Performance measurement and decomposition of value added

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Performance measurement and decomposition of value added

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Abstract. In this paper, we benchmark an investment actively managed (e.g., fund, portfolio) against a reference portfolio passively managed replicating the investment’s cash flows in order to measure the value added by the active investment and decompose it according to the influence of the investment choices (i.e., selection and allocation of assets) made in the various periods. The active investment choices are reflected in the investment’s returns as opposed to the benchmark returns earned by the passive strategy. We precisely quantify the impact of the holding period rates on the value added and rank them accordingly, in order to identify the most (and the least) influential ones. The analysis is performed by applying the Finite Change Sensitivity Index (FCSI) method (Borgonovo 2010a, 2010b), a recently-conceived technique of sensitivity analysis, which we refine by means of a duplication-clearing procedure which allows a perfect (i.e., with no residue) decomposition of the value added.

We conduct the analysis for a given contribution-and-distribution policy, characterized by a fixed sequence of deposits and withdrawals. We show that, if the contribution-and-distribution policy changes, the effect of the investment choices made in the various periods on the value added changes as well.

Keywords. Value added, performance measurement, investment policy, sensitivity analysis.

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1 Introduction

A number of metrics are used in practice for measuring the performance of an investment (portfolio of assets, fund, etc.) and a substantial amount of contributions have recently dealt with pros and cons of various metrics from several points of view, all of which taking into account the role of a benchmark return in assessing the investment’s value added (see Long and Nickels 1996, Gredil et al 2014, Magni 2014, Altshuler and Magni 2015, Jiang 2017, Cuthbert and Magni 2018). However, despite the considerable attention drawn on the appropriateness of a performance criterion, the problem of measuring the impact of the investment choices (i.e., selection and allocation of assets) made in a period on the investment’s value added have been neglected. Since decisions about selection and allocation of assets in a given period generate a well-determined holding period rate, this problem boils down to measuring the effect of each investment’s holding period rate on the investment’s value added.

This paper is a first attempt to fill the need of measuring the impact of the investment policy on the value added. As anticipated, we use the effect of a holding period rate on the value added to measure the effect of the decisions made in that period on the overall investment’s value added. We measure the impact of each rate and rank the rates according to their impact on value added, thereby identifying the ones that have been most influential. In this way, the analysis enables measuring the effect of the investment decisions made in every period on the investment’s performance and understand in which periods the most important (and less important) decisions have been made. To accomplish this objective, we assume that the contribution-and-distribution policy is given (i.e., we assume the sequence of deposits and withdrawals is given) and describe an investment’s value added as the change in the capital terminal value obtained by switching from a passive investment in a benchmark portfolio to an active investment generating returns which are different from the benchmark returns. Then, we make use of a recently-conceived technique of sensitivity analysis, which apportions a discrete change in a model output to the discrete changes in the model inputs: The so-called Finite Change Sensitivity Index (FCSI), introduced in Borgonovo (2010a, 2010b). We suitably supplement this technique with a fine-tuning of the FCSI procedure which enables achieving a perfect (i.e., with no residue) decomposition of the investment’s value added.

The remainder of the paper is structured as follows. Section 2 introduces the setting and, in particular, presents the benchmark portfolio and its role in the definition of an investment’s value added. Section 3 introduces the Finite Change Sensitivity Index and the way it triggers a decomposition of the finite change of an objective function. Since the FCSI duplicates the interaction effects, we fine-tune it with a simple duplication-clearing procedure and provide the Clean FCSI. Section 4 uses the Clean FCSI technique for apportioning the effect of the investment decisions made in the various periods to the investment’s value added. Section 5 illustrates the procedure with a numerical example. Some remarks conclude the paper.
2 Benchmark portfolio and value added

Following is a simple description of a model for the (discrete) evaluation of the investment, consisting of a portfolio of assets. An investor invests a capital $B_0$ at time $t = 0$. By selecting the assets and allocating them in every period, the portfolio’s value is increased or decreased. Furthermore, the investor makes decisions about capital contributions or distributions in the various periods, which increase or decrease the amount of capital invested in the portfolio.

We assume that the investment starts at time $t = 0$ and analyze its performance in the time interval $[0, n]$ where, for convenience, we assume that $n$ is the current date.

