. Let $H_i = h_{pL}(p_x) = \{s_{\theta_i}((x) | p), h(\gamma_i | p_i) | \gamma_i, p_i > \} (i = 1, 2, \dots, n)$ be collection of HPFLEs. Let $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ be weight vector of $H_i (i = 1, 2, \dots, n)$ with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, then

$$\begin{aligned}
 \text{HPFLWG} (H_1, H_2, \dots, H_n) &\leq \text{HPFLWA}(H_1, H_2, \dots, H_n) \\
 \text{HPFLG} (H_1, H_2, \dots, H_n) &\leq \text{HPFLA}(H_1, H_2, \dots, H_n)
 \end{aligned}$$

Proof. Using Lemma 1, we have following inequality for any $\gamma_i \in H_i (i = 1, 2, \dots, n)$,

$$\prod_{i=1}^n (\gamma_i)^{\omega_i} \leq \sum_{i=1}^n \gamma_i \omega_i = 1 - \sum_{i=1}^n (1 - \gamma_i) \omega_i \leq \prod_{i=1}^n (1 - \gamma_i)^{\omega_i}$$

Thus, we can obtain the following inequality:

$$\begin{aligned}
 &\left\langle \left[\bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, \dots, \gamma_n \in H_n} \left\{ \prod_{i=1}^n (\gamma_i)^{\omega_i} | p_1 p_2 \dots p_n \right\} \right] \right\rangle \leq \left\langle \left[\bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, \dots, \gamma_n \in H_n} \left\{ 1 - \prod_{i=1}^n (1 - \gamma_i)^{\omega_i} | p_1 p_2 \dots p_n \right\} \right] \right\rangle \\
 &\left\langle \left[\sum_{i=1}^n s_{\theta_i}^{\omega_i}((x) | p) \right], \left[\bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, \dots, \gamma_n \in H_n} \left\{ \prod_{i=1}^n (\gamma_i)^{\omega_i} | p_1 p_2 \dots p_n \right\} \right] \right\rangle \leq \left\langle \left[\sum_{i=1}^n \omega_i s_{\theta_i}((x) | p) \right], \left[\bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2, \dots, \gamma_n \in H_n} \left\{ 1 - \prod_{i=1}^n (1 - \gamma_i)^{\omega_i} | p_1 p_2 \dots p_n \right\} \right] \right\rangle \\
 &\quad \sum_{i=1}^{\#h} (\gamma_i p_i)
 \end{aligned}$$

Using definition of score function $S(h_{pL}(p_x)) = \frac{i-1}{\#h} (s_{\theta}(x)(p_k))$, we have $\text{HPFLWG} (H_1, H_2, \dots, H_n) \leq \text{HPFLWA}(H_1, H_2, \dots, H_n)$. Similarly, it can be proved that $\text{HPFLG} (H_1, H_2, \dots, H_n) \leq \text{HPFLA}(H_1, H_2, \dots, H_n)$. □

4.2. Hesitant Probabilistic Fuzzy Linguistic Ordered Weighted Aggregation Operators

Xu and Zhou [58] defined ordered weighted averaging and geometric aggregation operators to aggregate hesitant probabilistic fuzzy information for MCGDM problems. In this sub section we propose hesitant probabilistic fuzzy linguistic ordered weighted averaging (HPFLOWA) operator and hesitant probabilistic fuzzy linguistic ordered weighted geometric (HPFLOWG) operators.

Let $H_{PL}^i = h_{PL}(p_x) = \{s_{\theta_i}((x) | p), h(\gamma_i | p_i) | \gamma_i, p_i > \} (i = 1, 2, \dots, n)$ be collection of HPFLEs, $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is weight vector of with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. Let p_i is the probability of γ_i in the HPFLEs $H_i (i = 1, 2, \dots, n)$, $\gamma_{\sigma(i)}$ ith be the largest of H_i , $p_{\sigma(i)}$ is the probability of $\gamma_{\sigma(i)}$, and $\omega_{\sigma(i)}$ be the largest of ω . We develop the following two ordered weighted aggregation operators:

Definition 10. HPFLOWA operator is a mapping $H^n \rightarrow H_{PL}$ such that

$$\begin{aligned} \text{HPFLOWA} (H_1, H_2, \dots, H_n) &= \bigoplus_{i=1}^n (\omega_i H_{\sigma(i)}) \\ &= \left\langle \left[\sum_{i=1}^n \omega_i s_{\theta_{\sigma(i)}}((x) | p) \right], \left[\bigcup_{\gamma_{\sigma(1)} \in H_{\sigma(1)}, \gamma_{\sigma(2)} \in H_{\sigma(2)}, \dots, \gamma_{\sigma(n)} \in H_{\sigma(n)}} \left\{ 1 - \prod_{i=1}^n (1 - \gamma_{\sigma(i)})^{\omega_i} | p_{\sigma(1)} p_{\sigma(2)} \dots p_{\sigma(n)} \right\} \right] \right\rangle \end{aligned} \tag{13}$$

Definition 11. HPFLOWG operator is a mapping $H^n \rightarrow H_{PL}$ such that

$$\begin{aligned} \text{HPFLOWG} (H_1, H_2, \dots, H_n) &= \bigoplus_{i=1}^n (H_{\sigma(i)})^{\omega_i} \\ &= \left\langle \left[\sum_{i=1}^n s_{\theta_{\sigma(i)}}^{\omega_i}((x) | p) \right], \left[\bigcup_{\gamma_{\sigma(1)} \in H_{\sigma(1)}, \gamma_{\sigma(2)} \in H_{\sigma(2)}, \dots, \gamma_{\sigma(n)} \in H_{\sigma(n)}} \left\{ \prod_{i=1}^n (\gamma_{\sigma(i)})^{\omega_i} | p_{\sigma(1)} p_{\sigma(2)} \dots p_{\sigma(n)} \right\} \right] \right\rangle \end{aligned} \tag{14}$$

Similar to Theorem 1, the above ordered weighted operators have the relationship below:

$$\text{HPFLOWG} (H_1, H_2, \dots, H_n) \leq \text{HPFLOWA}(H_1, H_2, \dots, H_n)$$

4.3. Properties of Proposed Weighted and Ordered Weighted Aggregation Operators

Following are few properties of proposed weighted and ordered weighted aggregation operators that immediately follow from their definitions.

Property 1. (Monotonicity). Let (H_1, H_2, \dots, H_n) and $(H'_1, H'_2, \dots, H'_n)$ be two collections of HPFLNs, if $H'_i \leq H_i$ for all $i = 1, 2, \dots, n$, then

$$\begin{aligned} \text{HPFLWA} (H'_1, H'_2, \dots, H'_n) &\leq \text{HPFLWA}(H_1, H_2, \dots, H_n) \\ \text{HPFLWG} (H'_1, H'_2, \dots, H'_n) &\leq \text{HPFLWG}(H_1, H_2, \dots, H_n) \\ \text{HPFLOWA} (H'_1, H'_2, \dots, H'_n) &\leq \text{HPFLOWA}(H_1, H_2, \dots, H_n) \\ \text{HPFLOWG} (H'_1, H'_2, \dots, H'_n) &\leq \text{HPFLOWG}(H_1, H_2, \dots, H_n) \end{aligned}$$

Property 2. (Idempotency). Let $H_i = H'_i, (i = 1, 2, \dots, n)$, then

$$\text{HPFLWA}(H_1, H_2, \dots, H_n) = \text{HPFLWG}(H_1, H_2, \dots, H_n) = \text{HPFLOWA}(H_1, H_2, \dots, H_n) = \text{HPFLOWG}(H_1, H_2, \dots, H_n) = H$$

Property 3. (Boundedness). All aggregation operators lie between the max and min operators:

$$\begin{aligned} \min (H_1, H_2, \dots, H_n) &\leq \text{HPFLWA}(H_1, H_2, \dots, H_n) \leq \max (H_1, H_2, \dots, H_n) \\ \min (H_1, H_2, \dots, H_n) &\leq \text{HPFLWG}(H_1, H_2, \dots, H_n) \leq \max (H_1, H_2, \dots, H_n) \\ \min (H_1, H_2, \dots, H_n) &\leq \text{HPFLOWA}(H_1, H_2, \dots, H_n) \leq \max (H_1, H_2, \dots, H_n) \\ \min (H_1, H_2, \dots, H_n) &\leq \text{HPFLOWG}(H_1, H_2, \dots, H_n) \leq \max (H_1, H_2, \dots, H_n) \end{aligned}$$

5. Application of Hesitant Probabilistic Fuzzy Linguistic Set to Multi-Criteria Group Decision Making (MCGDM)

In this section, we propose a MCGDM method with hesitant probabilistic fuzzy linguistic information. Let $\{A_1, A_2, \dots, A_m\}$ be set of alternatives to be ranked by a group of DMs $\{D_1, D_2, \dots, D_k\}$ against criteria $\{C_1, C_2, \dots, C_n\}$. $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of criteria with the condition $0 \leq w_j \leq 1$ and $\sum_{j=1}^n w_j = 1$. $\tilde{H}^k = (H_{ij}^k)_{m \times n}$ is HPFL decision matrix where $H_{ij}^k = \{s_{\theta}((x) | p), h(\gamma_t | p_t) | \gamma_t, p_t\} (t = 1, 2, \dots, T)$ denotes HPFLE when alternative A_i is evaluated by k th DM under the criteria C_j . If two or more decision makers provide the same value, then the value comes only once in decision matrix. Algorithm of proposed HPFLS-based MCGDM method includes following steps:

Step 1: Construct HPFL decision matrices $\tilde{H}^k = (H_{ij}^k)_{m \times n}$ ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$), according to the preferences information provided by the DMs about the alternative A_i under the criteria C_j denoted by HPFLE $H = h_{PL}(p_x) = \{s_{\theta}((x) | p), h(\gamma_t | p_t) | \gamma_t, p_t\} (t = 1, 2, \dots, T)$.

