Methods of Economic Theory: Variables, Transactions and Expectations as Functions of Risks

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Abstract

This paper develops methods and framework of economic theory free from general equilibrium tools and assumptions. We model macroeconomics as system of agents those perform transactions with other agents under action of numerous expectations. Agents expectations are formed by economic and financial variables, transactions, expectations of other agents, other factors that impact macro economy. We use risk ratings of agents as their coordinates on economic domain and approximate description of economic variables, transactions and expectations of numerous separate agents by density functions of variables, transactions and expectations of aggregated agents on economic domain. Motion of separate agents on economic domain due to change of agents risk rating produce economic flows of variables, transactions and expectations. These risk flows define dynamics of economic variables and disturb any supposed market equilibrium states all the time. Permanent evolution of market supply-demand states due to risk flows makes general equilibrium concept too doubtful. As example we apply our methods to model assets pricing and return fluctuations.

Keywords: economic theory, risk ratings, economic flows, density functions

JEL: C00, C02, C10, E00

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1. Introduction

Economic policy and market regulation rely heavily on methods of general equilibrium theory (GE) (Arrow and Debreu, 1954; Tobin, 1969; Arrow, 1974; Kydland and Prescott, 1990; Starr, 2011) and DSGE (Fernández-Villaverde, 2010; Komunjer and Ng, 2011; Negro, et al, 2013; Farmer, 2017). Existing flaws and weaknesses of GE and DSGE may bring unjustified decisions and excess shocks to unsteady global economic and financial processes. Numerous papers discuss pro and contra of GE (Hazlitt, 1959; Morgenstern, 1972; Ackerman, 1999; Stiglitz, 2017). A special issue of Oxford Review of Economic Policy on "Rebuilding macroeconomic theory" discusses: "What new ideas are needed? What needs to be thrown away? What might a new benchmark model look like? Will there be a 'paradigm shift'?" (Vines and Wills, 2018a,b).

It is well known that methods of mainstream GE theory often give significant failure in forecasting of real macroeconomic processes. That doesn't for sure makes GE models incorrect as any theory has it's own accuracy level. Assumptions and approximations those ground any theory determine it's level of accuracy and that don't make theory good or bad. It seems much more important to compare economic modeling and forecasting based on GE methods with results based on methods that are different from mainstream. Such comparison may select cases were GE methods may give better forecasting and cases where different methods and tools should be more preferable. Such comparison requires sufficiently general methods that may describe a wide range of economic processes.

In this paper we present methods, models and equations of economic theory that don't use any GE tools and assumptions. We introduce here main issues of our approach and will make some comments on GE accuracy below.

Let's treat macroeconomics as a system of agents with economic and financial variables. Agents are engaged into numerous economic transactions with other agents. Agents perform their transactions under different expectations. Agents form their expectations on base of economic or financial variables, transactions, expectations of other agents, economic policy, technology or regulatory changes and so on. Economic activities of agents are always performed under risks. Moreover, activity of economic agents creates risks and no economic or financial development is possible without risks. We use risk ratings of agents as their coordinates and show that change of agents risk ratings due to any reasons cause risk flows of economic variables, transactions and expectations. These risk flows induce continuous changes of economic variables like investment and credits or market supply and demand in particular. Permanent evolution of market supply-demand states due to risk flows makes general equilibrium concept too doubtful. We develop methods, tools and equations that describe relations between variables, transactions and expectations and their flows.

The rest of the paper unfolds as follows. In Sec.2 we discuss assumptions, introduce economic domain, economic and financial variables, their flows and main equations. In Sec. 3 we discuss economic transactions as functions of risks. In Sec. 4 we introduce expectations as functions of risk. To show advantages of our approach in Sec. 5 we apply our methods to model assets price and return fluctuations. Conclusions in Sec. 6. In (Olkhov, 2016a-2019) we use our methods to describe wave propagation of disturbances of economic variables and transactions, model business and credit cycles and study hidden problems of classical Black-Scholes option pricing.

2. Assumptions, variables and equations

Let's discuss main assumptions starting with properties of economic and financial variables. Economic agents have many different variables like credits and debts, assets and investment, supply and demand and etc. Some variables are additive and some non-additive variables. For example sum of investment or credits of group of agents (without doubling) define investment and credits of entire group. Ratios of additive variables define non-additive variables like prices or financial rations. Inflation, indexes are determined as ratio of prices in different moments of time and are non-additive also. Thus agents additive financial and economic variables describe all economic and financial variables. Aggregation of agents additive variables define macro variables. For example sum of agents assets value (without doubling) determine macroeconomic assets value, sum of agents investment define investment of entire economics and etc. Thus agents additive variables are key factors that define macro economic variables and their evolution.

Some additive variables are involved into transactions between agents. Any transaction implies that seller transfer certain volume of additive variable like commodities, assets, service, investment, credits and etc., to buyer. Let's call additive variables involved into transactions between agents as additive variables of type 1. Let's call other additive variables as type 2 additive variables. For example sum of agents value-added define macroeconomic additive variable – GDP (Fox, et al, 2014). As well agents value-added are not subject of any transaction and are determined by accounting procedure. Sales and expenditures subject of buy-sell transactions and hence are type 1 variables. Thus type 1 variables sales and expenditures define type 2 additive variable value-added. Hence transactions between agents define type 1 additive variables and they determine type 2 additive variables and all non-additive variables. Transactions between agents are only factors that impact change of additive variables. Changes of market regulations, political, technology or climate impacts on markets have results only after certain transactions are performed. Available information about value and volume of the performed transactions, prices of the transactions may impact change of variables for all agents. Thus description of transactions between agents play key role for modeling all agents variables and all macroeconomic variables and is well known at least since Leontief's models (Leontief, 1941; 1955; Horowitz and Planting, 2006).

Now let's argue description of transactions. All agents perform transactions under different expectations. Agents expectations determine volume and value of transactions, choice of commodities, amount of credits and directing investment. Agents expectations as drivers of transactions are responsible for economic activity and hence impact evolution of macro variables. Agents form their expectations as their forecasts of economic variables, expectations of other agents, market regulatory trends, technology, climate and other changes. Thus expectations transfer impact of endogenous and exogenous factors on performance of transactions between agents and hence on evolution of economic variables.

Relations between economic variables, transactions and expectations establish core problem for macroeconomic modeling. In this paper we present methods and tools that describe evolution of variables, transactions and expectations under different approximations. We use bold italic to denote vectors and italic for scalars. Let's outline three main issues that determine our approach:

I. *Let's use risk ratings of economic agents as their coordinates*
II. *Let's describe variables, transactions and expectations as functions of risks*
III. *Changes of agents risk ratings produce collective flows of variables, transactions and expectations and we describe their impact on economic evolution*

Let's discuss these issues in details.

I. Risk ratings of economic agents as their coordinates

We use agents risk ratings as their coordinates (Olkhov, 2016a – 2017a). International rating agencies as S&P, Moody's, Fitch (Metz and Cantor, 2007; S&P, 2014; Fitch, 2018) for decades provide risk assessments for major banks, corporations, securities and etc., and deliver distributions of biggest banks by their risk ratings (Moody's, 2018; South and Gurwitz, 2018). These assessments are basis for investment expectations of biggest hedge funds, investors, traders etc. According to current risk assessment methodologies (Altman, 2010; Moody's, 2010; S&P&, 2016; Fitch, 2018) risk ratings take values of risk grades like *AAA, AA, BB, C* etc. Different rating agencies use different risk assessment methodologies and risk grades notions differs slightly.

Let's outline that risk grades *AAA, AA, BB, C* can be treated as points $x_1,...x_N$ of space that we call further as economic space. Risk assessment methodology use available economic statistics and determine number N of risk points. Let's propose that economic statistics and econometrics can provide sufficient data to assess risk ratings for all economic agents and for all risks that may hit macroeconomic evolution and growth. Let's assume that rating agencies assess risk ratings for all economic agents: for large corporations and banks and for small companies, firms and even households. Now let's assume that risk assessment methodologies can define continuous spectrum of risk grades on space R. Risk methodology always can take continuous risk grades as $[0,1]$ with point 0 as most secure and 1 as most risky grades. A lot of different risks can disturb macroeconomic processes (McNeil, Frey and Embrechts, 2005;). Assessments of single risk, like credit risk, distribute agents over range $[0,1]$ of 1-dimensional space R. Assessments of two or three risks, like credit, exchange rate and liquidity risks for example, distribute economic agents over unit square or cube. For given configuration of n macroeconomic risks, assessments of agents risk rating distribute agents by their risk coordinates $\boldsymbol{x}=(x_1,...x_n)$ over economic domain

$$0 \leq x_i \leq 1 \ , \ \ i = 1, \ldots n \tag{1}$$

of n-dimensional space R^n. Distribution of economic agents by their risk coordinates $\boldsymbol{x}=(x_1,...x_n)$ over economic domain (1) mean that all economic and financial variables of agents are also distributed on (1). Aggregation of similar variables for agents with coordinates near point $\boldsymbol{x}=(x_1,...x_n)$ of (1) define collective economic variables as functions of $\boldsymbol{x}$. Aggregations of similar transactions between agents with coordinates $\boldsymbol{x}$ and $\boldsymbol{y}$ determine collective transactions as functions of $\boldsymbol{x}$ and $\boldsymbol{y}$ on (1). As we show below this helps describe dynamics of macro variables, transactions and expectations by partial differential equations on economic domain.