Let $E_t$ be the end-of-period portfolio’s value and $B_t$ its beginning-of-period value. Let $F_t$ be the investor’s contribution/distribution into/from the portfolio at time $t = 0, 1, \ldots, n - 1$. From the point of view of the investor, a contribution is an outflow ($F_t < 0$), a distribution is an inflow ($F_t > 0$). In particular, at time 0, the contributed amount is an outflow, so $F_0 = -B_0 < 0$. Then, the following relations hold:

\[
B_t = E_t - F_t \\
i_t = \frac{E_t - B_{t-1}}{B_{t-1}} \\
E_t = B_{t-1} \cdot (1 + i_t)
\]

where $i_t$ denotes the rate of return in the period. The first equation says that the beginning-of-period value is obtained by deducting the capital call or adding the contribution made by the investor; the second relation says that the investment’s holding period rate expresses the relative increase in the capital value; the third relation says that the ending value is obtained from the beginning value by marking it up by the return rate $i_t$. The selection and allocation policy affects $i_t$, which in turn affects $E_t$ and, hence, $B_t$. The investor’s choices about withdrawals and deposits affects $B_t$ and, hence, $E_t$. Therefore, both types of policies affect the capital values, but only the investment policy affects $i_t$. The latter is then an appropriate measure of the effect on the value added of the investment policy in a given period.

Let us focus on the terminal date, $t = n$, and on its closing value, $E_n = B_{n-1}(1 + i_n)$.\(^1\)

Using (1) and solving for $t = n$, one can express $E_n$ as a function of the return rates and the cash flows prior to $n$:

\[
E_n = -\sum_{t=0}^{n-1} F_t (1 + i_{t+1})(1 + i_{t+2}) \cdots (1 + i_n).
\]

The above relation tells us that the terminal investment’s value is the compounded amount of the contributions (net of distributions) made by the investor.

Consider now a benchmark index whose holding period rate is denoted as $i^*_t$, and a reference (benchmark) portfolio which acts as the opportunity cost of capital for the investment. More precisely, let us consider what would have occurred if the investor had made the same

\(^1\)If the investment is liquidated at time $n$, then $E_n = F_n$. 

contributions/distributions in the benchmark portfolio. Under this assumption, the investor follows a passive strategy and replicates the investment’s cash flows: Every contribution to the investment is matched by an equal contribution in the benchmark portfolio and every distribution from the investment is matched by an equal distribution from the benchmark. In general, the benchmark portfolio’s value is different from the investment’s value at every date \( t \), which means that the holding period rates \( i_t \) and \( i_t^* \) are different. The difference between the two returns is determined by the active choices of asset selection and stock allocation in period \( t \). In such a way, the benchmark portfolio is a replica of the investment’s cash flows up to (and including) time \( n - 1 \). At time \( n \), the investment’s residual value will differ from the benchmark’s residual value.

Formally, let \( F_t^* = F_t \) be the cash flows in the reference portfolio, \( t = 0, \ldots, n - 1 \). We denote as \( B_t^* \) and \( E_t^* \) the beginning-of-period and end-of-period market value of this benchmark portfolio. Then, the following relations mimic the ones presented in (1):

\[
B_t^* = E_t^* - F_t^* \\
i_t^* = \frac{E_t^* - B_{t-1}^*}{B_{t-1}^*} \\
E_t^* = B_{t-1}^* \cdot (1 + i_t^*).
\]

In \( t = n \), the net value of the benchmark portfolio is \( E_n^* = B_{n-1}^* (1 + i_n^*) \). Analogously to eq. (2), the benchmark terminal net asset value \( E_n^* \) depends on the previous cash flows and the benchmark index return rates:

\[
E_t^* = - \sum_{t=0}^{n-1} F_t (1 + i_{t+1}^*) (1 + i_{t+2}^*) \ldots (1 + i_n^*). \tag{4}
\]

As the investment and the benchmark portfolio release the same sequence of inflows and outflows up to time \( n - 1 \), the investment outperforms the benchmark if and only if the terminal value of the fund is greater than the terminal value of the replicating portfolio: \( E_n > E_n^* \). The difference \( E_n - E_n^* \) is the value added, denoted as VA:

\[
VA = E_n - E_n^* = \sum_{t=0}^{n-1} F_t \cdot \left( (1 + i_{t+1}^*) (1 + i_{t+2}^*) \ldots (1 + i_n^*) - (1 + i_{t+1}) (1 + i_{t+2}) \ldots (1 + i_n) \right). \tag{5}
\]

Therefore, the investment outperforms the benchmark if and only if the value added is positive, \( VA > 0 \).

For a given sequence of injections and withdrawals \((F_0, F_1, \ldots, F_{n-1})\) and a given sequence of benchmark returns \((i_1^*, i_2^*, \ldots, i_n^*)\), the value added by such an investment depends on the active investment decisions, which is reflected in the return vector \((i_1, i_2, \ldots, i_n)\).
3 Finite Change Sensitivity Indices

Sensitivity analysis (SA) is the study of how the variance of the output of a model (numerical or otherwise) can be apportioned to different input key parameters (Saltelli et al. 2004). As such, it aims at quantifying how much of an output change is attributed to a given parameter or a set of parameters. It is widely employed in finance and management (Huefner 1972), for instance in analysing the value creation of industrial projects (Borgonovo and Peccati 2004, 2006; Borgonovo, Gatti, and Peccati 2010; Percoco and Borgonovo 2012; Marchioni and Magni 2018), the composition of optimal financial portfolios (Luo, Seco and Wu 2015), and the effects of corporate debt (Donders, Jara and Wagner 2018; Dèlèze and Korkeamäki 2018).

