Patterns of Competitive Interaction

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Abstract

We explore patterns of competitive interaction by studying mixed-strategy equilibrium pricing in oligopoly settings where consumers vary in the set of suppliers they consider for their purchase. In the case of "nested reach" we find equilibria, unlike those in existing models, in which price competition is segmented: small firms offer only low prices and large firms only offer high prices. We characterize equilibria in the three-firm case using correlation measures of competition between pairs of firms. We then contrast them with equilibria in the parallel model with capacity constraints. A theme of the analysis is how patterns of consumer consideration matter for competitive outcomes.

1 Introduction

In settings where consumers vary in the set of suppliers they consider for their purchase, how do outcomes depend on the *patterns* of competitive interactions? The simplest situation in which this question arises is a duopoly in which each firm has some captive customers, while non-captive customers are able to choose whichever firm's offer they like better. With more than two firms, richer patterns of consideration become possible. Some consumers may be captive to particular firms, some might consider the offers of all firms, while others can choose among the offers of various subsets of firms. Competitive outcomes, including patterns of price dispersion, then depend not only on the number and firms and their relative sizes, but also upon the pattern of consumer consideration of firms. The main aim of this paper is to explore this issue in an otherwise standard setting where firms compete in prices using mixed strategies.

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There are various reasons why different consumers have different sets of choices open to them. Perhaps following a prior stage of advertising by firms or search by consumers, some might become aware of a different set of suppliers than do other consumers. For instance, Honka *et al.* (2017, Table 1) document different levels of consumer awareness of various retail banks in a local market. Alternatively, as in Spiegler (2006), there might be horizontal product differentiation such that each consumer considers only a subset of products to be suitable. The set of firms who are currently active in the market might be uncertain (Janssen and Rasmusen (2002)) or the set of firms who choose to post prices on a comparison website might be uncertain (Baye and Morgan (2001)). Some consumers might be constrained in their choices by location, transport costs or switching costs. For instance, some models of spatial competition, such as Smith (2004), suppose that a consumer considers buying from those firms located within a specified radius of her. Consumers might also differ in their ability to make comparisons between offers, with confused consumers choosing randomly between suppliers or buying from a default seller (Piccione and Spiegler (2012), Chioveanu and Zhou (2013)). Our analysis does not take a view on the underlying reason why consumers have different consideration sets. Rather, it takes the distribution of consideration sets in the consumer population as given, and explores the consequences for competition.

A considerable literature has explored aspects of this question, and some settings are well understood—the case with symmetric sellers considered randomly, the case of independent reach, and duopoly. (These special cases are discussed in more detail in section 2.) As to the first of these, Rosenthal (1980) and Varian (1980) considered the situation in which some consumers are randomly captive to particular firms, while others compare the offerings of all firms and buy from the cheapest. There is a symmetric equilibrium with price dispersion, in which all firms choose prices according to the same mixed strategy. Burdett and Judd (1983, section 3.3) analyze a more general symmetric model, in which arbitrary fractions of consumers consider one random firm, two random firms, and so on. Provided some consumers consider just one firm and some consider more than one, the symmetric equilibrium involves price dispersion, and industry profit is proportional to the number of captive consumers who consider just one firm.

With independent reach, studied by Ireland (1993) and McAfee (1994), the fact that a consumer considers one firm does not affect the probability she considers any other firm.[1] Then the firm that reaches the most consumers also has the largest proportion of captive consumers among the consumers within its reach—i.e., the highest captive-to-reach ratio. In equilibrium all firms use the same minimum price, but the maximum price charged is lower for smaller firms. Since firms use the same minimum price, their profits are proportional to their reach. The same is true in duopoly, as analyzed by Narasimhan (1988).[2] In these situations with symmetry, independent reach or duopoly, firms compete head-to-head in price, in the sense that there is a range of prices chosen by all firms.

The aim of the present paper is to take further the analysis of asymmetric cases. In doing so, we discover equilibria with quite different characteristics from those in the literature.[3] In section 3 we consider *nested* reach, in which only the largest firm has any captive customers, and we find equilibria with an "overlapping duopoly" property if the increments between successive firm sizes are non-decreasing. There is an increasing sequence of prices $\{p_k\}$ such that the range of prices that the kth smallest firm might charge is an interval $[p_{k-1}, p_{k+1}]$. Hence small firms charge low prices while large firms charge high prices, so that price competition is segmented instead of head-to-head.

The paper goes on in section 4 to provides a general analysis of the three-firm case. Even with triopoly, a wide variety of patterns of competitive interactions is possible. We define a measure of the competitive interaction between a pair of firms, which reflects *correlation* between consideration of the two firms. When competitive interactions between pairs of firms are similar, as with independent reach, we show that all firms use a common lowest price and hence have profit proportional to their reach. In some of these cases, however, we find that the price support of the least competitive firm might not be an interval—the firm might price high and low but not in an intermediate range. By contrast, when one pair of firms is significantly more competitive than other pairs, the equilibrium has the

[1] Manzini and Mariotti (2014) study a choice model where an agent is aware of a particular option with specified independent probability. In an empirical study of the personal computer market, Sovinsky Goeree (2008) assumes that the reach of the various products is independent.

[2] With duopoly or independent reach, the largest firm chooses the maximum price with positive probability, which could be interpreted as its "regular" price. In Armstrong and Vickers (2019) we use Narasimhan's duopoly framework to investigate the impact of firms being able to offer different deals to captive and contested customers.

[3] An important early exception is the asymmetric model is Baye, Kovenock and De Vries (1992, Section V), where consumers either consider a single firm or all firms, but firms have different numbers of captive customers. They show that all but the two smallest firms choose the monopoly price for sure, while the two smallest firms compete using mixed strategies as in the Narasimhan duopoly model. This is an extreme case of the situation where large firms choose only high prices, which we discuss further at several points in the analysis to follow.

"overlapping duopoly" property—one firm prices low, one high, and one across the full price range. Intuitively, this pair mostly compete with each other, leaving the remaining firm with an incentive to set high prices. When the market changes so that one pair of firms has greater competitive interaction—e.g., if additional consumers consider both firms—this can induce the remaining firm to retreat to its captive base. The triopoly case also allows analysis of the effects of entry. While entry pushes down prices in some cases, there are natural patterns of competitive interaction where, counter-intuitively, the opposite happens and consumers are harmed by entry.

Another setting in which firms have limited reach and use mixed pricing strategies is when they have capacity constraints, as in the classic Bertrand-Edgeworth model—see, for example, Vives (1999, section 5.2) for an overview.[4] For comparison with our main model with consideration sets, section 5 presents the solution to the triopoly version of that model in a simplified setting with unit demand. The closest papers to our analysis are Hirata (2009) and De Francesco and Salvadori (2015), who show how a small firm might be unwilling to price as low as larger firms, and hence obtains a higher profit per unit of capacity than its larger rivals. We solve this capacity model using a similar method as we use in the consideration set model, although the analysis is considerably simplified since there is a clear-cut ordering of firms by capacity. In contrast to the consideration set model, segmented price competition is not possible in the capacity model, nor is it possible for entry by a third firm to harm consumers.

We conclude in section 6 by summarizing our main insights, and suggesting avenues for further research on this topic.

2 A model with consideration sets

There are n firms that costlessly supply a homogeneous product. There is a population of consumers of total measure normalized to 1, each of whom has unit demand and is willing to pay up to 1 for a unit of the product.[5] Consumers differ according to which firms they consider for their purchase, and for each subset $S \subset \{1, ..., n\}$ of firms (including the null

[4] Montez and Schutz (2019) study a duopoly model where both capacity constraints and heterogenous consideration sets play a role.

[5] The positive analysis which follows is not affected if each consumer has a downward-sloping demand function $x(p)$, provided revenue $px(p)$ is an increasing function up to the monopoly price. However, welfare analysis (for instance in our discussion of entry) requires adjustment with downward-sloping demand.

set) suppose that the fraction of consumers who consider exactly the subset S is α_S. (We slightly abuse notation, and write α_1 for the fraction who consider only firm 1, $\alpha_{12} = \alpha_{21}$ for the fraction who consider only firms 1 and 2, and so on.) When there are only few firms the pattern of consideration sets can be illustrated using a Venn diagram, and Figure 1 depicts the market with three firms.[6] Here, a consumer considers a particular subset of firms if she lies inside the "circle" of each of those firms. For instance, a fraction α_{12} of consumers consider the two firms 1 and 2.

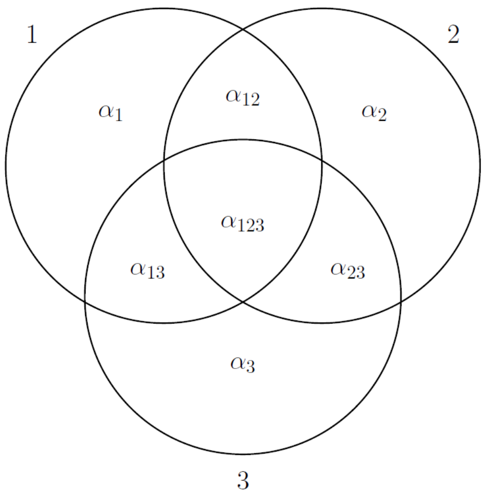

Figure 1: Consideration sets with three firms

A consumer is *captive* to firm i if she considers i but no other other firm, and there is a fraction α_i of such consumers. The *reach* of firm i is the set of consumers who consider the firm, and the fraction of such consumers is denoted σ_i, so that

$$\sigma_i = \sum_{S|i \in S} \alpha_S \ .$$

Finally, the *captive-to-reach* ratio of firm i is denoted ρ_i, where

$$\rho_i = \frac{\alpha_i}{\sigma_i} \ .$$

Firms compete in a one-shot Bertrand manner, and a consumer buys from the firm she considers which has the lowest price (provided this price is no greater than 1). If two or

[6] In a spatial context this Venn diagram has a more literal interpretation: if consumers only consider buying from a firm within a specified distance, then the locations of firms determine the centre of the circles on the diagram. With more firms (and a finite set of consumers), consideration sets can be conveniently depicted using a bipartite graph, where the two groups in the graph are the consumers and the firms, and a line connecting a consumer to a firm corresponds to the former considering the latter. In a very different context, Prat (2018) uses a model of consideration sets similar to that presented here.

more firms choose the same lowest price, suppose the consumer is equally likely to buy from any such firm. Since industry profit is a continuous function of the vector of prices chosen, Theorem 5 in Dasgupta and Maskin (1986) shows that an equilibrium exists. Since an individual firm's profit is usually discontinuous in the price vector, the equilibrium will usually involve mixed strategies for some firms. It is useful to rule out some extreme and uninteresting configurations. The first assumption requires that there be some competitive interaction between sellers:

Assumption 1: Some consumers consider at least two firms.

(If all customers were captive, each firm chooses $p \equiv 1$ for sure.) The second assumption prohibits the possibility that a subset of firms choose the competitive price $p \equiv 0$ for sure, as such firms play no important role in the analysis:

Assumption 2: Every non-empty subset of firms S contains at least one firm with consumers within its reach who consider no other firm in S.

For instance, this assumption rules out the situation where two firms reach precisely the same set of consumers. Intuitively, Assumption 2 ensures that no subset S of firms will set $p \equiv 0$, since there is a firm in S which has some customers with no overlap with other firms in S, and this firm can profitably raise its price above zero. These two assumptions together imply that there is no equilibrium in pure strategies, and at least some firms choose their price according to a mixed strategy.

When firm i chooses price $p \leq 1$ it will sell to a consumer when that consumer is within its reach and when none of the other firms the consumer considers offers a lower price. Therefore, when rival firms $j \neq i$ choose price according to the cumulative distribution function (CDF) $F_j(p)$, firm i's expected demand with price $p \leq 1$ is

$$q_i(p) \equiv \sum_{S|i \in S} \alpha_S \left(\prod_{j \in S/i} (1 - F_j(p)) \right) . \tag{1}$$

Here, the sum takes place over all consumer segments which consider firm i, and for each such segment the product takes place over all rivals for firm i in that segment. (If there are no such rivals, i.e., when the segment comprises firm i's captive customers, we use the convention that this product equals 1.[7]) Equilibrium occurs when each firm i obtains profit

[7]Expression (1) is written without taking into account the possibility of ties; however, Lemma 1 shows that ties do not occur with positive probability.

π_i, chooses price according to the CDF $F_i(p)$, and firm i's profit $pq_i(p)$ is equal to π_i for every price in firm i's support and no higher than π_i for any price outside its support.[8]

The following result collects a number of observations about the nature of equilibrium, some of which are familiar from the existing literature.[9]

Lemma 1 *In any equilibrium:*
(i) firm i obtains profit $\pi_i \geq \alpha_i$, with equality for at least one firm, and the minimum price in its support is no smaller than ρ_i;
(ii) each firm obtains positive profit (even if it has no captive customers) and p_0, the minimum price chosen by any firm, is positive;
(iii) each firm's price distribution is continuous (that is, has no "atoms") in the half-open interval $[p_0, 1)$;
(iv) each price in the interval $[p_0, 1]$ lies in the price support of at least two firms;
(v) if there are three or more firms, there is at least one price which lies in the support of three or more firms, and
(vi) p_0 lies weakly between the second lowest ρ_i and the highest ρ_i. If the firm with the highest ρ_i has p_0 in its support then p_0 is equal to the highest ρ_i.