II. Variables, transactions and expectations as functions of risks

Description of economic variables, transactions and expectations of separate agents of entire economics is too complex problem and don't helps for modeling evolution of macro economic variables. Up now macroeconomic modeling uses aggregations of economic variables of all agents as functions of time. For example, sum of investment of all agents define macro investment as function of time. In particular, GE and DSGE theories describe relations between macro variables as functions of time. We propose that such approach hides too much information about properties of and relations between economic and financial macro variables and that may be origin for numerous failures of GE and DSGE. We propose use distribution of agents on economic domain (1) as the tool for description of collective economic variables, transactions and expectations as functions of risks. Such approximation is much rougher then description relations between variables, transactions and expectations of separate agents and much more detailed then description as functions of time only. Description of variables and transactions of numerous separate agents is too complex and specific. We propose approximation that gives more rough description and requires significantly less econometric data. To do that let's collect variables, transactions or expectations of agents with risk coordinates inside small volume dV on economic domain (1) and then average them during certain time. Let's chose economic space scale d and time scale Δ. For n-dimensional economic space R^n let's take a unit volume $dV=d^n$ near point $\boldsymbol{x}$ of (1) and assume that scales $d<<1$ but many economic agents have risk coordinates inside this unit volume dV near point $\boldsymbol{x}$. Let's assume that time Δ is small to compare with time scale of the problem under consideration but many transactions are be performed during Δ. For example, let's estimate the number of agents in economics with population 10^8-10^9 as 10^8-10^9. Thus the scale $d\sim 10^{-2}$ on 2-dimensional economic domain (1) defines a unit volume $dV\sim 10^{-4}$ with around 10^4-10^5 agents inside it. Time scale $\Delta=1$ *week* is small to compare with time term one quarter or year. Let's assume that agents perform 1 transaction per second and hence there are about $6*10^5$ transactions per $\Delta=1$ *week.* Thus aggregation by scales $d\sim 10^{-2}$ and averaging by $\Delta=1$ *week* may approximate economic processes for time term one quarter, year or more. As example let's consider credits provided by agents inside dV near point $\boldsymbol{x}$ and average them during $\Delta=1$ *week.* Let's take that $C(t,\boldsymbol{x})$ equals sum of credits provided by agents in volume dV and averaged during time Δ. Function $C(t,\boldsymbol{x})$ has meaning of density of credits provided by agents from point $\boldsymbol{x}$ at moment t. Indeed, integral of $C(t,\boldsymbol{x})$ by $d\boldsymbol{x}$ over economic domain equals total credits provided by all agents in economics at moment t. Averaging over time Δ reduce high frequency fluctuations of the collected credits and makes this variable smooth. Introduction of scale d and scale Δ reduce accuracy of the model approximation. If one chose scale $d=1$ then volume dV will be equal economic domain (1) and sum of credits provided by agents inside (1) equals all credits provided in macroeconomics. Similar definitions allow introduce collective transactions between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ on (1) as density functions of two risk coordinates and density functions of expectations as functions of $\boldsymbol{x}$. Thus introduction of scales $d<<1$ establishes approximation that is intermediate between precise description of separate economic agents and too rough approximation based on aggregation of variables of all agents in the economy. Below we give formal definitions for define density functions for economic and financial variables, transactions and expectations.

It is obvious that one may aggregate agents and their variables, transactions and expectations on economic domain (1) by various economic groups with section by different industry sectors, wealth, gender, age or other economic or financial specification. Macroeconomic models based on aggregation of agents by various groups on economic domain may model relations between different industry sectors or describe influence of any specifications those define grouping agents. For such models one may use different sets of risks and different risk measures for different groups of agents. For example risk assessments may differ for different industry sectors, for different wealthy and etc. Any specific grouping and usage of different set of risks and risk measures induce additional complexity to the model. In current paper we describe simplest tools and framework without any additional specifications.

The most important factor that impact evolution of density functions of variables, transactions and expectations is determined by collective flows of variables, transactions and expectations induced by motion of agents on economic domain (1). Such economic flows result of motion of agents on economic domain (1) due to change of their risk rating.

III. Changes of agents risk ratings produce collective flows of variables, transactions and expectations.

Most economic and financial risks like credit or investment risks, market or tax policy risks are generated by activity of economic agents. Any economic development reproduces economic and financial risks. Economic activity without risks is impossible. Changes of agents risk ratings due to their economic activity, variation of economic environment and other reasons cause change of agents risk coordinates on economic domain (1). Let's model change of agents risks during time Δ by certain speed $\boldsymbol{v}$ on economic domain. Motion of agents with risk speed $\boldsymbol{v}$ indicates that agents carry their economic and financial variables, transactions and expectations. For example if certain agent provides credits C and moves with speed $\boldsymbol{v}$ then it carries credit flow $\boldsymbol{P}_C=C\boldsymbol{v}$. Flows of variables, transactions and expectations carried by agents due to change of their risk ratings have important impact on economic and financial evolution. Flows of variables and transactions induce continuous changes of economic variables. In particular that cause continuous changes of market supply-demand and that makes GE concept of supply-demand equilibrium very questionable. Recent papers on "Rebuilding macroeconomic theory" (Vines and Wills, 2018a) don't study this issue and we plan to argue it in details in forthcoming work. Collective flows of separate agents define economic flows of variables, transactions and expectations. Let's explain meaning of aggregative variables and collective flows in a more rigorous way.

Let's regard macroeconomics as system of numerous agents on *n*-dimensional economic domain (1) and state that agents at moment *t* have risk ratings coordinates $\boldsymbol{x}=(x_1,\ldots x_n)$ and velocities $\boldsymbol{\upsilon}=(\upsilon_1,\ldots\upsilon_n)$. Velocities $\boldsymbol{\upsilon}=(\upsilon_1,\ldots\upsilon_n)$ describe change of agents risk coordinates during time term Δ. Let's assume that scale $d<<1$ define a unit volume dV at point $\boldsymbol{x}$:

$$d \ll 1 \quad ; \quad dV = d^n \tag{2}$$

and volume dV contains many agents. Let's take only additive variables of agents and assume that econometric statistics select "independent" agents. Let's call agents as "independent" if sum of their additive variables equals same variable of the entire group. For example sum of credits of *k* agents equals credits of the group of these *k* agents. Let's define additive aggregate variable $A(t,\boldsymbol{x})$ at point $\boldsymbol{x}$ as sum of variables $A_i(t,\boldsymbol{x})$ of agents *i* with coordinates in a unit volume $dV(x)$ (2) and then average it during term Δ as:

$$A(t,x) = \sum_{i\in dV(x);\,\Delta} A_i(t,x) \tag{3}$$

$$\sum_{i\in dV(x);\,\Delta} A_i(t,x) = \frac{1}{\Delta}\int_t^{t+\Delta} d\tau \, \sum_{i\in dV(x)} A_i(\tau,x) \tag{4}$$

We use $i \in dV(x)$ to denote that risk coordinates $\boldsymbol{x}$ of agent *i* belong to unit volume $dV(\boldsymbol{x})$. For brevity we use left hand sum (4) to denote averaging during time Δ in a unit volume $dV(\boldsymbol{x})$. Scale Δ is small to compare with time scales of the problem under consideration but a lot of economic transactions between agents are performed during time Δ. Time averaging smooth changes of agents variables under numerous transactions during time Δ. We aggregate values of variables of numerous agents with risk coordinates inside volume $dV(\boldsymbol{x})$, smooth their changes during time Δ and denote result as density function of variable at point $\boldsymbol{x}$. Density function $A(t,\boldsymbol{x})$ describes financial or economic variable at point $\boldsymbol{x}$ on (1). For example let's take $A_i(t,\boldsymbol{x})$ as credits of agent *i*. Then density of credits $A(t,\boldsymbol{x})$ describe sum of credits issued by all agents with coordinates $\boldsymbol{x}$ inside a unit volume $dV(\boldsymbol{x})$ and averaged during time Δ. Then total credits $A(t)$ in economy equal integral (5) over (1):

$$A(t) = \int d\boldsymbol{x} \; A(t,\boldsymbol{x}) \tag{5}$$

Thus function $A(t,\boldsymbol{x})$ (3) can be treated as economic density of variable $A(t)$ (5) on (1). Now let's introduce collective flows $\boldsymbol{P}$ and collective velocities $\boldsymbol{\upsilon}$. We describe change of coordinates $\boldsymbol{x}_i=(x_1,\ldots x_n)$ of agent *i* with additive variable $A_i(t,\boldsymbol{x})$ during time Δ by velocities velocity $\boldsymbol{\upsilon}_i=(\upsilon_{1i},\ldots\upsilon_{ni})$. Thus each agent *i* carries flow $\boldsymbol{p}_{iA}(t,\boldsymbol{x})$:

$$\boldsymbol{p}_{iA}(t,\boldsymbol{x}) = A_i(t,\boldsymbol{x})\boldsymbol{\upsilon}_{\boldsymbol{i}}(t,\boldsymbol{x}) \tag{6}$$

Different agents induce different flows of economic variable A in different directions with different velocities. Let's collect flows of variable $A_i(t,\boldsymbol{x})$ in the direction of velocity $\boldsymbol{v}_i$ of agents i with coordinates in a unit volume $dV(\boldsymbol{x})$ (2) and then average this flow during time Δ similar to relations (3, 4). Let's define collective flow $\boldsymbol{P}_A(t,\boldsymbol{x})$ of variable $A(t,\boldsymbol{x})$ as:

$$\boldsymbol{P}_A(t,\boldsymbol{x}) = \sum_{i \in dV(x);\, \Delta} A_i(t,\boldsymbol{x})\boldsymbol{v}_i(t,\boldsymbol{x}) \tag{7}$$

Similar to (5) integral of (7) by $d\boldsymbol{x}$ over (1) define macro flow $\boldsymbol{P}_A(t)$ of variable $A(t)$ as:

$$\boldsymbol{P}_A(t) = \int d\boldsymbol{x}\; \boldsymbol{P}_A(t,\boldsymbol{x}) \tag{8}$$

Flow $\boldsymbol{P}_A(t,\boldsymbol{x})$ (7) of variable $A(t,\boldsymbol{x})$ (3) defines collective velocity $\boldsymbol{v}_A(t,\boldsymbol{x})$ of variable $A(t,\boldsymbol{x})$ as:

$$\boldsymbol{P}_A(t,\boldsymbol{x}) = A(t,\boldsymbol{x})\boldsymbol{v}_A(t,\boldsymbol{x}) \tag{9}$$

