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Abstract

This paper proposes a residual-based cointegration test in the presence of smooth structural changes approximated by a Fourier function. The test offers a simple way to accommodate unknown number and form of structural breaks and have good size and power properties in the presence of breaks.

Keywords: cointegration test; Fourier function; structural breaks.

JEL Codes: C12, E4.

[Declarations of interest: none]

1. Introduction

The performance of traditional residual-based cointegration tests that examine the null hypothesis of the nonexistence of a long-term relationship between economic variables is affected by structural changes that occur due to economic crises, technological changes, political regime shifts, and similar shocks. As noted by Gregory and Hansen (1996), if structural breaks exist, the power of cointegration tests, such as Engle-Granger (1987), which ignore these changes, will be reduced. Several cointegration tests have been introduced to take into account structural breaks in the cointegration relationship. While Gregory and Hansen (1996) considered one unknown shift, Hatemi-J (2008) extended this test to allow two structural breaks. Both of the tests capture the regime shifts incorporating dummy variables and assume the number of breaks a priori.

The study by Becker et al. (2005, BEL hereafter), who used Gallant's Flexible Fourier form to model unknown structural breaks, brought new depth to the unit root testing literature because previous studies generally used Perron's (1989) modeling strategy of structural breaks with dummy variables. BEL (2005) showed that a single frequency of Fourier approximation can mimic various breaks and unattended nonlinearity. Several unit root tests, such as those by BEL (2005), Enders and Lee (2012), and Rodriguez and Taylor (2012), who employed a variant of the Flexible Fourier form, have been introduced to the literature. The main advantage of using trigonometric terms is that the locations, numbers, and forms of the structural breaks do not need to be predetermined. In this study, we extend the residual-based cointegration test of Engle-Granger (1987) using Fourier approximation to test the existence of a cointegration relationship allowing unknown forms of breaks. The remainder of the paper is organized as follows: Section 2 defines the model and test statistic and contains the asymptotic distributions of the latter; Section 3 assesses the size and power of the suggested test statistic; and, finally, Section 4 presents the conclusion.

2. Model with Fourier Approximation and Test Statistics

We consider the following cointegration regression in this paper:

$$y_{1t} = d(t) + \beta' y_{2t} + u_t \tag{1}$$

where $t = 1, 2, \ldots, T$. The dependent variable y_t is a scalar, and $x_t = (x_{1t}, \ldots, x_{mt})'$ is a $(m \times 1)$ vector of independent variables. $d(t)$ is a deterministic function of t that can be approximated using the following Fourier expansion with a single-frequency component[1]:

$$d(t) = \alpha_0 + \gamma_k \sin\left(\frac{2\pi kt}{T}\right) + \delta_k \cos\left(\frac{2\pi kt}{T}\right)$$

where α_0 shows the traditional deterministic term including a constant with or without a linear term, T shows the number of observations, and k represents the Fourier frequency, the values of which are selected using the value that minimizes the sum of squared residuals (SSR). When $\gamma_k = \delta_k = 0$, there is no nonlinear trend, and the traditional Engle-Granger cointegration test emerges.

We obtain the following equation when we implement this function in Model 1:

$$y_{1t} = \alpha_0 + \gamma_1 \sin\left(\frac{2\pi kt}{T}\right) + \gamma_2 \cos\left(\frac{2\pi kt}{T}\right) + \beta' y_{2t} + u_t \tag{2}$$

To test the null hypothesis of no-cointegration, we apply the Augmented Dickey-Fuller unit root test to the residuals of Model 2. Hence, we estimate the following autoregression:

$$\Delta \hat{u}_t = \rho \hat{u}_{t-1} + \sum_{i=1}^{p} \gamma_i \hat{u}_{t-i} + \varepsilon_t$$

[1] It is also possible to use multiple frequencies in the Fourier expansion; to conserve space, the critical values and size–power properties are excluded but are available from the authors upon request.

Where $\varepsilon_t \sim i.i.d.(0, \sigma^2)$, we let τ_{FEG} show the t-statistics for the null hypothesis of no-cointegration that is defined as:

$$\tau_{FEG} = \frac{\hat{\rho}}{se(\hat{\rho})}$$

where $\hat{\rho}$ denotes the ordinary least squares estimator of ρ while $se(\hat{\rho})$ is the standard error of $\hat{\rho}$.

Critical values for the Fourier cointegration test (FEG) are obtained via simulations considering a different number of regressors (*n* = 1, 2, 3) and frequency values (*k* = 1, 2, 3, 4, 5). We report them in Table 1, considering both constant, and constant and trend cases for the sample sizes are t = 100, 500, and 1000.

