Saving and dissaving under Ramsey - Rawls criterion

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Abstract

This article studies an inter-temporal optimization problem using a criterion which is a combination between Ramsey and Rawls criteria. A detailed description of the saving behaviour through time is provided. The optimization problem under $\alpha-$*maximin* criterion is also considered with optimal solution characterized.

Keywords: maximin principle, $\alpha-$ maximin, Ralws criterion, Ramsey criterion, $\epsilon-$contamination.

JEL classification numbers: C61, D11, D90.

Contents

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1. Introduction

In the classical work "Theory of justice", Rawls [22] poses the following question: what would be the choice for the outcome of the society if one is cached behind a *veil of ignorance*? In the total lack of information about the condition under which he[1] will be born, the economic agent should choose the maximization of the least favoured person (or generation). For example, given a inter-temporal consumption streams, his evaluation criterion of inter-temporal utilities streams should be

$$U(c_0, c_1, c_2, \dots) = \inf_{t \geq 0} u(c_t),$$

where $u(c_t)$ is the utility of the t^{th} generation.

Naturally, numerous attempts, for example Arrow [2] or Calvo [6] have been done to study the evolution of the economy if this criterion is used to evaluate inter-temporal welfare. Arrow [2] assumes constant productivity. Calvo [6] studies the maximin problem with uncertain technology. The result is pessimistic: if the initial accumulation of capital is low, the economy remains in this low capital accumulation situation forever.

The first part of this article studies the same question but with the difference that we allow the possibility for a growth to infinity, by allowing for the case where the productivity of every level of capital accumulation is sufficiently high. We consider a generalisation the set up of Arrow [2] by imposing only the concavity to the production function. The result is the same: for any initial capital accumulation, the best choice to maximize the least favoured generation is stay the initial state forever. Equality is maximized at the cost of efficiency[2].

What happens then if we combine the famous Ramsey criterion, which evaluates the inter-temporal utilities streams using a constant discount rate $\beta \in (0, 1)$, and

[1]We use male pronouns as a convenient default.

[2]See Fleurbaye & Tungodden [12] for an analysis about the dilemma where "*one is forced to accept principle and give full priority to the worst-off even when a tiny gain to imposes a substantial sacrifice on arbitrarily many well-off*".

the Rawls criterion? Precisely, we can consider the evaluation criterion as follows:

$$U(c_0, c_1, c_2, \dots) = \sum_{t=0}^{\infty} \beta^t u(c_t) + a \inf_{t \geq 0} u(c_t),$$

with some positive parameter a, representing the importance of equality in the choice of the economic agent[3].

There are always sacrificed generations with the Ramsey criterion. If the level of productivity is high, the utility of the present generations will be lowered for a rapid accumulation of capital, and on the contrary, the generations in the distant future will be subject to the same decision. And if we combine Ramsey and Rawls criteria, by considering not only efficiency but also equality?

This is not the only motivation which urges us to study the Ramsey-Rawls combination problem.

The link between the results in decision theory and the time discounting literature is strong. The reason for this tight link is clear: by normalizing the time discounting system in order to obtain a probability and consider the set of time as the set of states, the inter-temporal choice is equivalent to an act in the world of Savage [23]. For example while the theorem of Savage [23] poses an axiomatic base for *mean expected utility*, the works of Koopmans in [16] and [17] provide the conditions for the inter-temporal representation in the later.

In recent decades, there is a vast literature which expands the world of Savage, by extending the theory in order to encompass the behaviours which do not satisfy Savage's famous *sure-thing principle*. The classical work of Gilboa & Schmeidler [14] formulates the notion of *ambiguity aversion*, representing the behaviour of an economic agent as always maximizing the worst scenario among the set of different possible probabilities.

In an parallel line of thinking, the same consideration can also be made in the time

[3] A similar combination (between a Ramsey part and an infinite part following the non-dictatorial criterion) is studied by Asheim & Ekeland [4].

discounting domain. Let Δ be the set of time discounting systems possibles:

$$\Delta = \left\{ \pi = (\pi_0, \pi_1, \dots) \text{ such that } \pi_s > 0, \forall s \text{ and } \sum_{s=0}^{\infty} \pi_s = 1 \right\}.$$

The inter-temporal evaluation of the economic agent, while having only a vague idea about the appropriate time discounting system to choose, but knowing that the appropriate time discount system must belong to D, a subset of Δ, can be represented as follows, in the same spirit of Gilboa & Schmeidler [14]:

$$U(c_0, c_1, c_2, \dots) = \inf_{\pi \in D} \left[\sum_{t=0}^{\infty} \pi_t u(c_t) \right].$$

Recently, Chambers & Echenique [8] established the axiomatic bases for the maximin criterion inter-temporal evaluation, with different discount rates. The corresponding set Δ in the set up of Chamber & Echenique [8] is a convex hull of a set of time discounting systems which are geometrical sequences[4].

Imagine a situation where the ambiguity is total, for example our agent is cached behind a *veil of ignorance*. Without any possible information to predict the future, the set of all possible time discounting systems should be Δ and the inter-temporal evaluation becomes

$$\begin{aligned} U(c_0, c_1, c_2, \dots) &= \inf_{\pi \in \Delta} \left[\sum_{t=0}^{\infty} \pi_t u(c_t) \right] \\ &= \inf_{t \geq 0} u(c_t). \end{aligned}$$

We can push further the question about the criterion with multiple possible time discounting systems. Suppose the agent is not completely ignorant but always has doubts about his choice of a time discounting system. Our agent has "opinion" that the good constant discount rate to choose is $\beta \in (0, 1)$ and the corresponding discount rates system is $\pi_t^* = (1 - \beta)\beta^t$, for all $t \geq 0$[5].

The word "opinion" is used in the same spirit as Kopylov [18], to define a state of

[4]Drugeon & al [10] prove that the solution of optimization problem under this criterion is monotonic and time consistent.

[5]The term $1 - \beta$ is just a normalizing term, to ensures that $\sum_{t=0}^{\infty} \pi_t^* = 1$.

mind that is less rigid than "belief". The economic agent thinks that π^* is a good choice, but there are reasons suggesting him that this conclusion could be hasty. He should also take into account all other time discounting systems. Precisely, he should consider the set $D = (1-\lambda)\pi^* + \lambda\Delta$, with some $0 \leq \lambda \leq 1$.

This formulation is very similar to the $\lambda-$*contamination* literature, with the axiomatic foundation established in Alon [1], and Kopylov [18]. The parameter λ represents the lack of confidence in the choice π^* of the agent. If $\lambda = 1$, the ambiguity is total. In contrast to this, if $\lambda = 0$, he believes without doubt that π^* is the good one.

Under the $\lambda-$*contamination* criterion, the inter-temporal evaluation becomes

$$\begin{aligned} U(c_0, c_1, c_2, \dots) &= \inf_{\pi \in D} \left[\sum_{t=0}^{\infty} \pi_t u(c_t) \right] \\ &= (1-\lambda) \sum_{t=0}^{\infty} (1-\beta)\beta^t u(c_t) + \lambda \inf_{\pi \in \Delta} \left[\sum_{t=0}^{\infty} \pi_t u(c_t) \right] \\ &= (1-\lambda) \sum_{t=0}^{\infty} (1-\beta)\beta^t u(c_t) + \lambda \inf_{t \geq 0} u(c_t). \end{aligned}$$

Taking $a = \frac{\lambda}{(1-\lambda)(1-\beta)}$, this is equivalent to the criterion being represented as

$$U(c_0, c_1, c_2, \dots) = \sum_{t=0}^{\infty} \beta^t u(c_t) + a \inf_{t \geq 0} u(c_t).$$

One more time, we find the Ramsey-Rawls combination. The second part of this work is devoted to the study of this problem.

We consider the following *modified* optimization problem: if we accept to lower the value of the Rawls part to ϵ, what is the best we can make for the Ramsey part? By lowering the former, we have more room to approve the later. And what is the optimal acceptable level ϵ? This modification allows us to study the classical optimization problems with an additional constraint. We can hence apply the usual techniques well-known in dynamic programming literature to circumvent the difficulties being posed by the time-inconsistency of the criterion. The optimal ϵ can be considered as the efficiency-equality trade-off cost.

If the level of productivity is high, the utility of the early dates (or generations) are reduced as much as possible, for the sake of a rapid accumulation of capital. It is worth to sacrifice even a litter bit the value of the equality criterion, in order to have a better accumulation level of capital.

Once the capital accumulation level is sufficiently high, the economy follows a Ramsey path which does not violate the equality constraints, and converges to a steady state, or infinity, if such steady state does not exist. Thanks to the constraints imposed by the equality criteria of Rawls, the difference of utility between early and later dates is not too high.

In the case of low productivity, the economy converges to a higher steady state than the one of the Ramsey problem. The difference between the lowest weighted dates (in distant future) and the highest weighted dates (in present) is diminished. The optimal choice in the long run behaves as that at a steady state of some Ramsey problem with a higher discount rate.

Moreover, if the pondering weight of the equality part is high, the optimal sequence coincides with the solution of Rawls problem. For a high importance of the equality part, the Ramsey part has no effect.

The Rawls criterion may be considered too severe, since it cares only about the worst generation. This can be considered as a special case of the $\alpha-$*maximin* formulation in Arrow & Hurwicz [3], Ghirardato & al [13], or Chateauneuf & al [9], balancing pessimism and optimism, which considers not only the worst case but also the best case[6]. Their generalization consists the criterion which is a convex combination of the worst and the best scenarios. Applying to the context of this article, the criterion becomes as follows

$$U(c_0, c_1, \dots) = \alpha \sup_{t \geq 0} u(c_t) + (1-\alpha) \inf_{t \geq 0} u(c_t),$$

for some $0 \leq \alpha \leq 1$. The Rawls criterion is equivalent to the case $\alpha = 0$. The parameter α can be considered as the *optimism* degree of the economic agent.

[6]Bossert & al [5] also take into account the worst and the best cases, but their formulation is much more different with the ones treated in our article. In this article we do not have ambition to make an exhaustive review of the ambiguity literature, which is very large. For a general review about formulation under uncertainty, see Etner, Jeleva and Tallon [11].

For small initial value, the economy has an infinite number of solutions. Every optimal path fluctuates between two different values determined by the fundamental parameters of the problem. For initial value high enough, there exists unique solution and this path takes constant value from the date $t = 1$.

The article is organized as follows. Section 2 considers the optimization under *Rawls* criterion, with a general production function and utility function. Section 3 analyses the *Ramsey-Rawls* problem. Using results of Section 3, Section 5 studies the problem with linear production function and logarithmic utility function. The proofs are given in Appendix.

2. Optimal solution under *Rawlsian* criterion

We consider the following optimization problem under the Rawls criterion:

$$\max \left[\inf_{t \geq 0} u(c_t) \right],$$

under the constraint $c_t + k_{t+1} \leq f(k_t)$ for all t, with $k_0 > 0$ given.

Let $\Pi(k_0)$ be the set of feasible paths $\{k_t\}_{t=0}^{\infty}$: $0 \leq k_{t+1} \leq f(k_t)$ for any t. This set is compact in the product topology. For each feasible sequence $\mathbf{k} = (k_0, k_1, k_2, \dots)$ in $\Pi(k_0)$, define

$$v(\mathbf{k}) = \inf_{t \geq 0} u\big(f(k_t) - k_{t+1}\big).$$

The upper semi-continuity of the Rawls criterion with respect to this topology requires only the continuity of the utility function and the production function.