There exist several SA techniques defined in the literature (see Borgonovo and Plischke 2016, Pianosi et al. 2016, Saltelli et al. 2008, 2004 for reviews of SA methods). Among others, the Finite Change Sensitivity Indices (FCFIs) have been recently conceived for analyzing the effect of the finite changes in the model inputs onto the finite changes of a model output. Formally, let $f$ be the objective function, which maps the vector of inputs (parameters, key drivers) $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ onto the model output $y(x)$:

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad y = f(x), \quad x = (x_1, x_2, \ldots, x_n). \quad (6)$$

Let the inputs vary from $x^0 = (x_1^0, \ldots, x_n^0)$, the so-called base value, to $x^1 = (x_1^1, x_2^1, \ldots, x_n^1)$, the realized value. The corresponding model outputs are $f(x^0)$ and $f(x^1)$, so that the output variation is $\Delta f = f(x^1) - f(x^0)$. Let $(x_j^1, x_{(j)}^0) = (x_1^0, x_2^0, \ldots, x_{j-1}^0, x_j^1, x_{j+1}^0, \ldots, x_n^0)$ be the vector consisting of all the inputs set at their base value $x^0$, except parameter $x_j$ which is given the realized value $x_j^1$. Analogously, let

$$(x_j^1, x_k^1, x_{(-j,k)}^0) = (x_1^0, x_2^0, \ldots, x_{j-1}^0, x_j^1, x_{j+1}^0, \ldots, x_{k-1}^0, x_k^1, x_{k+1}^0, \ldots, x_n^0)$$

be the input vector where $x_j$ and $x_k$ are set to the realized values, while the remaining $n - 2$ are set at their base value, and so forth for all $s$-tuples of inputs, $s = 1, 2, \ldots, n$.

Borgonovo (2010a, 2010b) defines two versions of FCSIs: First Order FCSI and Total Order FCSI. The First Order FCSI of parameter $x_j$ measures the individual effect of $x_j$ (Borgonovo 2010a), $\Delta^1_j f = f(x_j^1, x_{(-j)}^0) - f(x^0)$, and, in normalized version, $\Phi^1_j f = \frac{\Delta^1_j f}{\Delta f}$. On the other side, the Total Order FCSI quantifies the total effect of the parameter, including both its individual contribution and its interactions with other parameters. Let $\Delta_{j,k} f$ be the interaction between $x_j$ and $x_k$, that is the portion of $f(x_j^1, x_k^1, x_{(-j,-k)}^0) - f(x^0)$ not explained by the individual effects $\Delta^1_j f$ and $\Delta^1_k f$: $\Delta_{j,k} f = f(x_j^1, x_k^1, x_{(-j,-k)}^0) - f(x^0) - \Delta^1_j f - \Delta^1_k f$. Similarly, let $\Delta_{j,k,h} f$ be the interaction among the inputs $x_j, x_k$ and $x_h$, which is the portion of $f(x_j^1, x_k^1, x_h^1, x_{(-j,-k,-h)}^0) - f(x^0)$ not explained by the individual effects and by the interactions between any pair: $\Delta_{j,k,h} f = f(x_j^1, x_k^1, x_h^1, x_{(-j,-k,-h)}^0) - f(x^0) - \Delta^1_j f - \Delta^1_k f - \Delta^1_h f - \Delta_{j,k} f - \Delta_{j,h} f - \Delta_{k,h} f$ (analogously for a $s$-tuple, with $s > 3$). The variation of $f$ from $x^0$ to $x^1$ is equal to the sum of individual effects
and interactions, counted only once, between parameters and groups of parameters:

\[
\Delta f = \sum_{i=1}^{n} \Delta_i f + \sum_{j_1 < j_2} \Delta_{j_1,j_2} f + \sum_{j_1 < j_2 < j_3} \Delta_{j_1,j_2,j_3} f + \cdots + \sum_{j_1 < j_2 \cdots < j_s} \Delta_{j_1,j_2 \cdots ,j_s} f + \cdots + \sum_{j_1 < j_2 \cdots < j_n} \Delta_{j_1,j_2 \cdots ,j_n} f,
\]

where \( \sum_{j_1 < j_2 \cdots < j_s} \Delta_{j_1,j_2 \cdots ,j_s} f \) is the sum of the interactions between \( s \)-tuples.