Proof. This and subsequent proofs are contained in the appendix. ■

Various changes to the market can naturally be studied within this framework of consideration sets. For instance, entry by a new firm can be modelled as a new "circle" superimposed onto the existing Venn diagram. That is, entry does not affect which consumers consider the incumbent firms, and the reach of an incumbent firm is unaffected by entry, although its number of captive customers will weakly fall.[10] Since welfare (consumer surplus plus industry profit) is the total number of consumers reached, it follows that entry (if it is costless) will weakly increase welfare. Likewise, if entry reduces industry profit it will benefit consumers. Relatedly, an increase in a firm's reach is modelled as an expansion of its "circle", so that a larger subset of consumers consider it, while the consumers who

[8] As usual, the support of firm i's price distribution is defined to be the smallest closed set $\mathcal{P} \subset [0, 1]$ such that the probability that the firm chooses a price in $\mathcal{P}$ equals one.

[9] For instance, see McAfee (1994, page 28).

[10] In particular, there is no danger of "choice overload", whereby the number of consumers who compare prices falls when there are more firms, as discussed for instance in Spiegler (2011, page 150).

consider the other firms is unchanged. Mergers also have a natural set-theoretic interpretation in this framework: when two or more firms merge we assume that the merged entity sets the same price to all its customers, and that the set of consumers who consider the merged entity is the union of the sets of consumers who considered the separate firms.[11] Thus, a merger (if there are no accompanying cost synergies) has no impact on welfare, and harms consumers if and only if it increases industry profit. Note that the fraction of consumers reached by the merged firm is no greater than the sum of those reached by the separate firms, while the captive base of the merged firm is no lower than the sum of captives of the separate firms. Finally, a market expansion can be modelled as an increase in the fractions of consumers in each segment of the Venn diagram (taken from the consumer segment who previously considered no firm at all).

As discussed in the introduction, previous work has studied the special cases of duopoly, symmetry arising from random consideration, and independent reach, and we describe those cases here for later reference.

Duopoly: Lemma 1 determines the unique equilibrium when there are two firms, the situation studied by Narasimhan (1988). Suppose firms are labelled so $\sigma_2 \geq \sigma_1$ (which with duopoly implies $\alpha_2 \geq \alpha_1$ and $\rho_2 \geq \rho_1$). Then both firms have the same support for prices, $[p_0, 1]$, where $p_0 = \rho_2$, and firm i has profit $\pi_i = \sigma_i \rho_2$. Note that the smaller firm's profit weakly exceeds its captive profit α_1. The larger firm's profit necessarily increases when its reach increases, as its profit is equal to its fraction of captive customers, which weakly increases. However, the smaller firm's profit could fall when its reach increases, for instance if its own captive base does not change but it expands sufficiently into the rival's captive base to become the larger firm.

Industry profit in equilibrium is

$$\Pi = (\sigma_1 + \sigma_2)\rho_2 = \sigma_1 + \sigma_2 - \alpha_{12} - \alpha_{12}\frac{\sigma_1}{\sigma_2} \ . \tag{2}$$

One can check that industry profit increases with each fraction in the Venn diagram (i.e., with α_1, α_2 and α_{12}), so that any form of market expansion boosts industry profit. Total welfare is the total number of consumers reached, $W = \sigma_1 + \sigma_2 - \alpha_{12}$, and consumer surplus

[11] An alternative approach would be for the merged entity to maintain separate brands and to charge distinct prices for each brand.

is therefore

$$CS = W - \Pi = \alpha_{12}\frac{\sigma_1}{\sigma_2} \ .$$

Thus, keeping reaches constant, consumer surplus increases when the overlap α_{12} is larger, even though fewer consumers are then served. Likewise, consumer surplus decreases when the larger firm's set of captive customers expands, keeping the other regions of the Venn diagram unchanged, even though more consumers are served. A merger from duopoly to monopoly is always profitable, and so harms consumers.

Symmetric firms: Burdett and Judd (1983, section 3.3) study a market with $n \geq 2$ symmetric firms and where consumers consider firms at random (a specified fraction consider one random firm, a specified fraction consider two random firms, and so on). This model can be generalised so that firms are symmetric but consideration sets need not be random. Specifically, suppose that each firm has a_1 captive customers, a_2 consumers who consider exactly one other firm (not necessarily random), and in general a_m consumers who consider $m-1$ other firms for $m \leq n$. Let

$$\phi(x) \equiv a_1 + a_2 x + a_3 x^2 + ... + a_n x^{n-1}$$

be the probability generating function associated with the number of rivals faced by a firm. Here, $\phi(x)$ is convex and increasing, the number of captive customers for each firm is $\phi(0)$, each firm has reach is $\sigma = \phi(1)$ and captive-to-reach ratio $\rho = \phi(0)/\phi(1)$. Assumptions 1 and 2 entail $0 < \phi(0) < \phi(1)$.

In a symmetric market, the unique symmetric equilibrium (which is not necessarily the only equilibrium) is derived as follows. Each firm obtains equilibrium profit $\pi_i \equiv \phi(0)$ and has the minimum price ρ. When each of its rivals uses the CDF $F(p)$, a firm's demand with price $p \leq 1$ in (1) is $q(p) = \phi(1 - F(p))$. Since each firm makes profit $\phi(0)$, the symmetric equilibrium CDF satisfies

$$\phi(1 - F(p)) \equiv \frac{\phi(0)}{p} \ , \tag{3}$$

and the function $F(p)$ strictly increases from 0 to 1 as p increases from ρ to 1.

The models in Rosenthal (1980) and Varian (1980) are a special case of this framework, where consumers either consider one random firm or consider all firms, so that $a_m = 0$ for $1 < m < n$. With this "all-or-nothing" pattern of consideration, Baye *et al.* (1992) show that when $n \geq 3$ there are multiple equilibria (all of which involve the same profit for firms). For instance, all but two firms might choose $p = 1$ for sure, selling only to their captive customers, while the remaining two firms choose prices on the interval $[\rho, 1]$.

In general, entry by a new firm into a symmetric market has ambiguous effects on industry profit and consumer surplus, as we discuss in more detail in section 4. However, a merger between two or more firms in a symmetric market is always profitable. Before merger each firm obtained profit equal to its captive base, and a merger can only increase the merged entity's number of captive customers. A merger cannot decrease the profit of the non-merging firms (since they still obtain at least their captive profit), and so the merger increases industry profit and harms consumers.

Independent reach: Ireland (1993) and McAfee (1994) study the situation where each firm has an independent chance of being considered by a consumer. Specifically, firm i is considered by an independent fraction σ_i of the consumer population, where firms are labelled so that $\sigma_1 \leq \sigma_2 \leq ... \leq \sigma_n \leq 1$. The fraction of consumers who are captive to firm i is $\alpha_i = \sigma_i \Pi_{j \neq i}(1 - \sigma_j)$ and so this firm's captive-to-reach ratio is $\rho_i = \Pi_{j \neq i}(1 - \sigma_j)$. Thus, as with duopoly, the firm with the largest reach is also the firm with the highest captive-to-reach ratio.

If firm j chooses its price with the CDF $F_j(p)$, firm i sells to a consumer if it reaches that consumer (which occurs with probability σ_i) and no rival reaches that consumer with a lower price. The probability that firm j does reach the consumer with a lower price is $\sigma_j F_j(p)$. Therefore, firm i's demand with price $p \leq 1$ in (1) takes the multiplicatively separable form

$$q_i(p) = \sigma_i \prod_{j \neq i} (1 - \sigma_j F_j(p)) \ . \tag{4}$$

Ireland (1993) and McAfee (1994) show that the equilibrium is such that all firms have the same minimum price p_0, which from Lemma 1(vi) is equal to $\rho_n = \Pi_{j=1}^{n-1}(1 - \sigma_j)$, and the profit of firm i is $\pi_i = \sigma_i p_0$. (In particular, unless it is the largest firm, a firm's profit decreases with its reach σ_i when $\sigma_i \geq 1/2$.) Thus, firms' profits are proportional to their reaches, the profit of the largest firm is equal to its number of captive consumers, while the profit of smaller firms is weakly greater than their number of captive consumers. The CDFs which support these equilibrium profits are such that firm i chooses its price with interval support $[p_0, p_i]$, where firm i's maximum price p_i is smaller for smaller firms. The two largest firms choose prices with support $[p_0, 1]$, so that the maximum prices satisfy

$p_1 \leq p_2 \leq ... \leq p_{n-1} = p_n = 1$. Thus price supports are nested, so that smaller firms only offer low prices while the largest firms offer the full range of prices.[12]

With independent reach, industry profit is

$$\Pi = \left(\sum_{i=1}^{n} \sigma_i\right) p_0 = \left(\sum_{i=1}^{n} \sigma_i\right) \prod_{i=1}^{n-1} (1 - \sigma_i) \ . \tag{5}$$

Total welfare is the fraction of consumers who consider at least one firm, which is $1 - \Pi_{i=1}^{n}(1 - \sigma_i)$, and the difference between welfare and profit is consumer surplus

$$CS = 1 - \left(1 + \sum_{i=1}^{n-1} \sigma_i\right) \prod_{i=1}^{n-1} (1 - \sigma_i) \ . \tag{6}$$

Expression (6) can be interpreted as an index of the "competitiveness" of the market in this context. Consumer surplus does not depend on the reach of the largest firm, σ_n, but increases with the reach of each smaller firm.

One can verify that entry by a new firm, also with independent reach, will necessarily increase consumer surplus in (6). If two firms i and j merge, the merged entity has independent reach $\sigma_i + \sigma_j - \sigma_i \sigma_j$. If the merged entity is not the largest firm in the post-merger market, the minimum price p_0 is unaffected by the merger, and since the reach of the merged entity is below the sum of the individual reaches, it follows that the merger is unprofitable for the two firms. A merger which is profitable, therefore, has the merged entity being the largest firm in the market. One can check that this implies that the minimum price p_0 rises with the merger, in which case the non-merging firms also increase their profit after the merger. Therefore, with independent reach a profitable merger must increase industry profit, and hence reduce consumer surplus.

In each of these special cases of duopoly, symmetry and independence, the format of the equilibrium is similar: each firm chooses its price from an interval, all firms have the same minimum price p_0, and as a result a firm's profit is proportional to its reach. All firms compete "head-to-head" in prices, in the sense that there is a range of prices that all firms choose. In the remainder of the paper we show that other possibilities exist outside these special cases. We start in the next section by describing a radically different kind of equilibrium that can occur when firms have *nested* reach.

[12] This equilibrium was subsequently shown by Szech (2011) to be unique.

3 Nested reach

The situation with independent reach has all consumers being equally likely to be reached by a firm, regardless of which other firms they consider. At the other extreme one could envisage consideration sets as being *nested*, in the sense that if firm i reaches a greater fraction of consumers than firm j, all firm j's consumers also consider firm i. For example, an entrant's reach lies inside an incumbent's reach if only a subset of latter's existing customers are willing to consider buying from the entrant. Likewise, if consumers consider options in an ordered fashion, as may be the case with internet search results (where some consumers just consider the first result, others consider the first two, and so on), then the reach of a lower ranked option is nested inside that of a higher ranked option. Alternatively, if consumers only consider the firms whose product they find suits their tastes, then low-quality firms could supply a product which is found suitable by only a subset of the consumers who like the product of a higher-quality firm. With nested reach only the largest firm has any captive customers, and a smaller firm has positive demand only if its price is below all the prices of larger firms.

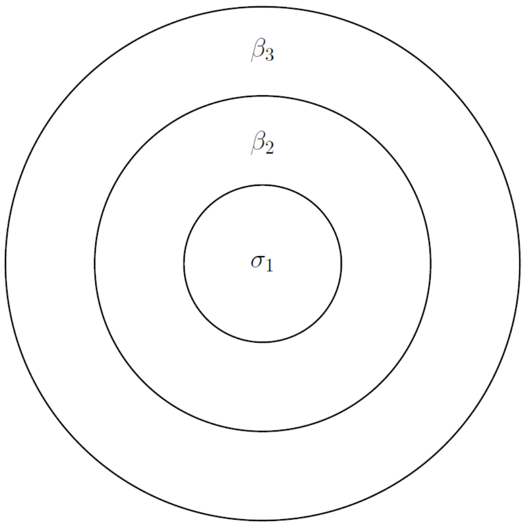

Figure 2: Three firms with nested reach

As depicted in Figure 2, suppose there are $n \geq 3$ firms with nested reach, let firm i have reach σ_i, where firms are ordered as $\sigma_1 < \sigma_2 < ... < \sigma_n$, and for $i \geq 2$ write $\beta_i = \sigma_i - \sigma_{i-1}$ for the incremental reach of firm i. While it is hard to find the equilibrium in all nested situations, the following result describes equilibrium in those cases where incremental reach is larger for larger firms. (This is the case, for instance, if there is a constant rate of attrition in consideration, so that the fraction of consumers who consider firm k is $\sigma_k = \delta^{n-k}$ for some $\delta < 1$.)

Proposition 1 *Suppose $n \geq 3$ firms have nested reach such that*

$$0 < \beta_2 \leq \ ... \ \leq \beta_n \ . \tag{7}$$

Then there is an equilibrium with price thresholds $p_1 < p_2 < ... < p_{n-1} < p_n = 1$ such that the price support of firm 1 is $[p_1, p_2]$, the support of firm n is $[p_{n-1}, p_n]$, and the support of firm $1 < i < n$ is $[p_{i-1}, p_{i+1}]$. Thus, only firms i and $i+1$ (where $1 \leq i < n$) choose prices in the interval (p_i, p_{i+1}). The thresholds are determined recursively by $p_2 = \frac{\sigma_1 + \beta_2}{\beta_2} p_1$ and for $1 < i < n$

$$p_{i+1} = p_i + \frac{\beta_i}{\beta_{i+1}} p_{i-1} \ , \tag{8}$$

where p_1 is chosen to make $p_n = 1$. The profit of firm 1 is $\pi_1 = \sigma_1 p_1$ and the profit of firm $i > 1$ is $\pi_i = \beta_i p_i$.