Thus (9) describes flow $\boldsymbol{P}_A(t,\boldsymbol{x})$ of variable $A(t,\boldsymbol{x})$ with velocity $\boldsymbol{v}_A(t,\boldsymbol{x})$. Relations (5) and (8) define macro velocity $\boldsymbol{v}_A(t)$ on (1) of macro variable $A(t)$ as:

$$\boldsymbol{P}_A(t) = A(t)\boldsymbol{v}_A(t) \tag{10}$$

One can obtain relations (8; 10) as sum of flows (6) of all agents of entire economics. Let's mention that due to (3; 5; 7-9 and 10) velocity $\boldsymbol{v}_A(t)$ is not equal to integral of velocity $\boldsymbol{v}_A(t,\boldsymbol{x})$ over economic domain (1). Due to (3-10) different variables A define different collective flows $\boldsymbol{P}_A(t,\boldsymbol{x})$ with different velocities $\boldsymbol{v}_A(t,\boldsymbol{x})$. In other words – motion of different additive variables $A(t,\boldsymbol{x})$ on (1) has different velocities $\boldsymbol{v}_A(t,\boldsymbol{x})$. For example flow $\boldsymbol{P}_C(t,\boldsymbol{x})$ of credits $C(t,\boldsymbol{x})$ has velocity $\boldsymbol{v}_C(t,\boldsymbol{x})$ that differs from velocity $\boldsymbol{v}_L(t,\boldsymbol{x})$ of loans $L(t,\boldsymbol{x})$ collective flow $\boldsymbol{P}_L(t,\boldsymbol{x})$ or from collective velocity $\boldsymbol{v}_I(t,\boldsymbol{x})$ that describe flow $\boldsymbol{P}_I(t,\boldsymbol{x})$ of investment $I(t,\boldsymbol{x})$ on (1). Flows (8; 10) for different variables are also different. For example, flows $P_S(t)$ and velocities $\boldsymbol{v}_S(t,\boldsymbol{x})$ of market supply are different from flows $P_D(t)$ and velocities $\boldsymbol{v}_D(t,\boldsymbol{x})$ of market demand for any commodities, assets or goods and any markets. That cause permanent change of supply and demand and makes existence of any market supply-demand equilibrium very doubtful. Lack of any assessments of time terms that may bring market to equilibrium state and ignore of risk and economic risk flows impact that disturb all imaginable equilibrium states make GE as a concept too questionable. Current discussion on "Rebuilding macroeconomic theory" (Vines and Wills, 2018a) doesn't study this important issue. We propose that further research on applicability of GE concept to economic modeling is required. Macroeconomic models should describe dynamics and mutual interactions between numerous variables and their flows. Properties of economic and financial flows are completely different from properties of any physical flows.

As we show below similar considerations define collective flows of transactions and expectations. To outline impact of collective flows of variables, transactions and expectations on macroeconomics let's argue equations that govern evolution of collective variables, transactions and expectations as functions of risk coordinates on (1). All equations have similar form and we derive them for credit density function $C(t,\boldsymbol{x})$ as example.

Credit density function $C(t,\boldsymbol{x})$ (3,4) describes collective credits issued by agents with coordinates inside small volume dV at point $\boldsymbol{x}$. Motion of agents inside volume dV induces collective credit flows $\boldsymbol{P}_C(t,\boldsymbol{x})=C(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})$ (7,9). Function $\boldsymbol{v}(t,\boldsymbol{x})$ describes velocity of flow of credit density $C(t,\boldsymbol{x})$. To describe change of credit density function $C(t,\boldsymbol{x})$ during time dt in a small volume dV on economic space let's take into account two factors of such change. The first one describes change of $C(t,\boldsymbol{x})$ in time dt in a small volume dV :

$$\int dV \, \frac{\partial}{\partial t} C(t,\boldsymbol{x})$$

The second factor is determined by credit flows $\boldsymbol{P}_C=C\boldsymbol{v}$ of agents that may flow in- or flow out- of small volume dV during time dt. Agents that flow in- a volume dV during dt with credit flow $\boldsymbol{P}_C=C\boldsymbol{v}$ increase credit density function $C(t,\boldsymbol{x})$ in a volume dV and agents that flow out of the volume dV with credit flow $\boldsymbol{P}_C=C\boldsymbol{v}$ decrease credit density function $C(t,\boldsymbol{x})$. Balance of credit flows in- and credit flows out- takes form of integral of credit flows $\boldsymbol{P}_C(t,\boldsymbol{x})=C(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})$ over the surface of a volume dV:

$$\oint ds \, \boldsymbol{P}_{\boldsymbol{C}}(t,\boldsymbol{x}) = \oint ds \, C(t,\boldsymbol{x}) \, \boldsymbol{v}(t,\boldsymbol{x})$$

Due to well-known divergence Gauss' Theorem (Strauss 2008, p.179), surface integral of the flows equals volume integral of the flows divergence over small volume dV:

$$\oint ds \, C(t,\boldsymbol{x}) \, \boldsymbol{v}(t,\boldsymbol{x}) = \int dV \;\; \nabla \cdot \big(\, C(t,\boldsymbol{x}) \, \boldsymbol{v}(t,\boldsymbol{x})\big) \tag{11.1}$$

Hence total change of credit density function during time dt in a small volume dV equals:

$$\int dV \left[\frac{\partial}{\partial t} C(t,\boldsymbol{x}) + \nabla \cdot \big(\, C(t,\boldsymbol{x}) \, \boldsymbol{v}(t,\boldsymbol{x})\big)\right]$$

Volume dV is arbitrary small thus equations on density functions (Olkhov, 2016a-2017a):

$$\frac{\partial}{\partial t} C(t,\boldsymbol{x}) + \nabla \cdot \big(\, C(t,\boldsymbol{x}) \, \boldsymbol{v}(t,\boldsymbol{x})\big) = F_C(t,\boldsymbol{x}) \tag{11.2}$$

Function $F_C(t,x)$ in the right side (11.2) describes any factors defined by variables, transactions or expectations and their flows on credit density function $C(t,\boldsymbol{x})$. Equation (11.2) depends on credit flow $\boldsymbol{P}_C(t,\boldsymbol{x})=C(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})$ and hence one should derive equation on it. Absolutely same considerations as above cause equations on flows $\boldsymbol{P}_C(t,\boldsymbol{x})=C(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})$ as:

$$\frac{\partial}{\partial t} \boldsymbol{P}_C(t,\boldsymbol{x}) + \nabla \cdot \big(\, \boldsymbol{P}_C(t,\boldsymbol{x}) \, \boldsymbol{v}(t,\boldsymbol{x})\big) = \boldsymbol{G}_C(t,\boldsymbol{x}) \tag{11.3}$$

Function $\boldsymbol{G}_C(t,\boldsymbol{x})$ describes any factors defined by variables, transactions and expectations and their flows on credit flows $\boldsymbol{P}_C(t,\boldsymbol{x})$. Due to (5) integral by $d\boldsymbol{x}$ of (11.2) over (1) equals:

$$\frac{d}{dt}C(t) + \int d\boldsymbol{x}\, \nabla\cdot\big(C(t,\boldsymbol{x})\,\boldsymbol{v}(t,\boldsymbol{x})\big) = \int d\boldsymbol{x}\; F_C(t,\boldsymbol{x}) = F_C(t) \quad (11.4)$$

Due to (11.1) integral in left side (11.4) equals zero as no agents, in- or out- flows exist on surface outside of domain (1). Thus (11.4) takes form of ordinary differential equation:

$$\frac{d}{dt}C(t) = F_C(t) \quad (11.5)$$

Complexities of (11.5) are hidden by function $F_C(t)$ determined by integral in (11.4). Function $F_C(t,\boldsymbol{x})$ may depend many variables, transactions, expectations and their flows and that may define $F_C(t)$ as very complex function. Thus time evolution of aggregate variables like macro credits $C(t)$ may depend on hidden dynamics of variables, transactions and expectations and their flows on domain (1). Due to (8; 10; 11.1) integral by $d\boldsymbol{x}$ for equations (11.3) over domain (1) defines ordinary differential equation on credit flows $\boldsymbol{P}_C(t)$:

$$\frac{d}{dt}\boldsymbol{P}_C(t) = \int d\boldsymbol{x}\; \boldsymbol{G}_C(t,\boldsymbol{x}) = \boldsymbol{G}_C(t) \quad (11.6)$$

Function $\boldsymbol{G}_C(t)$ in (11.6) as function $F_C(t)$ (11.4) may be a very complex function. Equations similar to (11.2; 11.3; 11.5; 11.6) are valid for other additive variables as investment, loans, demand and supply and etc., and their flows. Let's underline that each aggregate variable $A(t)$ as function of time defines different velocity $\boldsymbol{v}(t)$. Macroeconomic evolution requires description of motion of numerous financial variables with different velocities on (1) and that is a tough problem. Let's argue meaning of (11.6). Velocity $\boldsymbol{v}(t)$ of credit flow $\boldsymbol{P}_C(t)= C(t)\boldsymbol{v}(t)$ describes motion of credits $C(t)$ on (1). Economic domain (1) is bounded along each risk axes by most secure and most risky grades [0,1]. Thus motion of credits $C(t)$ with velocity $\boldsymbol{v}(t)$ along each risk axis from secure to risky direction should change by opposite motion from risky to secure area. Thus credit velocity $\boldsymbol{v}(t)$ should fluctuate in time and such fluctuations describe credit cycles. Similar fluctuations describe cycles of GDP, investment and etc.

Let's argue some consequences of our model. As we mention above equations similar to (11.2; 11.3) describe density functions and flows of numerous economic and financial variables, transactions and expectations. Thus equations similar to (11.2; 11.3) define macro model for each selected set of variables, transactions and expectations. Let's argue model determined by set of k different transactions like credit, investment, buy-sell transactions and etc. Each transaction defines change of variables of sellers and buyers. For example credit transaction change value of credits provided by creditor (seller) and amount of loans received by borrowers (buyers). Thus k transactions change $2k$ type 1 additive variables. Each transaction can be performed under different expectations. Let's assume that k transactions are performed under W expectations. To develop self-consistent model that describe macro model determined by $2k$ additive variables of type 1 and k transactions one should assume that all W expectations are determined by $2k$ additive variables, k selected transactions and their flows. In particular W expectations should depend on $2k$ additive variables or on non-additive variables that can be determined by $2k$ additive variables and their flows. If some expectations depend on exogenous factors then evolution of macroeconomic model reflect action of exogenous properties. Exogenous expectations approve transactions and thus transfer impact of exogenous factors on macroeconomic dynamics.