Borgonovo (2010a) defines the Total Order FCSI of \( x_j, \Delta_j^T f \), as the sum of First Order FCSI of \( x_j, \Delta_j^1 f \), and the interaction effect of \( x_j \), identified as \( \Delta_j^2 f \) and called Interaction FCSI. The latter is the sum of every interaction involving \( x_j \):

\[
\Delta_j^2 f = \sum_{j_1 < j_2} \Delta_{j_1,j_2} f + \sum_{j_1 < j_2 < j_3} \Delta_{j_1,j_2, j_3} f + \cdots + \sum_{j_1 < j_2 \cdots < j_n} \Delta_{j_1,j_2 \cdots ,j_n} f. \tag{7}
\]

Therefore,

\[
\Delta_j^T f = \Delta_j^1 f + \Delta_j^2 f = \Delta_j^1 f + \sum_{j_1 < j_2} \Delta_{j_1,j_2} f + \sum_{j_1 < j_2 < j_3} \Delta_{j_1,j_2, j_3} f + \cdots + \sum_{j_1 < j_2 \cdots < j_n} \Delta_{j_1,j_2 \cdots ,j_n} f. \tag{8}
\]

and, in normalized version, \( \Phi_j^T = \frac{\Delta_j^T f}{\Delta_j^T} \).

Computationally, the calculation of the Interaction FCSIs may be extremely burdensome if the model does not contain a very small number of inputs.\(^2\) Borgonovo (2010a, Proposition 1) shows that the following result holds:

\[
\Delta_j^T f = f(x^1_j) - f(x^0_{j(-j)}), \quad \forall j = 1, 2, \ldots, n, \tag{8}
\]

where \((x^0_j, x^1_{j(-j)})\) denotes the vector with each input equal to the realized value \( x^1 \), except for \( x_j \) which is set equal to \( x^0_j \). This enables computing the total FCSI of \( x_j \) with no need of summing the First Order FCSI of \( x_j \) and the Interaction FCSI of \( x_j \).

Unfortunately, the Total Order FCSI does not provide a complete decomposition of the output change:

\[
\sum_{l=1}^{n} \Delta_l^T f \neq \Delta f = f(x^1) - f(x^0) \quad \text{or, equivalently,} \quad \sum_{l=1}^{n} \Phi_l^T \neq 1.
\]

In other words, the sum of Total FCSIs explains less (or more) than 100% of the output change. To understand why, consider that, in the sum of the Interaction FCSIs, \( \sum_{l=1}^{n} \Delta_l^T f \), the pairwise interactions of \( x_j \) and \( x_k \) appear twice (in \( \Delta_j^T f \) and in \( \Delta_k^T f \)); the three-wise interactions of \( x_j \),

\(^2\)The number of interactions between parameters and groups of parameters is equal to \( 2^n - n \).
$x_k$, and $x_h$ appear three times (in $\Delta^f_j f$, in $\Delta^f_k f$, and in $\Delta^f_h f$); and so on for all the $s$-wise interactions, $s = 2, 3, \ldots, n$. This implies that the sum of Interaction FCSIs does not equate the overall interaction effects:

$$
\sum_{l=1}^{n} \Delta^f_l f \neq \sum_{j_1 < j_2} \Delta_{j_1,j_2} f + \sum_{j_1 < j_2 < j_3} \Delta_{j_1,j_2,j_3} f + \cdots + \sum_{j_1 < j_2 < \cdots < j_s} \Delta_{j_1,j_2,\ldots,j_s} f + \cdots + \Delta_{j_1,j_2,\ldots,j_n} f
$$

and, therefore, $\sum_{l=1}^{n} \Delta^f_l f \neq \Delta f$.

However, it is possible to introduce a duplication-clearing factor which eliminates the redundant, multiple interactions and allows a complete and exact decomposition of the output change. We define the Clean Interaction FCSI of $x_j$, $\Delta^f_I j f$, as the product of the Interaction FCSI $\Delta^f_j f$ and a suitable corrective factor:

$$
\Delta^f_I j f = \Delta^f_j f \cdot \frac{\sum_{j_1 < j_2} \Delta_{j_1,j_2} f + \cdots + \sum_{j_1 < j_2 < \cdots < j_s} \Delta_{j_1,j_2,\ldots,j_s} f + \cdots + \Delta_{j_1,j_2,\ldots,j_n} f}{\sum_{j=1}^{n} \Delta^f_j f}.
$$

(9)

Considering that $\Delta^f_j f = \Delta^f f - \Delta^1_j f$ and

$$
\sum_{j_1 < j_2} \Delta_{j_1,j_2} f + \cdots + \sum_{j_1 < j_2 < \cdots < j_s} \Delta_{j_1,j_2,\ldots,j_s} f + \cdots + \Delta_{j_1,j_2,\ldots,j_n} f = \Delta f - \sum_{j=1}^{n} \Delta^1_j f,
$$

one may reframe (9) as

$$
\Delta^f_I j f = \frac{\Delta^f f - \Delta^1 f}{\sum_{l=1}^{n} (\Delta^f_l f - \Delta^1_l f)} \cdot (\Delta f - \sum_{l=1}^{n} \Delta^1_l f).
$$

(10)

In other words, the Clean Interaction FCSI is computed by imputing a share of the overall true interaction effect ($\Delta f - \sum_{l=1}^{n} \Delta^1_l f$) to parameter $x_j$. This share is obtained as the ratio of the Interaction FCSI of $x_j$ and the sum of all Interaction FCSIs.