The format of this equilibrium consists of "overlapping duopolies", where each price is in the support of exactly two firms,[13] and where smaller firms only choose low prices while larger firms only choose high prices.[14] In this sense there is segmented price competition rather than head-to-head price competition, even though there is head-to-head competition in terms of consumer consideration (as firm 1's potential customers consider all firms). Nevertheless, the presence of large firms affects the profits of smaller firms, and (except for the very largest firm) *vice versa*. To illustrate, suppose that $\sigma_1 = \beta_2 = ... = \beta_n \equiv \beta$ so that reach is equally spaced. Then expression (8) implies that $p_{i+1} = p_i + p_{i-1}$, so that $p_i = p_1 \times \varphi_i$ where φ_i is the i^{th} number in the Fibonacci sequence (as given by 1, 2, 3, 5, 8, 13,...). Since $p_n = 1$, it follows that the lowest price is $p_1 = 1/\varphi_n$, in which case $p_i = \varphi_i / \varphi_n$ and the profit of firm i is $\pi_i = \beta \varphi_i / \varphi_n$.

Proposition 1 describes equilibrium only for cases where incremental reach weakly increases. In the next section we specialise the framework to triopoly, and there we will

[13] With the exception of the threshold prices $p_2, ..., p_{n-1}$, which are in the support of three firms.

[14] A similar pattern of segmented pricing is seen in Bulow and Levin (2006). They study a matching model where n heterogeneous firms each wish to hire a single worker from a pool with n heterogeneous workers, where the payoff from a match is (in the simplest version of their model) the product of qualities of the firm and worker. Firms choose wages which they must pay regardless of the quality of the worker eventually hired, workers care only about their wage, and higher quality workers choose their employer first. In equilibrium, firms offer wages according to mixed strategies, where higher quality firms offer wages in a higher range than lower quality firms.

obtain results that imply for the case of nested reach that (i) when $\beta_2 > \beta_1$ the equilibrium in Proposition 1 is unique and (ii) when $\beta_2 < \beta_1$ the equilibrium instead has all three firms using the same minimum price p_0. However, in the latter case we will see that the largest firm can sometimes have a gap in its price support, so that it uses high and low prices but not intermediate prices.

4 The three-firm problem

In the cases considered so far (duopoly, independent reach, and nested reach) there is a clear-cut ordering of the firms, in the sense that a firm with a larger reach also has a weakly higher captive-to-reach ratio. However, more generally those two ways to order firms need not always coincide. For instance, a "niche" firm could have limited reach but have a high proportion of its reach being captive. In this section we allow for general patterns of competitive interaction in the context of triopoly.

Consider the triopoly market shown on Figure 1. For each pair of firms i and j define

$$\gamma_{ij} = \frac{\alpha_{ij} + \alpha}{\sigma_i \sigma_j} ,$$

where to simplify notation we have written $\alpha = \alpha_{123}$. The parameter γ_{ij} reflects *correlation* in the reach of firms i and j: σ_i and σ_j are the respective probabilities that a consumer considers firm i and firm j while $(\alpha_{ij} + \alpha)$ is the probability she considers both firms, and so γ_{ij} is above or below 1 according to whether consideration of firm i is positively or negative correlated with consideration of firm j. With independent reach we have $\gamma_{ij} = 1$, while if the reach of firms i and j is disjoint then $\gamma_{ij} = 0$. The pair of firms with the largest γ_{ij} can be thought of having the "most competitive interaction" in the market, and the remaining firm can be considered to be the "least competitive firm". As we will see, if only two firms choose the lowest price p_0 in equilibrium, while the third firm only uses higher prices, they will be the firms with the largest γ_{ij}.

Similarly, write

$$\gamma = \frac{\alpha}{\sigma_1 \sigma_2 \sigma_3} ,$$

which is again equal to 1 with independent reach. Note that $\sigma_k \gamma \leq \gamma_{ij}$ for distinct i, j and k, with equality if and only if $\alpha_{ij} = 0$. For simplicity, if $F_i(p)$ is firm i's CDF for price in equilibrium write $G_i(p) \equiv \sigma_i F_i(p)$, so that G_i increases from zero to σ_i. Using this notation, firm i's demand at price p in (1) is

$$q_i = \alpha_i F_j F_k + \sigma_i(1 - F_j)(1 - F_k) + (\alpha_i + \alpha_{ij})(1 - F_j)F_k + (\alpha_i + \alpha_{ik})F_j(1 - F_k)$$

$$= \sigma_i + \alpha F_j F_k - (\alpha + \alpha_{ij})F_j - (\alpha + \alpha_{ik})F_k \tag{9}$$

$$= \sigma_i[1 + \gamma G_j G_k - \gamma_{ij} G_j - \gamma_{ik} G_k] \ . \tag{10}$$

Our main result in this section shows that the form of equilibrium depends on whether or not the competitive interactions between firms, measured by γ_{ij}, are similar or asymmetric.

Proposition 2 *Suppose that firms are labelled so that firms 2 and 3 are the most competitive pair of firms, i.e.,* $\gamma_{23} \geq \max\{\gamma_{12}, \gamma_{13}\}$.
(i) If

$$\gamma \min\{\sigma_2, \sigma_3\} < \gamma_{12} + \gamma_{13} - \gamma_{23} \tag{11}$$

then in equilibrium all firms have the same minimum price p_0*, which is the highest captive-to-reach ratio among the firms;*
(ii) If

$$\gamma \min\{\sigma_2, \sigma_3\} > \gamma_{12} + \gamma_{13} - \gamma_{23} \tag{12}$$

then equilibrium takes the form of "overlapping duopoly". In particular, if firms 2 and 3 are labelled so $\sigma_3 \leq \sigma_2$*, then there are prices* p_0 *and* p_1*, with* $p_0 < p_1 \leq 1$*, such that firm 3 has price support* $[p_0, p_1]$*, firm 2 has support* $[p_0, 1]$ *and firm 1 has support* $[p_1, 1]$*. (If* $\sigma_2 = \sigma_3$ *then* $p_1 = 1$ *and firm 1 chooses* $p \equiv 1$ *for sure.) Explicit expressions for the thresholds* p_0 *and* p_1*, as well as for the profits of the three firms, are given in the proof.*

This result shows that only limited kinds of pricing patterns can emerge in equilibrium. For example, it cannot be that two firms choose prices over a range $[p_0, 1]$ while the third firm only chooses from an intermediate or upper range of prices.

Clearly, part (i) of this result applies when the competitive interactions are similar across pairs of firms (and where some consumers consider exactly two firms so that $\gamma\sigma_k < \gamma_{ij}$), as is the case with independent reach. Indeed, part (i) applies if the two most competitive pairs are approximately equally competitive: if say $\gamma_{23} = \gamma_{13} \geq \gamma_{12}$ and there are some consumers who consider exactly two firms then condition (11) is satisfied. In particular, if in the statement of Proposition 2 there is a "tie" for which pair of firms is the most competitive, then part (i) must apply. With nested reach the two smallest firms are the most competitive pair and condition (11) requires that incremental reach is smaller for larger firms. Thus with three firms, the cases not covered by Proposition 1 have all firms using the same minimum price.

Part (ii) applies when one pair of firms has significantly more competitive interaction than other pairs. For instance, if firms 2 and 3 are considered by almost the same set of consumers (so their circles on the Venn diagram almost coincide), and if $\alpha_1 > 0$, then firms 2 and 3 are the most competitive pair and condition (12) is satisfied, and the least competitive firm 1 chooses price $p \approx 1$. Intuitively, when two firms reach nearly the same set of consumers, they compete fiercely between themselves, leaving the remaining firm to price at or near the monopoly level. Likewise, if firm 1 has a large captive base so that α_1 is large (and when firms 2 and 3 have some overlap), then firms 2 and 3 are the most competitive pair and condition (12) is satisfied. With nested reach, condition (12) requires that incremental reach is larger for larger firms, thus verifying Proposition 1. Another situation where (12) holds is the specification in Baye *et al.* (1992, Section V), where no consumer considers exactly two firms and $\sigma_1 > \sigma_2 \geq \sigma_3$, in which case $\gamma_{ij} = \alpha/(\sigma_i\sigma_j)$ and the two smaller firms 2 and 3 are the most competitive pair. Yet another configuration where part (ii) applies is when two firms have disjoint reach, so that $\gamma_{13} = \gamma = 0$ say, in which case (12) holds whenever $\gamma_{12} \neq \gamma_{23}$.

In the knife-edge case where

$$\gamma \min\{\sigma_2, \sigma_3\} = \gamma_{12} + \gamma_{13} - \gamma_{23} \; , \tag{13}$$

which is not covered by Proposition 2, there is the possibility that both kinds of equilibrium coexist. For instance, this is so in the symmetric Varian-type market where $\alpha_{12} = \alpha_{13} = \alpha_{23} = 0$ and $\alpha_1 = \alpha_2 = \alpha_3$, where there is a symmetric equilibrium where all firms price low and also asymmetric equilibria where one of the firms chooses $p \equiv 1$. (See Baye *et al.* (1992) for the full range of equilibria in this market.) Another example with multiple equilibria is when two firms have disjoint reach and each lies inside the reach of the third firm, where again (13) is satisfied.[15]

[15] Inderst (2002, section 3) presents a model where two symmetric firms each have reach which lies inside that of a larger firm. This configuration could fall into either part (i) or part (ii) of Proposition 2, depending on the extent of overlap between the smaller firms. The paper does not derive the equilibrium, but argues that the expected price chosen by the large firm is lower when there is overlap between entrants compared to when the entrants' reach is disjoint.

The impact of entry: As an application of this analysis, consider the impact of entry by a third firm E into a duopoly market with incumbents A and B. One immediate point is that the *external* impact of entry on consumers and incumbents cannot be positive. The impact on total welfare is the extra consumer segment reached by the entrant, which is the entrant's captive base α_E. However, the entrant's equilibrium profit must be at least α_E, and so the sum of incumbent profit and consumer surplus must weakly fall. As a corollary to this, if entry does not induce a fall in the market minimum price, p_0, then consumers must be harmed by entry. If p_0 does not fall then neither does an incumbent's profit (since it could choose price equal to the new p_0 to obtain profit $\sigma_i p_0$, but may do better than this, and $\sigma_i p_0$ is no lower than its profit before entry), and hence consumer surplus must weakly fall with entry.

In many situations, entry will induce the minimum price to fall. Consider for example a symmetric market where n firms each reach an independent fraction σ of consumers. Then (3) implies that each firm chooses price with CDF F satisfying

$$\left(\frac{1-\sigma F(p)}{1-\sigma}\right)^{n-1}=\frac{1}{p}\ .$$

This CDF increases with n, so the presence of one more firm causes each incumbent to reduce its price in the sense of first-order stochastic dominance. Such a change must benefit all consumers, including those who remain captive to incumbents after entry.

Other patterns of entry could be less "balanced", however, and might induce an incumbent to "retreat" to its captive base by raising its price, thereby harming its captive customers. To illustrate, consider an extreme case where the entrant's reach coincides exactly with the reach of one of the incumbents (a situation which does not satisfy Assumption 2). Then these firms will set $p \equiv 0$, while the other incumbent chooses $p \equiv 1$ and fully exploits its captive customers. Nevertheless, since entry of this form reduces industry profit, consumers in aggregate will benefit.

Finally, consider entry which does not induce a fall in the minimum price, and therefore harms consumers in aggregate. One situation where this happens is when incumbents are symmetric and the entrant is considered only by those consumers who already consider both incumbents, as illustrated on Figure 3. This pattern of consideration is reasonable if only "savvy" consumers consider buying from the entrant, and these are the consumers who are already willing to consider both incumbents. In this case part (i) of Proposition 2 applies to the post-entry market (provided the entrant's reach lies strictly inside the incumbents' overlap). The minimum price is equal to an incumbent's captive-to-reach ratio, which is unchanged with entry. Thus, entry of this form harms consumers. In fact, it is perfectly possible that even the consumers who consider all three firms are harmed by this form of entry, despite being able to choose among more firms, as the higher prices offered by incumbents leave the entrant relatively free to set high prices too.

