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Abstract

We introduce a class of differentiable, strictly increasing, concave utility functions exhibiting an explicit demand of a good which may have Giffen behavior. We provide a necessary and sufficient condition (bases on prices and consumers' preferences and income) under which this good is normal, inferior or Giffen.

JEL Classifications: D11, D50.
Keywords: Inferior good, Giffen good, equilibrium price.

1 Introduction

Inferior and Giffen goods have been mentioned in most microeconomics textbooks (see Mas-Colell et al. (1995), Jehle and Reny (2011), Varian (2014) for instance).[1] However, they are usually illustrated by pictures. In this paper, we present a class of differentiable, strictly increasing, concave utility functions exhibiting an explicit demand of a good which may have Giffen behavior. In our example, the consumption set is $\mathbb{R}^2_+$, and the demand function generated by our simple utility function has a closed-form. Thanks to this tractability, we provide a necessary and sufficient condition (based on prices and consumers' preferences and income) under which this good is normal, inferior or Giffen good. This helps us to analytically study income and prices

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[1] Jensen and Miller (2008) provide real evidences (in two provinces of China: Hunan and Gansu) of Giffen behavior.

effects. In particular, we show that the Giffen behavior arises when the price is not so high and the consumer's income is at the middle level. This is supported by empirical evidences in Jensen and Miller (2008): when the price of a staple good increases, the poor people responds by decreasing their demand of this good while the group in the middle increases demand.

The second part of our paper focuses on the general equilibrium effects. Our utility function leads to an interesting point in general equilibrium context: the price of a good may be an increasing function of the aggregate supply of this good. Moreover, we show that the Giffen behavior may arise in equilibrium when preferences or/and endowments of agents change.

In the existing literature, several examples of Giffen good have been proposed. However, in most of the cases, utility functions are piecewise-defined or demand functions are not explicit or the consumption set is restricted. Heijman and von Mouche (2012) provide a collection of papers studying Giffen goods, including the paper of Doi, Iwasa, and Shimomura (2009).

Here, we just mention two recent papers (Haagsma, 2012; Biederman, 2015). Haagsma (2012) presents a separable utility function generating Giffen behavior.[2] In this example, the consumption set is restricted (precisely, it is $(\gamma_1, \infty) \times [0, \gamma_2)$ with $\gamma_1 > 0, \gamma_2 > 0$) and the utility function is quasi-concave but not concave. Moreover, in Haagsma (2012), the good 1 demand c_1 is always decreasing in the income, denoted by w, whatever the prices and the consumer's income. However, in our model, the sign of $\frac{\partial c_1}{\partial w}$ depends on prices and the consumer's income. Recently, Biederman (2015) provides a concave utility function[3] and gives some numerical examples where Giffen behavior arises. However, the demand function is not explicit. In our paper, we can explicitly derive the demand function.

2 Individual demand

Assume that there are two goods and the consumption set is $\mathbb{R}^2_+$. Given prices $p_1 > 0$, $p_2 > 0$ and income $w > 0$, the consumer maximizes her utility $U(c_1, c_2)$ subject to the budget constraint $p_1c_1 + p_2c_2 \leq w$. We will study how the demand c_1 changes when the consumer's income w or/and price p_1 change.

Assume that the solution is interior and the utility function is strictly increasing, then we have $p_1c_1 + p_2c_2 = w$ and

$$p_2U_1(c_1, c_2) = p_1U_2(c_1, c_2) \tag{1}$$

where $U_i(c_1, c_2) \equiv \frac{\partial U}{\partial c_i}(c_1, c_2)$ for $i = 1, 2$. From this, we obtain the following result.

Lemma 1. *Assume that U is strictly increasing and in C^2. Let (c_1, c_2) be an interior solution and assume that $p_2^2U_{11}(c_1, c_2) - 2p_1p_2U_{12}(c_1, c_2) + p_1^2U_{22}(c_1, c_2) < 0$.*[4] *Then, we*

[2] The utility function is $u(c_1, c_2) = \alpha_1 ln(c_1 - \gamma_1) - \alpha_2 ln(\gamma_2 - c_2)$ where $0 < \alpha_1 < \alpha_2$ and $\gamma_1, \gamma_2 > 0$, with the domain $c_1 > \gamma_1$ and $0 \leq c_2 < \gamma_2$.

[3] Biederman (2015) considers the following utility function

$$u(c_1, c_2) = \begin{cases} \frac{(c_1+\alpha c_2)^{1-\sigma}}{1-\sigma} - Ae^{-\beta c_1} & \text{for } \sigma > 0, \sigma \neq 1 \\ ln(c_1 + \alpha c_2) - Ae^{-\beta c_1} & \text{for } \sigma = 0 \end{cases}.$$

[4] This condition holds if the function U is strictly concave.

have that:

$$\frac{\partial c_1}{\partial w} < 0 \quad \textit{if and only if} \quad \frac{p_1}{p_2}U_{22}(c_1,c_2) - U_{21}(c_1,c_2) > 0 \tag{2a}$$

$$\frac{\partial c_1}{\partial p_1} > 0 \quad \textit{if and only if} \quad \Big(\frac{p_1}{p_2}U_{22}(c_1,c_2) - U_{21}(c_1,c_2)\Big)c_1 > U_2(c_1,c_2). \tag{2b}$$

Consequently, $\frac{\partial c_1}{\partial p_1} > 0$ *implies* $\frac{\partial c_1}{\partial w} < 0$ *(i.e., if good* 1 *is Giffen, then it must be inferior).*

Proof. See Appendix A.1. □

We now introduce a class of utility function generating demand with Giffen behavior. Suggesting by (2b), we choose a function so that U_{21}/U_{22} is constant. Our utility function is the following

$$U(c_1,c_2) = c_1 + bc_2 + A\frac{(ac_1+c_2)^{1-\lambda}}{1-\lambda} \tag{3}$$

where $a > 0, b > 0, \lambda > 0, A > 0, \ \lambda \neq 1$.