Step 2: Use the proposed aggregation operators (HPFLWA and HPFLWG) given in Section 3, to aggregate individual hesitant probabilistic fuzzy linguistic decision matrix information provided by each decision maker into a single HPFL decision matrix $\tilde{H} = (H_{ij})_{m \times n}$ ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$).

Step 3: Calculate the overall criteria value for each alternative $A_i (i = 1, 2, \dots, m)$ by applying the HPFLWA and HPFLWG aggregation operator as follows:

$$H_i (i = 1, 2, \dots, m) = \left\langle \begin{array}{l} \text{HPFLWA } (C_{i1}, C_{i2}, \dots, C_{in}) \\ = \text{HPFLWA } ((C_{11}, C_{12}, \dots, C_{1n}), (C_{21}, C_{22}, \dots, C_{2n}), \dots, (C_{m1}, C_{m2}, \dots, C_{mn})) \end{array} \right\rangle$$

$$H_i (i = 1, 2, \dots, m) = \left\langle \begin{array}{l} \text{HPFLWG } (C_{i1}, C_{i2}, \dots, C_{in}) \\ = \text{HPFLWG } ((C_{11}, C_{12}, \dots, C_{1n}), (C_{21}, C_{22}, \dots, C_{2n}), \dots, (C_{m1}, C_{m2}, \dots, C_{mn})) \end{array} \right\rangle$$

Step 4: Use score or accuracy functions to calculate the score values $S(h_{PL}(p_x))$ and accuracy values $A(h_{PL}(p_x))$ of the aggregated hesitant probabilistic fuzzy linguistic preference values $H_i (i = 1, 2, \dots, m)$.

Step 5: Rank all the alternatives $A_i (i = 1, 2, \dots, m)$ in accordance with $S(h^i_{PL}(p_x))$ or $A(h^i_{PL}(p_x)) (i = 1, 2, \dots, m)$.

6. Illustrative Example

An example is undertaken in this section to understand the implementation methodology of proposed MCGDM method with HPFL information. Further, a real case study is done to rank organizations using proposed MCGDM method. We also compare proposed method with existing HPFL-based MCGDM methods proposed by Lin et al. [50] and Zhou et al. [68].

Example. Suppose that a group of three decision makers (D_1, D_2, D_3) intend to rank four alternatives (A_1, A_2, A_3, A_4) on the basis of three criteria (C_1, C_2, C_3). All DMs are considered equally important and equal weights are assigned to them. Each DM provides evaluation information of each alternative under each criterion in form of HPFLEs with following LTS:

$$S = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{fair}, s_4 = \text{good}, s_5 = \text{very good}, s_6 = \text{extremely good}\}.$$

Step 1: HPFL decision matrices are constructed according to preferences information provided by DMs D_1, D_2 and D_3 about the alternative A_i ($i = 1, 2, 3, 4$) under the criteria C_j ($j = 1, 2, 3$).

Tables 1–3 represent HPFL evaluation matrices provided by DMs D_1, D_2 and D_3 .

Table 1. Hesitant probabilistic fuzzy linguistic (HPFL) decision matrix \tilde{H}^1 provided by D_1 .

	C_1	C_2	C_3
A_1	$\{(s_1, 0.4 0.3, 0.5 0.7)\}$	$\{(s_2, 0.5 0.2, 0.6 0.8)\}$	$\{(s_1, 0.4 1.0)\}$
A_2	$\{(s_3, 0.3 0.5, 0.4 0.5)\}$	$\{(s_4, 0.4 0.4, 0.5 0.6)\}$	$\{(s_2, 0.2 0.6, 0.5 0.4)\}$
A_3	$\{(s_5, 0.1 0.5, 0.5 0.5)\}$	$\{(s_4, 0.3 0.3, 0.5 0.7)\}$	$\{(s_1, 0.1 0.4, 0.2 0.6)\}$
A_4	$\{(s_2, 0.4 0.6, 0.5 0.4)\}$	$\{(s_1, 0.2 1.0)\}$	$\{(s_3, 0.2 0.6, 0.5 0.4)\}$

Table 2. HPFL decision matrix \tilde{H}^2 provided by D_2 .

	C_1	C_2	C_3
A_1	$\{(s_1, 0.4 1.0)\}$	$\{(s_2, 0.2 0.4, 0.4 0.6)\}$	$\{(s_1, 0.8 1.0)\}$
A_2	$\{(s_4, 0.2 0.5, 0.4 0.5)\}$	$\{(s_5, 0.4 0.4, 0.5 0.6)\}$	$\{(s_2, 0.1 0.4, 0.5 0.6)\}$
A_3	$\{(s_2, 0.2 0.5, 0.5 0.5)\}$	$\{(s_4, 0.3 0.6, 0.4 0.4)\}$	$\{(s_2, 0.1 1.0)\}$
A_4	$\{(s_3, 0.3 0.5, 0.5 0.5)\}$	$\{(s_4, 0.5 1.0)\}$	$\{(s_3, 0.2 0.6, 0.5 0.4)\}$

Table 3. HPFL decision matrix \tilde{H}^3 provided by D_3 .

	C_1	C_2	C_3
A_1	$\{(s_2, 0.2 0.4, 0.4 0.6)\}$	$\{(s_4, 0.2 1.0)\}$	$\{(s_4, 0.5 1.0)\}$
A_2	$\{(s_4, 0.3 1.0)\}$	$\{(s_5, 0.3 0.4, 0.4 0.6)\}$	$\{(s_3, 0.3 0.6, 0.5 0.4)\}$
A_3	$\{(s_2, 0.3 0.5, 0.4 0.5)\}$	$\{(s_2, 0.2 0.5, 0.4 0.5)\}$	$\{(s_4, 0.5 1.0)\}$
A_4	$\{(s_2, 0.4 0.6, 0.5 0.4)\}$	$\{(s_4, 0.5 1.0)\}$	$\{(s_4, 0.3 0.5, 0.5 0.5)\}$

Step 2: Aggregate (\tilde{H}^1) , (\tilde{H}^2) and (\tilde{H}^3) into a single HPFL decision matrix $\tilde{H} = (H_{ij})_{4 \times 3}$ ($i = 1, 2, \dots, 4; j = 1, 2, \dots, 3$) using HPFLWA and HPFLWG operators.

Following is the sample computation process of aggregation of HPFLEs $h_{11}^1, h_{11}^2, h_{11}^3$ into a single H_{11} using proposed HPFLWA and HPFLWG operators.

$$H_{11} = \text{HPFLWA}(h_{11}^1, h_{11}^2, h_{11}^3) = [\{(s_1, 0.4|0.3, 0.5|0.7)\}, \{(s_1, 0.4|1.0)\}, \{(s_2, 0.2|0.4, 0.4|0.6)\}]$$

$$= \left[\begin{array}{l} \{(s_{(1+1+2)/3}, [1 - (((1 - 0.4)^{1/3}) * ((1 - 0.4)^{1/3}) * ((1 - 0.2)^{1/3}))] | (0.3 * 1 * 0.4)\}, \\ \{[1 - (((1 - 0.4)^{1/3}) * ((1 - 0.4)^{1/3}) * ((1 - 0.2)^{1/3}))] | (0.3 * 1 * 0.4)\}, \\ \{[1 - (((1 - 0.4)^{1/3}) * ((1 - 0.4)^{1/3}) * ((1 - 0.4)^{1/3}))] | (0.3 * 1 * 0.4)\}, \\ \{[1 - (((1 - 0.4)^{1/3}) * ((1 - 0.4)^{1/3}) * ((1 - 0.4)^{1/3}))] | (0.3 * 1 * 0.6)\}, \\ \{[1 - (((1 - 0.5)^{1/3}) * ((1 - 0.4)^{1/3}) * ((1 - 0.2)^{1/3}))] | (0.7 * 1 * 0.4)\}, \\ \{[1 - (((1 - 0.5)^{1/3}) * ((1 - 0.4)^{1/3}) * ((1 - 0.4)^{1/3}))] | (0.7 * 1 * 0.6)\}, \\ \{[1 - (((1 - 0.5)^{1/3}) * ((1 - 0.4)^{1/3}) * ((1 - 0.2)^{1/3}))] | (0.7 * 1 * 0.4)\}, \\ \{[1 - (((1 - 0.5)^{1/3}) * ((1 - 0.4)^{1/3}) * ((1 - 0.4)^{1/3}))] | (0.7 * 1 * 0.6)\} \end{array} \right]$$