We define the Clean Total Order FCSI of parameter $x_j$, $\Delta^T f j$, as the sum of individual contribution and Clean Interaction FCSI of $x_j$:

$$
\Delta^T f j = \Delta^1 f j + \Delta^f_I j f
$$

(11)

and, in normalized version, $\Phi^T_I j = \frac{\Delta^T f j}{\Delta f}$. It is easy to see that the Clean Total FCSIs completely explain the output variation:

$$
\sum_{l=1}^{n} \Delta^T f l = \Delta f,
$$

and, in normalized version, $\sum_{l=1}^{n} \Phi^T_I l = 1$.

The sign of a Clean Total FCSI, $\Delta^T f j$, signals the directional effect of an input change onto the output change: A positive (negative) index signals that the change in the input has the effect of increasing (decreasing) the output. The absolute value of the Clean Total FCSI quantifies the
magnitude of the effect; one may then rank the input factors according to their influence on the 
change in the objective function: Input \( x_j \) has higher rank than \( x_j \) if and only if \( |\Delta_j^T f| > |\Delta_j^T f| \).
We denote the rank of parameter \( x_j \) as \( R_j \). The rank vector is \( R = (R_1, R_2, \ldots, R_n) \).

4 Attribution of value added

Let \( x = (x_1, x_2, \ldots, x_n) \) be the vector of time-varying return rates of an investment with cash 
flows \( F_t \) from \( t = 0 \) to \( n - 1 \). Generalizing equations (2) and (4), the terminal net asset value 
implied by the return rates vector \( x \), denoted as \( f(x) \), is, for a given sequence of cash flows 
\( (F_0, F_1, \ldots, F_{n-1}) \), equal to
\[
    f(x) = -\sum_{t=0}^{n-1} F_t (1 + x_{t+1})(1 + x_{t+2}) \ldots (1 + x_n). 
\]
(12)

Let \( x^0 = i^* \) be the stream of benchmark returns (base value). The active investment policy 
followed in the various periods has the effect of moving the rates from \( x^0 = i^* \) to \( x^1 = i \) 
(realized case). This in turn has the effect of changing the terminal value from \( f(x^0) = f(i^*) \) 
to \( f(x^1) = f(i) \). However,
\[
    f(i^*) = E^*_n \quad \text{(13)} \\
    f(i) = E_n. \quad \text{(14)}
\]

Therefore, the value added by the investment may be written as
\[
    VA = E_n - E^*_n = f(i) - f(i^*). \quad \text{(15)}
\]

As a result, the value added is equal to a finite change of \( f \). Therefore, one may apply the 
FCSI technique integrated by the duplication-clearing procedure for decomposing \( VA \) in terms 
of period rates. It is then possible to identify the periods whose investment choices have most 
affected the investment’s performance. In particular, for any given sequence of contributions 
and distributions, the value added may be considered as the sum of all the effects of the active 
selection and allocation choices made in the various periods, as opposed to a passive strategy 
consisting in investing in a benchmark portfolio.

For accomplishing a complete, exact decomposition of the value added, we use the Clean 
FCSIs. Note that the piece of information provided by \( \Phi^T_j \) is not whether and how much 
the investment outperforms or underperforms the benchmark in period \( t \), but whether the 
investment decisions made in period \( t \) have contributed, overall, to outperform or underperform 
the benchmark in the time interval \([0, n]\) and how much of the value added is attributable to 
them. This piece of information necessarily takes account of the interactions with the decisions 
made in the other periods. The decisions made in period \( t \) determine \( i_t \), which measures the 
relative growth in the investment’s value at time \( t \) and, therefore, affect (along with the other
rates) the magnitude of the value added not only in period $t$, but also in the following periods $t + 1, t + 2, \ldots, n$. The Clean Total FCSI, $\Delta^T_j f$, precisely provides the amount of value added that is determined by the investment policy in period $t$.