Lemma 2. *The function* U *defined by (3) is strictly increasing, differentiable, concave. It is strictly quasi-concave if* $ab \neq 1$.

Proof. See Appendix A.2. □

When $ab = 1$, we have $ac_1 + c_2 = a(c_1 + bc_2)$. Hence $U(c_1,c_2) = c_1 + bc_2 + A\frac{(c_1+bc_2)^{1-\lambda}a^{1-\lambda}}{1-\lambda}$ which is increasing in c_1+bc_2. By consequence, maximizing the function $U(c_1,c_2)$ is equivalent to maximizing the function $c_1 + bc_2$. In this case, the demand for good 1 is given by

$$c_1 = \begin{cases} 0 & \text{if } bp_1 > p_2 \\ \in [0, \frac{w}{p_1}] & \text{if } bp_1 = p_2 \\ \frac{w}{p_1} & \text{if } bp_1 < p_2 \end{cases} \tag{4}$$

and hence inferior and Giffen behaviors do not arise.

In the following, we will focus on the case $ab \neq 1$. If (c_1,c_2) is an interior solution $(0 < c_1 < w/p_1)$, the FOC gives

$$A(ap_2 - p_1) = \Big(bp_1 - p_2\Big)\Big(\frac{(ap_2-p_1)c_1 + w}{p_2}\Big)^{\lambda} \tag{5}$$

Combining this condition with $ab \neq 1$, we get that $ap_2 - p_1 \neq 0$ and $bp_1 - p_2 \neq 0$. By consequence, $0 < c_1 < w/p_1$ implies that $(bp_1 - p_2)\big(\frac{w}{p_2}\big)^{\lambda} < A(ap_2 - p_1) < (bp_1 - p_2)\big(\frac{aw}{p_1}\big)^{\lambda}$ or equivalently

$$\frac{a + \frac{1}{A}\big(\frac{w}{p_2}\big)^{\lambda}}{1 + b\frac{1}{A}\big(\frac{w}{p_2}\big)^{\lambda}}p_2 > p_1 > \frac{a + \frac{1}{A}\big(\frac{aw}{p_1}\big)^{\lambda}}{1 + b\frac{1}{A}\big(\frac{aw}{p_1}\big)^{\lambda}}p_2 \tag{6}$$

From this observation, we can compute the demand for good 1.

Proposition 1. *Consider the utility function given by (3) with $ab \neq 1$. The demand function for good 1 is given by*

$$c_1 = \begin{cases} 0 & \text{if } \frac{a+\frac{1}{A}(\frac{w}{p_2})^\lambda}{1+b\frac{1}{A}(\frac{w}{p_2})^\lambda} p_2 \leq p_1 \\ \frac{p_2\left(A\frac{ap_2-p_1}{bp_1-p_2}\right)^{\frac{1}{\lambda}}-w}{ap_2-p_1} & \text{if } \frac{a+\frac{1}{A}(\frac{w}{p_2})^\lambda}{1+b\frac{1}{A}(\frac{w}{p_2})^\lambda} p_2 > p_1 > \frac{a+\frac{1}{A}(\frac{aw}{p_1})^\lambda}{1+b\frac{1}{A}(\frac{aw}{p_1})^\lambda} p_2 \\ \frac{w}{p_1} & \text{if } p_1 \leq \frac{a+\frac{1}{A}(\frac{aw}{p_1})^\lambda}{1+b\frac{1}{A}(\frac{aw}{p_1})^\lambda} p_2 \end{cases} \tag{7}$$

The demand function is continuous. Moreover, it is differentiable in $(w, p_1, p_2, a, b, \lambda)$ except points satisfying $p_1 = \frac{a+\frac{1}{A}(\frac{w}{p_2})^\lambda}{1+b\frac{1}{A}(\frac{w}{p_2})^\lambda} p_2$ or $p_1 = \frac{a+\frac{1}{A}(\frac{aw}{p_1})^\lambda}{1+b\frac{1}{A}(\frac{aw}{p_1})^\lambda} p_2$.

Proof. See Appendix A.3. □

Notice that the demand function in (7) is computed for all possible parameters, including prices and income. The consumer does not buy good 1 (resp., good 2) if the price of good 1 (resp., good 2) is high in the sense that $p_1 \geq \frac{a+\frac{1}{A}(\frac{w}{p_2})^\lambda}{1+b\frac{1}{A}(\frac{w}{p_2})^\lambda} p_2$ (resp., $p_2 > \frac{1+b\frac{1}{A}(\frac{aw}{p_1})^\lambda}{a+\frac{1}{A}(\frac{aw}{p_1})^\lambda} p_1$). Under condition (6), the solution is interior. This happens when prices and income have a middle level.

Proposition 1 allows us to identify conditions under which good 1 is normal, inferior or Giffen.

Proposition 2. *Let assumptions in Proposition 1 be satisfied. Consider the case of interior solution (i.e., condition (6) holds).*

1. *Good 1 is normal (i.e., $\partial c_1/\partial w > 0$) if and only if $ap_2 < p_1$.*
2. *Good 1 is inferior (i.e., $\partial c_1/\partial w < 0$) if and only if $ap_2 > p_1$.*
3. *Good 1 has Giffen behavior (i.e., $\partial c_1/\partial p_1 > 0$) if and only if*

$$(bp_1 - p_2)\left(\frac{w}{p_2}\right)^\lambda < A(ap_2 - p_1) < (bp_1 - p_2)\left(\frac{aw}{p_1}\right)^\lambda \tag{8a}$$

$$p_2\left(A\frac{ap_2 - p_1}{bp_1 - p_2}\right)^{\frac{1}{\lambda}}\left(1 - \frac{p_2(ab-1)}{\lambda(bp_1 - p_2)}\right) - w > 0. \tag{8b}$$

Moreover, there exists a positive list $(p_1, p_2, a, b, \lambda, A, w)$ such that (8a) and (8b) hold.