Importance of expectations is not reduced by their role as transmitter of exogenous shocks. As we argue above expectations may depend on flows of variables, transactions and other expectations. Dependence of expectations on financial flows makes them key factors that determine impact of flows on macro evolution. Dynamics of flows of variables, transactions and expectations and their mutual interactions on (1) establish a very complex picture. For example flows on domain (1) generate business cycles that describe slow oscillations of macro variables. On the other hand perturbations of flows cause generation, propagation and interaction of waves of disturbances of economic or financial variables, transactions and expectations those induce fast oscillations of economic parameters. We apply our methods to study approximations based on equations similar to (11.2; 11.3) that describe "simplified" model interactions between two variables (Olkhov, 2016a, 2016b), between two transactions (Olkhov, 2018a), model business cycles (Olkhov, 2017c; 2019) and wave propagation of disturbances of financial variables (Olkhov, 2016a-2017a) and transactions (Olkhov, 2018a) and surface-like waves (Olkhov, 2017b) on domain (1). In Sec.5 we use equations similar to (11.2; 11.3) to model price fluctuations induced by interactions between transactions and numerous expectations.

3. Transactions as functions of risks

In this Section we describe economic and financial transactions between agents as functions of risk coordinates. Let's take that agent i at point $\boldsymbol{x}$ sell amount Q_{ij} of variable E to agent j at point $\mathbf{y}$. Variable E may be commodities, credits, investment, assets, service and etc. For example let's take that agent i provide credits C to agent j. Such transactions between agents i and j change amount of credits C provided by i and amount of loans L received by j. Transaction of amount Q_{ij} cost certain value C_{ij} that should be paid by agent j as buyer to agent i as seller. Thus each transaction defines two variables – amount Q_{ij} and cost C_{ij} and price p_{ij} of economic or financial variable E. For agent i with risk coordinates $\boldsymbol{x}$ and agent j with coordinates $\boldsymbol{y}$ at moment t amount $Q_{ij}(t,\boldsymbol{x},\boldsymbol{y})$ and cost $C_{ij}(t,\boldsymbol{x},\boldsymbol{y})$ let's define transaction $\boldsymbol{bs}_{ij}(t,\boldsymbol{x},\boldsymbol{y})$ as:

$$\boldsymbol{bs}_{ij}(t,\boldsymbol{z}) = \left(Q_{ij}(t,\boldsymbol{z}); C_{ij}(t,\boldsymbol{z})\right) \quad ; \quad \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{12.1}$$

Then price $p_{ij}(t,z)$ of transaction (12.1) take obvious form:

$$p_{ij}(t,\boldsymbol{z}) = C_{ij}(t,\boldsymbol{z})/Q_{ij}(t,\boldsymbol{z}) \quad ; \quad \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{12.2}$$

we use bold for buy-sell transaction $\boldsymbol{bs}_{ij}(t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ to underline that each transaction defines two additive variables – amount Q_{ij} and cost C_{ij} of economic or financial variable *E*. Each transaction takes certain time *dt* and we consider transactions as rate or speed of change of corresponding variable *E* for agents involved into transaction. For example all buy-sell transactions of agent *i* at moment *t* during time [0, *t*] define change of variable *E* (Steel, Energy, Shares, Credits, Assets and etc.) owned by agent *i* during period [0, *t*]. Similar to transition from description of variables $A_i(t,\boldsymbol{x})$ of separate agents *i* to description of aggregate variable $A(t,\boldsymbol{x})$ (3,4) we move description of transactions $\boldsymbol{bs}_{ij}(t,\boldsymbol{x},\boldsymbol{y})$ between separate agents *i* at $\boldsymbol{x}$ and *j* at $\boldsymbol{y}$ to description of collective transactions $\boldsymbol{BS}(t,\boldsymbol{x},\boldsymbol{y})$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$.

Let's take that agents on (1) at moment *t* have coordinates $\boldsymbol{x}=(x_1,\dots x_n)$ and risk velocities $\boldsymbol{\upsilon}=(\upsilon_1,\dots\upsilon_n)$. Transactions between agents with risk coordinates $\boldsymbol{x}$ and agents with risk coordinates $\boldsymbol{y}$ are determined on $2n$-dimensional economic domain, $z=(\boldsymbol{x},\boldsymbol{y})$:

$$\boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \;;\; \boldsymbol{x} = (x_1 \dots x_n) \;;\; \boldsymbol{y} = (y_1 \dots y_n) \tag{12.3}$$

$$0 \le x_i \le 1 \;\;;\;\; 0 \le y_i \le 1\,, \quad i = 1, \dots n \tag{12.4}$$

Relations (12.3; 1.2) define $2n$-dimensional economic domain that is filled by pairs of agents with coordinates $z=(\boldsymbol{x},\boldsymbol{y})$. Let's take a unit volume $dV(z)$

$$dV(\boldsymbol{z}) = dV(\boldsymbol{x})dV(\boldsymbol{y}) \;;\; dV(\boldsymbol{x}) = d^n \;;\; dV(\boldsymbol{x}) = d^n \;;\; \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{12.5}$$

and assume that $dV(\boldsymbol{x})$ and $dV(\boldsymbol{y})$ follow relations (2) and their scales d<<1. Let's assume that each unit volume $dV(\boldsymbol{x})$ and $dV(\boldsymbol{y})$ contain a lot of agents with risk coordinates inside $dV(\boldsymbol{x})$ and $dV(\boldsymbol{y})$ and during time Δ agents inside $dV(\boldsymbol{x})$ and $dV(\boldsymbol{y})$ perform a lot of transactions. Let's define collective transaction $\boldsymbol{BS}(t,\boldsymbol{x},\boldsymbol{y})$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$ as sum of all transactions of agents *i* with coordinates in a unit volume $dV(\boldsymbol{x})$ and agents *j* with coordinates in a unit volume $dV(\boldsymbol{y})$ (12.5) and then average it during term Δ similar to (3,4) as:

$$\boldsymbol{BS}(t,\boldsymbol{x},\boldsymbol{y}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} \boldsymbol{bs}_{i,j}(t,\boldsymbol{x},\boldsymbol{y}) \tag{12.6}$$

$$\sum_{i\in dV(\boldsymbol{x});\,\Delta} \boldsymbol{bs}_{i,j}(t,\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\Delta}\int_t^{t+\Delta} d\tau \, \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})} \boldsymbol{bs}_{i,j}(t,\boldsymbol{x},\boldsymbol{y}) \tag{12.7}$$

$$\boldsymbol{BS}(t,\boldsymbol{z}) = \left(Q(t,\boldsymbol{z}); C(t,\boldsymbol{z})\right) \quad ; \quad \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{12.8}$$

$$Q(t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} \sum_{k_1} Q_{ij}(t,\boldsymbol{z}) \tag{12.9}$$

$$C(t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} \sum_{k_2} C_{ij}(t,\boldsymbol{z}) \tag{12.10}$$

and price $p(t,z)$ of aggregate transaction $\boldsymbol{BS}(t,z)$ take form:

$$C(t,\boldsymbol{z}) = p(t,\boldsymbol{z})Q(t,\boldsymbol{z}) \quad ; \quad \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{12.11}$$

Integral of transaction ***BS****(t,z)* (12.8) by $d\boldsymbol{y}$ over economic domain (12.3;12.4) defines all sells ***BS****(t,**x**)* of variable *E* performed by agents inside a unit volume $dV(\boldsymbol{x})$ at $\boldsymbol{x}$

$$\boldsymbol{BS}(t,\boldsymbol{x}) = \left(Q_S(t,\boldsymbol{x}); C_S(t,\boldsymbol{x})\right) \tag{13.1}$$

$$Q_S(t,\boldsymbol{x}) = \int d\boldsymbol{y}\, Q(t,\boldsymbol{x},\boldsymbol{y}) \;\; ; \;\; C_S(t,\boldsymbol{x}) = \int d\boldsymbol{y}\, C(t,\boldsymbol{x},\boldsymbol{y}) \tag{13.2}$$

Relations (13.1;13.2) define price $p_S(t,\boldsymbol{x})$ of sellers for transactions with variable *E* from $\boldsymbol{x}$:

$$C_S(t,\boldsymbol{x}) = p_S(t,\boldsymbol{x})Q_S(t,\boldsymbol{x}) \tag{13.3}$$

Integral of transaction ***BS****(t,z)* (12.8) by $d\boldsymbol{x}$ over (12.3;12.4) defines buyers price $p_B(t,\boldsymbol{y})$ at $\boldsymbol{y}$:

$$C_B(t,\boldsymbol{y}) = \int d\boldsymbol{x}\, C(t,\boldsymbol{x},\boldsymbol{y}) = p_B(t,\boldsymbol{y})Q_B(t,\boldsymbol{y}) \;\; ; \;\; Q_B(t,\boldsymbol{y}) = \int d\boldsymbol{x}\, Q(t,\boldsymbol{x},\boldsymbol{y}) \tag{13.4}$$

Integral of transaction ***BS****(t,z)* (12.8) by $d\boldsymbol{x}d\boldsymbol{y}$ over (12.3;12.4) define trading volume *Q(t)*, cost *C(t)* and price *p(t)* of transactions ***BS****(t)* in economy at moment *t:*

$$BS(t) = (Q(t); C(t)) \tag{13.5}$$

$$C(t) = \int d\boldsymbol{x}d\boldsymbol{y}\, C(t,\boldsymbol{x},\boldsymbol{y}) = p(t)Q(t) \;\; ; \;\; Q(t) = \int d\boldsymbol{x}d\boldsymbol{y}\, Q(t,\boldsymbol{x},\boldsymbol{y}) \tag{13.6}$$

For example, if *CI(t)* equals amount of cumulative investment made in economy during term [0,t] and *Q(t,**x**,**y**)* – amount of investment transactions ***BS****(t,**x**,**y**)* (12.6) made from $\boldsymbol{x}$ to $\boldsymbol{y}$ during time term *dt* then from (13.6):

$$\frac{d}{dt}CI(t) = Q(t) = \int d\boldsymbol{x}d\boldsymbol{y}\;\, Q(t,\boldsymbol{x},\boldsymbol{y}) \tag{13.7}$$

Hence transactions define time derivative of cumulative macro variables like investment, credits and etc. Let's call *BS(t,z)* as transactions density functions on *2n*-dimensional domain (12.3; 12.4) similar to variable density function *A(t,**x**)* (3;4) on (1). Relations (12.6-13.4) demonstrate that different levels of aggregation describe different meaning of price of transactions. Thus different levels of aggregation of price impact different ways for price evolution and price fluctuations.