The analysis above assumes that the policy of contributions and distributions is fixed and equal to $(F_0, F_1, \ldots, F_{n-1})$. Consider now a different sequence of contributions and distributions:

$$(G_0, G_1, \ldots, G_{n-1}) \neq (F_0, F_1, \ldots, F_{n-1})$$

and let

$$g(x) = -\sum_{t=0}^{n-1} G_t (1 + x_{t+1})(1 + x_{t+2}) \ldots (1 + x_n) \quad (16)$$

be the investment’s terminal value. In general, the functions $f(x)$ and $g(x)$ are different, which implies that the value added will be different as well: $f(i) - f(i^*) \neq g(i) - g(i^*)$. In addition, the Clean Total FCSIs of the parameters under $f$ and $g$ will generally be different, implying that the same choices about investments in a given period have a different impact on the value added depending on the choices about injections/withdrawals made by the investor. Therefore, it may occur the case where a given parameter $x_j$ triggered by a given investment policy in period $j$ has a substantial impact on value added for a contribution-and-distribution policy and a negligible impact on value added for a different contribution-and-distribution policy.

In the following section we present a worked example where we measure the impact of the period investment decisions under two different assumptions about contributions and distributions.

5 Worked example

We consider an investment management agreement whereby an investor endows a fund manager the capital amount $B_0 = -F_0 = 100$. The investment lasts $n = 8$ periods and is described in Table 1. The contribution and distribution policy is under full control of the investor, who determines the timing and amount of withdrawals and deposits from $t = 1$ to $t = 7$. The investment policy of the fund manager in period $t$ brings about a return rate equal to $i_t$ in period $t$, $t = 1, 2, \ldots, 8$. In the same period, the benchmark index’s return is $i^*_t$. From (2) and (4), the terminal values of the fund and of the replicating portfolio are $E_8 = 7.71$ and $E_8^* = 5.25$, respectively, implying that, given the sequence of contributions and distributions, the value added is $VA = 2.47 = 7.71 - 5.25 > 0$.

We now decompose the value added in terms of the influences of the active investment choices made in the various periods with respect to a passive investment earning the benchmark return with the same array of contributions and distributions. This is done by evaluating the effect of the change of the terminal value when the return vector is changed from the benchmark return
Table 1: Input data

<table>
<thead>
<tr>
<th>Time (t)</th>
<th>Fund’s cash flows (F_t)</th>
<th>Fund’s returns (i_t)</th>
<th>Benchmark’s returns (i*_t)</th>
</tr>
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<td>-100</td>
<td></td>
<td></td>
</tr>
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</tr>
<tr>
<td>8</td>
<td></td>
<td>4%</td>
<td>5%</td>
</tr>
</tbody>
</table>

vector, i*, to the fund’s return vector, i. To this end, we consider the objective function

\[ f(x) = -\sum_{t=0}^{7} F_t (1 + x_{t+1}) \ldots (1 + x_8) \]

with

\[ x^0 = i^* = (3\%, 4\%, 3\%, 6\%, 1\%, 2\%, 2\%, 5\%) \]

and

\[ x^1 = i = (4\%, 5\%, 2\%, 4\%, 3\%, 3\%, 5\%, 4\%) \]

Table 2 collects the results of the analysis. The first column collects the vector of input parameters, \((x_1, x_2, \ldots, x_8)\), which are determined by the investment choices made in the various periods. The second column describes the First Order FCSIs, the third column is the Total Order FCSI determined via eq. (8), the fourth one collects the Interaction FCSIs calculated as difference between third column and fourth column; the fifth column clears the duplications and supplies the Clean Interaction FCSI, which is computed as in (10); the sixth column represents the Clean Total Order FCSI as defined in (11); the seventh column reports the normalized Clean Total Order FCSI, and, finally the eight column shows the inputs’ ranking.

Table 2: Decomposition of the value added and inputs’ ranking

<table>
<thead>
<tr>
<th>(x_j)</th>
<th>(\Delta^1_j f)</th>
<th>(\Delta^2_j f)</th>
<th>(\Delta^3_j f)</th>
<th>(\Delta^4_j f)</th>
<th>(\Phi^T_j)</th>
<th>(R_j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
<td>1.25</td>
<td>1.29</td>
<td>0.04</td>
<td>-0.02</td>
<td>1.23</td>
<td>49.96%</td>
</tr>
<tr>
<td>(x_2)</td>
<td>0.88</td>
<td>0.91</td>
<td>0.03</td>
<td>-0.02</td>
<td>0.86</td>
<td>34.98%</td>
</tr>
<tr>
<td>(x_3)</td>
<td>-1.12</td>
<td>-1.18</td>
<td>-0.06</td>
<td>0.03</td>
<td>-1.09</td>
<td>-44.24%</td>
</tr>
<tr>
<td>(x_4)</td>
<td>-1.30</td>
<td>-1.38</td>
<td>-0.08</td>
<td>0.05</td>
<td>-1.25</td>
<td>-50.70%</td>
</tr>
<tr>
<td>(x_5)</td>
<td>1.14</td>
<td>1.17</td>
<td>0.03</td>
<td>-0.02</td>
<td>1.13</td>
<td>45.74%</td>
</tr>
<tr>
<td>(x_6)</td>
<td>0.89</td>
<td>0.91</td>
<td>0.03</td>
<td>-0.01</td>
<td>0.87</td>
<td>35.40%</td>
</tr>
<tr>
<td>(x_7)</td>
<td>0.77</td>
<td>0.81</td>
<td>0.04</td>
<td>-0.02</td>
<td>0.75</td>
<td>30.34%</td>
</tr>
<tr>
<td>(x_8)</td>
<td>-0.05</td>
<td>-0.07</td>
<td>-0.02</td>
<td>0.01</td>
<td>-0.04</td>
<td>-1.48%</td>
</tr>
</tbody>
</table>