Proof. See Appendix A.4. □

By combining Propositions 1 and 2, good 1 is normal if (1) the consumer only buys this good ($c_1 = w/p_1$) or (2) the solution is interior (condition (8a) holds) and the relative price is quite high (i.e., $ap_2 < p_1$). When the solution is interior, good 1 is inferior if and only if the relative price p_1/p_2 is low (i.e., $p_1 < ap_2$).

We now look at conditions under which Giffen behavior arises. Condition (8a) is to ensure that the optimal allocation is interior while condition (8b) means that

$\partial c_1/\partial p_1 > 0$. Observe that conditions (8a) and (8b) are satisfied if $bp_1 > p_2 > w > p_1/a$ and λ is high enough. In this case, we have Giffen effect. Point 3 of Proposition 2 suggests that Giffen behavior cannot arise if the income is high. When the income is very low, Proposition 2 indicates that the solution is at the corner and so the good 1 is normal. To sum up, Giffen behavior only arises when the income is at the middle level. This property is supported by the empirical evidences in Jensen and Miller (2008).

We illustrate our result by some examples.

Example 1 (inferior good)**.** *Let prices be such that $bp_1 > p_2$, $ap_2 > p_1$ and income w vary. According to (7), the demand for good 1 is*

$$c_1 = \begin{cases} \frac{w}{p_1} & \text{if } w \in (0, \underline{w}] \\ \frac{p_2\left(A\frac{ap_2-p_1}{bp_1-p_2}\right)^{\frac{1}{\lambda}}-w}{ap_2-p_1} & \text{if } w \in (\underline{w}, w^*) \\ 0 & \text{if } w \in [w^*, \infty) \end{cases} \quad \text{where } \begin{cases} \underline{w} \equiv \frac{p_1}{a}\left(A\frac{ap_2-p_1}{bp_1-p_2}\right)^{\frac{1}{\lambda}} \\ w^* \equiv p_2\left(A\frac{ap_2-p_1}{bp_1-p_2}\right)^{\frac{1}{\lambda}} \end{cases} \tag{9}$$

The good 1 is inferior iff the income has a middle level, i.e., $w \in (\underline{w}, w^*)$.

Example 2 (Giffen good)**.** *Take $p_2 = 2, a = 2, b = 3, A = 3, \lambda = 6, w = 1.1$ and let p_1 vary. In this case, we have $ab > 1$. Denote $c^{int}(p_1) \equiv \frac{2\left(3\frac{4-p_1}{3p_1-2}\right)^{\frac{1}{6}}-1.1}{4-p_1}$. We can compute the demand for good 1 as a function of p_1 (see Appendix A.5 for detailed proof)*

$$c_1(p_1) = \begin{cases} \frac{1.1}{p_1} & \text{if } p_1 \in (0, 2/3] \\ c^{int}(p_1) & \text{if } p_1 \in (2/3, 2.098) \\ \frac{1.1}{p_1} & \text{if } p_1 \in [2.098, 3.895] \\ c^{int}(p_1) & \text{if } p_1 \in (3.895, 3.91) \\ 0 & \text{if } p_1 \in [3.91, \infty) \end{cases} \tag{10}$$

We can verify that $c^{int}(p_1)$ is decreasing on $(2/3, 1.75]$ and increasing on $[1.75, 2.098)$. So, the demand is decreasing in p_1 on $(0, 1.75]$ or $[2.098, \infty)$ but increasing in p_1 on $[1.75, 2.098)$. Hence, Giffen behavior arises when the price p_1 runs from 1.75 to 2.098.

Remark 1 (Good 2)**.** We have so far focused on good 1. We now look at good 2. Observe that $\frac{U(c_1,c_2)}{b} = c_2 + \frac{1}{b}c_1 + \frac{Aa^{1-\lambda}}{b}\frac{(c_2+\frac{1}{a}c_1)^{1-\lambda}}{1-\lambda}$. Denote, $a' = 1/a, b' = 1/b, A' = a^{1-\lambda}A/b, p'_2 = p_1, p'_1 = p_2$. We see that the demand for good 2 corresponds to the demand for good 1 of the consumer having the utility function $c'_1 + b'c'_2 + A'\frac{(a'c'_1+c'_2)^{1-\lambda}}{1-\lambda}$ and facing budget constraint $p'_1c'_1 + p'_2c'_2 \leq w$. By consequence, good 2 may be normal, inferior or Giffen.

We focus on the interior solution case (condition (6) holds). According to Proposition 1, we have $c_2 = \frac{p_1\left(A\frac{ap_2-p_1}{bp_1-p_2}\right)^{\frac{1}{\lambda}}-aw}{p_1-ap_2}$. In this case, good 2 is inferior if and only if $p_1 > ap_2$ (in this case, good 1 is normal).

A natural issue is to study conditions under which good 2 is Giffen. First, we observe that these two goods cannot be Giffen at the same time. Indeed, if good 1 is Giffen, then Lemma 1 implies that it is inferior or equivalently $ap_2 > p_1$. If good 2 is Giffen, applying Lemma 1, it must be inferior and hence $p_1 < ap_2$, a contradiction. Second, we can prove that good 2 is Giffen if and only if

$$\frac{p_1}{a}\Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}}\Big(1-\frac{p_1(ab-1)}{a\lambda(bp_1-p_2)}\Big)-w>0. \tag{11}$$

Applying our above result, this happens if $p_2/b > p_1 > w > ap_2$ and λ is high enough. As for good 1, good 2 is Giffen only when the income has a middle level.