$$= [\{(s_{1.3}, 0.34|0.12, 0.34|0.12, 0.4|0.18, 0.4|0.18, 0.38|0.28, 0.44|0.42, 0.38|0.28, 0.44|0.42)\}]$$

$$H_{11} = [\{(s_{1.3}, 0.34|0.12, 0.4|0.18, 0.38|0.28, 0.44|0.42)\}]$$

$$H_{11} = \text{HPFLWG}(h_{11}^1, h_{11}^2, h_{11}^3) = [\{(s_1, 0.4|0.3, 0.5|0.7)\}, \{(s_1, 0.4|1.0)\}, \{(s_2, 0.2|0.4, 0.4|0.6)\}]$$

$$\begin{aligned}
 &= \left[\left\{ (s_{(1^{1/3})+1^{1/3}+2^{1/3}}), \{ [((0.4)^{1/3}) * ((0.4)^{1/3}) * ((0.2)^{1/3})] \mid (0.3 * 1 * 0.4) \}, \right. \right. \\
 &\quad \left. \left\{ [(((0.4)^{1/3}) * ((0.4)^{1/3}) * ((0.2)^{1/3}))] \mid (0.3 * 1 * 0.4) \}, \right. \right. \\
 &\quad \left. \left\{ [(((0.4)^{1/3}) * ((0.4)^{1/3}) * ((0.4)^{1/3}))] \mid (0.3 * 1 * 0.6) \}, \right. \right. \\
 &\quad \left. \left\{ [(((0.5)^{1/3}) * ((0.4)^{1/3}) * ((0.2)^{1/3}))] \mid (0.7 * 1 * 0.4) \}, \right. \right. \\
 &\quad \left. \left\{ [(((0.5)^{1/3}) * ((0.4)^{1/3}) * ((0.4)^{1/3}))] \mid (0.7 * 1 * 0.6) \}, \right. \right. \\
 &\quad \left. \left\{ [(((0.5)^{1/3}) * ((0.4)^{1/3}) * ((0.2)^{1/3}))] \mid (0.7 * 1 * 0.4) \}, \right. \right. \\
 &\quad \left. \left\{ [(((0.5)^{1/3}) * ((0.4)^{1/3}) * ((0.4)^{1/3}))] \mid (0.7 * 1 * 0.6) \} \right. \right] \\
 &= [\{ (s_{3,3}, 0.32 \mid 0.12, 0.32 \mid 0.12, 0.4 \mid 0.2, 0.4 \mid 0.2, 0.34 \mid 0.3, 0.43 \mid 0.38, 0.34 \mid 0.3, 0.43 \mid 0.38) \}] \\
 &H_{11} = [\{ (s_{3,3}, 0.32 \mid 0.12, 0.4 \mid 0.2, 0.3 \mid 0.3, 0.4 \mid 0.38) \}]
 \end{aligned}$$

Similarly other HPFLEs of HPFL decision matrices (Tables 1–3) are aggregated into the single HPFL decision matrix using HPFLWA and HPFLWG operators, and shown in Tables 4 and 5.

Table 4. Aggregated hesitant probabilistic fuzzy linguistic element (HPFLE) group decision matrix using hesitant probabilistic fuzzy linguistic weighted averaging (HPFLWA) operator.

	C ₁	C ₂	C ₃
A ₁	{(s _{1,3} , 0.34 0.12, 0.4 0.18, 0.38 0.28, 0.44 0.42)}	{(s _{2,7} , 0.32 0.08, 0.38 0.12, 0.37 0.32, 0.42 0.48)}	{(s ₂ , 0.61 1.0)}
A ₂	{(s _{3,7} , 0.27 0.25, 0.34 0.25, 0.3 0.25, 0.37 0.25)}	{(s _{4,7} , 0.37 0.08, 0.41 0.12, 0.4 0.08, 0.44 0.18, 0.41 0.096, 0.44 0.144, 0.47 0.216)}	{(s _{2,3} , 0.2 0.144, 0.35 0.22, 0.29 0.144, 0.42 0.144, 0.32 0.096, 0.39 0.064, 0.44 0.144, 0.5 0.096)}
A ₃	{(s ₃ , 0.2 0.125, 0.32 0.13, 0.24 0.125, 0.35 0.125, 0.38 0.125, 0.44 0.125, 0.47 0.125)}	{(s _{3,3} , 0.27 0.09, 0.31 0.06, 0.34 0.09, 0.37 0.06, 0.35 0.21, 0.41 0.21, 0.38 0.14, 0.44 0.14)}	{(s _{2,3} , 0.26 0.4, 0.29 0.6)}
A ₄	{(s _{2,3} , 0.37 0.18, 0.44 0.18, 0.41 0.18, 0.47 0.12, 0.41 0.12, 0.44 0.08, 0.47 0.12, 0.5 0.08)}	{(s ₃ , 0.42 1.0)}	{(s _{3,3} , 0.23 0.18, 0.35 0.12, 0.32 0.18, 0.42 0.12, 0.35 0.12, 0.42 0.12, 0.44 0.08, 0.5 0.08)}

Table 5. Aggregated HPFLE group decision matrix using hesitant probabilistic fuzzy linguistic weighted geometric (HPFLWG) operator.

	C ₁	C ₂	C ₃
A ₁	{(s _{3,3} , 0.32 0.12, 0.4 0.2, 0.3 0.3, 0.4 0.38)}	{(s _{4,1} , 0.27 0.08, 0.3 0.1, 0.3 0.3, 0.36 0.5)}	{(s _{3,6} , 0.54 1.0)}
A ₂	{(s _{4,6} , 0.26 0.25, 0.3 0.25, 0.3 0.3, 0.4 0.2)}	{(s ₅ , 0.36 0.08, 0.4 0.12, 0.4 0.22, 0.4 0.14, 0.42 0.1, 0.5 0.22)}	{(s ₄ , 0.18 0.14, 0.3 0.22, 0.2 0.14, 0.4 0.1, 0.2 0.1, 0.3 0.06)}
A ₃	{(s _{4,2} , 0.18 0.13, 0.2 0.13, 0.3 0.15, 0.3 0.13, 0.42 0.2, 0.5 0.23)}	{(s _{4,4} , 0.26 0.09, 0.3 0.06, 0.34 0.09, 0.37 0.06, 0.35 0.21, 0.41 0.21, 0.38 0.14, 0.44 0.14)}	{(s _{3,8} , 0.17 0.4, 0.2 0.6)}
A ₄	{(s ₄ , 0.36 0.18, 0.43 0.24, 0.39 0.18, 0.46 0.12, 0.42 0.08, 0.46 0.12, 0.5 0.08)}	{(s _{4,2} , 0.37 1.0)}	{(s _{4,5} , 0.23 0.18, 0.31 0.16, 0.27 0.28, 0.42 0.18, 0.42 0.12, 0.5 0.08)}

Step 3: Aggregate assessment of each alternative $A_i (i=1,2,3,4)$ against each criteria is calculated using the HPFLWA and HPFLWG aggregation operators with criteria weights $w_1 = 0.4, w_2 = 0.3, w_3 = 0.4$ as follows:

$$\begin{aligned}
 &H_1 = \text{HPFLWA} (C_{11}, C_{12}, C_{13}) \\
 &H_1 = [(s_{1,3}, 0.34 \mid 0.12, 0.4 \mid 0.18, 0.38 \mid 0.28, 0.44 \mid 0.42), (s_{2,7}, 0.32 \mid 0.08, 0.38 \mid 0.12, 0.37 \mid 0.32, 0.42 \mid 0.48), (s_2, 0.61 \mid 1.0)] \\
 &H_1 = \left[(s_{1,93}, 0.43 \mid 0.01, 0.44 \mid 0.014, 0.44 \mid 0.038, 0.45 \mid 0.058, 0.45 \mid 0.014, 0.46 \mid 0.022, 0.46 \mid 0.058, 0.47 \mid 0.086, \right. \\
 &\quad \left. 0.44 \mid 0.022, 0.45 \mid 0.034, 0.45 \mid 0.09, 0.47 \mid 0.134, 0.46 \mid 0.034, 0.47 \mid 0.05, 0.47 \mid 0.134, 0.49 \mid 0.202) \right] \\
 &H_1 = \text{HPFLWG} (C_{11}, C_{12}, C_{13})
 \end{aligned}$$

$$H_1 = [(s_{3,3}, 0.32 | 0.12, 0.4 | 0.2, 0.3 | 0.3, 0.4 | 0.38), (s_{4,1}, 0.27 | 0.08, 0.3 | 0.1, 0.3 | 0.3, 0.36 | 0.5), (s_{3,6}, 0.54 | 1.0)]$$

$$H_1 = \left[\begin{array}{l} (s_{1,93}, 0.4 | 0.01, 0.42 | 0.014, 0.41 | 0.038, 0.43 | 0.058, 0.42 | 0.014, 0.45 | 0.022, 0.44 | 0.058, 0.46 | 0.086, \\ 0.41 | 0.022, 0.44 | 0.034, 0.43 | 0.09, 0.45 | 0.134, 0.44 | 0.034, 0.46 | 0.05, 0.46 | 0.134, 0.48 | 0.202) \end{array} \right]$$

Similarly, other elements of HPFL decision matrices (Tables 4 and 5) are aggregated into the overall HPFL decision matrix using HPFLWA and HPFLWG operators and shown in Tables 6 and 7.