Lemma 2.1. *Assume that the utlity function u and the production function f are continuous:*

i) *The function v is upper semi-continuous for the product topology.*

ii) *There exists* $\boldsymbol{k}^* \in \Pi(k_0)$ *such that*

$$v\left(\boldsymbol{k}^*\right) = \max_{\boldsymbol{k} \in \Pi(k_0)} v\left(\boldsymbol{k}\right).$$

For the description of the solution of Rawls problem, we add the concavity of production function f, and the existence of a non-trivial feasible sequence.

Assumption A1. The utility function u is strictly concave, increasing and satisfies Inada condition. The production function f is concave, strictly increasing and satisfies $f'(0) > 1$[7].

Denote by $\overline{k}$ the solution to $f'(k) = 1$, which maximizes $f(k) - k$. In the case $f'(k) > 1$ for any $k \geq 0$, let $\overline{k} = +\infty$.

Under the continuity of utility function and the concavity of production function, we can prove that for k_0 smaller than $\overline{k}$, the optimal choice for is to remain in the *status quo*. For k_0 bigger than $\overline{k}$, there exists an infinite number of optimal paths, and the optimal value is $u\left(f(\overline{k}) - \overline{k}\right)$.

Proposition 2.1. i) *Consider the case* $0 \leq k_0 \leq \overline{k}$. *The problem has a unique solution* $\boldsymbol{k}^* = (k_0, k_0, \ldots)$ *and*

$$\max_{\boldsymbol{k} \in \Pi(k_0)} v(\boldsymbol{k}) = u\left(f(k_0) - k_0\right).$$

ii) *Consider the case* $\overline{k}$ *is finite and* $k_0 \geq \overline{k}$. *The problem has an infinite number of solutions and*

$$\max_{\boldsymbol{k} \in \Pi(k_0)} v(\boldsymbol{k}) = u\left(f(\overline{k}) - \overline{k}\right).$$

From now on, for the sake of simplicity, let $\hat{v}(k_0)$ be the best value possible for the Rawls criteria with initial state k_0:

$$\hat{v}(k_0) = \max_{\mathbf{k} \in \Pi(k_0)} v\left(\mathbf{k}\right).$$

[7]Otherwise every feasible sequence converges to zero, and the problem becomes trivial.

3. The dynamics under *Ramsey-Rawls* criterion

3.1 *Ramsey-Rawls* problem

We consider in this section the criterion which a a convex combination of the well-known criteria Ramsey and Rawls:

$$U(c_0, c_1, \dots) = \sum_{t=0}^{\infty} \beta^t u(c_t) + a \inf_{t\geq 0} u(c_t),$$

where a is a positive constant. We will use the term "Ramsey part" to denote the sum $\sum_{t=0}^{\infty} \beta^t u(c_t)$ and "Rawls part" to denote $\inf_{t\geq 0} u(c_t)$.

Consider the following optimization problem (P):

$$V(k_0) = \sup \left[\sum_{t=0}^{\infty} \beta^t u(c_t) + a \inf_{t\geq 0} u(c_t) \right]$$

s.c $c_t + k_{t+1} \leq f(k_t)$ for any $t \geq 0$,

$k_0 \geq 0$ is given.

From now on, for the sake of simplicity, we let $c_t = f(k_t) - k_{t+1}$, for any t and feasible set $\{k_t\}_{t=0}^{\infty}$.

In Section 2, we know that the Rawls part, $\nu(\mathbf{k}) = \inf_{t\geq 0} u\left(f(k_t) - k_{t+1}\right)$ is upper semi-continuous in respect to the product topology. It is well-known in the literature that under suitable conditions, the Ramsey part $\sum_{t=0}^{\infty} \beta^t u(c_t)$ is also upper semi-continuous. In order to simplify the exposition, we assume this upper semi-continuity property. Curious readers can refer to the work of Le Van & Morhaim [15] for the details of the conditions ensuring this property, with the most important one being the *tail-insensitivity* condition.

Assumption A2. Assume that for any feasible sequence $\{k_t\}_{t=0}^{\infty}$, the function $\sum_{t=0}^{\infty} \beta^t u(c_t)$ is determined and satisfies upper semi-continuity with respect to the product topology.

Under this assumption, the Ramsey part is also upper semi-continuous, and hence the same property is satisfied for the function U. Furthermore the problem (P) always has optimal solution and we can write:

$$V(k_0) = \max_{\mathbf{k}\in\Pi(k_0)} \left[\sum_{t=0}^{\infty} \beta^t u(c_t) + a \inf_{t\geq 0} u(c_t) \right].$$

The strictly concavity of utility function u ensures the uniqueness of the optimal solution.

3.2 *Ramsey* problem

In this subsection, we evoke some well-known results in the literature of the Ramsey model. Under Assumption A2, the Ramsey problem always has a solution. The strict concavity of the utility function u implies the uniqueness. Denote by v the value function and $\{\hat{k}_t\}_{t=0}^{\infty}$ the optimal solution of the Ramsey problem:

$$v(k_0) = \max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.c $c_t + k_{t+1} \leq f(k_t)$ for any $t \geq 0$,

k_0 is given.

By the uniqueness of optimal solution, and the well-known result that v is solution to an functional Bellman equation, there exists an optimal policy function σ which is strictly increasing such that $\hat{k}_{t+1} = \sigma(\hat{k}_t)$, for any t.

We recall here an important feature of the Ramsey problem. When the productivity is high ($f'(k_0) > \frac{1}{\beta}$), the economic agent prefers to sacrifice the welfare of the early dates (or early generations) for a rapid accumulation of capital. The economy saves. The consumption sequence is increasing in this case.

In contrast to this, when the productivity is low ($f'(k_0) < \frac{1}{\beta}$), the economy chooses to dissaving. The impatience imposed by the discount rate implies the welfare sacrifices of dates (ou generations) in a distant future. In this set up, the consumption sequence is decreasing.

Let k^s be a solution to

$$f'(k) = \frac{1}{\beta}.$$

If the solution is not unique, we can take any one in the set of solutions. If $f'(x) > \frac{1}{\beta}$ for all $x \geq 0$, let $k^s = \infty$, and if $f'(x) \leq \frac{1}{\beta}$ for all $x \geq 0$, let $k^s = 0$.

LEMMA 3.1. i) *If $k_0 \leq k^s$, then the consumption sequence $\{\hat{c}_t\}_{t=0}^{\infty}$ and capital accumulation $\{\hat{k}_t\}_{t=0}^{\infty}$ sequence are increasing, and converge respectively to $c^s = f(k^s) - k^s$ and k^s. As a consequence of this,*

$$v(\hat{\boldsymbol{k}}) = u(\hat{c}_0).$$

ii) *If $k_0 \geq k^s$, the consumption sequence $\{\hat{c}_t\}_{t=0}^{\infty}$ and capital accumulation $\{\hat{k}_t\}_{t=0}^{\infty}$ sequence are decreasing, and converge respectively to $c^s = f(k^s) - k^s$ and k^s. As a consequence of this,*

$$v(\hat{\boldsymbol{k}}) = u\left(f(k^s) - k^s\right).$$

3.3 *Ramsey-modified* problem

For $\epsilon \geq 0$, we first consider the following intermediary problem (P^{ϵ}):

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.c } c_t + k_{t+1} \leq f(k_t), \forall\ t \geq 0,$$

$$u(c_t) \geq \hat{v}(k_0) - \epsilon, \forall\ t \geq 0,$$

$$k_0 \text{ is given.}$$

The intuition for studying this problem runs as follows. We already know that the maximum value possible for the Rawls part is $\hat{v}(k_0)$. Naturally, the following question rises: if we accept a lower value of the Rawls part up to ϵ, what is the best improvement we can obtain for the Ramsey part? And which is the optimal acceptable sacrifice level ϵ? This optimal value represents the cost of the trade-off between efficiency and equality.

In order to respond to these questions, we study the problem (P^ϵ). The Proposition 3.1 states that the optimal solution of (P) is also the optimal solution of (P^ϵ), for some optimal value ϵ.

PROPOSITION 3.1. *For any $k_0 \geq 0$,*

$$V(k_0) = \max_{\epsilon \geq 0} \left[W(\epsilon) + a\left(\hat{v}(k_0) - \epsilon\right) \right].$$

By the Proposition 3.1, in order to understand the behavior of the optimal solution of initial problem (P), we study the behavior of the optimal solution of problems (P^ϵ), with $\epsilon \geq 0$.

For the sake of simplicity, from now on, we will use the term "equality constraint" to denote the constraint $u(c_t) \geq \hat{v}(k_0) - \epsilon$. Let $W(\epsilon)$ be the value of the problem (P^ϵ) and $\{c_t^\epsilon, k_{t+1}^\epsilon\}_{t=0}^\infty$ be its optimal solution. By the strict concavity of u, this sequence is unique.

It is obvious that, if ϵ is sufficiently big, the solution of Ramsey problem satisfied also the equality constraints, and solving problem (P^ϵ) becomes trivial task. Let $\tilde{\epsilon}$ be the critical value for this property: if we accept to lower the Rawls part to $\tilde{\epsilon}$, the solution of Ramsey problem satisfies also the constraint of the Ramsey-modified problem, and becomes solution of the later one.

Define

$$\tilde{\epsilon} = \begin{cases} u\big(f(k_0) - k_0\big) - u\big(f(k_0) - \sigma(k_0)\big) \text{ if } 0 \leq k_0 \leq k^s, \\ u\big(f(k_0) - k_0\big) - u\big(f(k^s) - k^s\big) \text{ if } k^s \leq k_0 \leq \overline{k} \\ u\big(f(\overline{k}) - \overline{k}\big) - u\big(f(k^s) - k^s\big) \text{ if } k_0 \geq \overline{k}. \end{cases}$$

The proof of Lemma 3.2 is easy, based on the fact that the solution of Ramsey problem satisfies the constraints of Ramsey-modified one for sufficiently high ϵ.

LEMMA 3.2. *Assume that $\epsilon \geq \tilde{\epsilon}$.*

i) *The optimal solution of problem (P^ϵ) coincides with the solution of Ramsey problem.*

ii) $W(\epsilon) = W(\tilde{\epsilon}) = v(k_0)$.

If $\epsilon = 0$, by Proposition 2.1, the optimal solution is $(k_0, k_0, \ldots)$. We consider now the interesting case, where $0 < \epsilon \leq \tilde{\epsilon}$.

If $0 \leq k_0 \leq k^s$, the equality constraints are binding in the early dates and the optimal solution behaves as a solution of Ramsey problem when the accumulation of capital reaches a sufficiently high level.

If $k_0 \geq k^s$, the equality constraints are binding from some date T sufficiently big and in the long run, every date (or generation) has the same utility level, which is equal exactly the lowest level acceptable.