[Table 1 about here]

As can be seen in Table 1, the asymptotic distribution of the test statistic depends on the frequency (*k*) and number of regressors (*n*). While ceteris paribus, an increase in *k* and/or *n* creates a decrease in critical values.

3. Size and power properties

We analyzed the finite sample properties of the suggested test by considering the following data generation process (DGP):

$$\Delta y_{1t} = \alpha_0 + \gamma_1 \sin\left(\frac{2\pi kt}{T}\right) + \gamma_2 \cos\left(\frac{2\pi kt}{T}\right) + \delta_1\left(y_{1,t-1} - \beta y_{2t}\right) + \phi_1 \Delta y_{2t} + \varepsilon_{1t},$$

$$\Delta y_{2t} = \psi' \Delta y_{2t-1} + \varepsilon_{2t},$$

$$\Omega = E\left(\varepsilon_t \varepsilon_t'\right) = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}.$$

where $\varepsilon_t = \left(\varepsilon_{1t}, \varepsilon_{2t}\right)$, $\sigma_1^2 = \sigma_{\Delta y_{1t}}^2$ and $\sigma_2^2 = \sigma_{\Delta y_{2t}}^2$. We assume that $\beta = 1$ and $\sigma_{12} = \sigma_{21} = \theta$. Similar DGPs have been used in several prior studies [see Banerjee and Smith (1986), Lee et al. (2015), Banerjee et al. (2017) among others.]. We conducted simulations using 20,000 replications at the 5% significance level. We examined the performance of the test from different perspectives:

- we let the persistent measure ψ change in the range {0, 0.9};
- we set $\sigma_1^2 = 1$, while letting the σ_2^2 vary along with $\{1,16\}$; and,
- we also evaluate two sets of γ_1, and γ_2 as $\gamma_1 = \{0,3\}$ and $\gamma_2 = \{0,5\}$.

We report the results in Table 2.

[Table 2 about here]

The results show that, as the sample size increases, the power of the test also increases[2]. Besides, in the case of $\gamma_1 \neq \gamma_2 \neq 0$, when the magnitude of the break increases, the power of the FEG test increases, and becomes 1, in most cases. When the persistent measure ψ increases, the power of the test seems to decrease in the case of $\gamma_1 = \gamma_2 = 0$. In this case, an increase in σ_2^2 causes an increase in the power of the test, in most cases. Overall, the test seems to have small size distortions and good power properties in the existence of breaks.

[2] Conventional Engle-Granger (1987) cointegration test loses power, and suffers from size distortions in presence of breaks. Results are available from author upon request.

4. Conclusion

In this paper, we proposed a residual-based test for cointegration with a Fourier approximation using a single-frequency component, allowing multiple smooth breaks. Thanks to the FEG cointegration test, we do not need to know the exact location, form, or number of breaks a priori. The only value to be determined is the frequency value, which is found by minimizing SSR. The suggested test can prevent potential loss of power in the cointegration tests that allow structural breaks by adding dummy variables in the testing equations. Simulation results show that the FEG test has small size distortions and good power properties, especially in the existence of breaks.

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Table 1: Critical Values of FEG Cointegration Test