Now similar to (6; 7) let's introduce transactions flows that are induced by change of risk coordinates of agents at point $\boldsymbol{x}$ and $\boldsymbol{y}$. Indeed, motion of agents due to change of their risk coordinates induces flows of transactions that change amount and cost of transactions in a small volume (12.3-12.5). Let's define flows $\boldsymbol{p}_{ij}(t,z)$ (14.1;14.2) of transactions $\boldsymbol{bs}_{ij}(t,z)$ between agents *i* and *j* similar to (6):

$$\boldsymbol{p}_{ij}(t,\boldsymbol{z}) = \left(\boldsymbol{p}_{Qij}(t,\boldsymbol{z}), \boldsymbol{p}_{Cij}(t,\boldsymbol{z})\right) \tag{14.1}$$

$$\boldsymbol{p}_{Qij}(t,\boldsymbol{z}) = \left(\boldsymbol{p}_{Qijx}(t,\boldsymbol{z});\, \boldsymbol{p}_{Qijy}(t,\boldsymbol{z})\right) \;;\;\; \boldsymbol{p}_{Cij}(t,\boldsymbol{z}) = \left(\boldsymbol{p}_{Cijx}(t,\boldsymbol{z});\, \boldsymbol{p}_{Cijy}(t,\boldsymbol{z})\right) \tag{14.2}$$

$$\boldsymbol{p}_{Qijx}(t,\boldsymbol{z}) = Q_{i,j}(t,\boldsymbol{z})\boldsymbol{v}_{ix}(t,\boldsymbol{x}) \quad ; \quad \boldsymbol{p}_{Qijy}(t,\boldsymbol{z}) = Q_{i,j}(t,\boldsymbol{z})\boldsymbol{v}_{jy}(t,\boldsymbol{y}) \tag{14.3}$$

$$\boldsymbol{p}_{Cijx}(t,\boldsymbol{z}) = C_{i,j}(t,\boldsymbol{z})\boldsymbol{v}_{ix}(t,\boldsymbol{x}) \quad ; \quad \boldsymbol{p}_{Cijy}(t,\boldsymbol{z}) = C_{i,j}(t,\boldsymbol{z})\boldsymbol{v}_{jy}(t,\boldsymbol{y}) \tag{14.4}$$

Flows $\boldsymbol{p}_{ij}(t,z)$ (14.1) define flows $\boldsymbol{p}_{Qij}(t,z)$ (14.3) that carry amount Q_{ij} and flows $\boldsymbol{p}_{Qij}(t,z)$ (14.4) that carry cost C_{ij} of transaction $\boldsymbol{bs}_{ij}(t,z)$ (12.1). Aggregate flows $\boldsymbol{P}(t,z)$ over all agents i at $\boldsymbol{x}$ inside $dV(\boldsymbol{x})$ and all agents j at $\boldsymbol{y}$ inside $dV(\boldsymbol{y})$ define transactions flows between points $\boldsymbol{x}$ and $\boldsymbol{y}$ similar to (7) as:

$$\boldsymbol{P}(t,\boldsymbol{z}) = \left(\boldsymbol{P}_Q(t,\boldsymbol{z}), \boldsymbol{P}_C(t,\boldsymbol{z})\right) \; ; \; \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{15.1}$$

$$\boldsymbol{P}_Q(t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} \boldsymbol{p}_{Qij}(t,\boldsymbol{z}) \; ; \; \boldsymbol{P}_C(t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} \boldsymbol{p}_{Cij}(t,\boldsymbol{z}) \tag{15.2}$$

$$\boldsymbol{P}_Q(t,\boldsymbol{z}) = \left(\boldsymbol{P}_{xQ}(t,\boldsymbol{z}); \boldsymbol{P}_{yQ}(t,\boldsymbol{z})\right) \; ; \; \boldsymbol{P}_C(t,\boldsymbol{z}) = \left(\boldsymbol{P}_{xC}(t,\boldsymbol{z}); \boldsymbol{P}_{yC}(t,\boldsymbol{z})\right) \tag{15.3}$$

$$\boldsymbol{P}_{xQ}(t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} Q_{ij}(t,\boldsymbol{z})\boldsymbol{v}_i(t,\boldsymbol{x}) = Q(t,\boldsymbol{z})\boldsymbol{v}_{xQ}(t,\boldsymbol{z}) \tag{15.4}$$

$$\boldsymbol{P}_{yQ}(t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} Q_{ij}(t,\boldsymbol{z})\boldsymbol{v}_j(t,\boldsymbol{y}) = Q(t,\boldsymbol{z})\boldsymbol{v}_{yQ}(t,\boldsymbol{z}) \tag{15.5}$$

$$\boldsymbol{P}_{xC}(t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} C_{ij}(t,\boldsymbol{z})\boldsymbol{v}_i(t,\boldsymbol{x}) = C(t,\boldsymbol{z})\boldsymbol{v}_{xC}(t,\boldsymbol{z}) \tag{15.6}$$

$$\boldsymbol{P}_{yC}(t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} C_{ij}(t,\boldsymbol{z})\boldsymbol{v}_j(t,\boldsymbol{y}) = C(t,\boldsymbol{z})\boldsymbol{v}_{yC}(t,\boldsymbol{z}) \tag{15.7}$$

$$\boldsymbol{v}(t,\boldsymbol{z}) = \left(\boldsymbol{v}_Q(t,\boldsymbol{z}); \; \boldsymbol{v}_C(t,\boldsymbol{z})\right) \tag{15.8}$$

$$\boldsymbol{v}_Q(t,\boldsymbol{z}) = \left(\boldsymbol{v}_{xQ}(t,\boldsymbol{z})\,; \boldsymbol{v}_{yQ}(t,\boldsymbol{z})\right) \; ; \; \boldsymbol{v}_C(t,\boldsymbol{z}) = \left(\boldsymbol{v}_{xC}(t,\boldsymbol{z})\,; \boldsymbol{v}_{yC}(t,\boldsymbol{z})\right) \tag{15.9}$$

Flows of transactions $\boldsymbol{P}(t,z)$ (15.1-15.6) between points $\boldsymbol{x}$ and $\boldsymbol{y}$ describe amounts of transactions $\boldsymbol{BS}(t,z)$ (13.1) carried by transactions velocities $\boldsymbol{v}(t,z)$ (15.7-15.9) through *2n*-dimensional domain (12.3;12.4). Let's underline that velocities $\boldsymbol{v}_Q(t,z)$ (15.8) of that define motion of amount of transactions may be different from velocities $\boldsymbol{v}_C(t,z)$ (15.9) that describe motion of transactions costs. These distinctions add additional perturbations for price of transactions (12.11). Similar to (8; 9) integrals of flows $\boldsymbol{P}(t,z)$ (15.1-15.6) and (15.7-15.9) over (12.3;12.4) by $d\boldsymbol{x}d\boldsymbol{y}$ define macro flows of transactions $\boldsymbol{BS}(t)$ (4.1) with velocity $\boldsymbol{v}(t)$ as:

$$\boldsymbol{P}(t) = (\boldsymbol{P}_Q(t); \boldsymbol{P}_C(t)) \tag{16.1}$$

$$\boldsymbol{P}_Q(t) = Q(t)\boldsymbol{v}_Q(t) = \int d\boldsymbol{z} \; Q(t,\boldsymbol{z})\boldsymbol{v}_Q(t,\boldsymbol{y}) \tag{16.2}$$

$$\boldsymbol{P}_C(t) = C(t)\boldsymbol{v}_C(t) = \int d\boldsymbol{z} \; C(t,\boldsymbol{z})\boldsymbol{v}_C(t,\boldsymbol{y}) \tag{16.3}$$

$$\boldsymbol{v}(t) = \left(\boldsymbol{v}_Q(t)\,; \boldsymbol{v}_C(t)\,\right) \tag{16.4}$$

$$\boldsymbol{v}_Q(t,\boldsymbol{z}) = \left(\boldsymbol{v}_{xQ}(t)\,; \boldsymbol{v}_{yQ}(t)\right) \; ; \; \boldsymbol{v}_C(t,\boldsymbol{z}) = \left(\boldsymbol{v}_{xC}(t)\,; \boldsymbol{v}_{yC}(t)\right) \tag{16.5}$$

For example let's take $\boldsymbol{BS}(t)$ as investment transactions with amount of investment $Q(t)$ in economy at moment t. Then relations (16.2) describe flow of amount of investment with velocity $\boldsymbol{v}_Q(t)$ on (12.3; 12.4). Components $\boldsymbol{v}_{xQ}(t)$ and $\boldsymbol{v}_{yQ}(t)$ describe motion of collective investors and recipients of investments. Positive or negative values of components of velocity $\boldsymbol{v}_{xiQ}(t)$ along axis x_i of (12.3;12.4) describe motion of investors in risky of safer directions. Positive values of components of velocity $\boldsymbol{v}_{yjQ}(t)$ along axis y_j of (12.3;12.4) describe motion of recipients of investments in risky direction and negative $\boldsymbol{v}_{yj}(t)$ describes decline of risks of recipients of investments along axis y_j. Collective investors and recipients of investments may move only inside bounded domain (12.3;12.4). Thus velocities (16.4; 16.5) can't be constant and must change signature and fluctuate as borders of domain (12.3; 12.4) reduce motion along each risk axes. Fluctuations of velocities (16.4; 16.5) describe motion of investors and recipients of investments from safer to risky areas and back from risky to safer areas and describe investment cycles. Credit transactions, buy-sell transactions and etc., induce similar macro transactions flows (16.1-16.5) and describe corresponding credit cycles, buy-sell cycles and etc., (Olkhov, 2017c; 2019).