Proposition 3.2. i) *Consider the case* $0 < k_0 < k^s$. *If* $0 < \epsilon \leq \tilde{\epsilon}$, *there exists* T *such that:*

a) *For* $0 \leq t \leq T$, $u(c_t^\epsilon) = \hat{v}(k_0) - \epsilon$.

b) *For* $t \geq T+1$, $u(c_t^\epsilon) > \hat{v}(k_0) - \epsilon$.

c) *The sequence* $\{k_t^\epsilon\}_{t=T+1}^{\infty}$ *is the solution of Ramsey problem with initial state* k_{T+1}^ϵ.

ii) *Consider the case* $k_0 > k^s$. *If* $0 < \epsilon \leq \tilde{\epsilon}$, *there exists* T *such that*

a) *For* $0 \leq t \leq T$, $u(c_t^\epsilon) > \hat{v}(k_0) - \epsilon$.

b) *For* $t \geq T+1$, $u(c_t^\epsilon) = \hat{v}(k_0) - \epsilon$.

For the case $k_0 \geq k^s$, define $\tilde{k}$ as the solution to

$$u\left(f(\tilde{k}) - \tilde{k}\right) = \hat{v}(k_0) - \epsilon.$$

It is easy to verify that the $k_t^\epsilon = \tilde{k}$ for T sufficiently high. Let $\tilde{\beta}$ be the discount rate satisfying

$$f'(\tilde{k}) = \frac{1}{\tilde{\beta}}.$$

By Proposition 2.1 and the choice of $\tilde{\epsilon}$, we have $k^s < \tilde{k} < \overline{k}$. Hence $\tilde{\beta} > \beta$. In the long run, the optimal solution for the case $k_0 \geq k^s$ behaves as a solution of a Ramsey problem with discount rate $\tilde{\beta}$, greater than β.

Lemma 3.3 is direct consequence of Proposition 3.2. The function W is strictly concave in respect to ϵ belonging to $[0, \tilde{\epsilon}]$. This concavity implies the existence of the right derivative of W at 0 and the left derivative of W at $\tilde{\epsilon}$. In Section 3.4, these two values will play the role of critical thresholds for the equality parameter a. The behavior of the optimal solution depends strongly in the relative position of a and $W'(0)$, $W'_{-}(\tilde{\epsilon})$. The details will be presented in subsection 3.4.

For instance, we give some results about $W'(0)$ and $W'_{-}(\tilde{\epsilon})$.

LEMMA 3.3. i) *For any k_0, the function W is strictly concave on* $[0, \tilde{\epsilon}]$.

ii) *If* $0 \leq k_0 < k^s$, *then* $W'(0) = +\infty$ *and* $W'(\tilde{\epsilon}) = 0$.

iii) *If* $k_0 > k^s$, *then* $W'(0) < +\infty$.

3.4 Optimal solution of *Ramsey-Rawls* problem

Denote by ϵ^* the optimal level in Proposition 3.1:

$$\epsilon^* = \underset{\epsilon \geq 0}{\operatorname{argmax}} \left[W(\epsilon) + a \left(\hat{v}(k_0) - \epsilon \right) \right].$$

Let $\{k_t^*\}_{t=0}^{\infty}$ be the corresponding optimal solution of the *Ramsey-modified* problem. By Proposition 3.1, the sequence $\{k_t^*\}_{t=0}^{\infty}$ is also the solution of *Ramsey-Rawls* problem.

In the case the productivity is high ($f'(k_0) > \frac{1}{\beta}$), the utility of the early dates (or generations) are lowered as much as possible, for the sake of a rapid accumulation of capital. It is worth to sacrifice even a litter bit the value of the equality part, in order to have a better accumulation level of capital.

Once the capital accumulation level is sufficiently high, the economy follows a Ramsey path which does not violate the equality constraints, and converges to the steady state k^s. Thanks to the constraints imposed by the equality criterion of Rawls, the difference in utility between early dates and the later dates in distant future is not too high. This difference depends negatively on the equality parameter a, which imposes a trade-off between equality and the speed of convergence to the steady state.

Proposition 3.3. *Consider the case* $0 < k_0 \leq k^s$. *For any* $a > 0$, *we have* $0 < \epsilon^* < \tilde{\epsilon}$ *and there exists* T *such that*

i) *For* $0 \leq t \leq T$, $u(c_t^*) = u\big(f(k_0) - k_0\big) - \epsilon^*$.

ii) *For* $t \geq T + 1$, $u(c_t^*) > u\big(f(k_0) - k_0\big) - \epsilon^*$.

iii) *The sequence* $\{k_t^*\}_{t=T+1}^{\infty}$ *is the solution of Ramsey problem with initial state* k_{T+1}^*.

In the case of low productivity ($f'(k_0) < \frac{1}{\beta}$), the equality part (if sufficiently high) causes the economy to converge to a higher steady state than the one of Ramsey problem. The difference between the lowest dates (in distant future) and the highest dates (in present) is diminished. The optimal choice in long term behaves as at a steady state of some Ramsey problem with a value of discount rate $\tilde{\beta}$ higher than β.

Moreover, there exists a threshold for equality parameter a. Beyond this threshold, the optimal sequence remains the same and every date (or generations) enjoys the same utility level.

If the equality parameter a is too low, there is no change in the behavior of the economy, comparing with the Ramsey problem.

Proposition 3.4. *Consider the case* $k_0 \geq k^s$.

i) *For* $W'(\tilde{\epsilon}) < a < W'(0)$, *we have* $0 < \epsilon^* < \tilde{\epsilon}$ *and there exists* T *such that:*

   a) *For* $0 \leq t \leq T$, $u(c_t^*) > \hat{v}(k_0) - \epsilon^*$.

   b) *For* $t \geq T + 1$, $u(c_t^*) = \hat{v}(k_0) - \epsilon^*$.

ii) *For* $a \geq W'(0)$, $\epsilon^* = 0$ *and for any* t, $k_t^* = k_0$.

iii) *For* $0 \leq a \leq W'(\tilde{\epsilon})$, *we have* $\epsilon^* = \tilde{\epsilon}$ *and the optimal solution of problem* (P) *coincides with the solution of the Ramsey problem with initial state* k_0.

4. Optimisation under $\alpha-$*maximin* criterion

4.1 The $\alpha-$*maximin* problem and the *sup*-modified problem

Consider the following problem,

$$\mathcal{V}(k_0) = \sup \left[\alpha \sup_{t \geq 0} u(c_t) + (1-\alpha) \inf_{t \geq 0} u(c_t) \right],$$

$$\text{s.c. } c_t + k_{t+1} \leq f(k_t) \text{ for all } t \geq 0,$$

$$k_0 \text{ is given.}$$

Observe that the *supremum* part is not upper semi-continuous with respect to the production topology [8].

The idea is similar to the one the previous section. In order to determine the supremum value of the optimisation problem, consider the following *sup*-modified problem: For $\epsilon > 0$, define

$$\mathcal{W}(\epsilon) = \max \left[\sup_{t \geq 0} u(c_t) \right],$$

$$\text{s.c } c_t + k_{t+1} \leq f(k_t), \text{ for all } t \geq 0,$$

$$c_t \geq \hat{v}(k_0) - \epsilon, \text{ for all } t \geq 0.$$

Let $\Pi^\epsilon(k_0)$ be the set of feasible paths of this problem.

If we accept to reduce the value of the *infimum* part to ϵ, we have more room to optimize the other part, and what is the best we can do? The resolution of the Denote by $\Pi^\epsilon(k_0)$ be the set of feasible paths of the *sup*-modified problem. problem will help us the response for the initial one, as stated in the Lemma 4.1.

[8] For example, consider a set of feasible sequence $\{k_t^n\}_{t=0}^\infty$ such that for any $0 \leq t \leq n$, $c_t^n = f(k_t^n) - k_{t+1}^n = 0$, and for $t \geq n+1$, $c_t^n = f(k_t^n) - k_{t+1}^n = c^* > 0$. While in the product topology, the sequence $\mathbf{c}^n = \{c_t^n\}_{t=0}^\infty$ converges to $(0, 0, \dots)$, the limit of $u(c_t^n)$ is

$$\lim_{n \to \infty} \sup_{t \geq 0} u(c_t^n) = u(c^*) > u(0).$$

Lemma 4.1. *We have*

$$\mathscr{V}(k_0) = \max_{\epsilon \geq 0} \left[\alpha \mathscr{W}(\epsilon) + (1-\alpha)\left(\hat{v}(k_0) - \epsilon\right) \right].$$

4.2 Solution of the *sup*-modified problem

With Lemma 4.1, we solve the modified problem, with some $\epsilon > 0$. Let $\underline{x}^\epsilon$ be the solution in $[0, k_0]$ to the equation

$$u\left(f(x) - x\right) = u\left(f(k_0) - k_0\right) - \epsilon.$$

If ϵ is big such that $u\left(f(x) - x\right) \geq u\left(f(k_0) - k_0\right) - \epsilon$ for any $0 \leq x \leq k_0$, let $\underline{x}^\epsilon = 0$. Similarly, let $\overline{x}^\epsilon$ be the solution in $[k_0, +\infty)$ solution to the same equation. If $u\left(f(x) - x\right) \geq u\left(f(k_0) - k_0\right) - \epsilon$ for any $x \geq k_0$, let $\overline{x}^\epsilon = +\infty$.

For the case $0 \leq k_0 \leq \overline{k}$, there exist an infinite number of optimal paths, and every optimal path fluctuates between $\underline{x}^\epsilon$ and $\overline{x}^\epsilon$.

Proposition 4.1. *Consider the case* $0 \leq k_0 \leq \overline{k}$.

i) *For any* $\epsilon \geq 0$,

$$\mathscr{W}(\epsilon) = u\left(f(\underline{x}^\epsilon) - \overline{x}^\epsilon\right).$$

ii) *For any optimal path* $\{k_t\}_{t=0}^{\infty}$, *we have*

$$\underline{x}^\epsilon < k_t < \overline{x}^\epsilon.$$

Moreover,

$$\liminf_{t \longrightarrow \infty} k_t = \underline{x}^\epsilon,$$
$$\limsup_{t \longrightarrow \infty} k_t = \overline{x}^\epsilon.$$

The case $\overline{k}$ is finite and $k_0 \geq \overline{k}$ deserves a slightly change in the treatment. The optimal value does not depend on k_0 and we have $\hat{v}(k_0) = u\left(f(\overline{k}) - \overline{k}\right)$ for any

$k_0 \geq \overline{k}$.

If for any $0 \leq x \leq \overline{k}$, $u\left(f(x) - x\right) \geq u\left(f(\overline{k}) - \overline{k}\right) - \epsilon$, let $\underset{\sim}{x}^\epsilon = 0$. Otherwise, let $\underset{\sim}{x}^\epsilon$ be the unique solution in $[0, \overline{k}]$ to

$$u\left(f(x) - x\right) = u\left(f(\overline{k}) - \overline{k}\right) - \epsilon.$$

Since $\overline{k}$ is finite, we have $f'(\infty) < 1$, and there is unique $\tilde{x}^\epsilon$ in $[\overline{k}, +\infty)$ solution to

$$u\left(f(x) - x\right) = u\left(f(\overline{k}) - \overline{k}\right) - \epsilon.$$

Note that contrary to the case $k_0 \leq \overline{k}$, in this case the values $\underset{\sim}{x}^\epsilon$ and $\tilde{x}^\epsilon$ are independent with k_0. If $k_0 \leq \tilde{x}^\epsilon$, then there exists an infinite number of solutions, and every optimal path fluctuates between $\underset{\sim}{x}^\epsilon$ and $\tilde{x}^\epsilon$.

Only for the case k_0 is sufficiently big, there exists unique solution and this solution is constant from the date $t = 1$.