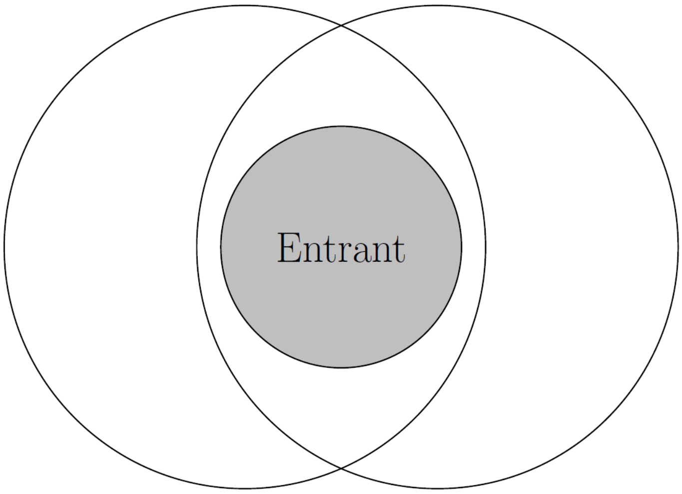

Figure 3: Entry into the contested market

This result is related to Rosenthal (1980), where entry by a new firm causes the average price paid by both captive and informed consumers to rise. However, in his model the entrant arrives with its own new pool of captive customers, thus raising welfare, whereas the effect arises in our scenario despite the entrant having none.[16]

The impact of market expansion and of mergers: Another useful comparative statics exercise is to consider the impact of a market expansion. An old intuition is that an increase in the number of comparison shoppers—consumers who compare prices from several firms—induce firms to lower their prices, which benefits all consumers including captives. This is true in a duopoly setting or in the "all-or-nothing" consideration pattern in a Varian-type model, but is less clear more generally. In particular, if the competitive interaction between

[16]Relatedly, in a setting with differentiated products, Chen and Riordan (2008) show how entry to a monopoly market can induce the incumbent to raise its price. For instance, entry by generic pharmaceuticals might cause a branded incumbent to raise its price, as it prefers to focus on those "captive customers" who care particularly about its brand. Closer to the consideration set framework is Chen and Riordan (2007), who study a model with symmetric firms, where consumers either consider a single random firm or consider a random pair of firms. Among other results, they show that the equilibrium price can increase when an additional firm enters.

one pair of firm increases disproportionately, this could give a third firm an incentive to *raise* its price, thereby harming its captive customers. To illustrate, starting from a symmetric triopoly market, if we increase α_{23} then part (ii) of Proposition 2 will eventually apply, in which case firm 1 will focus on exploiting its captive base and choose $p \equiv 1$. Thus, increased competition between two firms can harm the captives of a third firm.[17]

Consider next the impact of a market expansion on industry profit. With duopoly, we have seen that an increase in any or all of the three parameters α_1, α_2 and α_{12} must increase industry profit (although it might reduce one firm's profit). With duopoly, increasing the size of the overlap region α_{12} will intensify competition (in the sense that the minimum price p_0 is reduced), but this is outweighed by impact on each firm's reach so that $(\sigma_1+\sigma_2)p_0$ rises. With triopoly, by contrast, increasing the fractions in some regions of the Venn diagram can intensify competition to an extent that outweighs the market expansion effect, so that industry profit falls. To see this, consider a triopoly market where part (i) of Proposition 2 applies, in which case industry profit is

$$\Pi = (\sigma_1 + \sigma_2 + \sigma_3)p_0 \ , \tag{14}$$

where p_0 is the highest captive-to-reach ratio. If firm 1 has the highest captive-to-reach ratio, then a small increase in that firm's overlap regions α_{12}, α_{13} or α will keep the form of the equilibrium unchanged, but the minimum price p_0 will fall. Firm 1's profit is unchanged (since it obtains its captive profit regardless), and one can calculate that the impact on industry profit (14) of a small increase in α_{12} or α_{13} is negative if $\sigma_1 < \sigma_2 + \sigma_3$, while a small increase in α reduces profit if $2\sigma_1 < \sigma_2 + \sigma_3$.

Such situations can be adapted to show how a merger which profits the merging parties might lower industry profit, and hence benefit consumers. Suppose three firms, 1, 2 and 3, serve a population of consumers, and that firm 1 obtains exactly its captive profit. Suppose as above that adding a set of consumers C to this market, all of whom lie inside firm 1's reach, reduces industry profit. (However, firm 1's profit cannot fall with this change, since it obtained its captive profit before, and its number of captives does not fall.) Next, in this expanded population suppose there are two further firms, 4 and 5, which (departing from

[17] A similar effect can occur when the fraction of consumers who consider all three firms rises. For instance, suppose consumers segments are (proportional to) $\alpha_1 = 3$ and $\alpha_2 = \alpha_3 = \alpha_{12} = \alpha_{13} = \alpha_{23} = 1$, then for any α firms 2 and 3 are the most competitive pair, and for small α part (i) of the proposition applies, while if α is increased part (ii) eventually applies in which case firm 1 chooses $p \equiv 1$. Here, a increase in α affects the interaction between firms 2 and 3 disproportionately, and tilts the market towards segmented pricing.

Assumption 2) both reach exactly this set C of consumers. Since these two firms reach the same consumers, they will charge $p \equiv 0$ for sure, and for firms 1, 2 and 3 the market is as if the C consumers were absent. Now consider a merger between the three firms 1, 4 and 5. The effect of this merger on industry profit is the same as the effect of introducing the C consumers into the original three-firm situation, which is negative by assumption. Thus the merger is beneficial for consumers. It is also profitable for the merging parties because firm 1 made its captive profit before the merger while firms 4 and 5 made zero profit. This example shows that not all profitable mergers in our setting are detrimental to consumers.

But such mergers appear to be relatively rare. For instance, consider a triopoly market where part (i) of Proposition 2 applies. As with our discussion of mergers with independent reach in section 2, for a merger between two of the firms to be profitable, the minimum price p_0 must rise after the merger, and this benefits the non-merging firm too. Such a merger will therefore harm consumers.

Equilibrium strategies when all firms use the same minimum price: Proposition 2 provided much information about equilibria in this model—it characterises equilibrium profit and consumer surplus in the two regimes, and it describes equilibrium strategies when part (ii) applies. However, it does not describe equilibrium strategies for part (i), and the equilibrium patterns of prices turn out to have interesting economic properties.

In the earlier version of this paper (Armstrong and Vickers, 2018, Proposition 2) we calculated an equilibrium whenever part (i) applied (without showing if it was unique), and this took one of two forms: either (a) the three firms were active in a lower price range and then two were active in range of higher prices, or (b) the three firms were active in a lower price range, then only the most competitive pair were active in an intermediate price range, and then another pair of firms were active in a higher range. In particular, in situation (b) the least competitive firm used low and high prices, but not intermediate prices.

The general analysis was complicated, and the main insights can be obtained more transparently in the simpler case with nested reach, as presented in this result.

Proposition 3 *Suppose three firms have nested reach, where firm 1 has reach σ_1, firm 2 has reach $\sigma_2 = \sigma_1 + \beta_2$, and firm 3 has reach $\sigma_3 = \sigma_2 + \beta_3$.*
(i) If $\beta_3/\beta_2 \geq 1$ then firm 1 has support $[p_0, p_1]$, firm 2 has support $[p_0, 1]$ and firm 3 has

support $[p_1, 1]$*, where*

$$p_0 = \frac{\beta_2\beta_3}{\sigma_2\beta_3 + \beta_2^2}$$

and

$$p_1 = \frac{\sigma_2\beta_3}{\sigma_2\beta_3 + \beta_2^2} > \frac{1}{2} \ . \tag{15}$$

(ii) If $\beta_2/\sigma_2 < \beta_3/\beta_2 < 1$ *then firm 1 has support* $[\rho_3, p_1]$*, firm 2 has support* $[\rho_3, 1]$ *and firm 3 has support* $[\rho_3, \hat{p}] \cup [p_1, 1]$*, where* $\rho_3 = \beta_3/\sigma_3$ *is the highest captive-to-reach ratio and*

$$\hat{p} = 1 - p_1 = \frac{\beta_2^2}{\sigma_2\beta_3 + \beta_2^2} < \frac{1}{2} \ . \tag{16}$$

(iii) If $\beta_3/\beta_2 \leq \beta_2/\sigma_2$ *then firm 1 has support* $[\rho_3, p_1]$ *and firms 2 and 3 have support* $[\rho_3, 1]$*.*

The case of three nested firms can therefore exhibit three distinct patterns of price competition, depending on the relative sizes of demand increments. If the largest firm's captive portion is relatively small, firms compete head-to-head as in the case with independent reach—i.e., all price low and two price high. If the largest firm's captive portion is relatively large, it only prices high and we have overlapping duopoly pricing. In between are equilibria in which the largest firm prices low and high but not in a mid range.

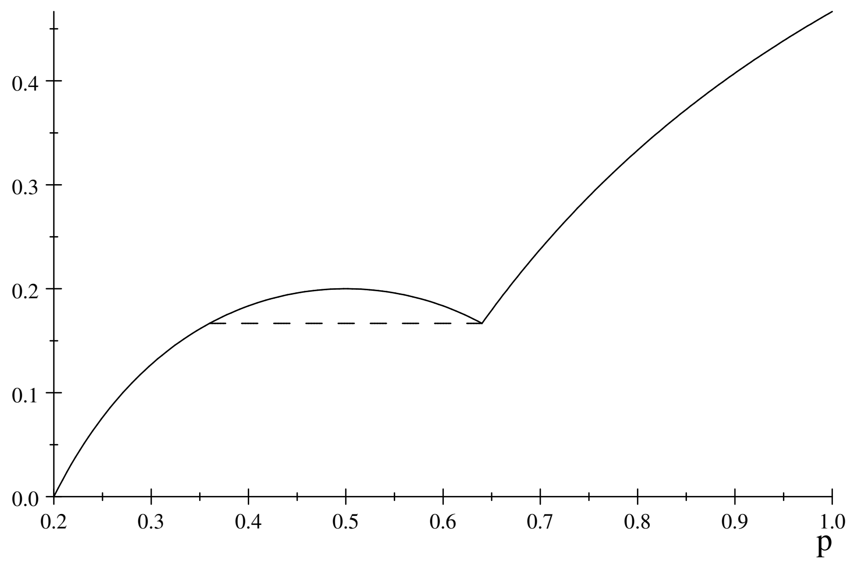

Figure 4: "Ironing" in nested market with $\sigma_1 = 1/2$, $\sigma_2 = 4/5$, $\sigma_3 = 1$

The reason why the largest firm has non-convex price support can be explained as follows. When all firms price low in equilibrium, so that part (i) of Proposition 2 applies, one can calculate that the three CDFs increase in p for prices just above p_0, the minimum price. (This is ensured by condition (11).) One can also calculate the smallest price, p_1 say, at which some CDF reaches 1 and above which the two remaining firms compete as duopolists for prices up to 1. (In the nested case, it is the smallest firm's CDF which first reaches 1, although in the general model more detailed analysis is required to determine which firm first drops out.)

However, in some cases—in the nested case those covered by part (ii) of Proposition 3—the least competitive firm's candidate CDF (i.e., when we ignore the monotonicity constraint on the CDF) starts to *decrease* in p before the largest CDF reaches 1, which cannot therefore be a valid CDF. Figure 4 illustrates an example with nested reach where $\sigma_1 = 1/2$, $\sigma_2 = 4/5$ and $\sigma_3 = 1$, and the solid curve depicts the largest firm's candidate CDF if we ignored its monotonicity constraint. The correct CDF for the largest firm is then obtained by "ironing" this curve as shown on the figure, so that the largest firm does not choose prices in the interval denoted by the dashed line, which from (15)–(16) is the interval $(9/25, 16/25)$ in this example. This procedure is valid as long as the decreasing candidate CDF does not become negative before the largest CDF reaches 1, and this is ensured by the condition $\beta_2 > \beta_3$ in Proposition 3 (or more generally by condition (11) in Proposition 2). As $\beta_3/\beta_2 \to 1$, this gap in the least competitive firm's support widens, until eventually this firm does not compete using low prices at all.

The equilibria with "ironing"—when one firm's price support has a gap in the middle—provide insight into the relationship between the two parts of Proposition 2. As the nested example in Proposition 3 illustrates, as parameters move from satisfying (11) towards satisfying (12), an equilibrium with ironing emerges, and the lower element of the price support of the firm in question shrinks until it disappears, leaving an equilibrium of the overlapping duopoly form.

5 A model with capacity constraints

As discussed in the introduction, another circumstance in which firms have limited reach is when they have capacity constraints, as in the Bertrand-Edgeworth model of competition. A natural question is how equilibria in this scenario compare with equilibria in our main model with consideration sets. To address this question in the most direct way we assume there are three firms and that consumers have unit demands and homogenous valuations (which avoids the need to posit a particular rationing rule). As we explain, for some configurations of capacities (and always when there are just two firms), equilibria in the Bertrand-Edgeworth model resemble those that arise in the model with consideration sets. But for other configurations they are quite unlike any such equilibria.

Suppose there is a continuum population of identical consumers of measure 1 who each consider all prices and are willing to pay 1 for a unit of homogeneous product. Firm $i = 1, 2, 3$ can costlessly supply any quantity up to its capacity κ_i but cannot supply beyond this. A consumer tries to buy at the lowest available price, but is not always able to do so: once the capacity of the cheapest firm is exhausted, remaining consumers then try to buy from the second cheapest firm, and then any remaining consumers buy from the third firm.

We make the following assumptions about capacities:

$$0 < \kappa_3 \leq \kappa_2 \leq \kappa_1 < 1 \ , \tag{17}$$

$$\mu \equiv \kappa_1 + \kappa_2 + \kappa_3 - 1 > 0 \ , \tag{18}$$

$$\kappa_2 + \kappa_3 < 1 \ . \tag{19}$$

Condition (17) reflects our labelling convention in this section, and has the substantive assumption that no firm can supply all consumer demand on its own.[18] Here, κ_i is firm i's supply when it offers a price below both its rivals, and corresponds to "reach" in our main model. In (18) μ is the excess of total capacity over demand, and unless it is positive there is no competition between firms and the equilibrium price for each firm is $p \equiv 1$. Firm i's supply if it offers a higher price than both its rivals is $1 - \kappa_j - \kappa_k$ if this is positive, and this represents the firm's captive customers. (Since $\mu > 0$, a firm is not capacity constrained when undercut by both rivals, and can only supply its residual demand $1 - \kappa_j - \kappa_k$, if any.) Firm i has captive customers if and only if $\kappa_i > \mu$, and (19) ensures that the largest firm has captive customers (otherwise equilibrium involves all firms choosing the price $p \equiv 0$).