Relations (12.6-12.10; 14.1-15.9) allow derive equations on transactions $\boldsymbol{BS}(t,z)$ and transactions flows $\boldsymbol{P}(t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ on $2n$-dimensional domain (12.3;12.4) similar to equations (11.2; 11.3) on density and flows of variables (3; 4; 7). To derive equations on transactions density $\boldsymbol{BS}(t,z)$ (12.8-12.10) and flows $\boldsymbol{P}(t,z)$ (15.1-15.9) let's describe their change in a small unit volume $dV(z)$ (12.3-12.5). Let's take equations on amount of transactions Q (12.9) and its flows $\boldsymbol{P}_Q$. Equations on cost of transaction take similar form. Two factors change amount of transaction $Q(t,z)$ in a unit volume $dV(z)$. The first change $Q(t,z)$ in time as:

$$\int d\boldsymbol{z}\ \frac{\partial}{\partial t}Q(t,\boldsymbol{z}) \tag{17.1}$$

The second factor describes change of $Q(t,z)$ due to flows $\boldsymbol{P}_Q(t,z)$: amount of $Q(t,z)$ in a unit volume $dV(z)$ (12.3-12.5) can grow up or decrease due to in- or out- flows $\boldsymbol{P}_Q(t,z)$ during time dt. If in-flows $\boldsymbol{P}_Q(t,z)$ exceed out-flows then $Q(t,z)$ grow up in a volume $dV(z)$. To calculate balance of in- and out-flows let's take integral of flow $\boldsymbol{P}_Q(t,z)$ over the surface of $dV(z)$:

$$\oint ds\, \boldsymbol{P}_Q(t,\boldsymbol{z}) = \oint ds\ \ Q(t,\boldsymbol{z})\boldsymbol{v}_Q(t,\boldsymbol{z}) \tag{17.2}$$

Due to divergence theorem (Strauss 2008, p.179) surface integral (17.2) of the flow $\boldsymbol{P}_Q(t,z)=Q(t,z)\boldsymbol{v}_Q(t,z)$ equals its volume integral by divergence of the flow:

$$\oint ds\, Q(t,\boldsymbol{z})\boldsymbol{v}_Q(t,\boldsymbol{z}) = \int d\boldsymbol{z}\ \nabla\cdot(Q(t,\boldsymbol{z})\boldsymbol{v}_Q(t,\boldsymbol{z})) \tag{17.3}$$

Relations (17.1; 17.3) give total change of amount of transactions $Q(t,z)$ in $dV(z)$:

$$\int d\boldsymbol{z}\ \left[\frac{\partial}{\partial t}Q(t,\boldsymbol{z}) + \nabla\cdot\Big(Q(t,\boldsymbol{z})\ \boldsymbol{v}_Q(t,\boldsymbol{z})\Big)\right]$$

As a unit volume $dV(z)$ is arbitrary one can take equations on economic density $Q(t,z)$ as

$$\frac{\partial}{\partial t}Q(t,\boldsymbol{z}) + \nabla\cdot\Big(Q(t,\boldsymbol{z})\ \boldsymbol{v}_Q(t,\boldsymbol{z})\Big) = F(t,\boldsymbol{z}) \tag{17.4}$$

Same considerations are valid for the flow $\boldsymbol{P}_Q(t,z)$:

$$\frac{\partial}{\partial t}\boldsymbol{P}_Q(t,\boldsymbol{z}) + \nabla\cdot\Big(\boldsymbol{P}_Q(t,\boldsymbol{z})\ \boldsymbol{v}_Q(t,\boldsymbol{z})\Big) = \boldsymbol{G}(t,\boldsymbol{z}) \tag{17.5}$$

Similar to (11.4; 11.5) integrals of (17.4; 17.5) by $dz=(d\boldsymbol{x},d\boldsymbol{y})$ over economic domain (12.3; 12.4) give:

$$\int d\boldsymbol{z}\left[\frac{\partial}{\partial t}Q(t,\boldsymbol{z})+\nabla\cdot\left(Q(t,\boldsymbol{z})\,\boldsymbol{v}_Q(t,\boldsymbol{z})\right)\right]=\frac{d}{dt}Q(t)=F(t)=\int d\boldsymbol{z}\,F(t,\boldsymbol{z}) \qquad (18.1)$$

$$\int d\boldsymbol{z}\left[\frac{\partial}{\partial t}\boldsymbol{P}_Q(t,\boldsymbol{z})+\nabla\cdot\left(\boldsymbol{P}_Q(t,\boldsymbol{z})\,\boldsymbol{v}_Q(t,\boldsymbol{z})\right)\right]=\frac{d}{dt}\boldsymbol{P}_Q(t)=\boldsymbol{G}(t)=\int d\boldsymbol{z}\,\boldsymbol{G}(t,\boldsymbol{z}) \qquad (18.2)$$

Relations (18.1; 18.2) illustrate that operators in the left hand of (17.4; 17.5) for $Q(t,z)$ and flows $\boldsymbol{P}_Q(t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ on $2n$-dimensional domain (12.3;12.4) play role alike to ordinary derivative by time t for amount of transactions $Q(t)$ (12.9; 18.1) and flows $\boldsymbol{P}_Q(t)$ (15.3; 18.2). Equations similar to (17.4; 17.5; 18.1; 18.2) are valid to cost of transactions $C(t,z)$ and cost flows $\boldsymbol{P}_C(t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$, but velocities $\boldsymbol{v}_C(t,z)$ of cost flows $\boldsymbol{P}_C(t,z)$ are different from velocities $\boldsymbol{v}_Q(t,z)$ of amount flows $\boldsymbol{P}_Q(t,z)$. Each component of each transaction has different velocities are described by different operators (17.4; 17.5; 18.1; 18.2) with different functions $F(t,z)$ and $\boldsymbol{G}(t,z)$. Such variety of flows and velocities on domain (12.3; 12.4) establish very complex picture of economic and financial processes and their evolution. Form of functions $F(t,z)$ and $\boldsymbol{G}(t,z)$ and the question - what factors impact equations on transactions (17.4; 17.5; 18.1; 18.2) reflect main complexity for modeling economic transactions and variables.

Various expectations impact agents to perform transactions $\boldsymbol{BS}(t,z)$ with other agents. Equations (17.4; 17.5) define evolution of amount Q (12.9) of transactions and similar equations describe cost C (12.10) of transactions. We propose that functions $F(t,z)$ and $\boldsymbol{G}(t,z)$ in the right-hand side of (17.4; 17.5) describe action of expectations of agents involved into transactions $\boldsymbol{BS}(t,z)$. Expectations may depend on economic and financial variables, transactions, expectations of other agents, market and tax regulation, technology trends and forecasts. That permits study evolution of financial systems in a different approximations.

Expectations are very numerous and different. Agents may go into same transactions under various expectations. For example different agents at point $\boldsymbol{x}$ may take decisions on amount Q (12.9) of the same transaction under inflation expectations, return expectations, professional macroeconomic forecasters (these expectations are mentioned by Mansky, 2017) and etc. It is clear that composition of different expectations as inflation expectations, return expectations or professional macroeconomic forecasters don't help establish collective expectation that may explain aggregate amount $Q(t,z)$ (12.9) of transaction made from point $\boldsymbol{x}$ to point $\boldsymbol{y}$, $z=(\boldsymbol{x},\boldsymbol{y})$. To describe collective impact of heterogeneous expectations let's introduce definitions of expectations similar to macro variables and transactions.

4. Expectations as functions of risks

Expectations are the most "etheric" substance of economics and finance. Expectations are treated as factors that govern economic transactions, price and return at least since Keynes (1936), Muth (1961) and Lucas (1972) and in numerous further publications (Sargent and Wallace, 1976; Hansen and Sargent, 1979; Kydland and Prescott, 1980; Blume and Easley, 1984; Brock and Hommes, 1998; Manski, 2004; Brunnermeier and Parker, 2005; Dominitz and Manski, 2005; Klaauw et al, 2008; Janžek and Ziherl, 2013; Greenwood and Shleifer, 2014; Lof, 2014; Manski, 2017; Thaler, 2018).

Expectations concern inflation and demand, exchange and bank rates, price trends and etc. There are a lot of studies on expectations measurements (Manski, 2004; Dominitz and Manski, 2005; Klaauw et al, 2008; Stangl, 2009; Janžek and Ziherl, 2013; Manski, 2017; Tanaka et al, 2018). Due to Manski (2004) "It would be better to measure expectations as - subjective probabilities". Dominitz and Manski (2005) "analyze probabilistic expectations of equity returns". Stangl (2009) suggests that "Visual Analog Scale (VAS) enables scores between categories, and the respondent can express not only the direction of his attitude but also its magnitude on a 1-to-100 point scale, which comes close to an interval scale measurement". Measurement of such "etheric" economic substance as expectations of separate agents is a really tough problem. Let's propose that it is possible to measure expectations of separate agents. How to establish collective expectations that collective transactions taken by agents at point $\boldsymbol{x}$? Indeed, aggregate transactions (12.6-12.10) are performed under collective expectations of agents on economic domain (12.3; 12.4) at point $\boldsymbol{z}$ in a unit volume (12.5). It is impossible collect different expectations like "inflation expectations, return expectations, professional macroeconomic forecasters" in one aggregate expectation. To define collective expectations let's simplify the problem. Let's assume that all different expectations are measured as index. It is clear, that scale of index is not important. Measure of expectations may take values between 0 and 100 or 0 and 1. Let's state that all expectations are measured by same measure with same scale. For certainty let's take interval [0,1] as measure of expectations. Let's assume that each economic agent may have $j=1,..K$ different expectations to take decisions on transactions $\boldsymbol{bs}_{ij}(t,\boldsymbol{x},\boldsymbol{y})$ (12.1) and each $j=1,..K$ particular expectation has particular measure on interval [0,1].

Now let's argue the problem: how to define measure of collective expectations that impact performance of aggregate transactions (12.6-12.10).

To aggregate value and importance of agents expectations let's state that financial or economic value of particular agent's expectation should be proportional to value of transactions made under this expectation. Indeed, if particular transactions amount 90% of all deals and are made under expectation 1 then expectation 1 is ninety times more important then expectation 2 that is responsible for only 1% of same deals. Thus to aggregate expectations of agents at point $\boldsymbol{x}$ one should collect expectations weighted by value of transactions made under these expectations.