Proposition 4.2. *Consier the case $k_0 \geq \overline{k}$.*

i) *If $\overline{k} \leq k_0 \leq \tilde{x}^\epsilon$, then*

$$\mathcal{W}(\epsilon) = u\left(f(\tilde{x}^\epsilon) - \underset{\sim}{x}^\epsilon\right).$$

Moreover, there is an infinite number of solutions. Every optimal paths $\{k_t\}_{t=0}^\infty$ satisfies

$$\underset{\sim}{x}^\epsilon < k_t < \tilde{x}^\epsilon,$$

and

$$\liminf_{t \to \infty} k_t = \underset{\sim}{x}^\epsilon,$$
$$\limsup_{t \to \infty} k_t = \tilde{x}^\epsilon.$$

ii) *If $k_0 \geq \tilde{x}^\epsilon$, then*

$$\mathcal{W}(\epsilon) = u\left(f(k_0) - \underset{\sim}{x}^\epsilon\right).$$

4.3 Optimal solution for the $\alpha-$*maximin* problem

With Lemma 4.1, Propositions 4.1 and 4.2, we can solve the $\alpha-$*maximin* problem. For low value of k_0, there exists an infinite number of solutions and the every optimal path fluctuates between two different level. For high value of k_0, there exists unique solution, and this solution is constant from the date $t = 1$.

Proposition 4.3. *For any $k_0 \geq 0$, there exists $\epsilon^* \geq 0$ such that*

$$\mathcal{V}(k_0) = \alpha W(\epsilon^*) + (1-\alpha)\left(\hat{v}(k_0) - \epsilon^*\right).$$

Moreover, there exists $\underline{k}_0 \leq \overline{k}_0$ such that

i) *For $0 \leq k_0 \leq \underline{k}_0$, every optimal path $\{k_t\}_{t=0}^{\infty}$ fluctuates between two different values and never converges:*

$$\liminf_{t\to\infty} k_t < \limsup_{t\to\infty} k_t.$$

ii) *For $k_0 \geq \overline{k}_0$, there exists unique solution and this solution is constant from the date $t = 1$.*

5. Constant productivity and logarithmic utility function

In this section, we provide some computations for the case the productivity is constant ($f(k) = Ak$) and the utility function is logarithmic $u(c) = \ln c$. The optimal policy function is[9]

$$\sigma(k) = \beta Ak.$$

Assume that $A > 1$. Hence $\overline{k} = \infty$.

[9]See Stokey & Lucas, with Prescott [21].

By induction, one has

$$\begin{aligned}
\hat{k}_t &= (\beta A)^t\, k_0, \\
\hat{c}_t &= A(1-\beta)\,(\beta A)^t\, k_0.
\end{aligned}$$

The value function is defined as

$$\begin{aligned}
v(k_0) &= \sum_{t=0}^{\infty} \beta^t \ln c_t \\
&= \frac{\ln A + \ln(1-\beta) + \ln k_0}{1-\beta} + (\ln\beta + \ln A) \sum_{t=0}^{\infty} t\beta^t.
\end{aligned}$$

1. Consider the case $A > \frac{1}{\beta}$. For this case, $k^s = \infty$. Hence for any k_0 we have $0 < k_0 < k^s$. By Lemma 3.3, $W'(0) = \infty$ and $W'(\bar{\epsilon}) = 0$. For any a there is an optimal sacrifice level ϵ^* satisfying $W'(\epsilon^*) = a$. There is T such that for $0 \le t \le T$,

$$u\left(f(k_t^*) - k_{t+1}^*\right) = u\left(f(k_0) - k_0\right) - \epsilon,$$

which is equivalent to

$$\ln\left(Ak_t^* - k_{t+1}^*\right) = \ln(A-1) + \ln k_0 - \epsilon.$$

For $0 \le t \le T$,

$$k_{t+1}^* = Ak_t^* - \frac{(A-1)k_0}{e^{\epsilon}}.$$

The value T is the smallest positive integer such that

$$u\left(f(k_{T+1}^*) - \sigma(k_{T+1}^*)\right) \ge u\left(f(k_0) - k_0\right) - \epsilon,$$

which is equivalent to

$$\ln\left(Ak_{T+1}^* - \beta Ak_{T+1}^*\right) \ge \ln\left(Ak_0 - k_0\right) - \epsilon.$$

This is equivalent to

$$\ln A + \ln(1-\beta) + \ln k^*_{T+1} \geq \ln(A-1) + \ln k_0 - \epsilon.$$

The value T is the first integer number satisfying

$$k^*_{T+1} \geq \frac{A-1}{A(1-\beta)} \times \frac{k_0}{e^\epsilon}.$$

The sequence $\{k^*_{T+t}\}_{t=0}^{\infty}$ is the solution of Ramsey problem with initial state k^*_{T+1}.

2. Consider the case $A < \frac{1}{\beta}$. In this case, $k^s = 0$ and every solution of Ramsey problem converges to zero. The critical value $\tilde{\epsilon}$ is then

$$\begin{aligned}\tilde{\epsilon} &= u\left(f(k_0) - k_0\right) - u(0)\\ &= \infty.\end{aligned}$$

We will then determine $W'(0)$. For ϵ close to zero, the critical time T from which $u(c_t^\epsilon) = u\left(f(k_0) - k_0\right) - \epsilon$ is $T = 1$.

The capital level k_1^ϵ is solution to

$$u\left(f(k_1) - k_1\right) = u\left(f(k_0) - k_0\right) - \epsilon.$$

This implies

$$\ln\left(Ak_1^\epsilon - k_1^\epsilon\right) = \ln(A-1) + \ln k_0 - \epsilon.$$

Hence

$$k_1^\epsilon = \frac{k_0}{e^\epsilon}.$$

We have

$$W(\epsilon) = u\left(f(k_0) - k_1^\epsilon\right) + \frac{\beta}{1-\beta}\left(u\left(f(k_0) - k_0\right) - \epsilon\right)$$

$$= \ln\left(Ak_0 - \frac{k_0}{e^{\epsilon}}\right) + \frac{\beta}{1-\beta}\left(\ln\left(Ak_0 - k_0\right) - \epsilon\right)$$
$$= \ln\left(A - \frac{1}{e^{\epsilon}}\right) + \frac{\beta}{1-\beta}\left(\ln(A-1) + \ln k_0 - \epsilon\right).$$

Hence for ϵ close to zero,

$$W'(\epsilon) = \frac{e^{-\epsilon}}{A - e^{-\epsilon}} - \frac{\beta}{1-\beta}.$$

Let ϵ converges to zero, we get

$$W'(0) = \frac{1-\beta A}{(A-1)(1-\beta)}.$$

We then have the following Proposition. The equality parameter has strong effect if it is sufficiently high. Otherwise, there is no difference between the behaviour following Ramsey-Rawls criterion and the one following Rawls criterion.

PROPOSITION 5.1. i) *For* $a \leq \frac{1-\beta A}{(A-1)(1-\beta)}$, *we have* $\epsilon^* \geq 0$, *and there exists* T *such that:*

a) *For* $0 \leq t \leq T$, $u(c_t^*) > \ln(A-1) + \ln k_0 - \epsilon^*$.

b) *For* $t \geq T+1$, $u(c_t^*) = \ln(A-1) + \ln k_0 - \epsilon^*$.

ii) *For* $a \geq \frac{1-\beta A}{(A-1)(1-\beta)}$, $\epsilon^* = 0$. *The optimal path is constant:* $k_t^* = k_0$ *for any* $t \geq 0$.

6. CONCLUSION

In this article we establish the solution of saving problems under Ramsey-Rawls and *maximin* criteria. The optimisation of the *inf* part leads to a *status-quo* situation. In order to circumvent the difficulties created by the *inf* part, we study the modified problems, by considering what is the best choice to do if we accept to lower the value of the *inf* part to some ϵ. Lowering this part gives us rooms to improve the value of the Ramsey part or the *sup* part.

We must do attention that though the modified problems have time-consistant solutions (for each given ϵ), it is not the same for the original problems. The reason is that the optimal value of ϵ depends on the initial state k_0. Moreover, the Ramsey-Rawls and *maximin* criteria are not consistent in time. Without no commitment between dates, or generations, it is possible that the in the future the economic agent desires to revise the past decision.

As a response to this time-inconsistency challenge, in our opinion, an approach by considering the *markovian rules*, as presented in the seminar work of Phelps & Pollack [20] may be a good idea. Phelps & Pollack [20] consider the existence and properties of linear *stationary markov equilibria* in the context of *quasi-hyperbolic discounting*. For general equilibria, this question becomes difficult and complicated, even in the case of constant productivity, as pointed out in the work of Krusell & Smith [19]. For a review of this literature, and an excellent analysis about saving and dissaving under *quasi-hyperbolic discounting* criterion, see Cao & Werming [7].

A. Proof of Lemma 2.1

(i) Consider the sequence of feasible paths $\mathbf{k}^n$ which converges to $\mathbf{k}$ in the the product topology.

Fix any $\epsilon > 0$. By the definition of $\nu(\mathbf{k})$, there exists T such that $u(c_T) < \inf_{t\geq 0} u(c_t) + \epsilon$.

By the convergence of the sequence $\{\mathbf{k}^n\}_{n=0}^{\infty}$ in product topology, we get $\lim_{n\to\infty} c_T^n = c_T$.

Hence for n sufficient large, it is true that $u(c_T^n) < u(c_T) + \epsilon$. This implies

$$
\begin{aligned}
\inf_{t\geq 0} u(c_t^n) &\leq u(c_T^n) \\
&< u(c_T) + \epsilon \\
&< \inf_{t\geq 0} u(c_t) + 2\epsilon.
\end{aligned}
$$

Thus

$$\limsup_{n\to\infty} v(\mathbf{k}^n) < v(\mathbf{k}) + 2\epsilon.$$

Let ϵ converges to zero, we get the upper semi-continuity of v.

(ii) The part (i) is a consequence of the upper semi-continuity of v and the compactness of $\Pi(k_0)$ in respect to product topology.

B. Proof of Proposition 2.1

(i) Denote $\mathbf{k}^*$ as a solution to the problem. For any $t \geq 0$,

$$\begin{aligned} u\left(f(k_t^*) - f(k_{t+1}^*)\right) &\geq v(\mathbf{k}^*) \\ &\geq v(k_0, k_0, \dots) \\ &= u\left(f(k_0) - k_0\right). \end{aligned}$$

We then have $f(k_0) - k_1^* \geq f(k_0) - k_0$, which is equivalent to $k_1^* \leq k_0$.

Suppose that $k_t^* \leq k_0$ for some t. Then

$$\begin{aligned} k_0 - k_{t+1}^* &\geq f(k_0) - f(k_t^*) \\ &\geq f'(k_0)(k_0 - k_t^*) \\ &\geq k_0 - k_t^*, \end{aligned}$$

which implies $k_{t+}^* \leq k_t^*$. By induction, $k_0 \geq k_t^*$ for all t. Furthermore, the sequence (k_t^*) is decreasing and then converges to $\hat{k} \leq k_0$.

From the continuity of f, we have that $f(\hat{k}) - \hat{k} \geq f(k_0) - k_0$. But the function $f(x) - x$ is increasing in $[0, \overline{k}]$, thus, $f(\hat{k}) - \hat{k} \leq f(k_0) - k_0$, then $\hat{k} = k_0$, and $k_t^* = k_0$ for all t, because $k_0 \geq k_t^*$ and the sequence $\{k_t^*\}_{t=0}^{\infty}$ is decreasing to k_0.