We end this section by providing some useful observations when finding utility functions generating inferior goods as well as Giffen behavior.

1. Assume that the utility function is separable, i.e., $U(c_1, c_2) = u(c_1) + v(c_2)$.

   If both u and v are concave, then good 1 is normal. Indeed, we have $U_{12} = 0$. So, Lemma 1 implies that: $\frac{\partial c_1}{\partial w} < 0$ if and only if $\frac{p_2^2}{p_1^2}\frac{u''(c_1)}{v''(c_2)} + 1 < 0$. This cannot happen because both u and v are concave. Therefore, good 1 is normal. So, if we want to have inferior or Giffen goods, u or v must not be concave.[5]

2. We can obtain Giffen behavior with simple utility functions by restricting the consumption set in another way. Indeed, assume that $U(c_1, c_2) = c_1 + bc_2$ with $b > 0$ and the consumption set is $\{(c_1, c_2) \in \mathbb{R}^2_+ : c_1 + c_2 \geq 1\}$. $c_1 + c_2 \geq 1$ is interpreted as survival condition. We can verify that: if $p_1 < p_2 < bp_1$ and $w < p_2$, then $c_1 = \frac{p_2-w}{p_2-p_1}$ which is increasing in price p_1 and decreasing in income w.

3. In the case of Leontief utility $U(c_1, c_2) = \min(u(c_1), v(c_2))$ where u, v are increasing, c_1 is increasing in w. However, Sorensen (2007) considers the function $U(c_1, c_2) = \min(u(c_1, c_2), v(c_1, c_2))$ and show that this function may generate Giffen behavior.

3 Equilibrium

We now look at equilibrium properties. We consider a pure exchange economy with two goods. Assume that there are m agents with the same utility function $U(c_1, c_2) = c_1 + bc_2 + A\frac{(ac_1+c_2)^{1-\lambda}}{1-\lambda}$, where $a > 0, b > 0, \lambda > 0$, $\lambda \neq 1$. The consumption set is $\mathbb{R}^2_+$ and the endowments of agent i are $w_1^i > 0, w_2^i > 0$ for goods $1, 2$, respectively.

We firstly investigate the equilibrium prices. The income of agent i is $w^i \equiv p_1w_1^i + p_2w_2^i$. We focus on interior equilibrium: $c_1^j \in (0, w^i/p_1)\ \forall i$. According to Proposition

[5] We may have examples with inferior good or Giffen behavior if u or v is not concave. Indeed, we firstly present an example generating inferior good. Assume that the consumption set is $\mathbb{R}^2_+$ and $U(c_1, c_2) = Aln(c_1) + \frac{c_2^2}{2}$. In this case, the demand for good 1 is $c_1 = \frac{w}{p_1}$ if $w^2 \leq 4Ap_2^2$ and $c_1 = \frac{w-\sqrt{w^2-4Ap_2^2}}{2p_1}$ if $w^2 > 4Ap_2^2$. So, the good 1 is normal if $w^2 \leq 4Ap_2^2$ and inferior if $w^2 > 4Ap_2^2$. Second, Haagsma (2012) considers a separable function $u(c_1, c_2) = \alpha_1 ln(c_1 - \gamma_1) - \alpha_2 ln(\gamma_2 - c_2)$ where the second term is convex in c_2. In this case, he shows that good 1 may be Giffen. Note that the consumption set is $(\gamma_1, \infty) \times [0, \gamma_2)$ which is restricted.

1, we have that

$$(ap_2 - p_1)c_1^i = p_2\Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}} - w^i. \tag{12}$$

From this and the market clearing condition $\sum_i c_j^i = \sum_i w_j^i \ \forall j = 1, 2$, we can compute the relative price $\bar{p}_1 \equiv p_1/p_2$.

Proposition 3. *Assume that $ab \neq 1$. Denote $p^* \equiv \frac{a+\frac{(aw_1+w_2)^\lambda}{A}}{1+b\frac{(aw_1+w_2)^\lambda}{A}}$ and $w_j \equiv \sum_{i=1}^m w_j^i/m$ for $j = 1, 2$.*

If $(aw_1 + w_2)min(1, \frac{p^*}{a}) < p^*w_1^i + w_2^i < (aw_1 + w_2)max(1, \frac{p^*}{a})$, then there exists an interior equilibrium with the relative price

$$\frac{p_1}{p_2} = p^*. \text{ Moreover, } \frac{\partial(p_1/p_2)}{\partial b} < 0 < \frac{\partial(p_1/p_2)}{\partial a} \tag{13}$$

1. *If $ab > 1$, then $p_1/p_2 \in (1/b, a)$ and is decreasing in w_1, w_2 but increasing in A.*

2. *If $ab < 1$, then $p_1/p_2 \in (a, 1/b)$ and is increasing in w_1, w_2 but decreasing in A.*

Proof. See Appendix A.6. □

According to Proposition, our utility function (3) generates a property: the price of good 1 (resp., good 2) is increasing in its aggregate supply $W_1 \equiv \sum_i w_1^i$ (resp., $W_2 \equiv \sum_i w_2^i$) if $ab < 1$ (resp., $ab > 1$). This point may illustrate ideas presented in Section 17.E "Anything goes: the theorem Sonnenschein-Martel-Debreu" in Mas-Colell et al. (1995).