It is convenient to focus on a firm's "contested" customers, defined to be its capacity minus is captive customers, and for firm i denote this by

$$\beta_i = \kappa_i - \max\{1 - \kappa_j - \kappa_k, 0\} = \min\{\kappa_i, \mu\} \ .$$

[18]The situation where one firm has capacity to serve all demand is analyzed as Case 1 in Hirata (2009), who shows there is an indeterminacy in the equilibrium price distributions for the smaller firms.

Note that $\beta_3 \leq \beta_2 \leq \beta_1 = \mu$. Firm i's captive-to-reach ratio is $1 - \beta_i/\kappa_i$, so that $\rho_3 \leq \rho_2 \leq \rho_1$, and unlike the consideration set framework here firms are necessarily ordered so that firms with large reach have a large captive-to-reach ratio. Dasgupta and Maskin (1986) ensures existence of equilibrium in this Bertrand-Edgeworth market, while our earlier Lemma 1 continues to apply.

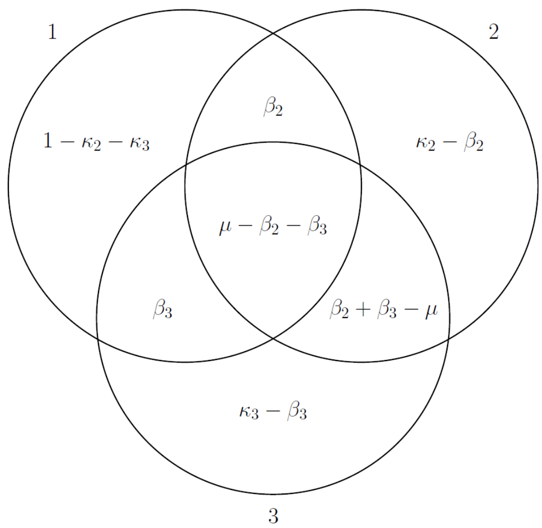

Figure 5: Interpreting the capacity model in terms of consideration sets

When its rivals use CDFs F_j and F_k to choose their prices, firm i's expected sales with price $p \leq 1$ is

$$\begin{aligned} q_i &= F_j F_k(\kappa_i - \beta_i) + (1 - F_j)(1 - F_k)\kappa_i \\ &\quad + (1 - F_j)F_k \min\{\kappa_i, 1 - \kappa_k\} + F_j(1 - F_k)\min\{\kappa_i, 1 - \kappa_j\} \ . \end{aligned}$$

For instance, if firm j undercuts firm i and firm k does not, firm i can supply the residual demand $1 - \kappa_j$ or its capacity κ_i, whichever is the smaller. Noting that

$$\min\{\kappa_i, 1 - \kappa_j\} = \kappa_i - \mu + \min\{\mu, 1 - \kappa_j - (\kappa_i - \mu)\} = \kappa_i - \mu + \beta_k \ ,$$

we can rewrite this expression for q_i as

$$q_i = \kappa_i + [2\mu - \beta_1 - \beta_2 - \beta_3]F_j F_k - [\mu - \beta_j]F_j - [\mu - \beta_k]F_k \ .$$

Comparing this expression to (9) shows that this market is equivalent to a market with consideration sets where firm i has $\kappa_i - \beta_i$ captive customers, $[\beta_i + \beta_j - \mu]$ customers who also consider firm j, $[\beta_i + \beta_k - \mu]$ customers who also consider firm k, and $[2\mu - \beta_1 - \beta_2 - \beta_3]$

customers who consider both rivals. Noting that $\beta_1 = \mu$ and that $\kappa_1 - \beta_1 = 1 - \kappa_2 - \kappa_3$, this demand system can be depicted as the Venn diagram shown on Figure 5, where the weights in the segments sum to total demand 1.

Here, the term $\beta_2 + \beta_3 - \mu$ is strictly positive.[19] Therefore, the consumer segment which "considers" all three firms has *negative* weight $\mu - \beta_2 - \beta_3 < 0$, and this crucial difference with the consideration set model can affect the structure of equilibrium. In particular, with three firms the capacity model is never isomorphic to a model with consideration sets.

As with expression (10), firm i's demand can be written succinctly as

$$q_i(p) = \kappa_i[1 + \hat{\gamma} G_j G_k - \hat{\gamma}_{ij} G_j - \hat{\gamma}_{ik} G_k] \ , \tag{20}$$

where $G_j(p) \equiv \kappa_j F_j(p)$, and

$$\hat{\gamma}_{12} = \frac{\mu - \beta_3}{\kappa_1 \kappa_2} \ , \ \hat{\gamma}_{13} = \frac{\mu - \beta_2}{\kappa_1 \kappa_3} \ , \ \hat{\gamma}_{23} = 0 \text{ and } \hat{\gamma} = \frac{\mu - \beta_2 - \beta_3}{\kappa_1 \kappa_2 \kappa_3} \ . \tag{21}$$

Here, $\hat{\gamma}_{12} \geq \hat{\gamma}_{13} \geq \hat{\gamma}_{23} = 0$. Therefore, using the terminology from section 4, it is the two largest firms which have the greatest competitive interaction. The following result is analogous to Proposition 2, and characterizes when it is an equilibrium for all firms to price low.

Proposition 4 *(i) If $\hat{\gamma}_{12} = \hat{\gamma}_{13}$ then in equilibrium all firms have the same minimum price $p_0 = (1 - \kappa_2 - \kappa_3)/\kappa_1$, which is the captive-to-reach ratio of the largest firm.*
(ii) If $\hat{\gamma}_{12} > \hat{\gamma}_{13}$ then in equilibrium only the two largest firms offer the lowest price p_0, which again is the captive-to-reach ratio of the largest firm.

There are just two ways to achieve the condition $\hat{\gamma}_{12} = \hat{\gamma}_{13}$. Either $\hat{\gamma}_{12} = \hat{\gamma}_{13} = 0$, in which case all three firms have captive customers.[20] Alternatively, $\hat{\gamma}_{12} = \hat{\gamma}_{13} > 0$, when neither firm 2 or 3 has captive customers, which requires $\kappa_2 = \kappa_3$ so that the two smaller firms are exactly the same size. Therefore, if firm 3 has no captive customers and is strictly smaller than firm 2, part (ii) of the proposition applies.

Part (ii) does not characterize the smallest firm's profit or the equilibrium strategies. However, in the earlier version of this paper (Armstrong and Vickers, 2018, Proposition 3)

[19] Since $\beta_i = \min\{\kappa_i, \mu\}$, the only way the term could be negative is if both κ_2 and κ_3 were below μ, in which case $\beta_2 + \beta_3 - \mu = \kappa_2 + \kappa_3 - \mu$, which is positive since $\kappa_1 < 1$.

[20] It is straightforward to extend this result—that when even the smallest firm has captive customers the equilibrium has all firms pricing low—to an arbitrary number of firms.

we calculated an equilibrium (for which we did not show uniqueness) whenever part (ii) applied. In that equilibrium the two largest firms have price support in the whole range $[p_0, 1]$, while the smallest firm chooses it price from a strictly interior interval $[p', p'']$, where $p_0 < p' < p'' < 1$. Thus the smallest firm obtains strictly greater profit per unit of capacity than its larger rivals. This pattern of pricing is not possible in the main model with consideration sets, where the only possibilities were for all firms to price low or for there to be overlapping duopoly. Conversely, one can show that the overlapping duopoly pattern is *not* possible in this capacity model.[21] Thus the segmented price competition sometimes seen in the consideration sets framework does not appear with Bertrand-Edgeworth competition.[22]

Another contrast with the main model is that here it is not possible that entry into a duopoly market can harm consumers. To see this, consider two incumbents, A and B, with respective capacities κ_A and $\kappa_B \leq \kappa_A$. If $\kappa_A + \kappa_B \leq 1$ then there is no competition between these firms, consumers have zero surplus, and entry can only improve consumer surplus. Suppose then that $\kappa_A + \kappa_B > 1$, so that the incumbents cover the market, in which case industry profit without entry (as in expression (2) above) is

$$(\kappa_A + \kappa_B)\frac{1 - \kappa_B}{\kappa_A} \ .$$

Suppose a third firm enters, with capacity κ_E. Since demand was already met, entry leaves welfare unchanged and consumers are harmed if and only if industry profit rises. If $\kappa_E \geq 1 - \kappa_B$ then no firm has any captive customers after entry, equilibrium price is $p \equiv 0$ and consumers benefit from entry. Otherwise, if $\kappa_E < 1 - \kappa_B$ firm A has captive demand but firm E does not. In the knife-edge case where $\kappa_E = \kappa_B$, part (i) applies, and a direct calculation shows that industry profit falls. If $\kappa_E < \kappa_B$, so that the entrant is the smallest firm, part (ii) applies with minimum price $p_0 = (1 - \kappa_B - \kappa_E)/\kappa_A$. If π_E denotes

[21] If an overlapping duopoly equilibrium did exist, part (ii) of Proposition 4 applies so firm 3 has no captive customers and firms 1 and 2 price low. There would then be a threshold price p_1 which is firm 2's highest price and firm 3's lowest price. Since firm 3 has no captive customers, its demand at $p = p_1$ is $\kappa_3(1 - F_1(p_1))$, and since it cannot be better off with price $p = p_0$, we require that $1 - F_1(p_1) \geq p_0/p_1$. However, the fact that firm 2 is willing to choose p_1 implies that $1 - F_1(p_1) < p_0/p_1$, which is a contraction.

[22] Unlike our main model with consideration sets, in the capacity framework our assumption of unit demands makes a significant difference to—and simplifies—the analysis. De Francesco and Salvadori (2015) have studied triopoly in the richer and more complex situation where aggregate demand is downward sloping under the assumption of an efficient rationing rule, and show that additional possibilities can then arise in equilibrium. For example, the smallest firm might have an atom at its maximum price, with the result that the two larger firms do not choose prices immediately above this maximum price and there are gaps in the set of prices offered in the market.

the entrant's profit, the change in profit due to entry is

$$(\kappa_A+\kappa_B)p_0+\pi_E-(\kappa_A+\kappa_B)\frac{1-\kappa_B}{\kappa_A}=\pi_E-\kappa_E\frac{\kappa_A+\kappa_B}{\kappa_A}\ .$$

However, the entrant cannot make profit greater than κ_E (which is its profit if it could supply its capacity at price $p=1$), and so the above change in profit is negative and consumers benefit from entry. Finally, if $\kappa_B<\kappa_E<\kappa_A$, so the entrant is the middle firm, then firm B also has no captive customers, and part (ii) applies with the same minimum price $p_0=(1-\kappa_B-\kappa_E)/\kappa_A$. The change in profit due to entry is now

$$\begin{aligned}(\kappa_A+\kappa_E)p_0+\pi_B-(\kappa_A+\kappa_B)\frac{1-\kappa_B}{\kappa_A}&\le(\kappa_A+\kappa_E)p_0+\kappa_B-(\kappa_A+\kappa_B)\frac{1-\kappa_B}{\kappa_A}\\&=\frac{(1-\kappa_A-\kappa_B)(\kappa_E-\kappa_B)-\kappa_E^2}{\kappa_A}<0\ .\end{aligned}$$

Here, the first inequality follows since $\pi_B\le\kappa_B$, and the second inequality follows since the entrant has no captive customers.

More generally, our main model with consideration sets allows for richer patterns of competition interaction than the Bertrand-Edgeworth model. In the former framework, entry can occur without reducing the number of captive customers, a firm can have different "overlap" with similarly-sized rivals, and a small firm can have a high proportion of its reach captive, none of which are possible in the capacity framework.

6 Conclusions

The aim of this paper has been to explore, in a parsimonious framework with price-setting firms and homogeneous products, how patterns of consumer consideration matter for competitive outcomes, in particular the nature of price dispersion in mixed-strategy equilibria. The analysis has yielded a number of results that we did not initially expect. First, whereas in existing models all firms are direct competitors over a range of prices, we found equilibria with segmented pricing patterns, i.e., with some firms only pricing high and others only pricing low.

Second, in the three-firm case we established generically either that all firms set the same minimum price (in which case their profit was proportional to reach), or that pricing was segmented (so that one firm only set low prices and one set only high prices). In prior literature multiplicity of equilibria has gained considerable attention, and all such cases lie on the knife edge between these two regimes. Third, the key to determining which of the two regimes applies was found to be the proximity or otherwise of the pairwise correlation measures of competitive interaction, γ_{ij}, and when one pair of firms had significantly greater competitive interaction than other pairs then segmented pricing ensued. Fourth, for some parameter configurations we found equilibria with a gap in one firm's price support, so that that firm sometimes prices high, and sometimes low, but never in between.

Fifth, we found plausible patterns of consumer consideration in which entry is detrimental to consumers because it softens competition between incumbents, leading them to retreat to exploit their captive consumers. Likewise, there were situations where an increase in the number of consumers who consider one pair of firms causes a third firm to retreat towards its captive base, showing that search externalities need not benefit all consumers. Sixth, our model of competition with consumer consideration sets can differ radically from the familiar Bertrand-Edgeworth model of competition with capacity constraints. Indeed in the three-firm case the latter can be interpreted as consideration set model in which a *negative* proportion of consumers consider all of the firms. This difference implies that overlapping duopoly pricing is not a feature of the Bertrand-Edgeworth model.

The analysis could be extended in two broad directions. One would be to settings beyond nested reach and the three-firm case that we have analysed in detail. For example, one could seek more general conditions for equilibrium to take the overlapping duopoly form, or one could try to establish that all firms use the same minimum price when pairwise competitive interactions are similar enough. The other approach would be to endogenise the pattern of consideration sets, beyond our analysis of entry and mergers, by introducing search by consumers and/or advertising by firms.[23] For instance, one could study a model of non-sequential search where a consumer can choose her consideration set S firms by incurring a specified up-front search cost (increasing in S). Such a framework would generalize Burdett and Judd (1983, section 3.2) to allow firms to be asymmetric and for consumers to target specific firms for consideration.