Table 6. Collective HPFLE group decision matrix using HPFLWA operator.

A_1	{(s _{1,93} , 0.43 0.01, 0.44 0.014, 0.44 0.038, 0.45 0.058, 0.45 0.014, 0.46 0.022, 0.46 0.058, 0.47 0.086, 0.44 0.022, 0.45 0.034, 0.45 0.09, 0.47 0.134, 0.46 0.034, 0.47 0.05, 0.47 0.134, 0.49 0.202}
A_2	{(s _{3,57} , 0.282 0.003, 0.321 0.004, 0.307 0.003, 0.345 0.006, 0.295 0.003, 0.332 0.005, 0.332 0.008, 0.323 0.004, 0.36 0.006, 0.347 0.004, 0.383 0.01, 0.335 0.005, 0.37 0.008, 0.37 0.012, 0.306 0.003, 0.344 0.004, 0.33 0.003, 0.367 0.006, 0.318 0.003, 0.354 0.005, 0.354 0.008, 0.345 0.003, 0.381 0.004, 0.368 0.003, 0.403 0.006, 0.357 0.003, 0.391 0.005, 0.391 0.008, 0.315 0.002, 0.353 0.003, 0.339 0.002, 0.375 0.004, 0.327 0.002, 0.363 0.003, 0.363 0.005, 0.338 0.001, 0.374 0.002, 0.361 0.001, 0.396 0.003, 0.35 0.002, 0.384 0.002, 0.384 0.003, 0.354 0.003, 0.39 0.004, 0.377 0.003, 0.411 0.006, 0.366 0.003, 0.399 0.005, 0.399 0.008, 0.375 0.002, 0.41 0.003, 0.397 0.002, 0.431 0.004, 0.387 0.002, 0.419 0.003, 0.419 0.005,
A_3	{(s _{2,9} , 0.241 0.005, 0.287 0.005, 0.256 0.005, 0.301 0.005, 0.312 0.005, 0.341 0.005, 0.354 0.005, 0.252 0.003, 0.298 0.003, 0.268 0.003, 0.312 0.003, 0.323 0.003, 0.351 0.003, 0.364 0.003, 0.262 0.005, 0.307 0.005, 0.277 0.005, 0.321 0.005, 0.332 0.005, 0.359 0.005, 0.372 0.005, 0.274 0.003, 0.318 0.003, 0.288 0.003, 0.332 0.003, 0.342 0.003, 0.369 0.003, 0.382 0.003, 0.266 0.011, 0.31 0.011, 0.281 0.011, 0.324 0.011, 0.335 0.011, 0.362 0.011, 0.375 0.011, 0.287 0.011, 0.33 0.011, 0.301 0.011, 0.344 0.011, 0.354 0.011, 0.381 0.011, 0.393 0.011, 0.277 0.007, 0.321 0.007, 0.292 0.007, 0.335 0.007, 0.345 0.007, 0.372 0.007, 0.385 0.007, 0.298 0.007, 0.34 0.007, 0.312 0.007, 0.354 0.007, 0.364 0.007, 0.39 0.007, 0.402 0.007, 0.241 0.007, 0.287 0.007, 0.256 0.007, 0.301 0.007, 0.312 0.007, 0.341 0.007, 0.354 0.007, 0.252 0.007, 0.298 0.007, 0.321 0.007, 0.332 0.007, 0.359 0.007, 0.372 0.007, 0.274 0.005, 0.318 0.005, 0.288 0.005, 0.332 0.005, 0.342 0.005, 0.369 0.005, 0.382 0.005, 0.266 0.016, 0.31 0.016, 0.281 0.016, 0.324 0.016, 0.335 0.016, 0.362 0.016, 0.375 0.016, 0.287 0.016, 0.33 0.016, 0.301 0.016, 0.344 0.016, 0.354 0.016, 0.381 0.016, 0.393 0.016, 0.277 0.011, 0.321 0.011, 0.292 0.011, 0.335 0.011, 0.345 0.011, 0.372 0.011, 0.385 0.011, 0.298 0.011, 0.34 0.011, 0.312 0.011, 0.354 0.011, 0.364 0.011, 0.39 0.011, 0.402 0.011)}
A_4	{(s _{2,83} , 0.346 0.032, 0.375 0.032, 0.362 0.032, 0.39 0.022, 0.362 0.022, 0.377 0.014, 0.39 0.022, 0.405 0.014, 0.376 0.022, 0.404 0.022, 0.391 0.022, 0.418 0.014, 0.391 0.014, 0.406 0.01, 0.418 0.014, 0.432 0.01, 0.368 0.032, 0.396 0.032, 0.383 0.032, 0.41 0.022, 0.383 0.022, 0.398 0.014, 0.41 0.022, 0.424 0.014, 0.397 0.022, 0.423 0.022, 0.411 0.022, 0.437 0.014, 0.411 0.014, 0.426 0.01, 0.437 0.014, 0.451 0.01, 0.405 0.014, 0.431 0.014, 0.419 0.014, 0.445 0.01, 0.419 0.01, 0.433 0.006, 0.445 0.01, 0.458 0.006, 0.425 0.014, 0.45 0.014, 0.438 0.014, 0.463 0.01, 0.438 0.01, 0.452 0.006, 0.463 0.01, 0.476 0.006,)}

Table 7. Collective HPFLE group decision matrix using HPFLWG operator.

A_1	{(s _{1,93} , 0.4 0.01, 0.42 0.014, 0.41 0.038, 0.43 0.058, 0.42 0.014, 0.45 0.022, 0.44 0.058, 0.46 0.086, 0.41 0.022, 0.44 0.034, 0.43 0.09, 0.45 0.134, 0.44 0.034, 0.46 0.05, 0.46 0.134, 0.48 0.202}
A_2	{(s _{3,57} , 0.27 0.003, 0.31 0.004, 0.29 0.003, 0.32 0.006, 0.28 0.003, 0.31 0.005, 0.31 0.008, 0.32 0.004, 0.36 0.006, 0.34 0.004, 0.38 0.01, 0.33 0.005, 0.37 0.008, 0.36 0.012, 0.31 0.003, 0.34 0.004, 0.33 0.003, 0.36 0.006, 0.31 0.003, 0.35 0.005, 0.34 0.008, 0.34 0.003, 0.38 0.004, 0.36 0.003, 0.41 0.006, 0.35 0.003, 0.39 0.005, 0.38 0.008, 0.31 0.002, 0.35 0.003, 0.34 0.002, 0.37 0.004, 0.32 0.002, 0.36 0.003, 0.35 0.005, 0.33 0.001, 0.37 0.002, 0.36 0.001, 0.39 0.003, 0.34 0.002, 0.38 0.002, 0.37 0.003, 0.34 0.003, 0.39 0.004, 0.37 0.003, 0.41 0.006, 0.35 0.003, 0.39 0.005, 0.39 0.008, 0.36 0.002, 0.4 0.003, 0.38 0.002, 0.42 0.004, 0.37 0.002, 0.41 0.003, 0.4 0.005,
A_3	{(s _{2,9} , 0.24 0.005, 0.29 0.005, 0.26 0.005, 0.3 0.005, 0.31 0.005, 0.32 0.005, 0.33 0.005, 0.25 0.003, 0.3 0.003, 0.27 0.003, 0.31 0.003, 0.32 0.003, 0.34 0.003, 0.35 0.003, 0.25 0.005, 0.3 0.005, 0.27 0.005, 0.32 0.005, 0.33 0.005, 0.35 0.005, 0.36 0.005, 0.26 0.003, 0.31 0.003, 0.28 0.003, 0.33 0.003, 0.34 0.003, 0.36 0.003, 0.37 0.003, 0.26 0.011, 0.31 0.011, 0.28 0.011, 0.32 0.011, 0.33 0.011, 0.35 0.011, 0.36 0.011, 0.27 0.011, 0.32 0.011, 0.29 0.011, 0.34 0.011, 0.35 0.011, 0.37 0.011, 0.38 0.011, 0.26 0.007, 0.32 0.007, 0.28 0.007, 0.33 0.007, 0.34 0.007, 0.36 0.007, 0.37 0.007, 0.28 0.007, 0.33 0.007, 0.3 0.007, 0.34 0.007, 0.35 0.007, 0.37 0.007, 0.38 0.007, 0.24 0.007, 0.29 0.007, 0.26 0.007, 0.3 0.007, 0.31 0.007, 0.32 0.007, 0.33 0.007, 0.25 0.005, 0.3 0.005, 0.27 0.005, 0.31 0.005, 0.32 0.005, 0.34 0.005, 0.35 0.005, 0.25 0.007, 0.3 0.007, 0.27 0.007, 0.32 0.007, 0.33 0.007, 0.35 0.007, 0.36 0.007, 0.26 0.005, 0.31 0.005, 0.28 0.005, 0.33 0.005, 0.34 0.005, 0.36 0.005, 0.37 0.005, 0.26 0.016, 0.31 0.016, 0.28 0.016, 0.32 0.016, 0.33 0.016, 0.35 0.016, 0.36 0.016, 0.27 0.016, 0.32 0.016, 0.29 0.016, 0.34 0.016, 0.35 0.016, 0.37 0.016, 0.38 0.016, 0.26 0.011, 0.32 0.011, 0.28 0.011, 0.33 0.011, 0.34 0.011, 0.36 0.011, 0.37 0.011, 0.28 0.011, 0.33 0.011, 0.34 0.011, 0.35 0.011, 0.37 0.011, 0.38 0.011)}
A_4	{(s _{2,83} , 0.33 0.032, 0.36 0.032, 0.35 0.032, 0.37 0.022, 0.35 0.022, 0.36 0.014, 0.37 0.022, 0.38 0.014, 0.37 0.022, 0.4 0.022, 0.39 0.022, 0.41 0.014, 0.39 0.014, 0.4 0.01, 0.41 0.014, 0.42 0.01, 0.36 0.032, 0.39 0.032, 0.38 0.032,