(ii) First, consider the sequence $\mathbf{k} = (k_0, \overline{x}, \overline{x}, \dots)$ which is feasible. We have

$$\max_{\mathbf{k}\in\Pi(k_0)} v(\mathbf{k}) \geq f(\overline{k}) - \overline{k}.$$

Let $\mathbf{k}^s$ be an optimal solution. Since for all $t \geq 0$, $f(k_t^*) - k_{t+1}^* \geq f(\overline{k}) - \overline{k}$,

$$\begin{aligned}
\overline{k} - k_{t+1}^* &\geq f(\overline{k}) - f(k_t^*) \\
&\geq f'(\overline{k})(\overline{k} - k_t^*) \\
&= \overline{k} - k_t^*.
\end{aligned}$$

This implies $k_{t+1}^* \leq k_t^*$ for any t. The sequence $\mathbf{k}^s$ is decreasing and converges to some $\hat{k}$. By the continuity of f, $f(\hat{k}) - \hat{k} \geq f(\overline{k}) - \overline{k}$. Since $\overline{k}$ maximizes $f(x) - x$, this implies $\hat{k} = \overline{k}$. Hence

$$\hat{v}(k_0) = f(\overline{k}) - \overline{k}.$$

Since $k_0 > \overline{x}$, by induction, we can construct a sequence $\mathbf{k}$ which satisfies: for all t, $\overline{k} < k_{t+1} < f(k_t) - f(\overline{x}) + \overline{k}$. With this sequence, we have $f(k_t) - k_{t+1} > f(\overline{k}) - \overline{k}$, and $k_{t+1} < k_t$, since $f(k_t) - k_t < f(\overline{k}) - \overline{k}$. So the sequence $\{k_t\}_{t=0}^{\infty}$ converges to $\overline{k}$ and $\hat{v}(k_0) = f(\overline{k}) - \overline{k}$. We have an infinity number of sequences satisfying this property. The problem has an infinite number of solutions.

C. Proof of Proposition 3.1

Recall that for $0 \leq k_0 \leq \overline{k}$, for any feasible sequence $\{k_t\}_{t=0}^{\infty}$,

$$\inf_{t \geq 0} u\big(f(k_t) - k_{t+1}\big) \leq \hat{v}(k_0).$$

Let $\{k_t^*\}_{t=0}^{\infty}$ be the optimal solution of problem (P). Define

$$\epsilon^* = \hat{v}(k_0) - \inf_{t \geq 0} u(c_t^*).$$

We have

$$V(k_0) = \sum_{t=0}^{\infty} \beta^t u(c_t^*) + a \inf_{t \geq 0} u(c_t^*)$$

$$
\begin{aligned}
&= \sum_{t=0}^{\infty} \beta^t u(c_t^*) + a\left(\hat{v}(k_0) - \epsilon^*\right) \\
&\leq W(\epsilon^*) + a\left(\hat{v}(k_0) - \epsilon^*\right).
\end{aligned}
$$

Conversely, for any $\epsilon \geq 0$,

$$
\begin{aligned}
W(\epsilon) + a\left(\hat{v}(k_0) - \epsilon\right) &= \sum_{t=0}^{\infty} \beta^t u\left(c_t^\epsilon\right) + a\left(\hat{v}(k_0) - \epsilon\right) \\
&\leq \sum_{t=0}^{\infty} \beta^t u\left(c_t^\epsilon\right) + a \inf_{t \geq 0} u\left(c_t^\epsilon\right) \\
&\leq V(k_0).
\end{aligned}
$$

The proof is completed.

D. Proof of Proposition 3.2

Obviously W is increasing. The concavity of W comes from the concavity of utility function u and production function f.

We consider first the case $0 \leq k_0 \leq \overline{k}$. For each $\epsilon > 0$, let $x^*(\epsilon)$ be the smallest $x \geq k_0$ such that

$$
u\big(f(x) - x\big) \geq u\big(f(k_0) - k_0\big) - \epsilon.
$$

If the strict inequality is satisfied for any $x \geq k_0$, let $x^*(\epsilon) = \infty$.

(i) We consider the case

$$
f'(k_0) > \frac{1}{\beta}.
$$

First, observe that $k^s > 0$ and $k_0 < k^s$.

Since the function $x - \sigma(x)$ is strictly increasing in (k_0, k^s), either $x^*(\epsilon) = \infty$, either x^* is finite and $0 < x^*(\epsilon) < k^s$. Indeed, it is obvious that $k^s = \infty$ implies $x^* = \infty$. Suppose that k^s is finite. Then $k^s = \sigma(k^s)$ and hence $u\big(f(k^s) - \sigma(k^s)\big) < u\big(f(k_0) - k_0\big) - \epsilon$. This implies x^* is finite and $k_0 < x^* < k^s$.

We will prove the following claim: for any t, $k_t^\epsilon < k^s$. This is true if for any t, $k_t^\epsilon < x^*(\epsilon)$. Consider the case there exists T satisfying: $k_t^\epsilon < x^*(\epsilon) \leq k_{t+1}^\epsilon$.

We have

$$\begin{aligned} u\big(f(k_t^\epsilon) - k_{t+1}^\epsilon\big) &\geq u\big(f(k_0) - k_0\big) - \epsilon \\ &> u\big(f(k^s) - k^s\big). \end{aligned}$$

Then $k_{t+1}^\epsilon < k^s$.

Let $\{\check{k}_t\}_{t=T+1}^\infty$ be the solution of Ramsey problem with initial state k_{t+1}^ϵ. Since $k_{t+1}^\epsilon < k^s$, $\check{k}_t < k^s$ for any $t \geq T+1$ and

$$\begin{aligned} \inf_{t\geq T+1} u(\check{c}_t) &= u\big(f(k_{t+1}^\epsilon) - \sigma(k_{t+1}^\epsilon)\big) \\ &\geq u\big(f(x^*(\epsilon)) - \sigma(x^*(\epsilon))\big) \\ &= u\big(f(k_0) - k_0\big) - \epsilon. \end{aligned}$$

Hence the sequence $\{k_0, k_1^\epsilon, \ldots, k_t^\epsilon, k_{t+1}^\epsilon, \check{k}_{T+2}, \check{k}_{T+2}, \ldots\}$ is the optimal solution for the problem (P^ϵ), or $\check{k}_t = k_t^\epsilon$ for any $t \geq T+1$. The prove that $k_t^\epsilon < k^s$ for any t is completed.

Consider the Lagrangian:

$$\begin{aligned} \mathscr{L} = \sum_{t=0}^\infty \beta^t u(c_t) - \sum_{t=0}^\infty \beta^t \lambda_t \big[c_t + k_{t+1} - f(k_t)\big] \\ - \sum_{t=0}^\infty \beta^t \mu_t \big[u\big(f(k_0) - k_0\big) - \epsilon - u(c_t)\big]. \end{aligned}$$

By the Inada condition of u, at optimal the consumption and capital level are strictly positive. The Lagrangian parameters for these constraints are hence zero. For any t:

$$\begin{aligned} (1 + \mu_t) u'(c_t^\epsilon) &= \lambda_t, \\ \lambda_t &= \beta \lambda_{t+1} f'(k_{t+1}^\epsilon). \end{aligned}$$

This implies for any t:

$$\begin{aligned}(1+\mu_t)u'(c_t^\epsilon) &= \beta(1+\mu_{t+1})u'(c_{t+1}^\epsilon)f'(k_{t+1}^\epsilon)\\ &\geq \beta f'(k_{t+1}^\epsilon)u'(c_{t+1}^\epsilon).\end{aligned}$$

Suppose that $u\left(c_T^\epsilon\right) > u\big(f(k_0)-k_0\big)-\epsilon$. The constraint does not bind and hence $\mu_T = 0$.

Since $f'(k_{T+1}^\epsilon) \geq \frac{1}{\beta}$, then $u'(c_T^\epsilon) \geq u'(c_{T+1}^\epsilon)$, and hence $c_{t+1}^\epsilon \geq c_T^\epsilon$. The $(T+1)^{th}$ constraint also does not bind: $u\left(c_{T+1}^\epsilon\right) > u\big(f(k_0)-k_0\big)-\epsilon$.

By induction, for any $t \geq T+1$, $u\left(c_t^\epsilon\right) > u\big(f(k_0)-k_0\big)-\epsilon$ and $\mu_t = 0$. The sequence $\{(c_t^\epsilon, k_{t+1}^\epsilon)\}_{t=T}^\infty$ is increasing and satisfies Euler equations. Hence $\{k_t^\epsilon\}_{t=T}^\infty$ is the solution for Ramsey problem with initial state k_T^ϵ. We also have $\lim_{t\to\infty} k_t^\epsilon = k^s$.

(ii) Consider the case

$$1 < f'(k_0) < \frac{1}{\beta}.$$

Necessary condition for this is $0 \leq k^s < \infty$. Recall that we are working in the case $k_0 \leq \overline{k}$. We first prove that $k_t^\epsilon > k^s$ for any $t \geq 0$. Assume that there exists T such that $k_T^\epsilon \leq k^s$. We have

$$\begin{aligned}u\big(f(k_T^\epsilon)-k_{T+1}^\epsilon\big) &\geq \hat{v}(k_0)-\epsilon\\ &= u\big(f(k_0)-k_0\big)-\epsilon\\ &> u\big(f(k_0)-k_0\big)-\tilde{\epsilon}\\ &= u\big(f(k^s)-k^s\big),\end{aligned}$$

which implies $k_{T+1}^\epsilon < k_T^\epsilon < k^s$, since $f(x)-x$ is strictly increasing in $(0,k^s)$. By induction, the sequence $\{k_{T+t}^\epsilon\}_{t=0}^\infty$ is decreasing and converges to $\underline{k} < k^s$. Taking the limit, we get

$$\begin{aligned}u\big(f(k^s)-k^s\big) &> u\big(f(\underline{k})-\sigma(\underline{k})\big)\\ &\geq \hat{v}(k_0)-\epsilon\end{aligned}$$

$$
\begin{aligned}
&\geq u\big(f(k_0) - k_0\big) - \epsilon \\
&> u\big(f(k^s) - k^s\big),
\end{aligned}
$$

a contradiction.