[23] For instance, in the context of advertising, Ireland (1993) and McAfee (1994) study a sequential model where firms first invest in reach and then compete in price, while Butters (1977) studies the situation where firms choose their reach and price simultaneously. (In each case reach is assumed to be independent.)

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Technical Appendix

Sketch proof of Lemma 1: We first discuss arguments to do with deletion of dominated prices. In any equilibrium we have $\pi_i \geq \alpha_i$, since firm i can ensure at least this profit by choosing price equal to 1 and serving its captive customers. For this reason, no firm would ever offer a price below ρ_i, its captive-to-reach ratio, since if it did so it would obtain profit below α_i even if it supplied its entire reach.

To see that every firm makes positive profit we invoke Assumption 2. There is at least one firm i which has captive customers, and which will not set price below $\rho_i > 0$. (Clearly this firm makes positive profit.) From the remaining firms, at least one firm j has captive customers in the subset of firms excluding i, and so this firm can set price ρ_i and be sure to obtain positive profit. Firm j therefore also has a positive lower bound on its prices. Following the same argument, a firm in the subset of firms excluding both i and j can obtain positive profit, and so on until the set of firms is exhausted. In particular, each firm's minimum price is strictly above zero and hence so is p_0. This proves part (ii).

If price $p < 1$ is in firm i's support then $q_i(\cdot)$ in (1) cannot be flat for prices just above p, for otherwise the firm would obtain strictly greater profit by raising its price above p. This implies that this price must be in the support of at least one other firm. More precisely, if price $p < 1$ is in firm i's support it must be in the support of at least one of its "potential competitors", where in a given equilibrium we say that firm j is a "potential competitor" for firm i at price p if firm i's expected demand falls when j slightly undercuts

i at price p given the equilibrium strategies followed by firms other than i and j. (This then implies that i is a potential competitor for j.) If for all duopoly segments we have $\alpha_{ij} > 0$, then every firm is a potential competitor for every other firm. However, two firms might have disjoint reaches, and so cannot be potential competitors. More generally, the overlap between i and j might be contained within a third firm's reach, and if in the equilibrium the third firm always chooses price below p, then i and j do not compete at price p. If price p in firm i's support was *not* in the support of at least one of its potential competitors, firm i's demand would be flat (and positive) in this neighbourhood of p, which is not compatible with p maximizing the firm's profit.

We next turn to arguments concerning the possibility of "atoms" in the price distributions. First observe that two firms cannot both have an atom at price p if they are potential competitors at this price (for otherwise each would have an incentive to undercut the price p and gain a discrete jump in demand).

To see that each firm's price distribution is continuous in the interval $[p_0, 1)$, suppose by contrast that firm i has an atom at some price $0 < p < 1$ in its support. We claim that firm's i demand in (1) must then be locally flat above p. As noted above, there cannot be a potential competitor to i at price p which also has an atom at p, and so q_i does not jump down discretely at p. In addition, any potential competitor to i at p obtains a discrete increase in demand if it slightly undercuts p, and so such a firm would never choose a price immediately above p. Since no potential competitor chooses a price immediately above p, firm i loses no demand if it raises its price slightly above p, which is not compatible with p maximizing the firm's profit. Therefore, firm i cannot have an atom below 1, and this completes the proof of part (iii). This implies that each firm's demand (1) is continuous in the interval $[p_0, 1)$.

Similarly, if p_0 is the minimum price ever chosen in the market, then all prices in the interval $[p_0, 1]$ are sometimes chosen. If p is in firm i's support but no firm is active in an interval (p, p') above p, then firm i has flat demand over the range (p, p'), and this cannot occur in equilibrium. This completes the proof of part (iv).

Suppose now that there are at least three firms. Let P_{ij} denote the set of prices in $[p_0, 1]$ which are in the supports of both firm i and firm j, which is a closed set. Part (iv) implies that the collection $\{P_{ij}\}$ covers the interval $[p_0, 1]$, and since each firm participates, at least two of the sets in $\{P_{ij}\}$ are non-empty. If there were no price in the support of three or more firms then the collection $\{P_{ij}\}$ would consist of disjoint sets. However, since $[p_0, 1]$ is connected it cannot be covered by two or more disjoint closed sets, and we deduce that at least two sets in $\{P_{ij}\}$ must overlap, which proves part (v).

Firms can have an atom at the reservation price $p = 1$. However, as noted above, if firm i has an atom at $p = 1$ then no potential competitor can also have an atom at 1, which implies that when firm i chooses $p = 1$ it sells *only* to its captive customers and so its profit is precisely $\pi_i = \alpha_i$. If no firm has an atom at $p = 1$ then any firm with $p = 1$ in its support (and there must be at two such firms from part (iv)) has profit equal to α_i. This completes the proof for part (i).

Let firm j be a firm which obtains profit equal to α_j. Then the minimum price ever chosen, p_0, must be no higher than ρ_j (for otherwise firm j could obtain more profit by choosing $p = p_0$), and so p_0 cannot exceed the highest ρ_i. Since no firm sets a price below its ρ_i, the minimum price p_0 (which from part (iv) is sometimes chosen by at least two firms) must be weakly above the second lowest ρ_i. Finally, if the firm with the highest ρ_i has p_0 in its support, then p_0 cannot be strictly lower than this highest ρ_i, and so must equal this highest ρ_i. This completes the proof for part (vi).

Proof of Proposition 1: We construct an equilibrium of the stated form. The profit of the largest firm n is $\pi_n = \beta_n$, its number of captive customers, and denote the profit of smaller firms by π_i. In the highest interval $[p_{n-1}, 1]$ used by the two largest firms, these firms are sure to be undercut by all smaller rivals, and so in this price range their CDFs must satisfy

$$\beta_n + \beta_{n-1}(1 - F_{n-1}(p)) = \frac{\beta_n}{p} \ ; \ \beta_{n-1}(1 - F_n(p)) = \frac{\pi_{n-1}}{p} \ .$$

Since $F_n(p_{n-1}) = 0$ it follows that p_{n-1} and π_{n-1} are related as

$$\pi_{n-1} = \beta_{n-1} p_{n-1} \ .$$

We have $F_{n-1}(1) = 1$, while the largest firm has an atom at $p = 1$ with probability $1 - F_n(1) = \pi_{n-1}/\beta_{n-1} = p_{n-1}$.

In the lowest interval $[p_1, p_2]$ used by the two smallest firms, these firms are sure to undercut all larger rivals, and so in this range their CDFs must satisfy

$$\beta_2 + \sigma_1(1 - F_1(p)) = \frac{\pi_2}{p} \ ; \ \sigma_1(1 - F_2(p)) = \frac{\pi_1}{p}$$

and since $F_1(p_1) = F_2(p_1) = 0$ it follows that

$$\pi_1 = \sigma_1 p_1 \ ; \ \pi_2 = (\sigma_1 + \beta_2)p_1 \ .$$

Since $F_1(p_2) = 1$ we have $\pi_2 = \beta_2 p_2$, which combined with the previous expression for π_2 implies that

$$p_2 = \frac{\sigma_1 + \beta_2}{\beta_2} p_1 \ . \tag{22}$$

If there are just three firms, these are the two price intervals in the equilibrium. With more than three firms there are intermediate intervals, and in the interval $[p_i, p_{i+1}]$, where $1 < i < n-1$, firms i and $i+1$ are active and will be undercut by smaller rivals and undercut their larger rivals. Therefore, in this range their CDFs must satisfy

$$\beta_{i+1} + \beta_i(1 - F_i(p)) = \frac{\pi_{i+1}}{p} \ ; \ \beta_i(1 - F_{i+1}(p)) = \frac{\pi_i}{p} \ . \tag{23}$$

Since $F_{i+1}(p_i) = 0$ it follows that

$$\pi_i = \beta_i p_i \ .$$

An intermediate firm i, where $2 \leq i \leq n-1$, is active in both the intervals $[p_{i-1}, p_i]$ and $[p_i, p_{i+1}]$, and its CDF F_i needs to be continuous across the threshold price p_i. At the price p_i we therefore require that

$$\frac{\pi_{i-1}}{\beta_{i-1} p_i} = 1 - F_i(p_i) = \frac{1}{\beta_i}\left(\frac{\pi_{i+1}}{p_i} - \beta_{i+1}\right) \ , \tag{24}$$

where in the case of $i = 2$ we have written $\beta_1 = \sigma_1$. If we write $p_n = 1$ then we have $\pi_i = \beta_i p_i$ for all firms $1 \leq i \leq n$, and so for $2 \leq i \leq n-1$ expression (24) entails expression (8). This is a second-order difference equation in p_i where p_1 is free, p_2 is given in (22), and the terminal condition $p_n = 1$ serves to pin down p_1. It is clear from (22) and (8) that the sequence $p_1, p_2, p_3, ...$ is an increasing sequence of price thresholds. This completes the description of the candidate equilibrium.

We next show that no firm has an incentive to deviate from its described strategy. By construction, firm i is indifferent between choosing any price in the interval $[p_{i-1}, p_{i+1}]$, assuming its rivals follow the stated strategies. We need to check that a firm's profit is no higher if it chooses a price outside this interval. Consider first an upward price deviation, which is only relevant if $i < n-1$. If $i < n-2$ and firm i chooses a price above p_{i+2} is has no demand since firm $i+1$ is sure to set a lower price and all firm i's potential customers also consider firm $(i+1)$'s price. Suppose then that $i < n-1$ and firm i chooses a price

$p \in [p_{i+1}, p_{i+2}]$, in which case it has demand β_i if its price is below the prices of both rivals $i+1$ and $i+2$. Therefore, from (23) its profit with such a price is

$$p\beta_i[1 - F_{i+1}(p)][1 - F_{i+2}(p)] = \frac{\beta_i \pi_{i+1}}{\beta_{i+1}^2}\left(\frac{\pi_{i+2}}{p} - \beta_{i+2}\right) = p_{i+1}\frac{\beta_i \beta_{i+2}}{\beta_{i+1}}\left(\frac{p_{i+2}}{p} - 1\right) .$$

This profit decreases from $\pi_i = \beta_i p_i$ at $p = p_{i+1}$ to zero at $p = p_{i+2}$. We deduce that firm i cannot increase its profit by choosing a price above p_{i+1}.

Next consider a downward price deviation, so that firm i chooses a price below p_{i-1} (which is only relevant when $i > 2$). Suppose that this firm chooses a price in the interval $[p_j, p_{j+1}]$, where $j \leq i - 2$. The firm will undercut all firms larger than firm $j + 1$, and so obtain demand at least $\beta_{j+2} + ... + \beta_i$. It will also serve the segment β_{j+1} if it undercuts firm $j + 1$ and it will additionally serve the segment β_j if it undercuts both firms j and $j + 1$. Putting this together implies that the firm's profit with price $p \in [p_j, p_{j+1}]$ is

$$p\left\{\beta_{j+2} + ... + \beta_i + (1 - F_{j+1}(p))(\beta_{j+1} + \beta_j(1 - F_j(p))\right\} . \tag{25}$$

Given the CDFs in (23), this profit is a convex function of p and so must be maximized in this range either at p_j or at p_{j+1}. Therefore, we can restrict our attention to deviations by firm $i > 2$ to the threshold prices $\{p_1, p_2, ..., p_{i-2}\}$. If it chooses price p_j where $2 \leq j \leq i-2$, expression (25) implies its profit is

$$p_j\left\{\beta_{j+1} + ... + \beta_i + \beta_j(1 - F_j(p_j))\right\} .$$

Expression (24) implies that $\beta_j(1 - F_j(p_j))$ is equal to $\beta_{j+1}(\frac{p_{j+1}}{p_j} - 1)$, in which case the above deviation profit with price p_j is

$$p_j\left(\beta_{j+1} + ... + \beta_i + \beta_{j+1}(\frac{p_{j+1}}{p_j} - 1)\right) = \beta_{j+1}p_{j+1} + (\beta_{j+2} + ... + \beta_i)p_j . \tag{26}$$

One can check that expression (26) holds also for $j = 1$. We need to show that (26) is no higher than firm i's equilibrium profit, which is $\pi_i = \beta_i p_i$. We do this in two steps: (i) we show that (26) is increasing in j given i, so that $j = i - 2$ is the most tempting of these deviations for firm i, and (ii) we show (26) is below $\beta_i p_i$ when $j = i - 2$.

To show (i), suppose that $i \geq 4$, which is the only relevant case, and suppose that $1 \leq j \leq i - 3$. Then firm i's deviation profit with price p_{j+1} from (26) is

$$\beta_{j+2}p_{j+2} + (\beta_{j+3} + ... + \beta_i)p_{j+1} = \beta_{j+1}p_j + \beta_{j+2}p_{j+1} + (\beta_{j+3} + ... + \beta_i)p_{j+1}$$

$$\geq \beta_{j+1}p_j + \beta_{j+2}p_{j+1} + (\beta_{j+3} + ... + \beta_i)p_{j+1} - (\beta_{j+2} - \beta_{j+1})(p_{j+1} - p_j)$$

$$= \beta_{j+1}p_{j+1} + \beta_{j+2}p_j + (\beta_{j+3} + ... + \beta_i)p_{j+1} \geq \beta_{j+1}p_{j+1} + (\beta_{j+2} + ... + \beta_i)p_j$$

where the final expression is the firm's deviation profit with price p_j, which proves claim (i). (Here, the first equality follows from (8), the first inequality follows from (7) and the fact that $\{p_j\}$ is an increasing sequence, while the final inequality follows from $\{p_j\}$ being an increasing sequence.)