0.4|0.022, 0.38|0.022, 0.39|0.014, 0.4|0.022, 0.41|0.014, 0.4|0.022, 0.42|0.022, 0.41|0.022, 0.44|0.014, 0.41|0.014, 0.43|0.01, 0.44|0.014, 0.45|0.01, 0.37|0.022, 0.4|0.022, 0.39|0.022, 0.41|0.014, 0.39|0.014, 0.4|0.01, 0.41|0.014, 0.42|0.01, 0.4|0.022, 0.42|0.022, 0.41|0.022, 0.44|0.014, 0.41|0.014, 0.43|0.01, 0.44|0.014, 0.45|0.01, 0.4|0.014, 0.43|0.014, 0.42|0.014, 0.44|0.01, 0.42|0.01, 0.43|0.006, 0.44|0.01, 0.46|0.006, 0.42|0.014, 0.45|0.014, 0.43|0.014, 0.46|0.01, 0.43|0.01, 0.45|0.006, 0.46|0.01, 0.47|0.006,))

Step 4: The score values $S(h^i_{PL}(p_x)) (i = 1, 2, 3, 4)$ of the alternatives $A_i (i = 1, 2, 3, 4)$ are calculated and shown as follows (Table 8):

Table 8. Score values for the alternatives using HPFLWA and HPFLWG operators.

Score	HPFLWA	HPFLWG
$S(h^1_{PL}(p_x))$	S _{0.05652}	S _{0.05409}
$S(h^2_{PL}(p_x))$	S _{0.00559}	S _{0.00545}
$S(h^3_{PL}(p_x))$	S _{0.00745}	S _{0.00723}
$S(h^4_{PL}(p_x))$	S _{0.01901}	S _{0.01874}

Step 5. Finally, alternatives $A_i (i = 1, 2, 3, 4)$ are ranked in accordance with score values $S(h^i_{PL}(p_x))$ and shown in Table 9.

Table 9. Ranking of alternatives using proposed HPFLWA and HPFLWG operators.

Method	Ranking	Best/Worst
Using HPFLWA operator	$A_1 > A_4 > A_3 > A_2$	A_1/A_2
Using HPFLWG operator	$A_1 > A_4 > A_3 > A_2$	A_1/A_2

Table 9 confirms that using both proposed HPFLWA and HPFLWG operators best and worst alternatives are A_1 and A_2 respectively.

6.1. A Real Case Study

A real case study is undertaken to rank seven organizations; State Bank of India (A_1), InfoTech Enterprises (A_2), ITC (A_3), H.D.F.C. Bank (A_4), Tata Steel (A_5), Tata Motors (A_6) and Bajaj Finance (A_7) on the basis of their performance against following four criteria.

1. Earnings per share (EPS) of company (C_1)
2. Face value (C_2)
3. Book value (C_3)
4. P/C ratio (Put-Call Ratio) of company (C_4)

In this real case study, $C_1, C_2,$ and C_3 are benefit criteria while C_4 is cost criterion. Real data for each alternative against each criterion are retrieved from <http://www.moneycontrol.com> from date 20.7.2017 to 27.7.2017. Table 10 shows their average values.

Table 10. Average of actual numerical value of criteria.

	C_1	C_2	C_3	C_4
A_1	13.15	1.00	196.53	19.27
A_2	61.18	5.00	296.12	14.98
A_3	8.54	1.00	37.31	30.52
A_4	59.07	2.00	347.59	28.50
A_5	22.25	2.00	237.82	5.98
A_6	35.47	1.12	511.31	7.95
A_7	36.64	2.00	174.60	45.39

To construct hesitant fuzzy decision matrix (Table 11), we use the method proposed by Bisht and Kumar [69] and fuzzify Table 10 using triangular and Gaussian membership functions.

Table 11. Hesitant fuzzy decision matrix

	C ₁	C ₂	C ₃	C ₄
A ₁	0.3784, 0.3029	0.6065, 0.50	0.7545, 0.6247	0.997, 0.9614
A ₂	0.9676, 0.8718	0.6065, 0.50	0.8964, 0.7662	0.696, 0.5743
A ₃	0.1534, 0.0318	0.6065, 0.50	0.1368, 0.0027	0.778, 0.6457
A ₄	0.9997, 0.9959	0.6065, 0.50	0.8122, 0.6775	0.7748, 0.6429
A ₅	0.949, 0.8382	0.6065, 0.50	0.8655, 0.7312	0.2278, 0.14
A ₆	0.7445, 0.6159	0.7491, 0.62	0.9843, 0.9111	0.512, 0.4214
A ₇	0.8197, 0.6847	0.6065, 0.50	0.933, 0.8138	0.3055, 0.23

Probabilities are associated with elements of hesitant fuzzy decision matrix (Table 11) to convert it into probabilistic hesitant fuzzy decision matrix $I = [I_{Pij} = (\mu_{ij} | p_{ij})]_{m \times n}$. Probabilities which are associated with first row of hesitant fuzzy decision matrix (Table 11) are as follows:

$$\mu(p_{11}^1) = \frac{0.3784}{0.3784 + 0.3029} = 0.5554, \mu(p_{11}^2) = \frac{0.3029}{0.3784 + 0.3029} = 0.4446$$

$$\mu(p_{12}^1) = \frac{0.6065}{0.6065 + 0.5} = 0.5481, \mu(p_{12}^2) = \frac{0.5}{0.6065 + 0.5} = 0.4519$$

$$\mu(p_{14}^1) = \frac{0.9614}{0.997 + 0.9614} = 0.9285, \mu(p_{14}^2) = \frac{0.997}{0.9614 + 0.997} = 0.0715$$

Similarly all elements of hesitant fuzzy decision matrix are associated with probabilities and probabilistic hesitant fuzzy decision matrix (Table 12) is obtained.

Table 12. Probabilistic Hesitant fuzzy decision matrix.

	C ₁	C ₂	C ₃	C ₄
A ₁	{{(0.3784 0.5554), (0.3029 0.4446)}}	{{(0.6065 0.5481), (0.5 0.4519)}}	{{(0.7545 0.5471), (0.6247 0.4529)}}	{{(0.997 0.0715), (0.9614 0.9285)}}
A ₂	{{(0.9676 0.5261), (0.8718 0.4739)}}	{{(0.6065 0.5481), (0.5 0.4519)}}	{{(0.8964 0.5392), (0.7662 0.4608)}}	{{(0.696 0.4166), (0.5743 0.5834)}}
A ₃	{{(0.1534 0.8283), (0.0318 0.1717)}}	{{(0.6065 0.5481), (0.5 0.4519)}}	{{(0.1368 0.9806), (0.0027 0.0194)}}	{{(0.778 0.3852), (0.6457 0.6148)}}
A ₄	{{(0.9997 0.501), (0.9959 0.499)}}	{{(0.6065 0.5481), (0.5 0.4519)}}	{{(0.8122 0.5452), (0.6775 0.4548)}}	{{(0.7748 0.3867), (0.6429 0.6133)}}
A ₅	{{(0.949 0.531), (0.8382 0.469)}}	{{(0.6065 0.5481), (0.5 0.4519)}}	{{(0.8655 0.542), (0.7312 0.458)}}	{{(0.2278 0.4731), (0.14 0.5269)}}
A ₆	{{(0.7445 0.5473), (0.6159 0.4527)}}	{{(0.7492 0.5472), (0.62 0.4528)}}	{{(0.9843 0.5193), (0.9111 0.4807)}}	{{(0.512 0.4575), (0.4214 0.5425)}}
A ₇	{{(0.8197 0.5449), (0.6847 0.4551)}}	{{(0.6065 0.5481), (0.5 0.4519)}}	{{(0.933 0.5341), (0.8138 0.4659)}}	{{(0.3055 0.4742), (0.23 0.5258)}}

Following table (Table 13) shows hesitant probabilistic fuzzy linguistic decision matrix.