We now look at the demand for good 1 of agent i to understand when Giffen behavior arises. According to (12) and (13), we can compute

$$c_1^i(\bar{p}_1) = \frac{aw_1 + w_2 - w_2^i - \bar{p}_1 w_1^i}{a - \bar{p}_1}. \tag{14}$$

By consequence, we have the following result.

Corollary 1. *We have*

$$\frac{\partial c_1^i}{\partial \bar{p}_1} = \frac{aw_1 - aw_1^i + w_2 - w_2^i}{(a - \bar{p}_1)^2}. \tag{15}$$

This result leads to an implication: the Giffen behavior arises when preferences of agents change. Indeed, without the loss of generality, assume that $ab > 1$. We also assume that agent i's endowments are low in the sense that $aw_1 + w_2 > aw_1^i + w_2^i$. In this case, when A increases or b decreases, the relative price $\bar{p}_1$ increases. By consequence, the demand for good 1 of this agent increases in the relative price p_1/p_2.

Notice that the Giffen behavior can also arise when agents' endowments change. Indeed, let us consider a simple case where there are identical agents and $ab < 1$. In this case, $c_1^i = w_1 \ \forall i$ and the relative price p_1/p_2 is increasing in w_1. So, the good 1 consumption $c_1^i = w_1$ is increasing in p_1/p_2. In this case, the good 2 consumption is decreasing in p_2/p_1.

Remark 2. Nachbar (1998) introduces another definition of Giffen good in a general equilibrium context. According to Nachbar (1998), the good 1 is Giffen in our exchange economy if the (endogenous income) aggregate demand C_1 for good 1 is increasing in its price p_1. We can compute the aggregate demand and the aggregate excess demand $Z_1(p)$

$$\begin{aligned}
C_1(p) &\equiv \sum_i c_1^i(p, p_1 w_1^i + p_2 w_2^i) = \sum_i \frac{p_2\big(A\frac{ap_2-p_1}{bp_1-p_2}\big)^{\frac{1}{\lambda}} - (p_1 w_1^i + p_2 w_2^i)}{ap_2 - p_1} \\
&= \frac{m}{ap_2 - p_1}\Big(p_2\big(A\frac{ap_2-p_1}{bp_1-p_2}\big)^{\frac{1}{\lambda}} - (p_1 w_1 + p_2 w_2)\Big) \\
Z_1(p) &\equiv \sum_i \Big(c_1^i(p, p_1 w_1^i + p_2 w_2^i) - w_1^i\Big) \\
&= \frac{m}{ap_2 - p_1}\Big(p_2\big(A\frac{ap_2-p_1}{bp_1-p_2}\big)^{\frac{1}{\lambda}} - p_2(a w_1 + w_2)\Big).
\end{aligned}$$

We can check that $\frac{\partial C_1(p)}{\partial p_1} = \frac{\partial Z_1(p)}{\partial p_1}$. Moreover, we compute

$$\frac{\partial C_1(p)}{\partial p_1} = \frac{mp_2}{(ap_2-p_1)^2}\Big[\Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}}\Big(1 - \frac{p_2(ab-1)}{\lambda(bp_1-p_2)}\Big) - (aw_1 + w_2)\Big]. \tag{16}$$

At equilibrium (i.e., $p_1/p_2 = p^*$), we have $\big(A\frac{ap_2-p_1}{bp_1-p_2}\big)^{\frac{1}{\lambda}} = aw_1 + w_2$. We also see that $\frac{ab-1}{bp_1-p_2} > 0$.[6] By consequence, $\frac{\partial C_1(p)}{\partial p_1} < 0$. So, the good 1 is not Giffen in the sense of Nachbar (1998). This result is in line with that in Remark 3 in Nachbar (1998).

To sum up, with our utility function (3), good 1 may be Giffen in the standard sense but it is not Giffen in the sense of Nachbar (1998).

Remark 3 (price tâtonnement)**.** Without loss of generality, we can normalize by setting $p_2 = 1$. The dynamic price equation (17.H.1) in Mas-Colell et al. (1995) is in our case of one-dimension $\frac{\partial p_1(t)}{\partial t} = c_1 Z_1(p_1(t))$. According to Remark 2, we have $\frac{\partial Z_1(p_1)}{\partial p_1} < 0$ at equilibrium. So, our equilibrium price is locally stable.[7]

A Appendix

A.1 Proof of Lemma 1

By taking the derivatives of both sides of the equation (1) with respect to w and noting that $c_2 = \frac{w - p_1 c_1}{p_2}$, we have

$$\begin{aligned}
&\Big(p_2 U_{11}(c_1,c_2) - p_1 U_{12}(c_1,c_2)\Big)\frac{\partial c_1}{\partial w} + U_{12}(c_1,c_2) \\
=&\Big(p_1 U_{21}(c_1,c_2) - \frac{p_1^2}{p_2}U_{22}(c_1,c_2)\Big)\frac{\partial c_1}{\partial w} + \frac{p_1}{p_2}U_{22}(c_1,c_2)
\end{aligned}$$

[6] If $ab > 1$, we have $p^* > 1/b$. Thus, $bp_1 > p_2$ and therefore $\frac{ab-1}{bp_1-p_2} > 0$. If $ab < 1$, we have $p^* < 1/b$. Thus, $bp_1 < p_2$ and therefore $\frac{ab-1}{bp_1-p_2} > 0$.