Once the property that $k_t^\epsilon > k^s$ for any $t \geq 0$ established, we re-utilise the Lagrangian:

$$
\begin{aligned}
\mathscr{L} = &\sum_{t=0}^{\infty} \beta^t u(c_t) - \sum_{t=0}^{\infty} \beta^t \lambda_t \big[c_t + k_{t+1} - f(k_t)\big] \\
&- \sum_{t=0}^{\infty} \beta^t \mu_t \big[u\big(f(k_0) - k_0\big) - \epsilon - u(c_t)\big].
\end{aligned}
$$

For any t:

$$
\begin{aligned}
(1 + \mu_t) u'(c_t^\epsilon) &= \lambda_t, \\
\lambda_t &= \beta \lambda_{t+1} f'(k_{t+1}^\epsilon).
\end{aligned}
$$

This implies for any t:

$$
\begin{aligned}
u'(c_t^\epsilon) &\leq (1 + \mu_t) u'(c_t^\epsilon) \\
&= \beta (1 + \mu_{t+1}) u'(c_{t+1}^\epsilon) f'(k_{t+1}^\epsilon).
\end{aligned}
$$

If $u\left(c_T^\epsilon\right) > u\big(f(k_0) - k_0\big) - \epsilon$, then the constraint does not bind, and $\mu_T = 0$. Since $f(k_T^\epsilon) < \frac{1}{\beta}$, we get $u'(c_{T-1}^\epsilon) < u'(c_T^\epsilon)$, which implies $c_{T-1}^\epsilon > c_T^\epsilon$, with the direct consequence

$$
u\left(c_{T-1}^\epsilon\right) > u\big(f(k_0) - k_0\big) - \epsilon.
$$

By induction, we get for any $0 \leq t \leq T$,

$$
u\left(c_t^\epsilon\right) > u\big(f(k_0) - k_0\big) - \epsilon.
$$

If this property is ensured for any $t \geq 0$, the the sequence $\{k_t^\epsilon\}_{t=0}^{\infty}$ satisfies Euler equations and transversality condition, hence it is the optimal solution for Ramsey problem and converges to k^s: a contradiction, since

$$u\big(f(k^s) - k^s\big) < u\big(f(k_0) - k_0\big) - \epsilon.$$

Hence there exists T such that for any $t \geq T$,

$$u\left(c_T^\epsilon\right) = u\big(f(k_0) - k_0\big) - \epsilon.$$

Obviously, for any $t \geq 0$, we have

$$u\left(c_{T+t}^\epsilon\right) = u\big(f(k_0) - k_0\big) - \epsilon,$$

otherwise using the same arguments in the induction, we get $u\left(c_T^\epsilon\right) > u\big(f(k_0) - k_0\big) - \epsilon$, a contradiction.

For the lase case $f'(k_0) \leq 1$, or $k_0 \geq \overline{k}$, we use the same arguments as for the case $1 \leq f'(k_0) \leq \frac{1}{\beta}$, with the observation that the value of $\hat{v}(k_0)$ is $u\left(f(\overline{k}) - \overline{k}\right)$ and $f(\overline{k}) - \overline{k} \geq f(k^s) - k^s$.

E. Proof of Lemma 3.3

(i) We prove that $W'(0) = +\infty$. Consider $T(\epsilon)$ in the proof of Proposition 3.2.

For any $0 \leq t \leq T(\epsilon)$:

$$\begin{aligned}
\epsilon &= u\big(f(k_0) - k_0\big) - u\big(f(k_t^\epsilon) - k_{t+1}^\epsilon\big) \\
&\geq u'\big(f(k_0) - k_0\big)\left(f(k_0) - k_0 - f(k_t^\epsilon) + k_{t+1}^\epsilon\right) \\
&\geq u'\big(f(k_0) - \sigma(k_0)\big)\left(f'(k_0)(k_0 - k_t^\epsilon) + k_{t+1}^\epsilon - k_0\right).
\end{aligned}$$

This implies

$$k_{t+1}^\epsilon - k_0 \leq \frac{\epsilon}{u'\big(f(k_0) - k_0\big)} + f'(k_0)(k_t^\epsilon - k_0).$$

By induction, we get for any $t \geq 0$,

$$k^{\epsilon}_{t+1} - k_0 \leq \frac{\left[f'(k_0)\right]^{t+1} - 1}{f'(k_0) - 1} \times \frac{\epsilon}{u'\big(f(x^*) - x^*\big)}.$$

Hence

$$\begin{aligned}
x^*(\epsilon) - k_0 &\leq k_{T(\epsilon)+1} - k_0 \\
&\leq \frac{\left[f'(k_0)\right]^{T(\epsilon)+1} - 1}{f'(k_0) - 1} \times \frac{\epsilon}{u'\big(f(k_0) - k_0\big)}.
\end{aligned}$$

$$\begin{aligned}
W(\epsilon) &= \sum_{t=0}^{T(\epsilon)} \beta^t u\left(c^{\epsilon}_t\right) + \sum_{t=T(\epsilon)+1}^{\infty} \beta^t u\left(c^{\epsilon}_t\right) \\
&= \big(u\big(f(k_0 - k_0\big) - \epsilon\big) \sum_{t=0}^{T(\epsilon)} \beta^t + \beta^{T(\epsilon)+1} v(k^{\epsilon}_{T(\epsilon)}).
\end{aligned}$$

Hence

$$\begin{aligned}
W(\epsilon) - W(0) &= -\epsilon \sum_{t=0}^{T(\epsilon)} \beta^t + \beta^{T(\epsilon)+1}\left(v\left(k^{\epsilon}_{T(\epsilon)}\right) - \frac{u\big(f(k_0) - k_0\big)}{1-\beta}\right) \\
&= -\epsilon \frac{1 - \beta^{T(\epsilon)+1}}{1-\beta} + \beta^{T(\epsilon)+1}\left(v\left(k^{\epsilon}_{T(\epsilon)}\right) - \frac{u\big(f(k_0) - k_0\big)}{1-\beta}\right).
\end{aligned}$$

Now we prove that

$$\lim_{\epsilon \to 0} \frac{\beta^{T(\epsilon)}}{\epsilon} = +\infty.$$

Indeed, recall that

$$\frac{\left[f'(k_0)\right]^{T(\epsilon)+1} - 1}{f'(k_0) - 1} \times \frac{\epsilon}{u'\big(f(k_0) - k_0\big)} \sim x^*(\epsilon) - k_0.$$

This implies

$$\big(f'(k_0)\big)^{T(\epsilon)} \epsilon \sim O(1).$$

Hence

$$T(\epsilon)\ln\big(f'(k_0)\big) \sim -\ln(\epsilon),$$

which is equivalent to

$$T(\epsilon) \sim -\frac{\ln(\epsilon)}{\ln\big(f'(k_0)\big)}.$$

We have

$$\begin{aligned}
\beta^{T(\epsilon)} &\sim \left(e^{\ln\beta}\right)^{-\frac{\ln(\epsilon)}{\ln\left(f'(k_0)\right)}} \\
&\sim \epsilon^{-\frac{\ln\beta}{\ln\left(f'(k_0)\right)}} \\
&\sim \epsilon^{\frac{\ln\left(\frac{1}{\beta}\right)}{\ln\left(f'(k_0)\right)}}.
\end{aligned}$$

Since $f'(k_0) > \frac{1}{\beta}$, we have

$$\lim_{\epsilon\to 0}\frac{\beta^{T(\epsilon)}}{\epsilon} = \lim_{\epsilon\to 0}\epsilon^{\frac{\ln\left(\frac{1}{\beta}\right)}{\ln\left(f'(k_0)\right)}-1} = \infty,$$

which implies $W'(0) = +\infty$.

(ii) First assume that $k^s < k_0 \le \overline{k}$. Now we prove that $W'(0) < +\infty$. For ϵ small:

$$\begin{aligned}
W(\epsilon) - W(0) &= \sum_{t=0}^{\infty}\beta^t\left[u\big(f(k_t^\epsilon) - k_{t+1}^\epsilon\big) - u\big(f(k_0) - k_0\big)\right] \\
&\le u'\big(f(k_0) - k_0\big)\sum_{t=0}^{\infty}\beta^t\left[f(k_t^\epsilon) - f(k_0) - k_{t+1}^\epsilon + k_0\right] \\
&\le u'\big(f(k_0) - k_0\big)\sum_{t=0}^{\infty}\beta^t\left[f'(k_0)(k_t^\epsilon - k_0) - k_{t+1}^\epsilon + k_0\right] \\
&\le u'\big(f(k_0) - k_0\big)\sum_{t=0}^{\infty}\beta^t\left[f'(k_0)(k_t^\epsilon - k_0)\right] \\
&\le u'\big(f(k_0) - k_0\big)f'(k_0)\sum_{t=0}^{\infty}\beta^t\left[k_t^\epsilon - k_0\right]
\end{aligned}$$

$$\leq u'\big(f(k_0)-k_0\big)f'(k_0)\sum_{t=0}^{\infty}\beta^t\frac{\left[f'(k_0)\right]^{t+1}-1}{f'(k_0)-1}\times\frac{\epsilon}{u'\big(f(k_0)-k_0\big)}$$
$$=f'(k_0)\sum_{t=0}^{\infty}\beta^t\frac{\left[f'(k_0)\right]^{t+1}-1}{f'(k_0)-1}\times\epsilon$$
$$=O(\epsilon),$$

since $\beta f'(k_0)<1$.

This implies $W(\epsilon)-W(0)=O(\epsilon)$, or $W'(0)<+\infty$.

Now assume that $\overline{k}$ is finite and $k_0\geq\overline{k}$. We use exactly the same arguments in the proof of part (ii), by changing the constrains $u(c_t)\geq u\left(f(k_0)-k_0\right)-\epsilon$ by $u(c_t)\geq u\left(f(\overline{k})-\overline{k}\right)$.

Now we prove that $W'(\tilde{\epsilon})=0$. For ϵ close enough to $\tilde{\epsilon}$, the critical time $T(\epsilon)$ from which the optimal path behaves as a solution of Ramsey problem with initial state $k^{\epsilon}_{T(\epsilon)}$ is $T(\epsilon)=1$. We then have

$$u\left(f(k_0)-k^{\epsilon}_1\right)=u\left(f(k_0)-k_0\right)-\epsilon,$$

and the sequence $\{k^{\epsilon}_{1+t}\}_{t=0}^{\infty}$ is the solution of Ramsey problem with initial state k^{ϵ}_1. This implies

$$W(\epsilon)=u\left(f(k_0)-k_0\right)-\epsilon+\beta v(k^{\epsilon}_1),$$

and

$$W'(\epsilon)=-1+\beta v'(k^{\epsilon}_1)\times\frac{dk'_1}{d\epsilon}.$$

By the implicite function theorem, we have

$$\frac{dk^{\epsilon}_1}{d\epsilon}=\frac{1}{u'\left(f(k_0)-k^{\epsilon}_1\right)}.$$

Observe that by letting ϵ converges to $\tilde{\epsilon}$ we have

$$\lim_{\epsilon \to \tilde{\epsilon}} k_1^{\epsilon} = \sigma(k_0).$$

This implies

$$W'_{-}(\tilde{\epsilon}) = -1 + \beta v'(\sigma(k_0)) \times \frac{1}{u'\left(f(k_0) - \sigma(k_0)\right)}.$$

Recall that it is well-known in dynamic programming literature that

$$\begin{aligned} v(k_0) &= \max_{0 \le k_1 \le f(k_0)} \left[u\left(f(k_0) - k_1\right) + \beta v(k_1)\right] \\ &= u\left(f(k_0 - \sigma(k_0))\right) + \beta v\left(\sigma(k_0)\right). \end{aligned}$$

Combining with Inada condition, this implies

$$-u'\left(f(k_0) - \sigma(k_0)\right) + \beta v'(\sigma(k_0)) = 0,$$

which is equivalent to

$$W'(\tilde{\epsilon}) = 0.$$

(iii) Since for any $0 \le \epsilon \le \tilde{\epsilon}$, there exists T such that the equality constraint corresponding to T bind. Hence the solutions corresponding to difference values of ϵ are different. Combining this with the strict concavity of u, we get W is strictly concave in $[0, \tilde{\epsilon}]$.