Table 13. Hesitant probabilistic fuzzy linguistic decision matrix.

	C ₁	C ₂	C ₃	C ₄
A ₁	{s ₁ , (0.3784 0.5554), (0.3029 0.4446)}	{s ₁ , (0.6065 0.5481), (0.5 0.4519)}	{s ₂ , (0.7545 0.5471), (0.6247 0.4529)}	{s ₃ , (0.997 0.0715), (0.9614 0.9285)}
A ₂	{s ₃ , (0.9676 0.5261), (0.8718 0.4739)}	{s ₁ , (0.6065 0.5481), (0.5 0.4519)}	{s ₂ , (0.8964 0.5392), (0.7662 0.4608)}	{s ₂ , (0.696 0.4166), (0.5743 0.5834)}

A_3	$\{s_0, (0.1534 0.8283), (0.0318 0.1717)\}$	$\{s_1, (0.6065 0.5481), (0.5 0.4519)\}$	$\{s_0, (0.1368 0.9806), (0.0027 0.0194)\}$	$\{s_2, (0.778 0.3852), (0.6457 0.6148)\}$
A_4	$\{s_3, (0.9997 0.501), (0.9959 0.499)\}$	$\{s_1, (0.6065 0.5481), (0.5 0.4519)\}$	$\{s_2, (0.8122 0.5452), (0.6775 0.4548)\}$	$\{s_2, (0.7748 0.3867), (0.6429 0.6133)\}$
A_5	$\{s_3, (0.949 0.531), (0.8382 0.469)\}$	$\{s_1, (0.6065 0.5481), (0.5 0.4519)\}$	$\{s_3, (0.8655 0.542), (0.7312 0.458)\}$	$\{s_0, (0.2278 0.4731), (0.14 0.5269)\}$
A_6	$\{s_2, (0.7445 0.5473), (0.6159 0.4527)\}$	$\{s_2, (0.7492 0.5472), (0.62 0.4528)\}$	$\{s_3, (0.9843 0.5193), (0.9111 0.4807)\}$	$\{s_1, (0.512 0.4575), (0.4214 0.5425)\}$
A_7	$\{s_2, (0.8197 0.5449), (0.6847 0.4551)\}$	$\{s_1, (0.6065 0.5481), (0.5 0.4519)\}$	$\{s_3, (0.933 0.5341), (0.8138 0.4659)\}$	$\{s_0, (0.3055 0.4742), (0.23 0.5258)\}$

Step 2: Assessment of each alternative $A_i (i = 1, 2, 3, 4, 5, 6, 7)$ against each criteria $C_j (j = 1, 2, 3, 4)$ is aggregated using HPFLWA aggregation operator (Equation (9)) as follows:

$$H_1 = \text{HPFLWA}(C_{11}, C_{12}, C_{13}, C_{14})$$

$$H_1 = \left[\begin{array}{l} \{s_1, (0.3784|0.5554), (0.3029|0.4446)\}, \{s_1, (0.6065|0.5481), (0.5|0.4519), \\ \{s_2, (0.7545|0.5471)\}, (0.6247|0.4529), \{s_3, (0.997|0.0715), (0.9614|0.9285)\} \end{array} \right]$$

$$H_1 = \left[\begin{array}{l} \{s_{1.75}, (0.884|0.012), (0.871|0.01), (0.877|0.1055), (0.8811|0.0095), (0.8678|0.0079), \\ (0.3029|0.4446), (0.3784|0.5554), (0.3029|0.4446), (0.3784|0.5554), (0.3029|0.4446), \\ (0.3784|0.5554), (0.3029|0.4446), (0.3784|0.5554), (0.3029|0.4446), (0.3784|0.5554), \\ (0.3029|0.4446) \end{array} \right]$$

In the aggregation of assessment of the alternatives, all criteria are considered of equal weight of 0.25. Similarly other elements of HPFL decision matrix (Table 13) are aggregated and following collective HPFL decision matrix (Table 14) is obtained.

Table 14. Collective hesitant probabilistic fuzzy linguistic decision matrix

A_1	$\{s_{1.75}, (0.884 0.012), (0.871 0.01), (0.877 0.1055), (0.8811 0.0095), (0.8678 0.0079), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446)\}$
A_2	$\{s_2, (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446)\}$
A_3	$\{s_{0.75}, (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446)\}$
A_4	$\{s_2, (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446)\}$
A_5	$\{s_{1.75}, (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446)\}$
A_6	$\{s_2, (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446)\}$
A_7	$\{s_{1.5}, (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446), (0.3784 0.5554), (0.3029 0.4446)\}$

Step 3: The score values $S(h^i_{PL}(p_x)) (i = 1, 2, 3, 4, 5, 6, 7)$ of the alternatives $A_i (i = 1, 2, 3, 4, 5, 6, 7)$ are calculated using Equation (7) and are shown as follows:

$$S(h^1_{PL}(p_x)) = S_{0.0902}, S(h^2_{PL}(p_x)) = S_{0.092}, S(h^3_{PL}(p_x)) = S_{0.019}$$

$$S(h^4_{PL}(p_x)) = S_{0.101}, S(h^5_{PL}(p_x)) = S_{0.076}, S(h^6_{PL}(p_x)) = S_{0.094}$$

$$S(h^7_{PL}(p_x)) = S_{0.063}$$

Step 4: Finally, alternatives $A_i (i = 1, 2, 3, 4, 5, 6, 7)$ are ranked as $A_4 > A_6 > A_2 > A_1 > A_5 > A_7 > A_3$ in accordance with score values $S(h^i_{PL}(p_x))$.

6.2. Comparative Analysis

In this section, we compare proposed HPFL-based MCGDM methods with existing HFL-based methods. We apply the proposed method on two different problems which are adapted from Zhou et al. (2016) and Lin et al. (2014) and compare the ranking results. In order to apply the proposed HPFL-based MCGDM on the examples taken by both Lin et al. (2014) and Zhou et al. (2016), we have considered probability of each element of HFL decision matrices as unity.

6.2.1. Comparison 1

In comparison 1, methodology of proposed HPFL-based MCGDM method is applied on the following HFL decision matrix (Table 15) of the problem taken by Lin et al. [50].

Table 15. Hesitant fuzzy linguistic decision matrix ([50]) .

	G ₁	G ₂	G ₃	G ₄
A ₁	<s ₅ , (0.3, 0.5)>	<s ₃ , (0.6, 0.7, 0.8)>	<s ₂ , (0.7, 0.8)>	<s ₄ , (0.8, 0.9)>
A ₂	<s ₂ , (0.3, 0.4, 0.5)>	<s ₅ , (0.6, 0.9)>	<s ₃ , (0.6, 0.7)>	<s ₅ , (0.4, 0.5)>
A ₃	<s ₆ , (0.4, 0.6)>	<s ₂ , (0.7, 0.8)>	<s ₅ , (0.3, 0.5, 0.7)>	<s ₃ , (0.6, 0.7)>
A ₄	<s ₅ , (0.7, 0.9)>	<s ₁ , (0.3, 0.4)>	<s ₇ , (0.5, 0.7)>	<s ₂ , (0.3, 0.5)>
A ₅	<s ₄ , (0.2, 0.3)>	<s ₂ , (0.6, 0.7)>	<s ₄ , (0.5, 0.6)>	<s ₂ , (0.7, 0.8, 0.9)>

Following table (Table 16) shows the ranking results of the alternatives which are obtained using proposed HPFL and existing HFL-based MCDM method of Lin et al. [50].

Table 16. Comparison of ranking of alternatives.

Method	Ranking	Best Alternative/Worst Alternative
Proposed	$A_4 > A_1 > A_3 > A_2 > A_5$	A ₄ /A ₅
Lin et al. [50]	$A_4 > A_3 > A_1 > A_2 > A_5$	A ₄ /A ₅

On applying the proposed MCGDM method on ranking problem which is adapted from Lin et al. (2014), A₄ and A₅ are ranked again as the best and the worst alternatives respectively.

6.2.2. Comparison 2

In comparison 2, the methodology of proposed HPFL-based MCGDM method is applied on the following HFL decision matrix (Table 17) of the problem taken by Zhou et al. [67].

Table 17. The special linguistic hesitant fuzzy decision matrix ([67]).