[7] Following Mas-Colell et al. (1995) (page 621), an equilibrium price (p_1, p_2) is locally stable if, whenever the initial price vector is sufficiently close to it, the dynamic trajectory causes relative prices to converge to the equilibrium relative price p_1/p_2.

from which we get that

$$\Big(p_2^2 U_{11}(c_1,c_2) - 2p_1p_2U_{12}(c_1,c_2) + p_1^2U_{22}(c_1,c_2)\Big)\frac{\partial c_1}{\partial w} = p_1U_{22}(c_1,c_2) - p_2U_{12}(c_1,c_2).$$

Condition $p_2^2 U_{11}(c_1,c_2) - 2p_1p_2U_{12}(c_1,c_2) + p_1^2U_{22}(c_1,c_2) < 0$ implies (2a).

By taking the derivatives of both sides of (1) with respect to p_1, we get

$$\begin{aligned} &\Big(p_2U_{11}\Big(c_1, \tfrac{w-p_1c_1}{p_2}\Big) - p_1U_{12}\Big(c_1, \tfrac{w-p_1c_1}{p_2}\Big)\Big)\tfrac{\partial c_1}{\partial p_1} - c_1U_{12}(c_1, \tfrac{w-p_1c_1}{p_2}) \\ = \;& U_2(c_1, \tfrac{w-p_1c_1}{p_2}) + p_1\Big(U_{21}(c_1, \tfrac{w-p_1c_1}{p_2}) - \tfrac{p_1}{p_2}U_{22}(c_1, \tfrac{w-p_1c_1}{p_2})\Big)\tfrac{\partial c_1}{\partial p_1} - c_1\tfrac{p_1}{p_2}U_{22}(c_1, \tfrac{w-p_1c_1}{p_2}) \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} &\tfrac{\partial c_1}{\partial p_1}\Big(p_2U_{11}(c_1,c_2) - 2p_1U_{12}(c_1,c_2) + \tfrac{p_1^2}{p_2}U_{22}(c_1,c_2)\Big) \\ &= U_2(c_1,c_2) + c_1U_{12}(c_1,c_2) - c_1\tfrac{p_1}{p_2}U_{22}(c_1,c_2). \end{aligned}$$

which implies (2b).

A.2 Proof of Lemma 2

It is easy to see that the function U is strictly increasing and differentiable. It is concave because both functions $c_1 + bc_2$ and $A\frac{(ac_1+c_2)^{1-\lambda}}{1-\lambda}$ are concave. It is strictly quasi-concave because

$$\begin{vmatrix} 0 & U_1 \\ U_1 & U_{11} \end{vmatrix} = -(U_1)^2 < 0 \text{ and } \begin{vmatrix} 0 & U_1 & U_2 \\ U_1 & U_{11} & U_{12} \\ U_2 & U_{12} & U_{22} \end{vmatrix} = \lambda A(ac_1+c_2)^{-\lambda-1}(1-ab)^2 > 0 \qquad \text{(A.1)}$$

where $U_i(x_1,x_2) \equiv \frac{\partial U}{\partial x_i}$ and $U_{ij} \equiv \frac{\partial}{\partial x_j}\big(\frac{\partial U}{\partial x_i}\big)$.

A.3 Proof of Proposition 1

The budget constraint must be binding: $p_1c_1 + p_2c_2 = w$. Since the feasible set is convex, concave and the function U is strictly quasi-concave and strictly increasing, there exists a unique solution. We write FOCs

$$U_1(c_1,c_2) + \kappa_1 = p_1\mu, \quad \kappa_1 \geq 0, \kappa_1c_1 = 0 \qquad \text{(A.2a)}$$

$$U_2(c_1,c_2) + \kappa_2 = p_2\mu, \quad \kappa_2 \geq 0, \kappa_2c_2 = 0. \qquad \text{(A.2b)}$$

We have $U_1(c_1,c_2) = 1 + aA(ac_1+c_2)^{-\lambda}$ and $U_2(c_1,c_2) = b + A(ac_1+c_2)^{-\lambda}$. We consider different cases.

1. $c_1 = 0, c_2 = w/p_2$. In this case, $\kappa_2 = 0$ and then $\frac{U_2(c_1,c_2)}{p_2} = \mu \geq \frac{U_1(c_1,c_2)}{p_1}$. This means that

$$\begin{aligned} &p_2\Big(1 + aA(ac_1+c_2)^{-\lambda}\Big) \leq p_1(b + A(ac_1+c_2)^{-\lambda}) \Leftrightarrow (\frac{w}{p_2})^{-\lambda}(ap_2 - p_1)A \leq bp_1 - p_2 \\ &\Leftrightarrow A(ap_2 - p_1) \leq \Big(bp_1 - p_2\Big)\Big(\frac{w}{p_2}\Big)^{\lambda} \Leftrightarrow \frac{a + \frac{1}{A}(\frac{w}{p_2})^{\lambda}}{1 + b\frac{1}{A}(\frac{w}{p_2})^{\lambda}}p_2 \leq p_1. \end{aligned}$$

   It is easy to verify that: this condition holds if and only if $(c_1,c_2) = (0, w/p_2)$.

2. $c_1 = w/p_1, c_2 = 0$. In this case, $\kappa_1 = 0$ and then $\frac{U_2(c_1,c_2)}{p_2} \leq \mu = \frac{U_1(c_1,c_2)}{p_1}$. This means that

$$p_2\Big(1 + aA(ac_1 + c_2)^{-\lambda}\Big) \geq p_1(b + A(ac_1 + c_2)^{-\lambda}) \Leftrightarrow (\frac{aw}{p_1})^{-\lambda}(ap_2 - p_1)A \geq bp_1 - p_2$$

$$\Leftrightarrow A(ap_2 - p_1) \geq \Big(bp_1 - p_2\Big)\Big(\frac{aw}{p_1}\Big)^{\lambda} \Leftrightarrow p_1 \leq \frac{a + \frac{1}{A}(\frac{aw}{p_1})^{\lambda}}{1 + b\frac{1}{A}(\frac{aw}{p_1})^{\lambda}} p_2$$

It is easy to verify that: this condition holds if and only if $(c_1, c_2) = (w/p_1, 0)$.