F. Proof of Proposition 3.3

For any $0 \le \epsilon \le \tilde{\epsilon}$, the optimal solution satisfies the following property: there exists t such that $u\left(c_t^{\epsilon}\right) = u\big(f(k_0) - k_0\big)$. Hence the solutions corresponding to difference values of ϵ are also different. Combining with the strictly concavity of u, the function W is strictly concave in $[0, \tilde{\epsilon}]$. This implies the existence of an unique left derivative of W.

Since for any $a > 0$, we have $W'_-(\tilde{\epsilon}) = 0 < a < W'(0) = \infty$, there exists unique $0 < \epsilon < \tilde{\epsilon}$ such that $W'(\epsilon^*) = a$. The statement of the Lemma is a consequence of Propositions 3.1 and 3.2.

G. Proof of Lemma 4.1

Consider any feasible sequence $\{k_t\}_0^\infty \in \Pi(k_0)$, with $c_t = f(k_t) - k_{t+1}$, define

$$\hat{\epsilon} = \hat{v}(k_0) - \inf_{t\geq 0} u\left(c_t\right).$$

Obviously,

$$\begin{aligned}(1-\alpha)\sup_{t\geq 0} u(c_t) + \alpha \inf_{t\geq 0} u(c_t) &= \alpha \sup_{t\geq 0} u(c_t) + (1-\alpha)\left(v(k_0) - \hat{\epsilon}\right)\\ &\leq \alpha \mathscr{W}(\hat{\epsilon}) + (1-\alpha)\left(v(k_0) - \hat{\epsilon}\right)\\ &\leq \sup_{\epsilon\geq 0}\left[(1-\alpha)\mathscr{W}(\epsilon) + \alpha\left(v(k_0) - \epsilon\right)\right].\end{aligned}$$

Now consider any feasible sequence $\{k_t\}_0^\infty \in \Pi(k_0)$ satisfying the constraints of the modified problem.

$$\begin{aligned}\alpha \sup_{t\geq 0} u(c_t) + (1-\alpha)\left(v(k_0) - \epsilon\right) &\leq \alpha \sup_{t\geq 0} u(c_t) + (1-\alpha)\inf_{t\geq 0} u(c_t)\\ &\leq \mathscr{V}(k_0).\end{aligned}$$

Taking the supremum in the left hand side, the proof of Lemma is completed.

H. Proof of Proposition 4.1

i) We prove that for any feasible sequence $\{k_t\}_{t=0}^\infty$ of the modified problem, we have for any $t \geq 0$,

$$\underline{x}^\epsilon \leq k_t < \overline{x}^\epsilon.$$

Assume the contrary of the first inequality, that for some T, $k_T < \underline{x}^\epsilon$. Since the function $f(x) - x$ is strictly increasing in $[0, k_0]$, se have

$$\begin{aligned} u\left(f(k_T) - k_T\right) &< u\left(f(\underline{x}^\epsilon) - \underline{x}^\epsilon\right) \\ &= u\left(f(k_0) - k_0\right) - \epsilon \\ &\leq u\left(f(k_T) - k_{T+1}\right). \end{aligned}$$

This implies that $k_{T+1} \leq k_T < \underline{x}^\epsilon$. By induction, the sequence $\{k_{T+t}\}_{t=0}^\infty$ is decreasing and converge to some $0 \leq k^* < \underline{x}^\epsilon$. Hence

$$\begin{aligned} u\left(f(k^*) - k^*)\right) &< u\left(f(\underline{x}^\epsilon) - \underline{x}^\epsilon\right) \\ &= u\left(f(k_0) - k_0\right) - \epsilon, \end{aligned}$$

a contradiction.

Consider the sequence $\{\overline{k}_t\}_{t=0}^\infty$ determined as

$$\begin{aligned} \overline{k}_0 &= k_0, \\ u\left(f(\overline{k}_t) - \overline{k}_{t+1}\right) &= u\left(f(k_0) - k_0\right) - \epsilon. \end{aligned}$$

It is easy to verify that the sequence $\{\overline{k}_t\}_{t=0}^\infty$ is increasing and converges to $\overline{x}^\epsilon$, whether this value is finite ou infinite.

Fix any feasible sequence $\{k_t\}_{t=0}^\infty$ of the modified problem. Assume that for some T, $k_T \leq \overline{k}_T$. As a consequence,

$$\begin{aligned} u\left(f(\overline{k}_t) - \overline{k}_{t+1}\right) &= u\left(f(k_0) - k_0\right) - \epsilon \\ &\leq u\left(f(k_T) - k_{T+1}\right) \\ &\leq u\left(f(\overline{k}_T) - k_{T+1}\right), \end{aligned}$$

which implies $k_{T+1} \leq \overline{k}_T$. By induction, for any $t \geq 0$,

$$k_{T+t} \leq \overline{k}_{T+t} < \overline{x}^\epsilon.$$

For any t, $\underline{x}^\epsilon \leq k_t < \overline{x}^\epsilon$, hence for any feasible sequence:

$$\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) \leq u\left(f(\overline{x}^\epsilon) - \underline{x}^\epsilon\right).$$

Now we prove that there exists feasible path $\{k_t\}_{t=0}^\infty$ such that $\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) = u\left(f(\overline{x}^\epsilon) - \underline{x}^\epsilon\right)$. Fix any two sequences $\{\underline{x}_n\}_{n=0}^\infty$ and $\{\overline{x}_n\}_{n=0}^\infty$ such that the former one is strictly decreasing and converges to $\underline{x}^\epsilon$ and the later one is strictly increasing and converges to $\overline{x}^\epsilon$.

$$\underline{x}_0 > \underline{x}_1 > \cdots > \underline{x}_n > \cdots \longrightarrow \underline{x}^\epsilon,$$
$$\overline{x}_0 < \overline{x}_1 < \cdots < \overline{x}_n < \cdots \longrightarrow \overline{x}^\epsilon.$$

We construct the sequence $T_0 < T_1 < \cdots < T_n$ and the sequence $\{k_t\}_{t=0}^\infty$ as follows. For $0 \leq t \leq T_0$,

$$u\left(f(k_t) - k_{t+1}\right) = u\left(f(k_0) - k_0\right) - \epsilon.$$

If we continue to use this equation to define k_{t+1} from k_t to infinity, the sequence will converges to $\overline{x}^\epsilon$. Hence the exists index T_0 which is the smallest index t satisfying $k_{T_0} > \overline{x}_0$. Let $k_{T_0+1} = \underline{x}_0$. We have

$$u\left(f(k_{T_0}) - k_{T_0+1}\right) = u\left(f(\overline{x}_0) - \underline{x}_0\right).$$

For $T_0 + 1 \leq T_1$, define the sequence as

$$u\left(f(k_t) - k_{t+1}\right) = u\left(f(k_0) - k_0\right) - \epsilon.$$

Using the same argument for the definition of T_0, there exists T_1 the smallest index satisfying $k_t > \overline{x}_1$. Let $k_{T_1+1} = \overline{x}_1$. We have

$$u\left(f(k_{T_1}) - k_{T_1+1}\right) = u\left(f(\overline{x}_1) - \underline{x}_1\right).$$

And we define in the same way, by induction T_{n+1} in function of T_n. For any

$n \geq 0$ we have

$$u\left(f(k_{T_n}) - k_{T_n+1}\right) = u\left(f(\overline{x}_n) - \underline{x}_n\right).$$

Let n converges to infinity,

$$\lim_{n\to\infty} u\left(f(k_{T_n}) - k_{T_n+1}\right) = u\left(f(\overline{x}^{\epsilon}) - \underline{x}^{\epsilon}\right).$$

Hence

$$\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) = u\left(f(\overline{x}^{\epsilon}) - \underline{x}^{\epsilon}\right).$$

Since the two sequence two sequences $\{\underline{x}_n\}_{n=0}^{\infty}$ and $\{\overline{x}_n\}_{n=0}^{\infty}$ can be defined arbitrarily, there exist an infinite number of optimal solution.

Consider any optimal path $\{k_t\}_{t=0}^{\infty}$. It is easy to verify that if $k_T = \underline{x}^{\epsilon}$, by the constraint, $k_{T+t} = \underline{x}^{\epsilon}$ for any $t \geq 0$, which implies $\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) < u\left(f(\overline{x}^{\epsilon}) - \underline{x}^{\epsilon}\right)$, a contradiction. Hence for any t, $\underline{x}^{\epsilon} < k_t < \overline{x}^{\epsilon}$. Moreover, there exist an infinite number $T_0 < T_1 < \cdots < T_n < \ldots$ such that

$$\lim_{n\to\infty} u\left(f(k_{T_n t}) - k_{T_n+1}\right) = u\left(f(\overline{x}^{\epsilon}) - \underline{x}^{\epsilon}\right).$$

Hence we have

$$\lim_{n\to\infty} k_{T_n} = \overline{x}^{\epsilon},$$
$$\lim_{n\to\infty} k_{T_n+1} = \underline{x}^{\epsilon}.$$

ii) This part is direct consequence of the proof of the first part.

iii) This is consequence of the first part and Lemma 4.1.

I. Proof of Proposition 4.2

i) First, we prove that for any $\overline{k} \leq k_0 \leq \tilde{x}^\epsilon$, any feasible path $\{k_t\}_{t=0}^\infty \in \Pi^\epsilon(k_0)$, we have

$$\underset{\sim}{x}^\epsilon \leq k_t \leq \tilde{x}^\epsilon.$$

Assume that there is some T such that $k_T < \underset{\sim}{x}^\epsilon$. Then

$$\begin{aligned} u\left(f(k_T) - k_T\right) &< u\left(f(\underset{\sim}{x}) - \underset{\sim}{x}\right) \\ &= u\left(f(\overline{k}) - \overline{k}\right) - \epsilon \\ &\leq u\left(f(k_T) - k_{T+1}\right), \end{aligned}$$

which implies $k_{T+1} \leq k_T < \underset{\sim}{x}^\epsilon$. By induction, the sequence $\{k_{T+t}\}_{t=0}^\infty$ is decreasing and converges to some $k^* < \underset{\sim}{x}^\epsilon$, and

$$\begin{aligned} u\left(f(k^*) - k^*\right) &< u\left(f(\underset{\sim}{x}) - \underset{\sim}{x}\right) \\ &= u\left(f(\overline{k}) - \overline{k}\right) - \epsilon, \end{aligned}$$

a contradiction.

Assume that $\underset{\sim}{x}^\epsilon \leq k_T \leq \tilde{x}^\epsilon$. Since

$$\begin{aligned} u\left(f(\tilde{x}^\epsilon) - \tilde{x}^\epsilon\right) &\leq u\left(f(k_T) - k_{T+1}\right) \\ &\leq u\left(f(\tilde{x}^\epsilon) - k_{T+1}\right), \end{aligned}$$

we have $k_{T+1} \leq \tilde{x}^\epsilon$. This property is satisfied by k_0, by induction, $k_t \leq \tilde{x}^\epsilon$ for all $t \geq 0$.