	C ₁	C ₂	C ₃	C ₄
X ₁	<s ₅ , (0.3, 0.4)>	<s ₆ , (0.2, 0.4)>	<s ₅ , (0.5, 0.7)>	<s ₄ , (0.4)>
X ₂	<s ₃ , (0.5, 0.6)>	<s ₅ , (0.3, 0.5)>	<s ₄ , (0.7)>	<s ₅ , (0.4, 0.6)>
X ₃	<s ₄ , (0.4, 0.6)>	<s ₅ , (0.4)>	<s ₇ , (0.7, 0.8)>	<s ₃ , (0.6)>
X ₄	<s ₃ , (0.3, 0.4, 0.5)>	<s ₄ , (0.6)>	<s ₃ , (0.4, 0.7)>	<s ₃ , (0.8)>
X ₅	<s ₆ , (0.5, 0.7)>	<s ₆ , (0.5, 0.6)>	<s ₄ , (0.6, 0.8)>	<s ₅ , (0.7)>

Following table (Table 18) shows the ranking results of the alternatives which are obtained using proposed HPFL and existing HFL-based MCDM method of Zhou et al. [67].

Table 18. Comparison of ranking of alternatives.

Method	Ranking	Best Alternative/Worst Alternative
Proposed	$X_5 > X_3 > X_1 > X_2 > X_4$	X_5/X_4
Zhou et al. [67]	$X_5 > X_3 > X_1 > X_2 > X_4$	X_5/X_4

On applying the proposed MCGDM method on ranking problem adapted from Zhou et al. [67], X_5 and X_4 are ranked again as the best and the worst alternatives respectively.

As there is no change found in the ranking results of the alternatives in both the comparisons, it confirms that the proposed HPFL-based MCGDM method is also suitable with HFL information.

7. Conclusions

Uncertainties due to randomness and fuzziness both occur in the system simultaneously. In certain decision making problem, DMs prefer to analyze the alternatives against decision criteria qualitatively using linguistic terms. In this paper, we have proposed hesitant probabilistic fuzzy linguistic set (HPFLS) to integrate hesitant fuzzy linguistic information with probability theory. Prominent characteristic of HPFLS is to associate occurring probabilities to HFLEs which makes it more effective than HFLS. We have investigated the expected mean, variance, score and accuracy function, and basic operations for HPFLEs. We have also defined HPFLWA, HPFLWG, HPFLOWA and HPFLOWG aggregation operators to aggregate hesitant probabilistic fuzzy linguistic information. A novel MCGDM method using HPFLWA, HPFLWG, HPFLOWA and HPFLOWG is also proposed in the present study. Advantage of proposed HPFLS-based MCGDM method is that it associates probabilities to HFLE which makes it competent enough to handle both stochastic and non-stochastic uncertainties with hesitant information using both qualitative and quantitative terms. Another advantage of proposed MCGDM method is that it allows DMs to use their intuitive ability to judge alternatives against criteria using probabilities. This is also important to note that the proposed method can also be used with HFL information if DMs associate equal probabilities to HFLE. Methodology of proposed HPFL-based MCGDM method is illustrated by an example. A real case study to rank the organizations is also undertaken in the present work.

Even though, proposed HPFL-based MCGDM method includes both stochastic and non-stochastic uncertainties along with hesitation, but to determine probabilities of membership grades in linguistic fuzzy set is very difficult in real life problem of decision making. Proposed HPFL-based MCGDM method will be effective when either DMs are expert of their field or they have pre-defined probability distribution function so that the appropriate probabilities could be assigned. Applications of proposed HPFLS with Pythagorean membership grades can also be seen as the scope of future research in decision making problems as an enhancement of the methods proposed by Garg [49].

Author Contributions: Dheeraj Kumar Joshi and Sanjay Kumar defined HPFLS and studied its properties. They together developed MCGDM method using HPFL information. Ismat Beg contributed in verifying the proof of Theorem 1 and the properties of aggregation operators. All authors equally contributed in the research paper.

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References

1. Meghdadi, A.H.; Akbarzadeh-T, M.R. Probabilistic fuzzy logic and probabilistic fuzzy systems. In Proceedings of the 10th IEEE International Conference on Fuzzy Systems, Melbourne, Australia, 2–5 December 2001; Volume 3, pp. 1127–1130.
2. Valavanis, K.P.; Saridis, G.N. Probabilistic modeling of intelligent robotic systems. *IEEE Trans. Robot. Autom.* **1991**, *7*, 164–171.
3. Pidre, J.C.; Carrillo, C.J.; Lorenzo, A.E.F. Probabilistic model for mechanical power fluctuations in asynchronous wind parks. *IEEE Trans. Power Syst.* **2003**, *18*, 761–768.
4. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353.

5. Zadeh, L.A. Fuzzy logic and approximate reasoning. *Synthese* **1975**, *30*, 407–428.
6. Lee, L.W.; Chen, S.M. Fuzzy decision making and fuzzy group decision making based on likelihood-based comparison relations of hesitant fuzzy linguistic term sets. *J. Intell. Fuzzy Syst.* **2015**, *29*, 1119–1137.
7. Wang, H.; Xu, Z. Admissible orders of typical hesitant fuzzy elements and their application in ordered information fusion in multi-criteria decision making. *Inf. Fusion* **2016**, *29*, 98–104.
8. Liu, J.; Chen, H.; Zhou, L.; Tao, Z. Generalized linguistic ordered weighted hybrid logarithm averaging operators and applications to group decision making. *Int. J. Uncertain. Fuzz. Knowl.-Based Syst.* **2015**, *23*, 421–442.
9. Liu, J.; Chen, H.; Xu, Q.; Zhou, L.; Tao, Z. Generalized ordered modular averaging operator and its application to group decision making. *Fuzzy Sets Syst.* **2016**, *299*, 1–25.
10. Yoon, K.P.; Hwang, C.L. *Multiple Attribute Decision Making: An Introduction*; Sage Publications: New York, NY, USA, 1995; Volume 104.
11. Mardani, A.; Nilachi, M.; Zavadskas, E.K.; Awang, S.R.; Zare, H.; Jamal, N.M. Decision making methods based on fuzzy aggregation operators: Three decades of review from 1986 to 2018. *Int. J. Inf. Technol. Decis. Mak.* **2017**, doi:10.1142/S021962201830001X.
12. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96.
13. Atanassov, K.T.; Gargov, G. Interval-valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *31*, 343–349.
14. Yager, R.R. Pythagorean membership grades in multicriteria decision making. *IEEE Trans. Fuzzy Syst.* **2014**, *22*, 958–965.
15. Majumdar, P. Neutrosophic Sets and Its Applications to Decision Making. In *Computational Intelligence for Big Data Analysis. Adaptation, Learning, and Optimization*; Acharjya, D., Dehuri, S., Sanyal, S., Eds.; Springer: Cham, Switzerland, 2015; Volume 19.
16. Torra, V.; Narukawa, Y. On hesitant fuzzy sets and decision. In Proceedings of the 18th IEEE International Conference on Fuzzy Systems, Jeju Island, Korea, 20–24 August 2009; pp. 1378–1382.
17. Torra, V. Hesitant fuzzy sets. *Int. J. Intell. Syst.* **2010**, *25*, 529–539.
18. Xia, M.; Xu, Z. Hesitant fuzzy information aggregation in decision making. *Int. J. Approx. Reason.* **2011**, *52*, 395–407.
19. Farhadinia, B.; Xu, Z. Distance and aggregation-based methodologies for hesitant fuzzy decision making. *Cogn. Comput.* **2017**, *9*, 81–94.
20. Qian, G.; Wang, H.; Feng, X. Generalized hesitant fuzzy sets and their application in decision support system. *Knowl.-Based Syst.* **2013**, *37*, 357–365.
21. Peng, J.J.; Wang, J.Q.; Wang, J.; Yang, L.J.; Chen, X.H. An extension of ELECTRE to multi-criteria decision-making problems with multi-hesitant fuzzy sets. *Inf. Sci.* **2015**, *307*, 113–126.
22. Chen, S.W.; Cai, L.N. Interval-valued hesitant fuzzy sets. *Fuzzy Syst. Math.* **2013**, *6*, 38–44.
23. Yu, D. Triangular hesitant fuzzy set and its application to teaching quality evaluation. *J. Inf. Comput. Sci.* **2013**, *10*, 1925–1934.
24. Zhu, B.; Xu, Z.; Xia, M. Dual hesitant fuzzy sets. *J. Appl. Math.* **2012**, *2012*, 879629.
25. Zhang, Z. Interval-valued intuitionistic hesitant fuzzy aggregation operators and their application in group decision-making. *J. Appl. Math.* **2013**, *2013*, 670285.
26. Joshi, D.; Kumar, S. Interval-valued intuitionistic hesitant fuzzy Choquet integral based TOPSIS method for multi-criteria group decision making. *Eur. J. Oper. Res.* **2016**, *248*, 183–191.
27. Garg, H. Hesitant Pythagorean fuzzy sets and their aggregation operators in multiple attribute decision making. *Int. J. Uncertain. Quantif.* **2018**, doi:10.1615/Int.J.UncertaintyQuantification.2018020979.
28. Qi, X.-W.; Zhang, J.-L.; Zhao, S.-P.; Liang, C.-Y. Tackling complex emergency response solutions evaluation problems in sustainable development by fuzzy group decision making approaches with considering decision hesitancy and prioritization among assessing criteria. *Int. J. Environ. Res. Public Health* **2017**, *14*, 1165, doi:10.3390/ijerph14101165.