3. Let us consider an interior solution $0 < c_1 < w/p_1$. We will prove that this is the case if and only if condition (8a) hold, i.e.,

$$(bp_1 - p_2)(\frac{w}{p_2})^{\lambda} < A(ap_2 - p_1) < (bp_1 - p_2)(\frac{aw}{p_1})^{\lambda}. \tag{A.3}$$

The FOC becomes $\frac{U_1}{p_1} = \frac{U_2}{p_2}$, or equivalent

$$p_2\Big(1 + aA(ac_1 + c_2)^{-\lambda}\Big) = p_1(b + A(ac_1 + c_2)^{-\lambda})$$
$$\Leftrightarrow A(ap_2 - p_1) = \Big(bp_1 - p_2\Big)\Big(\frac{(ap_2 - p_1)c_1 + w}{p_2}\Big)^{\lambda} \tag{A.4}$$

Since $ab \neq 1$, condition (A.4) implies that $ap_2 - p_1 \neq 0$ and $bp_1 - p_2 \neq 0$. So, the equation (A.4) has a unique solution (because $\lambda > 0$).

(a) Case 1: $ap_2 - p_1 > 0$ which implies that $bp_1 - p_2 > 0$. The above equation has a unique solution c_1 in $(0, w/p_1)$ if and only if (A.3) holds.

(b) Case 2: $ap_2 - p_1 < 0$ which implies that $bp_1 - p_2 < 0$. The right hand side is an increasing function of c_1. So, the equation (A.4) has a unique solution c_1 in $(0, w/p_1)$ if and only if (A.3) holds.

Summing up, the equation (A.4) has a unique solution c_1 in $(0, w/p_1)$ if and only if (A.3) holds. In such case, $ap_2 - p_1 \neq 0$ and $bp_1 - p_2 \neq 0$ and we find that

$$\Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}} = \frac{(ap_2 - p_1)c_1 + w}{p_2} \Leftrightarrow (ap_2 - p_1)c_1 = p_2\Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}} - w. \tag{A.5}$$

Continuity. We now prove the continuity of the demand function. Observe that the utility function is continuous and the budget correspondence

$$B(p_1, p_2) \equiv \{(c_1, c_2) \in \mathbb{R}^2_+ : p_1c_1 + p_2c_2 \leq w\}$$

is continuous. From the maximum theorem, the demand correspondence is upper semi continuous. Since we have proven above that it is single valued, it is in fact a continuous function. We can also prove the continuity of the demand function by using the following properties

$$\lim_{A(ap_2-p_1)-(bp_1-p_2)(\frac{w}{p_2})^{\lambda}\to 0} \frac{p_2\big(A\frac{ap_2-p_1}{bp_1-p_2}\big)^{\frac{1}{\lambda}} - w}{ap_2 - p_1} = 0,$$

$$\lim_{A(ap_2-p_1)-(bp_1-p_2)(\frac{aw}{p_1})^{\lambda}\to 0} \frac{p_2\big(A\frac{ap_2-p_1}{bp_1-p_2}\big)^{\frac{1}{\lambda}} - w}{ap_2 - p_1} = \frac{w}{p_1}.$$

Differentiability. If $A(ap_2 - p_1) \neq (bp_1 - p_2)\big(\frac{w}{p_2}\big)^\lambda$ and $A(ap_2 - p_1) \neq (bp_1 - p_2)\big(\frac{aw}{p_1}\big)^\lambda$, then the demand determined by (7) is differentiable. Indeed, there are only three cases:

If $A(ap_2 - p_1) < (bp_1 - p_2)\big(\frac{w}{p_2}\big)^\lambda$, then $c_1 = 0$ which is differentiable.

If $A(ap_2 - p_1) > (bp_1 - p_2)\big(\frac{aw}{p_1}\big)^\lambda$, then $c_1 = w/p_1$ which is differentiable.

If $(bp_1 - p_2)\big(\frac{w}{p_2}\big)^\lambda < A(ap_2 - p_1) < (bp_1 - p_2)\big(\frac{aw}{p_1}\big)^\lambda$, then we have $ap_1 - p_1 \neq 0$ and $bp_1 - p_2 \neq 0$. Then $c_1 = \frac{p_2\big(A\frac{ap_2-p_1}{bp_1-p_2}\big)^{\frac{1}{\lambda}} - w}{ap_2-p_1}$ is well defined and differentiable.

A.4 Proof of Proposition 2

Points 1 and 2 are obvious. We now look at the Giffen behavior. We have

$$c_1 = \frac{1}{ap_2 - p_1}\Big(p_2\Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}} - w\Big)$$

$$\frac{\partial c_1}{\partial p_1} = \frac{1}{(ap_2 - p_1)^2}\Big(p_2\Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}} - w\Big) + \frac{A^{\frac{1}{\lambda}}p_2}{ap_2 - p_1}\frac{1}{\lambda}\Big(\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}-1}\frac{p_2(1-ab)}{(bp_1 - p_2)^2}.$$