Since for any t, $\underset{\sim}{x}^\epsilon \leq k_t \leq \tilde{x}^\epsilon$,

$$\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) \leq u\left(f(\tilde{x}^\epsilon) - \underset{\sim}{x}^\epsilon\right).$$

In order to prove that the left hand side is equal to the right hand side in the above inequality, and there exists an infinite number of solution for the modified problem, we prove that for any $\underset{\sim}{x}^{\epsilon} \le k_0 \le \tilde{x}^{\epsilon}$, the sequence $\{\tilde{k}_t\}_{t=0}^{\infty}$ defined as below is increasing and converges to $\tilde{x}^{\epsilon}$:

$$\begin{aligned}
\tilde{k}_0 &= k_0, \\
u\left(f(\tilde{k}_t) - \tilde{k}_{t+1}\right) &= u\left(f(\overline{k}) - \overline{k}\right) - \epsilon, \text{ for all } t \ge 0.
\end{aligned}$$

Indeed, using the same arguments above, we have for any t, $\underset{\sim}{x}^{\epsilon} \le \tilde{k}_t \le \tilde{x}^{\epsilon}$. Then

$$\begin{aligned}
u\left(f(\tilde{k}_t) - \tilde{k}_{t+1}\right) &= u\left(f(\tilde{x}^{\epsilon}) - \tilde{x}^{\epsilon}\right) \\
&= u\left(f(\overline{k}) - \overline{k}\right) \\
&\le u\left(f(\tilde{k}_t) - \tilde{k}_t\right).
\end{aligned}$$

This implies $\tilde{k}_t \le \tilde{k}_{t+1}$ and the sequence $\{\tilde{k}_t\}_{t=0}^{\infty}$ is increasing and converges to the solution of $u\left(f(x) - x\right) = u\left(f(\overline{k}) - \overline{k}\right)$, or

$$\lim_{t \to \infty} \tilde{k}_t = \tilde{x}^{\epsilon}.$$

Now fix two sequences $\{\underset{\sim}{x}_n\}_{n=0}^{\infty}$ which is strictly decreasing and converges to $\underset{\sim}{x}^{\epsilon}$, and $\{\tilde{x}_n\}_{n=0}^{\infty}$ which is strictly increasing and converges to $\tilde{x}^{\epsilon}$.

Using the same arguments as in the Proof of Proposition 4.1, we can construct a feasible sequence $\{k_t\}_{t=0}^{\infty} \in \Pi^{\epsilon}(k_0)$ and a sequence of index $T_0 < T_1 < \cdots < T_n < \ldots$ such that for any n,

$$u\left(f(k_{T_n}) - k_{T_n+1}\right) = u\left(f(\tilde{x}_n) - \underset{\sim}{x}_n\right).$$

And we have

$$\begin{aligned}
\sup_{t \ge 0} u\left(f(k_t) - k_{t+1}\right) &= \lim_{n \to \infty} u\left(f(\tilde{x}_n) - \underset{\sim}{x}_n\right) \\
&= u\left(f(\tilde{x}^{\epsilon}) - \tilde{x}^{\epsilon}\right).
\end{aligned}$$

The two sequences $\{\underset{\sim}{x}_n\}_{n=0}^{\infty}$ and $\{\tilde{x}_n\}_{n=0}^{\infty}$ being chosen arbitrarily, there exists an infinite number of optimal paths.

Consider any optimal path $\{k_t\}_{t=0}^{\infty}$. It is easy to verify that if $k_T = \underset{\sim}{x}^{\epsilon}$, by the constraint, $k_{T+t} = \underset{\sim}{x}^{\epsilon}$ for any $t \geq 0$, which implies $\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) < u\left(f(\tilde{x}^{\epsilon}) - \underset{\sim}{x}^{\epsilon}\right)$, a contradiction. Hence for any t, $\underset{\sim}{x}^{\epsilon} < k_t < \tilde{x}^{\epsilon}$. Moreover, there exist an infinite number $T_0 < T_1 < \cdots < T_n < \ldots$ such that

$$\lim_{n\to\infty} u\left(f(k_{T_n t}) - k_{T_n+1}\right) = u\left(f(\tilde{x}^{\epsilon}) - \underset{\sim}{x}^{\epsilon}\right).$$

Hence we have

$$\lim_{n\to\infty} k_{T_n} = \tilde{x}^{\epsilon},$$
$$\lim_{n\to\infty} k_{T_n+1} = \underset{\sim}{x}^{\epsilon}.$$

ii) Consider now the case $k_0 \geq \tilde{x}^{\epsilon}$. Take any feasible sequence $\{k_t\}_{t=0}^{\infty} \in \Pi^{\epsilon}(k_0)$. We claim that for any $t \geq 0$,

$$\underset{\sim}{x}^{\epsilon} \leq k_t \leq k_0.$$

Using the same arguments as in the proof of the part (i), we have $k_t \geq \underset{\sim}{x}^{\epsilon}$ for any $t \geq 0$. Now we prove by induction that $k_t \leq k_0$ for any t. Indeed, this is true for $t = 0$. Assume that $k_t \leq k_0$ for any $0 \leq t \leq T$. If $k_T \geq \tilde{x}^{\epsilon}$, then

$$\begin{aligned} u\left(f(k_T) - k_T\right) &\leq u\left(f(\tilde{x}^{\epsilon}) - \tilde{x}^{\epsilon}\right) \\ &= u\left(f(\overline{k}) - \overline{k}\right) - \epsilon \\ &= u\left(f(k_T) - k_{T+1}\right), \end{aligned}$$

which implies $k_{T+1} \leq k_T \leq k_0$. The claim is proved, hence for any t,

$$\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) \leq u\left(f(k_0) - \underset{\sim}{x}^{\epsilon}\right).$$

It is easy to verify that the sequence $\{k_t^*\}_{t=0}^{\infty} = (k_0, \underset{\sim}{x}^{\epsilon}, \underset{\sim}{x}^{\epsilon}, \underset{\sim}{x}^{\epsilon}, \dots)$ is feasible and

$$\sup_{t\geq 0} u\left(f(k_t^*) - k_{t+1}^*\right) = u\left(f(k_0) - \underset{\sim}{x}^{\epsilon}\right).$$

To prove that this sequence is unique solution, take any feasible sequence $\{k_t\}_{t=0}^{\infty}$. Assume that $k_1 > \underset{\sim}{x}^{\epsilon}$. Hence

$$u\left(f(k_0) - k_1\right) < u\left(f(k_0) - \underset{\sim}{x}^{\epsilon}\right).$$

If $k_1 \geq \tilde{x}^{\epsilon}$, then

$$\begin{aligned}\sup_{t\geq 1} u\left(f(k_t) - k_{t+1}\right) &\leq u\left(f(k_1) - \underset{\sim}{x}^{\epsilon}\right) \\ &< u\left(f(k_0) - \underset{\sim}{x}^{\epsilon}\right).\end{aligned}$$

If $k_1 \leq \tilde{x}^{\epsilon}$, then

$$\begin{aligned}\sup_{t\geq 1} u\left(f(k_t) - k_{t+1}\right) &\leq u\left(f(\tilde{x}^{\epsilon}) - \underset{\sim}{x}^{\epsilon}\right) \\ &< u\left(f(k_0) - \underset{\sim}{x}^{\epsilon}\right).\end{aligned}$$

Combining these inequalities, we get

$$\begin{aligned}\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) &= \max\left\{u\left(f(k_0) - k_1\right), \sup_{t\geq 1} u\left(f(k_t) - k_{t+1}\right)\right\} \\ &< u\left(f(k_0) - \underset{\sim}{x}^{\epsilon}\right).\end{aligned}$$

For the case $k_1 = \underset{\sim}{x}^{\epsilon}$, in order to keep the path being feasible, we must have $k_t = \underset{\sim}{x}^{\epsilon}$ for any $t \geq 1$. The uniqueness of the optimal solution is proved.

J. Proof of Proposition 4.1

The first statement of the proposition is obvious, by Lemma 4.1. It is the same for the statement (i), by Proposition 4.1.

For the part (i), observe that the optimal value ϵ^* must satisfy

$$\begin{aligned}\epsilon^* &\leq \hat{v}(k_0)\\ &\leq u\left(f(\overline{k}) - \overline{k}\right).\end{aligned}$$

Hence there exists an upper bounde for $\tilde{x}^\epsilon$. Let $\overline{k}_0$ be this upper bound. For any $k_0 \geq \overline{k}_0$, we have $k_0 \geq \tilde{x}^{\epsilon^*}$, and applying Proposition 4.2, the proof is completed.

References

[1] Alon, S. (2015): Worst-case expected utility. *Journal of Mathematical Economics* **60**: 43-48.

[2] Arrow, K. J. (1973): Rawls's Principle of Just Savings. *The Swedish Journal of Economics* **75**, 323-335.

[3] Arrow, K., & L. Hurwicz (1972): An optimality criterion for decision making under ignorance, *Uncertainty and Expectations in Economics*, ed. by C. Carter and J. Ford, 1-11, *Blackwell.*

[4] Asheim, G. B., I. Ekeland (2016): Resource conservation across generations in a Ramsey - Chichilnisky model. *Economic Theory.*

[5] Bossert, W., P. K. Pattanaik & Y. Xu (2000): Choice under complete uncertainty: axiomatic characterizations of some decision rules. *Economic Theory* **16**: 295-312.

[6] Calvo, G. A. (1977): Optimal Maximin Accumulation With Uncertain Future Technology. *Econometrica* **45**: 317-327.

[7] Cao, D. & I. Werming (2018): Saving and dissaving with hyperbolic discounting. *Econometrica* **86**: 805-857.

[8] Chambers, C. & F. Echenique (2018): On Multiple Discount Rates. *Econometrica* **86**: 1325-1346.

[9] Chateauneuf, A., F. Maccheroni, M. Marinacci, & J. M. Tallon (2005): Monotone continuous multiple priors. *Economic Theory* **26**: 973-982.

[10] Drugeon, J., P., T. Ha-Huy & T. D. H. Nguyen (2018): On maximin dynamic programming and the rate of discount. *Economic Theory.*

[11] Etner, J., M. Jeleva, & J. M. Tallon (2012): Decision theory under ambiguity. *Journal of Economic Survey* **26**: 234-270.

[12] Fleurbaey, M., B. Tungodden (2010): The tyranny of non-aggregation versus the tyranny of aggregation in social choices: a real dilemma. *Economic Theory* **44**: 399 - 414.

[13] Ghirardato, P., F. Maccheroni & M. Marinacci (2004): Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory* **118**: 133-173.

[14] Gilboa, I. & D. Schmeidler (1989): Maxmin Expected utility with non-unique prior. *Journal of mathematical economics* **18**, 141-153.

[15] Le Van, C., L. Morhaim (2002): Optimal Growth Models with Bounded or Unbounded Returns: A Unifying Approach. *Journal of Economic Theory* **105**, 157-187.

[16] Koopmans, T. J. (1960): Stationary Ordinal Utility and Impatience, *Econometrica* **28**: 287-309.

[17] Koopmans, T. J. (1972): Representation of Preference Orderings over Time. In: *Decision and Organisation.* McGuire, C. and R. Radner, eds. Amsterdam: North-Holland.

[18] Kopylov, I. (2006): A parametric model of ambiguity hedging. *Working paper.*

[19] Krusell, P., & A. Smith Jr (2003): Consumption-Saving decisions with quasi-geometric discounting. *Econometrica*, **71**: 53-73.

[20] Phelps, P., & R. Pollack (1968): On Second-best National Saving and Game-Equilibrium Growth. *Review of Economics Studies* **35**: 185-199.

[21] Stokey, N., L. & R. Lucas Jr with E. Prescott (1989): Recursive methods in Economic Dynamics. *Harvard University Press.*

[22] Rawls, J. (1971): A Theory of Justice. *Oxford, England: Clarendon.*

[23] Savage (1954): The foundation of statistics. *Dover publication.*