Let's explain this statement using transactions $\boldsymbol{bs}_{ij}(t,z)$ (12.1) as example. Let's remind that each transaction $\boldsymbol{bs}_{ij}(t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ (12.1) defines quantity Q_{ij} and cost C_{ij} of transaction. Decisions on quantity Q_{ij} and cost C_{ij} may be done under different expectations. Seller and buyer may take decisions on transaction between them under different expectations too. Thus even single transaction $\boldsymbol{bs}_{ij}(t,z)$ (12.1) may be performed under four different expectations: two expectations of seller on quantity Q_{ij} and cost C_{ij} and two expectations of buyer. As example let's argue expectations that determine decisions of sellers on quantity Q_{ij} and cost C_{ij} of transaction (12.1). Let's denote seller's expectations $ex_{Qi}(k;t,\boldsymbol{x})$ of type $k=1,..K$ of agent i at point $\boldsymbol{x}$ to perform transaction of amount $Q_{ij}(k;t,z)$ as part of transactions of total amount $Q_{ij}(t,z)$ (12.1) of agent i with agent j under all expectations. Let's denote expectations $ex_{Ci}(l;t,\boldsymbol{x})$ of type $l=1,..K$ of same agent to perform transaction of amount $Q_{ij}(k;t,z)$ at the cost - $C_{ij}(l;t,z)$. It seems reasonable that decisions on quantity $Q_{ij}(k;t,z)$ depend on decisions on cost $C_{ij}(l;t,z)$ of transactions and vice versa. Thus amount Q_{ij} and cost C_{ij} of transactions preformed by seller should depend on both expectations k and l - $ex_{Qi}(k,l;t,\boldsymbol{x})$ and $ex_{Ci}(k,l;t,\boldsymbol{x})$. Let's denote volume Q_{ij} and cost C_{ij} of ***seller's*** transaction $\boldsymbol{bs}_{ij}(t,z)$ (12.1) as:

$$\boldsymbol{bs}_{ij}(t,\boldsymbol{z}) = \Big(Q_{ij}(\boldsymbol{k};t,\boldsymbol{z}); C_{ij}(\boldsymbol{k};t,\boldsymbol{z})\Big)\ ; \boldsymbol{k} = (k,l)\ ;\ k,l = 1,..K;\ \ \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \quad (19.1)$$

Same reasons allow denote volume Q_{ij} and cost C_{ij} of ***buyer's*** transaction $\boldsymbol{bs}_{ij}(t,z)$ (12.1) as:

$$\boldsymbol{bs}_{ij}(t,\boldsymbol{z}) = \Big(Q_{ij}(t,\boldsymbol{z};\boldsymbol{l}); C_{ij}(t,\boldsymbol{z};\boldsymbol{l})\Big)\quad ;\quad \boldsymbol{l} = (k,l);\ k,l = 1,..K \quad (19.2)$$

To define economic value of sellers expectations $ex_{Qi}(k,l;t,\boldsymbol{x})$ and $ex_{Ci}(k,l;t,\boldsymbol{x})$ let's introduce sellers expected transactions $\boldsymbol{et}_{ij}(\boldsymbol{k};t,\boldsymbol{x},\boldsymbol{y})$ as follows:

$$\boldsymbol{et}_{ij}(\boldsymbol{k};t,\boldsymbol{z}) = \Big(et_{Qij}(\boldsymbol{k};t,\boldsymbol{z})\ ;\ et_{Cij}(\boldsymbol{k};t,\boldsymbol{z})\Big)\ ;\ \ \boldsymbol{k} = (k,l) \quad (19.3)$$

$$et_{Qij}(\boldsymbol{k};t,\boldsymbol{z}) = ex_{Qi}(\boldsymbol{k};t,\boldsymbol{x})Q_{ij}(\boldsymbol{k};t,\boldsymbol{z})\ ;\ et_{Cij}(\boldsymbol{k};t,\boldsymbol{z}) = ex_{j}(\boldsymbol{k};t,\boldsymbol{y})C_{ij}(\boldsymbol{k};t,\boldsymbol{z})$$

Relations (19.3) describe sellers volume expected transactions $et_{Qi}(\boldsymbol{k};t,z)$ th*at* equal product of volume expectations $ex_{Qi}(\boldsymbol{k};t,\boldsymbol{x})$ of type k, $\boldsymbol{k}=(k,l)$ at point $\boldsymbol{x}$ weighted by quantity $Q_{ij}(\boldsymbol{k};t,z)$ of transaction performed between agents i at $\boldsymbol{x}$ as sellers and agents j at $\boldsymbol{y}$ as buyers. Cost expected transactions $et_{Ci}(\boldsymbol{k};t,z)$ (19.3) equal product of cost expectations $ex_{Ci}(\boldsymbol{k};t,\boldsymbol{x})$ of type l,

$\boldsymbol{k}$=(k,l) at point $\boldsymbol{x}$ weighted by cost $C_{ij}(\boldsymbol{k};t,z)$ of transaction performed between agents i at $\boldsymbol{x}$ as sellers and agents j at $\boldsymbol{y}$ as buyers. Similar considerations define buyers expected transactions:

$$\boldsymbol{et}_{ij}(t,\boldsymbol{z};\boldsymbol{l}) = \left(et_{Qij}(t,\boldsymbol{z};\boldsymbol{l})\,;\, et_{Cij}(t,\boldsymbol{z};\boldsymbol{l})\right) \;;\; \boldsymbol{l} = (k,l) \tag{19.4}$$

$$et_{Qij}(t,\boldsymbol{z};\boldsymbol{l}) = ex_{Qi}(t,\boldsymbol{x};l_1)Q_{ij}(t,\boldsymbol{z};\boldsymbol{l}) \;;\; et_{Cij}(t,\boldsymbol{z};\boldsymbol{l}) = ex_j(t,\boldsymbol{y};l_2)C_{ij}(t,\boldsymbol{z};\boldsymbol{l})$$

Let's move from description of transactions between agents to description of transactions between points of economic domain (12.3; 12.4) similar to (12.6 – 12.10) and define part $Q(\boldsymbol{k};t,z)$ of total amount $Q(t,z)$ and cost $C(\boldsymbol{k};t,z)$ of total cost $C(t,z)$ of transaction (12.8) performed under sellers expectations of type $\boldsymbol{k}$:

$$Q(\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} Q_{i,j}(\boldsymbol{k};t,\boldsymbol{z}) \;;\; \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{19.5}$$

$$C(\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} C_{i,j}(\boldsymbol{k};t,\boldsymbol{z}) \;;\; \boldsymbol{k} = (k,l) \tag{19.6}$$

Total amount $Q(t,z)$ (12.9) and total cost $C(t,z)$ of transactions (12.8) equal sum by all sellers expectations and by all buyers expectations:

$$Q(t,\boldsymbol{z}) = \sum_{kl} Q(k,l;t,\boldsymbol{z}) \;=\; \sum_{kl} Q(t,\boldsymbol{z};k,l) \tag{19.7}$$

$$C(t,\boldsymbol{z}) = \sum_{kl} C(k,l;t,\boldsymbol{z}) \;=\; \sum_{kl} C(t,\boldsymbol{z};k,l) \tag{19.8}$$

Now let's define sellers expected transactions $\boldsymbol{Et}_s(\boldsymbol{k};t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$ made under sellers expectations $\boldsymbol{k}$ =(k,l). Let's aggregate (19.3) alike to (15.4-15.7) as:

$$\boldsymbol{Et}_s(\boldsymbol{k};t,\boldsymbol{z}) = \left(Et_Q(\boldsymbol{k};t,\boldsymbol{z}) \;;\; Et_C(\boldsymbol{k};t,\boldsymbol{z})\right) \;;\; \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{20.1}$$

$$Et_Q(\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\Delta} ex_i(k_1;t,\boldsymbol{x})Q_{ij}(\boldsymbol{k};t,\boldsymbol{z}) \tag{20.2}$$

$$Et_C(\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\Delta} ex_j(k_2;t,\boldsymbol{y})C_{ij}(\boldsymbol{k};t,\boldsymbol{z}) \tag{20.3}$$

Relations (20.1-20.3) and functions $Q(\boldsymbol{k};t,z)$ (19.5) and $C(\boldsymbol{k};t,z)$ (19.6) define sellers expectations $Ex_Q(\boldsymbol{k};t,z)$ and $Ex_Q(\boldsymbol{k};t,z)$ of type $\boldsymbol{k}$ =(k,l) as:

$$Et_Q(\boldsymbol{k};t,\boldsymbol{z}) = Ex_Q(\boldsymbol{k};t,\boldsymbol{z})Q(\boldsymbol{k};t,\boldsymbol{z}) \tag{20.4}$$

$$Et_C(\boldsymbol{k};t,\boldsymbol{z}) = Ex_C(\boldsymbol{k};t,\boldsymbol{z})C(\boldsymbol{k};t,\boldsymbol{z}) \tag{20.5}$$

Let's underline that sellers expected transactions $Et_Q(\boldsymbol{k};t,z)$, $Et_C(\boldsymbol{k};t,z)$ (20.2; 20.3) and sellers expectations $Ex_Q(\boldsymbol{k};t,z)$, $Ex_C(\boldsymbol{k};t,z)$ (20.4; 20.5) are determined with respect to transactions (12.6-12.10) with selected financial or economic variable *E*. Transactions with different variables *E* – with commodities, service, assets and etc., - define different expectations $Ex_Q(\boldsymbol{k};t,z)$, $Ex_C(\boldsymbol{k};t,z)$. To define macro expectations of sellers $Ex_Q(\boldsymbol{k};t)$ and $Ex_C(\boldsymbol{k};t)$ at moment *t* let's take integrals over economic domain (12.3; 12.4):

$$Q(\boldsymbol{k};t,\boldsymbol{x}) = \int d\boldsymbol{y}\, Q(\boldsymbol{k};t,\boldsymbol{x};\boldsymbol{y}) \;;\; C(\boldsymbol{k};t,\boldsymbol{x}) = \int d\boldsymbol{y}\, C(\boldsymbol{k};t,\boldsymbol{x};\boldsymbol{y}) \tag{21.1}$$

$$Et_Q(\boldsymbol{k};t,\boldsymbol{x}) = \int d\boldsymbol{y}\, Et(\boldsymbol{k};t,\boldsymbol{x},\boldsymbol{y}) = Ex_Q(\boldsymbol{k};t,\boldsymbol{x})Q(\boldsymbol{k};t,\boldsymbol{x}) \tag{21.2}$$

$$Et_C(\boldsymbol{k};t,\boldsymbol{x}) = \int d\boldsymbol{y}\, Et(\boldsymbol{k};t,\boldsymbol{x},\boldsymbol{y}) = Ex_C(\boldsymbol{k};t,\boldsymbol{x})C(\boldsymbol{k};t,\boldsymbol{x}) \quad (21.3)$$

$$Q(\boldsymbol{k};t) = \int d\boldsymbol{x}d\boldsymbol{y}\, Q(\boldsymbol{k};t,\boldsymbol{x};\boldsymbol{y})\,;\; C(\boldsymbol{k};t) = \int d\boldsymbol{x}d\boldsymbol{y}\, C(\boldsymbol{k};t,\boldsymbol{x};\boldsymbol{y}) \quad (21.4)$$

$$Et_Q(\boldsymbol{k};t) = \int d\boldsymbol{x}d\boldsymbol{y}\, Et(\boldsymbol{k};t,\boldsymbol{x},\boldsymbol{y}) = Ex_Q(\boldsymbol{k};t)Q(\boldsymbol{k};t) \quad (21.5)$$

$$Et_C(\boldsymbol{k};t) = \int d\boldsymbol{x}d\boldsymbol{y}\, Et(\boldsymbol{k};t,\boldsymbol{x},\boldsymbol{y}) = Ex_C(\boldsymbol{k};t)C(\boldsymbol{k};t) \quad (21.6)$$