Therefore, we get that

$$(ap_2 - p_1)^2\frac{\partial c_1}{\partial p_1} = p_2\Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}} + \Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}}\frac{1}{\lambda}\frac{p_2^2(1-ab)}{bp_1 - p_2} - w \tag{A.6}$$

which implies point 3. We now prove that there exists a positive list $(p_1, p_2, a, b, \lambda, w, A)$ such that (8b) and (8a) hold, i.e., $\frac{\partial c_1}{\partial p_1} > 0$. Indeed, let $ap_2 - p_1 > 0, bp_1 - p_2 > 0$ and $p_2 > w > \frac{p_1}{a}$. These conditions imply that $0 < \frac{p_2 - w}{ap_2 - p_1} < \frac{w}{p_1}$. When $\lambda \to \infty$, we have

$$\frac{1}{ap_2 - p_1}\Big(p_2\Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}} - w\Big) \longrightarrow \frac{p_2 - w}{ap_2 - p_1} \in \Big(0, \frac{w}{p_1}\Big) \tag{A.7}$$

$$p_2\Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}} + \Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}}\frac{1}{\lambda}\frac{p_2^2(1-ab)}{bp_1 - p_2} - w \longrightarrow p_2 - w > 0. \tag{A.8}$$

A.5 Proof of Example 2

With our parameters, we have $\frac{a+\frac{1}{A}(\frac{w}{p_2})^\lambda}{1+b\frac{1}{A}(\frac{w}{p_2})^\lambda}p_2 = 3.91$.

1. According to Propositions 1, $c_1 = 0$ if $p_1 \geq 3.91$.
2. If $p_1 \leq 2/3$, then $bp_1 < p_2$ and $ap_2 > p_1$. According to Proposition 1, we have $c_1 = w/p_1 = 1.1/p_1$.
3. We now focus on the case $2/3 < p_1 < 3.91$. In this case, we have $bp_1 > p_2$ and $ap_2 = 4 > p_1$. Notice that

$$p_1 > \frac{a + \frac{1}{A}(\frac{aw}{p_1})^\lambda}{1 + b\frac{1}{A}(\frac{aw}{p_1})^\lambda}p_2 \Leftrightarrow A(ap_2 - p_1) < (bp_1 - p_2)\big(\frac{aw}{p_1}\big)^\lambda.$$

   Since $p_1 \in (2/3, 3.91)$, this happens if and only if $p_1 < 2.098$ or $p_1 > 3.895$. According to Proposition 1, we have $c_1 = c^{int}(p_1)$ if $p_1 \in (2/3, 2.098)$ or $p_1 \in (3.895, 3.91)$.

   When $p_1 \in [2.098, 3.895]$, we have $p_1 \leq \frac{a+\frac{1}{A}(\frac{aw}{p_1})^\lambda}{1+b\frac{1}{A}(\frac{aw}{p_1})^\lambda}p_2$. In this case, Proposition 1 implies that $c_1 = w/p_1 = 1.1/p_1$. We have just proved (10).

We now look at the case where $p_1 \in (2/3, 2.098)$. According to Proposition 2, good 1 is Giffen (i.e., $\partial c_1/\partial p_1 > 0$) if and only if $p_2\Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}}\Big(1-\frac{p_2(ab-1)}{\lambda(bp_1-p_2)}\Big) - w > 0$. This happens if and only if $p_1 > 1.75$.

A.6 Proof of Proposition 3

We have proved that if equilibrium is interior, the relative price must be

$$\frac{p_1}{p_2} = p^* \equiv \frac{a + \frac{(aw_1+w_2)^\lambda}{A}}{1 + b\frac{(aw_1+w_2)^\lambda}{A}}. \tag{A.9}$$

Note that $p^* \in \big(min(a, 1/b), max(a, 1/b)\big)$. We have to now prove that with this price, the allocation (c^i, c_2^i) given by

$$(ap_2 - p_1)c_1^i = p_2\Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}} - w^i, \quad p_1c_1^i + p_2c_2^i = p_1w_1^i + p_2w_2^i \tag{A.10}$$

is optimal for the agent i. To do so, it suffices to check the following condition (we apply Proposition 1),

$$\Big(bp_1 - p_2\Big)\Big(\frac{w^i}{p_2}\Big)^\lambda < A(ap_2 - p_1) < \Big(bp_1 - p_2\Big)\Big(\frac{aw^i}{p_1}\Big)^\lambda. \tag{A.11}$$

We prove this condition for the case $ab > 1$. The case $ab < 1$ is similar. Suppose $ab > 1$. Consider the function $f(x) = \frac{a+x}{1+bx}$. We have $f'(x) = \frac{1-ab}{(1+bx)^2} < 0$. Condition $a > 1/b$ implies that $\frac{p_1}{p_2} = \frac{a+\frac{(aw_1+w_2)^\lambda}{A}}{1+b\frac{(aw_1+w_2)^\lambda}{A}} \in (\frac{1}{b}, a)$. Since $ab > 1$, condition (A.11) is equivalent to

$$\begin{cases} bp_1 - p_2 > 0, ap_2 - p_1 > 0 \\ \Big(\frac{w^i}{p_2}\Big)^\lambda < A\frac{ap_2-p_1}{bp_1-p_2} < \Big(\frac{aw^i}{p_1}\Big)^\lambda \end{cases} \Leftrightarrow \begin{cases} bp_1 - p_2 > 0, ap_2 - p_1 > 0 \\ \frac{p_1w_1^i+p_2w_2^i}{p_2} < aw_1 + w_2 < \frac{a(p_1w_1^i+p_2w_2^i)}{p_1} \end{cases} \tag{A.12}$$

$$\Leftrightarrow \begin{cases} \frac{1}{b} < \frac{p_1}{p_2} < a \\ \frac{p_1}{p_2}w_1^i + w_2^i < aw_1 + w_2 < a(w_1^i + \frac{p_2}{p_1}w_2^i) \end{cases} \tag{A.13}$$

This condition holds because $ap_2 > p_1$ and we assume that

$$(aw_1 + w_2)min(1, \frac{p_1}{ap_2}) < \frac{p_1}{p_2}w_1^i + w_2^i < (aw_1 + w_2)max(1, \frac{p_1}{ap_2}) \tag{A.14}$$

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