		Model with a constant									Model with a constant and trend								
n	k	T=100			t=500			t=1000			T=100			t=500			t=1000		
		1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
1	1	-4.906	-4.302	-3.988	-4.756	-4.198	-3.898	-4.738	-4.175	-3.886	-5.354	-4.731	-4.423	-5.128	-4.576	-4.293	-5.074	-4.555	-4.274
	2	-4.665	-3.995	-3.648	-4.517	-3.912	-3.589	-4.503	-3.898	-3.579	-5.243	-4.582	-4.250	-4.995	-4.433	-4.136	-4.973	-4.410	-4.119
	3	-4.437	-3.743	-3.380	-4.333	-3.685	-3.349	-4.314	-3.686	-3.342	-5.002	-4.340	-3.997	-4.801	-4.230	-3.910	-4.804	-4.208	-3.901
	4	-4.285	-3.599	-3.252	-4.183	-3.554	-3.231	-4.172	-3.546	-3.221	-4.849	-4.175	-3.827	-4.697	-4.092	-3.767	-4.693	-4.088	-3.769
	5	-4.190	-3.520	-3.187	-4.091	-3.478	-3.165	-4.081	-3.477	-3.165	-4.774	-4.086	-3.739	-4.634	-3.997	-3.683	-4.593	-3.994	-3.677
2	1	-5.282	-4.655	-4.337	-5.067	-4.511	-4.220	-5.048	-4.487	-4.205	-5.641	-5.026	-4.705	-5.404	-4.855	-4.571	-5.367	-4.826	-4.550
	2	-5.168	-4.526	-4.189	-4.969	-4.394	-4.085	-4.949	-4.371	-4.065	-5.598	-4.954	-4.633	-5.329	-4.772	-4.480	-5.295	-4.748	-4.460
	3	-4.958	-4.283	-3.938	-4.804	-4.183	-3.870	-4.778	-4.172	-3.852	-5.450	-4.781	-4.436	-5.199	4.620	-4.313	-5.167	-4.597	-4.292
	4	-4.805	-4.122	-3.767	-4.647	-4.048	-3.722	-4.657	-4.040	-3.716	-5.294	-4.622	-4.271	-5.089	-4.487	-4.183	-5.065	-4.469	-4.158
	5	-4.708	-4.033	-3.689	-4.587	-3.964	-3.633	-4.536	-3.935	-3.629	-5.203	-4.508	-4.164	-5.006	-4.404	-4.086	-4.945	-4.370	-4.063
3	1	-5.596	-4.957	-4.640	-5.354	-4.796	-4.512	-5.315	-4.786	-4.497	-5.941	-5.294	-4.971	-5.638	-5.094	-4.814	-5.602	-5.070	-4.795
	2	-5.573	-4.918	-4.593	-5.330	-4.752	-4.460	-5.286	-4.727	-4.435	-5.926	-5.278	-4.961	-5.635	-5.078	-4.791	-5.590	-5.048	-4.762
	3	-5.393	-4.733	-4.394	-5.177	-4.597	-4.285	-5.150	-4.582	-4.277	-5.792	-5.141	-4.806	-5.515	-4.964	-4.659	-5.504	-4.940	-4.643
	4	-5.271	-4.605	-4.252	-5.071	-4.468	-4.148	-5.035	-4.134	-4.455	-5.698	-5.023	-4.681	-5.441	-4.843	-4.534	-5.404	-4.835	-4.529
	5	-5.155	-4.478	-4.127	-4.976	-4.378	-4.056	-4.959	-4.352	-4.042	-5.601	-4.905	-4.560	-5.361	-4.752	-4.436	-5.332	-4.743	-4.435

Note: n, k, and T show the number of the independent variables, number of frequencies of the Fourier function, and the sample size.

Table 2: Finite Sample Performance of FEG

				Model with a constant				Model with a constant and a trend			
				T=100		T=300		T=100		T=300	
ψ	σ_2^2	γ_1	γ_2	$\delta_1=0.0$	$\delta_1=-0.1$	$\delta_1=0.0$	$\delta_1=-0.1$	$\delta_1=0.0$	$\delta_1=-0.1$	$\delta_1=0.0$	$\delta_1=-0.1$
0	1	0	0	0.053	0.233	0.053	0.761	0.048	0.101	0.053	0.609
0	16	0	0	0.052	0.234	0.051	0.758	0.051	0.099	0.049	0.607
0.9	1	0	0	0.045	0.175	0.039	0.755	0.049	0.096	0.045	0.594
0.9	16	0	0	0.046	0.173	0.038	0.753	0.050	0.095	0.043	0.598
0	1	3	0	0.066	0.214	0.053	0.816	0.048	0.097	0.053	0.646
0	16	3	0	0.066	0.217	0.051	0.815	0.051	0.098	0.049	0.646
0.9	1	3	0	0.060	0.219	0.039	0.807	0.049	0.089	0.045	0.634
0.9	16	3	0	0.059	0.220	0.038	0.807	0.050	0.091	0.043	0.640
0	1	0	5	0.053	0.677	0.053	1	0.048	0.163	0.053	1
0	16	0	5	0.052	0.677	0.051	1	0.051	0.164	0.049	1
0.9	1	0	5	0.045	0.638	0.039	1	0.049	0.166	0.045	1
0.9	16	0	5	0.046	0.640	0.038	1	0.050	0.172	0.043	1
0	1	3	5	0.053	0.517	0.053	1	0.048	0.117	0.053	1
0	16	3	5	0.052	0.521	0.051	1	0.051	0.117	0.049	1
0.9	1	3	5	0.045	0.429	0.039	1	0.049	0.105	0.045	1
0.9	16	3	5	0.046	0.430	0.038	1	0.050	0.109	0.043	1

Note: $\delta_1=0.0$, and $\delta_1=-0.1$ represents the size, and power properties, respectively.