The most influential input on the value added is the return rate in period 4, \(x_4\), with \(\Delta^4_4 f = -1.25\) and \(\Phi^T_4 = -50.70\%\), implying that it has had a negative effect on the VA and that its magnitude is about half of the value added. In other words, the investment decisions made in the fourth period have overall contributed negatively to the fund’s performance and have had
the greatest impact on the value added.

It is worth noting that the individual contribution of $x_4$ to the value added is obtained with the following argument: Suppose the investor invests passively in the benchmark index from time $t = 0$ to time $t = 3$, then switches to the fund manager’s active investment at time $t = 3$ and then switches back to the benchmark index at time $t = 4$. This strategy results in the following terminal value:

$$E_8 = f(0.03, 0.04, 0.03, 0.01, 0.02, 0.02, 0.05)$$
$$= 100(1.03)(1.04)(1.03)(1.04)(1.01)(1.02)(1.05)$$
$$- 30(1.04)(1.03)(1.04)(1.01)(1.02)(1.05)$$
$$+ 20(1.03)(1.04)(1.01)(1.02)(1.05)$$
$$- 40(1.04)(1.01)(1.02)(1.05)$$
$$- 10(1.01)(1.02)(1.05)$$
$$+ 30(1.02)(1.05)$$
$$- 60(1.02)(1.05)$$
$$- 20(1.05) = 3.95.$$ 

If no switching occurs, the terminal capital value is

$$E_8^* = f(0.03, 0.04, 0.03, 0.06, 0.01, 0.02, 0.02, 0.05)$$
$$= 100(1.03)(1.04)(1.03)(1.06)(1.01)(1.02)(1.05)$$
$$- 30(1.04)(1.03)(1.06)(1.01)(1.02)(1.05)$$
$$+ 20(1.03)(1.06)(1.01)(1.02)(1.05)$$
$$- 40(1.06)(1.01)(1.02)(1.05)$$
$$- 10(1.01)(1.02)(1.05)$$
$$+ 30(1.02)(1.05)$$
$$- 60(1.02)(1.05)$$
$$- 20(1.05) = 5.25.$$ 

The difference, $\Delta_1 f = 3.95 - 5.25 = -1.3$, represents the individual contribution of $x_4$, that is, the impact of the decisions made in period 4 on the value added, taken in isolation from the other inputs. The clean interaction effect is calculated as in eq. (10) and supplies a partial compensating effect, $\Delta_4 f = 0.05$. Overall, the contribution to value of the active investment policy of the fourth period on the investment’s value added is $\Delta^{T}_4 = -1.25$. In relative terms, $x_4$’s weight is $\Phi^{T}_4 = -50.7\%$.

The second and third most influential inputs are the return rates in periods 1 and 5, $x_1$ and $x_5$, which have had a positive effect on value added. In particular, their total contributions are, respectively, $\Delta^{T}_1 = 1.23$ and $\Delta^{T}_5 = 1.13$. In relative terms, their weights are $\Phi^{T}_1 = 49.96\%$ and
$\Phi_S^T = 45.74\%$. Next come $x_3$ (negative impact), $x_6$, $x_2$, $x_7$ (positive impact) and $x_8$ (negative effect). The latter explains just $-1.48\%$ of VA. The Clean Total Order FCSIs exactly decompose the value added:

\[
\begin{align*}
\text{sum of Clean Total FCSIs} & = 1.23 + 0.86 - 1.09 - 1.25 + 1.13 + 0.87 + 0.75 - 0.04 = 2.47 \\
\text{sum of normalized Clean Total FCSI (percentage)} & = 49.96\% + 34.98\% - 44.24\% - 50.70\% + 45.74\% + 35.40\% + 30.34\% - 1.48\% = 100\%.
\end{align*}
\]