29. Garg, H.; Arora, R. Distance and similarity measures for dual hesitant fuzzy soft sets and their applications in multicriteria decision making problem. *Int. J. Uncertain. Quantif.* **2017**, *7*, 229–248.
30. Martínez, L.; Ruan, D.; Herrera, F.; Wang, P.P. Linguistic decision making: Tools and applications. *Inf. Sci.* **2009**, *179*, 2297–2298.
31. Xu, Z. An interactive procedure for linguistic multiple attribute decision making with incomplete weight information. *Fuzzy Optim. Decis. Mak.* **2007**, *6*, 17–27.
32. Herrera, F.; Martínez, L. A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Trans. Fuzzy Syst.* **2000**, *8*, 746–752.
33. Herrera, F.; Herrera-Viedma, E.; Alonso, S.; Chiclana, F. Computing with words and decision making. *Fuzzy Optim. Decis. Mak.* **2009**, *8*, 323–324.
34. Lan, J.; Sun, Q.; Chen, Q.; Wang, Q. Group decision making based on induced uncertain linguistic OWA operators. *Decis. Support Syst.* **2013**, *55*, 296–303.
35. Beg, I.; Rashid, T. TOPSIS for hesitant fuzzy linguistic term sets. *Int. J. Intell. Syst.* **2013**, *28*, 1162–1171.
36. Rodríguez, R.M.; Martínez, L.; Herrera, F. A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets. *Inf. Sci.* **2013**, *241*, 28–42.
37. Yuen, K.K.F. Combining compound linguistic ordinal scale and cognitive pairwise comparison in the rectified fuzzy TOPSIS method for group decision making. *Fuzzy Optim. Decis. Mak.* **2014**, *13*, 105–130.
38. Zhang, Z.; Wu, C. Hesitant fuzzy linguistic aggregation operators and their applications to multiple attribute group decision making. *J. Intell. Fuzzy Syst.* **2014**, *26*, 2185–2202.
39. Beg, I.; Rashid, T. Group decision making using comparative linguistic expression based on hesitant intuitionistic fuzzy sets. *Appl. Appl. Math. Int. J.* **2015**, *10*, 1082–1092.
40. Wang, J.Q.; Wang, D.D.; Zhang, H.Y.; Chen, X.H. Multi-criteria group decision making method based on interval 2-tuple linguistic information and Choquet integral aggregation operators. *Soft Comput.* **2015**, *19*, 389–405.
41. Merigó, J.M.; Palacios-Marqués, D.; Zeng, S. Subjective and objective information in linguistic multi-criteria group decision making. *Eur. J. Oper. Res.* **2016**, *248*, 522–531.
42. Beg, I.; Rashid, T. Hesitant 2-tuple linguistic information in multiple attributes group decision making. *J. Intell. Fuzzy Syst.* **2016**, *30*, 109–116.
43. Zhou, W.; Xu, Z. Generalized asymmetric linguistic term set and its application to qualitative decision making involving risk appetites. *Eur. J. Oper. Res.* **2016**, *254*, 610–621.
44. De Maio, C.; Fenza, G.; Loia, V.; Orciuoli, F. Linguistic fuzzy consensus model for collaborative development of fuzzy cognitive maps: A case study in software development risks. *Fuzzy Optim. Decis. Mak.* **2017**, in press, doi:10.1007/s10700-016-9259-3.
45. Gao, J.; Xu, Z.; Liao, H. A dynamic reference point method for emergency response under hesitant probabilistic fuzzy environment. *Int. J. Fuzzy Syst.* **2017**, *19*, 1261–1278.
46. Kobina, A.; Liang, D.; He, X. Probabilistic linguistic power aggregation operators for multi-criteria group decision making. *Symmetry* **2017**, *9*, 320, doi:10.3390/sym9120320.
47. Garg, H.; Kumar, K. Some aggregation operators for linguistic intuitionistic fuzzy set and its application to group decision-making process using the set pair analysis. *Arbian J. Sci. Eng.* **2017**, doi:10.1007/s13369-017-2986-0.
48. Liu, P.; Mahmood, T.; Khan, Q. Multi-attribute decision-making based on prioritized aggregation operator under hesitant intuitionistic fuzzy linguistic environment. *Symmetry* **2017**, *9*, 270, doi:10.3390/sym9110270.
49. Garg, H. Linguistic Pythagorean fuzzy sets and its applications in multi attribute decision making process. *Int. J. Intell. Syst.* **2018**, doi:10.1002/int.21979.
50. Lin, R.; Zhao, X.; Wei, G. Models for selecting an ERP system with hesitant fuzzy linguistic information. *J. Intell. Fuzzy Syst.* **2014**, *26*, 2155–2165.
51. Ren, F.; Kong, M.; Pei, Z. A new hesitant fuzzy linguistic topsis method for group multi-criteria linguistic decision making. *Symmetry* **2017**, *9*, 289, doi:10.3390/sym9120289.

52. Joshi, D.; Kumar, S. Trapezium cloud TOPSIS method with interval-valued intuitionistic hesitant fuzzy linguistic information. *Granul. Comput.* **2017**, doi:10.1007/s41066-017-0062-5.
53. Wu, Y.; Li, C.-C.; Chen, X.; Dong, Y. Group decision making based on linguistic distribution and hesitant assessment: Maximizing the support degree with an accuracy constraint. *Inf. Fusion* **2018**, *41*, 151–160.
54. Wang, R.; Li, Y. Generalized single-valued neutrosophic hesitant fuzzy prioritized aggregation operators and their applications to multiple criteria decision-making. *Information* **2018**, *9*, 10, doi:10.3390/info9010010.
55. Garg, H.; Nancy. Linguistic single-valued neutrosophic prioritized aggregation operators and their applications to multiple-attribute group decision-making. *J. Ambient Intell. Humaniz. Comput.* **2018**, doi:10.1007/s12652-018-0723-5.
56. Liang, P.; Song, F. What does a probabilistic interpretation of fuzzy sets mean? *IEEE Trans. Fuzzy Syst.* **1996**, *4*, 200–205.
57. Liu, Z.; Li, H.X. A probabilistic fuzzy logic system for modeling and control. *IEEE Trans. Fuzzy Syst.* **2005**, *13*, 848–859.
58. Xu, Z.; Zhou, W. Consensus building with a group of decision makers under the hesitant probabilistic fuzzy environment. *Fuzzy Optim. Decis. Mak.* **2017**, *16*, 481–503.
59. Hao, Z.; Xu, Z.; Zhao, H.; Su, Z. Probabilistic dual hesitant fuzzy set and its application in risk evaluation. *Knowl.-Based Syst.* **2017**, *127*, 16–28.
60. Zhou, W.; Xu, Z. Group consistency and group decision making under uncertain probabilistic hesitant fuzzy preference environment. *Inf. Sci.* **2017**, doi:10.1016/j.ins.2017.06.004, in press.
61. Zhou, W.; Xu, Z. Expected hesitant VaR for tail decision making under probabilistic hesitant fuzzy environment. *Appl. Soft Comput.* **2017**, *60*, 297–311.
62. Ding, J.; Xu, Z.; Zhao, N. An interactive approach to probabilistic hesitant fuzzy multi-attribute group decision making with incomplete weight information. *J. Intell. Fuzzy Syst.* **2017**, *32*, 2523–2536.
63. Li, J.; Wang, J.Q. Multi-criteria outranking methods with hesitant probabilistic fuzzy sets. *Cogn. Comput.* **2017**, *9*, 611–625.
64. Zhang, S.; Xu, Z.; He, Y. Operations and integrations of probabilistic hesitant fuzzy information in decision making. *Inf. Fusion* **2017**, *38*, 1–11.
65. Wang, Z.-X.; Li, J. Correlation coefficients of probabilistic hesitant fuzzy elements and their applications to evaluation of the alternatives. *Symmetry* **2017**, *9*, 259.
66. Xu, Z. A method based on linguistic aggregation operators for group decision making with linguistic preference relations. *Inf. Sci.* **2004**, *166*, 19–30.
67. Gou, X.; Liao, H.; Xu, Z.; Herrera, F. Double hierarchy hesitant fuzzy linguistic term set and MULTIMOORA method: A case of study to evaluate the implementation status of haze controlling measures. *Inf. Fusion* **2017**, *38*, 22–34.
68. Zhou, H.; Wang, J.Q.; Zhang, H.Y.; Chen, X.H. Linguistic hesitant fuzzy multi-criteria decision-making method based on evidential reasoning. *Int. J. Syst. Sci.* **2016**, *47*, 314–327.
69. Bisht, K.; Kumar, S. Fuzzy time series forecasting method based on hesitant fuzzy sets. *Expert Syst. Appl.* **2016**, *64*, 557–568.