To show claim (ii), suppose that $i \geq 3$ which is the only relevant case, and observe that

$$\begin{aligned} \beta_i p_i &= \beta_{i-1}p_{i-2} + \beta_i p_{i-1} \\ &\geq \beta_{i-1}p_{i-2} + \beta_i p_{i-1} - (\beta_i - \beta_{i-1})(p_{i-1} - p_{i-2}) \\ &= \beta_{i-1}p_{i-1} + \beta_i p_{i-2} \end{aligned}$$

where the final expression is (26) when $j = i - 2$. (Here, the first equality follows from (8) and the inequality follows from $\{\beta_i\}$ being an increasing sequence.) This completes the proof that the stated strategies constitute an equilibrium.

Proof of Proposition 2: Lemma 1 shows that in any equilibrium each price in the range $[p_0, 1]$ is chosen by at least two firms, where p_0 denotes the minimum price offered by any firm in the equilibrium. In particular, either two or all three firms have p_0 in their supports. The lemma also shows that there is at least one price in all three price supports. Let L and H denote respectively the lowest and highest price among the prices in all three supports. (The set of prices in all three supports is closed.)

(i) Suppose that an equilibrium has $L > p_0$, so that only two firms, say firms i and j, offer the minimum price p_0. These firms obtain profit $\pi_i = \sigma_i p_0$ and $\pi_j = \sigma_j p_0$ and in the interval $[p_0, L]$ where $G_k(p) = 0$ expression (10) implies

$$\gamma_{ij}G_j(p) = \gamma_{ij}G_i(p) = 1 - \frac{p_0}{p} \ . \tag{27}$$

This implies that $G_i \equiv G_j$ in this interval and let $\delta = G_i(L) = G_j(L) > 0$.

Firm k weakly prefers price L to price p_0, and so (10) implies

$$\sigma_k p_0 \leq \sigma_k L[1 - \gamma_{ik}G_i(L) - \gamma_{jk}G_j(L) + \gamma G_i(L)G_j(L)] \ . \tag{28}$$

(Here, the left-hand side is its profit if it chooses p_0, when it will serve its entire reach, while the right-hand side is its profit with the higher price.) This inequality can be written as

$$\gamma_{ik}\delta + \gamma_{jk}\delta - \gamma\delta^2 \leq 1 - \frac{p_0}{L} = \gamma_{ij}\delta$$

where the equality follows from (27). We can divide by $\delta > 0$ to obtain

$$\gamma_{ij} \geq \gamma_{ik} + \gamma_{jk} - \gamma\delta \ . \tag{29}$$

Since $\delta = G_i(L) \leq \sigma_i$ and $\delta = G_j(L) \leq \sigma_j$, the term $\gamma\delta$ is weakly below both γ_{ik} and γ_{jk}. Expression (29) therefore implies that γ_{ij} is weakly greater than both γ_{ik} and γ_{jk}, and so using the stated labelling for firms we have $k = 1$ and the two low-price firms are firms 2 and 3. Since $\delta \leq \min\{\sigma_2, \sigma_3\}$, expression (29) then implies

$$\gamma_{23} \geq \gamma_{12} + \gamma_{13} - \gamma\delta \geq \gamma_{12} + \gamma_{13} - \gamma \min\{\sigma_2, \sigma_3\} \ . \tag{30}$$

Therefore, if (11) holds the equilibrium cannot take the form where $L > p_0$, and the only alternative is that all three firms use the same minimum price p_0. Lemma 1 (vi) shows that this minimum price must then be the highest captive-to-reach ratio.

(ii) If condition (12) holds we will show that $L = H$ so there is only one price in all three supports. Either all three firms have the same minimum price p_0 or only two firms do, and in the latter case the proof for part (i) shows that it must be firms 2 and 3 which price low. In either case firms 2 and 3 use p_0, and in either case we have $G_2(L) = G_3(L) = \delta \geq 0$. Suppose by contradiction that in equilibrium we have $H > L$. Let i and j label firms 2 and 3 such that $G_i(H) \geq G_j(H)$. Then since we cannot have only firm 1 active in the open interval (L, H), one or both of 2 and 3 must choose prices in (L, H), and so $\delta = G_i(L) < G_i(H) \equiv g$.

Firms 2 and 3 obtain respective profits $p_0\sigma_2$ and $p_0\sigma_3$, and let π_1 denote firm 1's profit. Expression (10) shows that a price p in firm 1's support satisfies

$$\pi_1 = \sigma_1 p[1 - \gamma_{12}G_2(p) - \gamma_{13}G_3(p) + \gamma G_2(p)G_3(p)] \ ,$$

and setting $p = L, H$ in the above and subtracting implies that

$$\begin{aligned} \frac{\pi_1}{\sigma_1}\left(\frac{1}{L} - \frac{1}{H}\right) &= \gamma_{12}G_2(H) + \gamma_{13}G_3(H) - \gamma G_2(H)G_3(H) \\ &\quad -\gamma_{12}G_2(L) - \gamma_{13}G_3(L) + \gamma G_2(L)G_3(L) \\ &\leq \gamma_{12}g + \gamma_{13}g - \gamma g^2 - \gamma_{12}\delta - \gamma_{13}\delta + \gamma\delta^2 \\ &= (g - \delta)(\gamma_{12} + \gamma_{13} - \gamma(g + \delta)) \ . \end{aligned} \tag{31}$$

Here, the inequality follows since $\gamma_{12} \geq \gamma G_3(H)$ and $\gamma_{13} \geq \gamma G_2(H)$, and so the initial expression is weakly increased if we replace $G_j(H)$ by $g = G_i(H) \geq G_j(H)$. Likewise, and using that fact that $G_1(L) = 0$, for firm j we have

$$\begin{aligned} p_0\left(\frac{1}{L} - \frac{1}{H}\right) &= \gamma_{23}G_i(H) + \gamma_{1j}G_1(H) - \gamma G_1(H)G_i(H) - \gamma_{23}G_i(L) \\ &= \gamma_{23}g + \gamma_{1j}G_1(H) - \gamma g G_1(H) - \gamma_{23}\delta \\ &\geq \gamma_{23}(g - \delta) \ . \end{aligned}$$

Since $\pi_1 \geq \sigma_1 p_0$ (as firm 1 weakly prefers any price in its support to p_0) and $g - \delta > 0$, it follows that

$$\gamma_{23} \leq \gamma_{12} + \gamma_{13} - \gamma(g + \delta) \ . \tag{32}$$

If $\gamma = 0$ (so no consumers consider all three firms) this inequality contradicts (12), so we deduce that it is not possible to have $H > L$ when (12) holds and $\gamma = 0$. Therefore, suppose henceforth that $\gamma > 0$. Then since $g > 0$ the inequality (32) contradicts the first inequality in (30) which holds whenever $L > p_0$. We deduce that if $H > L$ then all three firms must have the same minimum price p_0 and hence $\delta = 0$.

We show next that if all three firms have the same minimum price, then (12) cannot hold. First suppose that $H < 1$, so that only two firms are active in the upper range $(H, 1]$. If firm 1 uses $p = 1$, then one of firms 2 or 3 has its maximum price at H, so that $G_2(H) = \sigma_2$ or $G_3(H) = \sigma_3$. Therefore $g = G_i(H) \geq \min\{\sigma_2, \sigma_3\}$, in which case (32) is inconsistent with (12).

Continue with the assumption that $H < 1$, but now suppose it is firms 2 and 3 which are active above H, so that $G_1(H) = \sigma_1$. Since all three firms have profit equal to p_0 multiplied by their reach, (10) implies that for firm 1 and firm j we have respectively

$$\begin{aligned} p_0 &= H\left[1 - \gamma_{12}G_2(H) - \gamma_{13}G_3(H) + \gamma G_2(H)G_3(H)\right] \\ p_0 &= H\left[1 - \gamma_{1j}\sigma_1 - \gamma_{23}G_i(H) + \gamma\sigma_1 G_i(H)\right] \ , \end{aligned}$$

and combining these yields

$$(\gamma_{23} - \gamma_{1i})G_i(H) = (\gamma_{1j} - \gamma G_i(H))(G_j(H) - \sigma_1) \ . \tag{33}$$

However, condition (12) implies $\gamma_{23} > \max\{\gamma_{12}, \gamma_{13}\}$, and since $G_i(H) > 0$ it follows that the right-hand side above is strictly positive, and in particular we have

$$\sigma_1 < \min\{G_2(H), G_3(H)\} \ . \tag{34}$$

Since firms 2 and 3 both use $p = H$ and $p = 1$, while $G_1(H) = \sigma_1$, for each $k = 2, 3$ we have

$$p_0 \left(\frac{1}{H} - 1 \right) = (\gamma_{23} - \sigma_1 \gamma)(G_k(1) - G_k(H)) \ . \tag{35}$$

Write $\eta \equiv G_2(1) - G_2(H) = G_3(1) - G_3(H) > 0$. Note (35) implies that $\gamma_{23} > \sigma_1 \gamma$ so that $\alpha_{23} > 0$ and there are some consumers who consider firms 2 and 3. As such, at most one of these firms can have an atom at $p = 1$. Since firm 1 weakly prefers $p = H$ to $p = 1$, we have

$$\begin{aligned} p_0 \left(\frac{1}{H} - 1 \right) \ \leq \ & \gamma_{12} G_2(1) + \gamma_{13} G_3(1) - \gamma G_2(1) G_3(1) \\ & -\gamma_{12} G_2(H) + \gamma_{13} G_3(H) - \gamma G_2(H) G_3(H) \end{aligned}$$

$$= \eta \left[\gamma_{12} + \gamma_{13} - \gamma(G_2(H) + G_3(H) + \eta) \right] \ .$$

Since $\eta > 0$, combining this inequality with (35) implies

$$\gamma_{23} - \sigma_1 \gamma \leq \gamma_{12} + \gamma_{13} - \gamma(G_2(H) + G_3(H) + \eta) \ ,$$

or

$$\gamma \min\{\sigma_2, \sigma_3\} \leq \gamma_{12} + \gamma_{13} - \gamma_{23} - \gamma(G_2(H) + G_3(H) + \eta - \sigma_1 - \min\{\sigma_2, \sigma_3\}) \ . \tag{36}$$

At most one of firms 2 and 3 has at atom at $p = 1$, so suppose that firm $k \in \{2, 3\}$ has no atom, so that

$$G_k(H) + \eta = G_k(1) = \sigma_k \geq \min\{\sigma_2, \sigma_3\} \ .$$

Combining this inequality with (34) and (36) then contradicts condition (12).

The final case to consider is when $H = 1$, so that all three firms use the highest price. If at most one of firms 2 and 3 has an atom at $p = 1$ then either $G_2(1) = \sigma_2$ or $G_3(1) = \sigma_3$ (or both). Therefore $g \geq \min\{\sigma_2, \sigma_3\}$, in which case (32) is inconsistent with (12). If both firms 2 and 3 have an atom at $p = 1$ then we must have $\alpha_{23} = 0$ otherwise the firms have an incentive to undercut one another at $p = 1$. It follows that $\gamma \sigma_1 = \gamma_{23}$, in which case (12) implies

$$\gamma(\sigma_1 + \min\{\sigma_2, \sigma_3\}) > \gamma_{12} + \gamma_{13} \geq \gamma(\sigma_2 + \sigma_3)$$

and so $\sigma_1 > \max\{\sigma_2, \sigma_3\}$. Since not all consumers are captive, when firms 2 and 3 each have an atom at $p = 1$, firm 1 cannot do so and $G_1(1) = \sigma_1$. Then the argument leading to the previous expression (34) applies, with $H = 1$, which contradicts our finding that $\sigma_1 > \max\{\sigma_2, \sigma_3\}$.

In sum, we have shown that when (12) holds, there is only one price in the support of all three firms, say p_1. In particular, only two firms offer the minimum price p_0, and these are firms 2 and 3. Clearly $p_0 < p_1$ and only firms 2 and 3 are active in the range $[p_0, p_1)$. If $p_1 = 1$ then the proof is complete. If $p_1 < 1$ then there is no price in $(p_1, 1]$ in the support of all firms, and so only two firms are active in this range, one of which must be firm 1. The remaining issue is which of firms 2 and 3 is the other firm active above p_1. Suppose henceforth that firms 2 and 3 are labelled so $\sigma_2 \geq \sigma_3$. Expression (27) implies that $\sigma_2 F_2(p) = \sigma_3 F_3(p)$ in the range $[p_0, p_1]$. If $\sigma_2 = \sigma_3$ then $F_2 = F_3$, and so one of these firms cannot drop out before the other and we must have $p_1 = 1$. If $\sigma_2 > \sigma_3$ then in the range $[p_0, p_1]$ we have $F_3 > F_2$ and so it is firm 3 which drops out first.