Consider now a different contribution and distribution policy, determined by the sequence $(G_0, G_1, \ldots, G_{n-1})$ such that $G_0 = F_0 = -100$ and $G_t = 0$ for $t = 1, 2, \ldots, 7$, and assume that the selection and allocation choices do not vary. The investment’s value added varies; in particular, using (16), the fund’s and the benchmark portfolio’s values at time 8 are, respectively,

\[E_8 = g(i) = 100(1.04)^3(1.05)^2(1.02)(1.03)^2 = 134.20\]

and

\[E_8^* = g(i^*) = 100(1.03)^2(1.04)(1.06)(1.01)(1.02)^2(1.05) = 129.04,\]

implying that the value added is

\[VA = g(i) - g(i^*) = 134.2 - 129.04 = 5.16.\]

The value added has increased with respect to the previous case. The FCSI analysis with duplication-clearing procedure is reported in Table 3, showing that, in the case of no interim contributions and distributions, the same investment choices have a very different impact on the value added. The most influential return rate is $x_7$ ($R_7 = 1$), which has a positive effect on VA. As previously seen, its rank in the case where $(F_0, F_1, \ldots, F_{n-1})$ represented the choices about deposits and withdrawals was only $R_7 = 7$. This means that investment decisions made by the manager in period 7 have the greatest impact if the investor does not make any interim contribution/distribution, whereas they have negligible effect in case of the timing and amounts of cash flows are $(F_0, F_1, \ldots, F_{n-1})$. Conversely, the first-period rate, $x_1$, which reflects the investment decisions made in period 1, has rank 6 ($R_1 = 6$), whereas it represented the second most influential parameter in the previous case.

### 6 Concluding remarks

This paper proposes a method for evaluating the effect of the investment policy on an investment’s performance, as measured by the value added. Specifically, we show how to quantify the part of the value added generated by the investment decisions made in the various periods, given a fixed sequence of cash flows (contributions and distributions). We compare an active investment strategy with a passive investment strategy in a benchmark portfolio and formalize it in terms of difference between terminal values in case of active investment and passive
Table 3: Decomposition of the value added and inputs’ ranking (no interim cash flows)

<table>
<thead>
<tr>
<th>$x_j$</th>
<th>$\Delta_j^1 f$</th>
<th>$\Delta_j^2 f$</th>
<th>$\Delta_j^3 f$</th>
<th>$\Delta_j^4 f$</th>
<th>$\Phi_j^T$</th>
<th>$R_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1.25</td>
<td>1.29</td>
<td>0.04</td>
<td>0.02</td>
<td>1.27</td>
<td>24.63%</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1.24</td>
<td>1.28</td>
<td>0.04</td>
<td>0.02</td>
<td>1.26</td>
<td>24.39%</td>
</tr>
<tr>
<td>$x_3$</td>
<td>-1.25</td>
<td>-1.32</td>
<td>-0.06</td>
<td>-0.03</td>
<td>-1.28</td>
<td>-24.86%</td>
</tr>
<tr>
<td>$x_4$</td>
<td>-2.43</td>
<td>-2.58</td>
<td>-0.15</td>
<td>-0.07</td>
<td>-2.50</td>
<td>-48.54%</td>
</tr>
<tr>
<td>$x_5$</td>
<td>2.56</td>
<td>2.61</td>
<td>0.05</td>
<td>0.02</td>
<td>2.58</td>
<td>49.99%</td>
</tr>
<tr>
<td>$x_6$</td>
<td>1.27</td>
<td>1.30</td>
<td>0.04</td>
<td>0.02</td>
<td>1.28</td>
<td>24.87%</td>
</tr>
<tr>
<td>$x_7$</td>
<td>3.80</td>
<td>3.83</td>
<td>0.04</td>
<td>0.02</td>
<td>3.81</td>
<td>73.91%</td>
</tr>
<tr>
<td>$x_8$</td>
<td>-1.23</td>
<td>-1.29</td>
<td>-0.06</td>
<td>-0.03</td>
<td>-1.26</td>
<td>-24.39%</td>
</tr>
</tbody>
</table>

investments, respectively. This difference, which equals the investment’s value added, depends on the relations between the sequence of benchmark returns and the sequence of investment’s returns. To accomplish the task, we make use of the Finite Change Sensitivity Index (FCSI) technique (Borgonovo 2010a, 2010b) suitably fine-tuned for clearing the double-counting of the interaction effects implied therein. This brings about the Clean Total FCSI which quantifies and ranks the efficacy of the investment policy via the ranking of the effect of the investment returns on the investment’s value added. We also find that, for a given investment policy, not only different contribution-and-distribution policies give rise to different performances but also the effect of the investment decisions have a different impact on the value added. This means that decisions about contributions and distributions and decisions about selection and allocation of assets are strictly intertwined. Further researches may be conducted to assess the degree and the direction of the interaction between investment policy and contribution/distribution policy.

References


Borgonovo, E., Peccati, L. (2004). Sensitivity analysis in investment project evaluation. Inter-