The final step in the proof is to determine the profits of the three firms, as well as the price thresholds p_0 and p_1. Since firms 2 and 3 have p_0 as their minimum price in this equilibrium, their profits are $\pi_2 = \sigma_2 p_0$ and $\pi_3 = \sigma_3 p_0$. In the range $[p_0, p_1]$ their CDFs are given by (27), and firm 3 drops out at price p_1, so that the ratio p_0/p_1 satisfies

$$\gamma_{23}\sigma_3 = 1 - \frac{p_0}{p_1} \ . \tag{37}$$

Expression (27) then implies that

$$G_2(p_1) = \sigma_3 \ . \tag{38}$$

Either firm 1 or 2 (or both) obtains exactly its captive profit.[24] Suppose first that firm 1 obtains its captive profit, so that $\pi_1 = \alpha_1$. For prices in the upper range $[p_1, 1]$ firms 1 and 2 compete and are sure to be undercut by firm 3, so from (10) firm 2's CDF satisfies

$$1 - \gamma_{12}G_2 - \gamma_{13}\sigma_3 + \gamma\sigma_3 G_2 = \frac{\rho_1}{p} \ ,$$

where recall that ρ_1 is firm 1's captive-to-reach ratio. In order for $G_2(\cdot)$ to be continuous at the threshold price p_1, (38) implies that

$$1 - \gamma_{12}\sigma_3 - \gamma_{13}\sigma_3 + \gamma\sigma_3^2 = \frac{\rho_1}{p_1} \ ,$$

[24]If one of these firms has no atom at $p = 1$ then the other obtains its captive profit when it chooses $p = 1$. If both have an atom at $p = 1$ then for neither to have an incentive to undercut the other we must have $\alpha_{12} = 0$, in which case both firms obtain their captive profit at $p = 1$.

which determines p_1. Expression (37) in turn implies that

$$p_0 = p_1(1-\gamma_{23}\sigma_3) = \frac{\rho_1(1-\gamma_{23}\sigma_3)}{1-\gamma_{12}\sigma_3-\gamma_{13}\sigma_3+\gamma\sigma_3^2} \ . \tag{39}$$

It is convenient to write P for the right-hand side above, so that

$$P = \frac{\rho_1(1-\gamma_{23}\sigma_3)}{1-\gamma_{12}\sigma_3-\gamma_{13}\sigma_3+\gamma\sigma_3^2} = \frac{\alpha_1(\alpha_2+\alpha_{12})}{\alpha_1\sigma_2+\alpha_{12}(\sigma_2-\sigma_3)} \ , \tag{40}$$

where the second equality follows by routine manipulation. Note from the first expression for P above that the condition $P > \rho_1$ is equivalent to (11), and $P < \rho_1$ corresponds to (12). Note also that $P \leq (\alpha_2+\alpha_{12})/\sigma_2$, and so a sufficient condition for overlapping duopoly to be the equilibrium is that

$$\frac{\alpha_2+\alpha_{12}}{\sigma_2} < \rho_1 \ .$$

In words, this condition states that the higher captive-to-reach ratio in the duopoly market with just firms 2 and 3 present is below firm 1's captive-to-reach ratio in the triopoly market. Expression (39) implies

$$p_1 = \frac{\alpha_1\sigma_2}{\alpha_1\sigma_2+\alpha_{12}(\sigma_2-\sigma_3)} \ . \tag{41}$$

Alternatively, suppose firm 2 obtains its captive profit, so that $\pi_2 = \alpha_2$. Since the firm has p_0 as its lowest price it follows that

$$p_0 = \rho_2 \ . \tag{42}$$

Expression (37) then implies that

$$p_1 = \frac{\alpha_2}{\alpha_2+\alpha_{12}} \ . \tag{43}$$

For prices in the upper range $[p_1, 1]$ firm 2's CDF now satisfies

$$1-\gamma_{12}G_2-\gamma_{13}\sigma_3+\gamma\sigma_3 G_2 = \frac{\pi_1}{\sigma_1 p} \ ,$$

where π_1 is firm 1's profit (to be determined). For G_2 to be continuous at $p = p_1 = \alpha_2/(\alpha_2+\alpha_{12})$, (38) implies that

$$1-\gamma_{12}\sigma_3-\gamma_{13}\sigma_3+\gamma\sigma_3^2 = \frac{\alpha_2+\alpha_{12}}{\alpha_2}\cdot\frac{\pi_1}{\sigma_1} \ ,$$

which determines π_1. This can be expressed as

$$\pi_1 = \frac{\alpha_1 \rho_2}{P} \tag{44}$$

where P is given in (40).

We next determine when it is that firm 1 or firm 2 obtains its captive profit. When firm 1 obtains its captive profit, firm 2's minimum price is P in (40), which must be no lower than ρ_2 if firm 2 is willing to offer this price. Therefore, if $P < \rho_2$ the equilibrium must instead have firm 2 obtaining its captive profit, in which case the threshold prices and firm 1's profit are given respectively by (42), (43) and (44). Conversely, when firm 2 obtains its captive profit, firm 1's profit is given in (44). This profit cannot be below its captive profit α_1, which therefore requires $P \leq \rho_2$. Therefore, if $P > \rho_2$ the equilibrium must involve firm 1 obtaining its captive profit, so $\pi_1 = \alpha_1$, and the threshold prices are given respectively by (40) and (41). Finally, in the knife-edge case where $P = \rho_2$ the two equilibria coincide, and firms 1 and 2 each obtain their captive profit. This completes the proof.

Proof of Proposition 3: Part (i) is just an instance of Proposition 1 specialized to triopoly. Now suppose $\beta_3 < \beta_2$, in which case part (i) of Proposition 2 shows that all firms choose the same lowest price $P_0 = \beta_3/\sigma_3$ and each firm's profit is $\pi_i = \sigma_i P_0$. In general (not just in the nested case), with demand in (10) the equilibrium condition $pq_i(p) \equiv \sigma_i P_0$ for a price in firm i's support can be written in factorized form

$$[\gamma_{ij} - \gamma G_k(p)][\gamma_{ik} - \gamma G_j(p)] = z_i(p) \tag{45}$$

where z_i is given by

$$z_i(p) \equiv \gamma_{ij}\gamma_{ik} - \gamma \left(1 - \frac{P_0}{p}\right) . \tag{46}$$

Expression (45) implies that for a price in the support of all three firms the function $[\gamma_{jk} - \gamma G_i(p)]z_i(p)$ is the same for each firm i, and so firm i's CDF is given by

$$[\gamma_{jk} - \gamma G_i(p)]^2 = \frac{z_j(p) z_k(p)}{z_i(p)} . \tag{47}$$

Specializing to the nested case, where

$$z_1 = z_2 = \frac{P_0}{\sigma_2\sigma_3}\frac{1}{p} , \; z_3 = \frac{P_0}{\sigma_2\sigma_3}\left(\frac{1}{p} - 1\right) ,$$

the expressions (47) simplify to

$$\left(1-\frac{\sigma_1}{\sigma_2}F_1\right)^2=(1-F_2)^2=\frac{\beta_3}{\sigma_2}\left(\frac{1}{p}-1\right) \tag{48}$$

$$(1-F_3)^2=\frac{\sigma_2\beta_3}{\sigma_3^2 p(1-p)}\ . \tag{49}$$

Here, F_1 and F_2 increase with p, while F_3 is hump-shaped and increases only in the range $p \leq 1/2$. Since CDFs cannot decrease, it follows that any interval of prices (rather than merely a single point) where all firms are active must be contained in the range $p \leq 1/2$. In particular, we must have $P_0 < 1/2$, which is indeed implied by the assumption $\beta_3 < \beta_2$. One can check that $F_1 \geq F_2 \geq F_3$ (where the second inequality requires $p \geq P_0$), so that F_1 will be the first CDF to reach $F_i = 1$. It will hit this constraint at price p_1 in (15).

If this price p_1 is no higher than 1/2, i.e., if

$$\frac{\beta_3}{\beta_2} \leq \frac{\beta_2}{\sigma_2}\ , \tag{50}$$

which is a stronger condition than $\beta_2 \geq \beta_3$, then the equilibrium involves all three firm active in the range $[p_0, p_1]$, and then only the two larger firms are active in the range $(p_1, 1]$. For prices above p_1, where firm 1 is sure to serve its reach, the CDFs for the two larger firms satisfy

$$1-F_2=\frac{\beta_3}{\beta_2}\left(\frac{1}{p}-1\right)\ ;\ 1-F_3=\frac{\sigma_2\beta_3}{\sigma_3\beta_2 p} \tag{51}$$

The only condition to check is that firm 1 has no incentive to choose a price above p_1. However, when its rivals use the CDFs in (51), one can check that firm 1's profit with price $p > p_1$ is lower than its profit with price p_1, and so this deviation is not profitable. This proves part (iii).

The remaining parameter region is when

$$\frac{\beta_2}{\sigma_2} < \frac{\beta_3}{\beta_2} < 1\ . \tag{52}$$

In this case the candidate for the CDF F_3 in (49) begins to decrease in p before F_1 reaches 1. Here, an equilibrium can be constructed by first deriving F_3 as in part (iii), ignoring the constraint that it needs to be increasing, and then "ironing" the result to eliminate the "hump". To illustrate, consider the example where $\sigma_1 = 5$, $\beta_2 = 3$ and $\beta_3 = 2$. Then $P_0 = 1/5$, in (15) $p_1 = 16/25$, and F_3 as given in (49) and (51) is depicted as the solid curve on Figure 4 in the main text. The dashed line shows the ironing procedure, so that F_3 is flattened to be no greater than the level $F_3(p_1) = 1/6$ for prices below p_1. The smaller root of $F_3 = 1/6$ in (49) is $\hat{p} = 9/25$. In this example, all three firms are active in the price range $[\frac{1}{5}, \frac{9}{25}]$, only firms 1 and 2 are active in the interior range $(\frac{9}{25}, \frac{16}{25})$, and then only firms 2 and 3 are active in the range $(\frac{16}{25}, 1)$. In the interior range $(\frac{9}{25}, \frac{16}{25})$, the other CDFs F_1 and F_2 also need modifying from (48) to reflect that they will be undercut by firm 3 with the constant probability $F_3(p_1) = 1/6$ in this range (in which case they have no demand), so that their CDFs satisfy

$$\frac{5}{6}(3 + 5(1 - F_1)) = \frac{16}{10p} \ ; \ \frac{5}{6}(1 - F_2) = \frac{1}{5p} \ .$$

With these CDFs, one can check that firm 1 does not gain by choosing a price in this interior range. As before, firm 1 has no incentive to choose a price above $p_1 = 16/25$, and so this is indeed an equilibrium.

Exactly the same procedure is valid with any case in the parameter region (52). One can check that F_3 in (49) evaluated at price p_1 in (15) is positive if and only if $\beta_2 > \beta_3$. From (49) it is clear that other price $\hat{p}$ which yields the same value for F_3 as p_1 is

$$\hat{p} = 1 - p_1 = \frac{\beta_2^2}{\sigma_2\beta_3 + \beta_2^2} \ ,$$

which is below $1/2$ given that p_1 is above $1/2$, and one can check that $\hat{p} > \beta_3/\sigma_3$ as well. Thus this equilibrium has all three firms active between β_3/σ_3 and $\hat{p}$, firms 1 and 2 are active in the interior region between $\hat{p}$ and p_1 (which is symmetric about $p = 1/2$), and then firms 2 and 3 are active between p_1 and 1. This completes the proof for part (ii).

Proof of Proposition 4: (i) The proof mirrors the corresponding proof in Proposition 2, and the argument leading to expression (29) shows that if only two firms price low it is necessary that

$$\hat{\gamma}_{ij} \geq \hat{\gamma}_{ik} + \hat{\gamma}_{jk} - \hat{\gamma}\delta \ . \tag{53}$$

Since $\hat{\gamma} < 0$, it follows that if only two firms price low it must be the pair of firms with the highest $\hat{\gamma}_{ij}$, i.e., firms 1 and 2. Expression (53) then implies

$$\hat{\gamma}_{12} \geq \hat{\gamma}_{13} + \hat{\gamma}_{23} - \hat{\gamma}\delta > \hat{\gamma}_{13} + \hat{\gamma}_{23} = \hat{\gamma}_{13} \ . \tag{54}$$

(The final equality follows since $\hat{\gamma}_{23} = 0$.) Therefore, if $\hat{\gamma}_{12} \leq \hat{\gamma}_{13}$ the equilibrium cannot take the form where only two firms price low, and the only alternative is that all three firms use the same minimum price p_0. However, since $\hat{\gamma}_{12} \geq \hat{\gamma}_{13}$, the only way to have $\hat{\gamma}_{12} \leq \hat{\gamma}_{13}$ is $\hat{\gamma}_{12} = \hat{\gamma}_{13}$ as stated.

(ii) Next suppose that all three firms have the same lowest price p_0, so that $\pi_i = \kappa_i p_0$. Prices just above p_0 are therefore in each firm's support, in which case expression (20) implies that for prices just above p_0 we have

$$\hat{\gamma}_{ij} G_j + \hat{\gamma}_{ik} G_k - \hat{\gamma} G_j G_k = 1 - \frac{p_0}{p} \ .$$

Since $\hat{\gamma}_{23} = 0$, the conditions for firms 2 and 3 become

$$G_1(\hat{\gamma}_{12} - \hat{\gamma} G_3) = 1 - \frac{p_0}{p} = G_1(\hat{\gamma}_{13} - \hat{\gamma} G_2) \ .$$

Since $G_1(p) > 0$ for p above p_0, this requires that

$$\hat{\gamma}_{12} - \hat{\gamma} G_3(p) = \hat{\gamma}_{13} - \hat{\gamma} G_2(p) \ .$$

However, since G_2 and G_3 are continuous and both equal zero when $p = p_0$, this condition requires $\hat{\gamma}_{12} = \hat{\gamma}_{13}$. Therefore, if $\hat{\gamma}_{12} > \hat{\gamma}_{13}$ the only possibility is for only two firms to price low. From part (i), these two firms must be firms 1 and 2. We know from Lemma 1 (vi) that p_0 is therefore equal to firm 1's captive-to-reach ratio, which completes the proof.