Relations (21.1) define amount $Q(\boldsymbol{k};t,\boldsymbol{x})$ of transactions (12.6) with economic variable E performed by sellers at $\boldsymbol{x}$ under their expectations with all buyers of entire economics. Functions $C(\boldsymbol{k};t,\boldsymbol{x})$ (21.1) define cost of sellers transactions of amount $Q(\boldsymbol{k};t,\boldsymbol{x})$ (21.1) with all buyers of the entire economics. Relations (21.2) define sellers expected transactions $Et_Q(\boldsymbol{k};t,\boldsymbol{x})$ of amount $Q(\boldsymbol{k};t,\boldsymbol{x})$ under sellers expectations $Ex_Q(\boldsymbol{k};t,\boldsymbol{x})$ with all buyers of entire economics. Relations (20.3) define sellers expected transactions $Et_C(\boldsymbol{k};t,\boldsymbol{x})$ of cost $C(\boldsymbol{k};t,\boldsymbol{x})$ under sellers expectations $Ex_C(\boldsymbol{k};t,\boldsymbol{x})$ with all buyers of entire economics. Relations (21.4) define volume $Q(\boldsymbol{k};t)$ of all transactions with economic variable E performed in economics under sellers expectations $Ex_Q(\boldsymbol{k};t)$ (20.5) of type $\boldsymbol{k}$ =(k,l). Relations (21.4) define cost $C(\boldsymbol{k};t)$ of all transactions with economic variable E performed in economics under sellers expectations $Ex_C(\boldsymbol{k};t)$ (21.6). Thus starting with definitions of sellers expected transactions (20.1-20.3) and definitions of $Q(\boldsymbol{k};t,z)$ (19.5) and $C(\boldsymbol{k};t,z)$ (19.6) we derive reasonable definitions of macro sellers expectations of volume $Ex_Q(\boldsymbol{k};t)$ (21.5) and cost $Ex_C(\boldsymbol{k};t)$ (21.6) for transactions with economic variable E. Relations similar to (21.1-21.6) are valid for buyers volume $Q(t,z;\boldsymbol{l})$ cost $C(t,z;\boldsymbol{l})$ and buyers expectations $Ex_Q(t;\boldsymbol{l})$ and $Ex_C(t;\boldsymbol{l})$. Let's outline that expectations of type $\boldsymbol{k}$ =(k,l) play different role for transactions with different economic variables E. That makes observations, measurements and description of economic and financial expectations a really complex problem.

Now let's explain and describe how expected transactions and expectations can flow on economic domain (12.3; 12.4) alike to flows of variables (6-10) and transactions (14.1-15.9). For brevity let's take flows of amounts $Q_{ij}(t,z)$ (12.1) of sellers transactions $\boldsymbol{bs}_{ij}(t,z)$ (12.1) only. Flows of sellers cost $C_{ij}(t,z)$ (12.1) and flows of buyers expected transactions are determined in the same way. Motion of agents i and j at points $\boldsymbol{x}$ and $\boldsymbol{y}$ with velocities $\boldsymbol{\upsilon}_i(t,\boldsymbol{x})$ and $\boldsymbol{\upsilon}_j(t,\boldsymbol{y})$ (14.3; 14.4) due to change of their risk ratings induce flows $\boldsymbol{p}_{Qij}(\boldsymbol{k};t,z)$ of sellers expected transactions $et_{Qij}(\boldsymbol{k};t,z)$ (19.3) alike to flows $\boldsymbol{p}_{Qijx}(t,z)$ (14.3) of amount $Q_{ij}(t,z)$ (12.1) of transactions $\boldsymbol{bs}_{ij}(t,z)$ (12.1) as:

$$\boldsymbol{pe}_{Qij}(\boldsymbol{k};t,\boldsymbol{z}) = et_{Qij}(\boldsymbol{k};t,\boldsymbol{z})\boldsymbol{\upsilon}_i(\boldsymbol{x}) = ex_i(\boldsymbol{k};t,\boldsymbol{x})Q_{i,j}(\boldsymbol{k};t,\boldsymbol{z})\boldsymbol{\upsilon}_i(\boldsymbol{x}) \quad (22.1)$$

Functions $\boldsymbol{pe}_{Qij}(\boldsymbol{k};t,z)$ (22.1) describe flows of expected transactions $et_{Qij}(\boldsymbol{k};t,z)$, $\boldsymbol{k}$=(k,l) carried by agent i with velocity $\boldsymbol{\upsilon}_i$. To define aggregate flows of sellers expected transactions let's collect flows $\boldsymbol{pe}_{Qij}(\boldsymbol{k};t,z)$ of expected transactions $et_{Qij}(\boldsymbol{k};t,z)$ (22.1) of agents i in a unit $dV(t,z)$ (12.5) and then average the sum during time term Δ similar to (12.6-12.9) as:

$$\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(x);j\in dV(y)} {}_{\Delta}\, et_{Qij}(\boldsymbol{k};t,\boldsymbol{z})\boldsymbol{v}_i(\boldsymbol{x}) \quad (22.2)$$

Sellers move along axes X (12.3; 12.4) and hence let's note flow of sellers expected transactions as $\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t,z)$. Let's note buyers flows as $\boldsymbol{Pe}_{Qy}(t,z;\boldsymbol{l})$.

$$\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t,\boldsymbol{z}) = Et_Q(\boldsymbol{k};t,\boldsymbol{z})\,\boldsymbol{ve}_{Qx}(\boldsymbol{k};t,\boldsymbol{z}) = Ex_Q(\boldsymbol{k};t,\boldsymbol{z})Q(\boldsymbol{k};t,\boldsymbol{z})\boldsymbol{ve}_{Qx}(\boldsymbol{k};t,\boldsymbol{z}) \quad (22.3)$$

Relations (22.2-3) define aggregated flows $\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t,z)$ and velocities $\boldsymbol{ve}_{Qx}(\boldsymbol{k};t,z)$ of sellers expected transactions $Ex_Q(\boldsymbol{k};t,z)$ of type $\boldsymbol{k}=(k,l)$. Similar to definitions of flows of variables (6-9) and flows of transactions (14.1 -15.9) integrals of (22.2-3) by $dz=\boldsymbol{dxdy}$ over domain (12.3; 12.4) define macro flows $\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t)$ and macroeconomic velocities $\boldsymbol{ve}_{Qx}(\boldsymbol{k};t)$ of expected transactions $Et_{Qx}(\boldsymbol{k};t)$ and expectations $Ex_{Qx}(\boldsymbol{k};t)$ (21.5) as:

$$\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t) = \int d\boldsymbol{z}\,\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t,\boldsymbol{z}) \quad (22.4)$$

$$\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t) = Et_{Qx}(\boldsymbol{k};t)\,\boldsymbol{ve}_{Qx}(\boldsymbol{k};t) = Ex_{Qx}(\boldsymbol{k};t)Q(\boldsymbol{k};t)\boldsymbol{ve}_{Qx}(\boldsymbol{k};t) \quad (22.5)$$

Relations (22.4-5) define sellers macro flows of $\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t)$ and velocities $\boldsymbol{ve}_{Qx}(\boldsymbol{k};t)$ of expected transaction $Et_{Qx}(\boldsymbol{k};t)$ and describe motion of expectations $Ex_{Qx}(\boldsymbol{k};t)$ of variable E. Borders of economic domain (12.3;12.4) reduce motion along risk axes and hence values and direction of flows $\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t)$ and velocities $\boldsymbol{ve}_{Qx}(\boldsymbol{k};t)$ should fluctuate. That induce time oscillations of expectations $Ex_{Qx}(\boldsymbol{k};t)$. Such fluctuations of expectations $Ex_{Qx}(\boldsymbol{k};t)$ should correlate with fluctuations of volume $Q(\boldsymbol{k};t)$ (21.4) and $Q(t)$ (13.7) of transactions induced by oscillations of flows $\boldsymbol{P}(t)$ (16.2) and velocities $\boldsymbol{v}(t)$ (16.4). We propose that fluctuations of macro variables, transactions and expectations induced by oscillations of their flows and velocities due to borders of economic domain (12.3; 12.4) should be treated as business cycles. We present business cycles models in Olkhov (2017c; 2019).

Let's underline that velocities of $\boldsymbol{v}_x(t)$ of sellers and velocities $\boldsymbol{v}_y(t)$ of buyers (15.9) differ from velocities $\boldsymbol{ve}_{Qx}(\boldsymbol{k};t)$ of sellers expectations $Ex_{Qx}(\boldsymbol{k};t)$ and velocities $\boldsymbol{ve}_{Qy}(t;\boldsymbol{l})$ of buyers expectations $Ex_{Qy}(t;\boldsymbol{l})$. Flows and velocities of expectations for different $\boldsymbol{k}=(k,l)$ are also different. Flows of different variables E, transactions and expectations have different velocities and their mutual interactions on economic domain (12.3; 12.4) reflect extreme complexity of real financial and economic processes.

Definitions of sellers expected transactions $Et_{Qx}(\boldsymbol{k};t,z)$ (20.2), their flows $\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t,z)$ and velocities $\boldsymbol{v}_{Qx}(\boldsymbol{k};t,z)$ (22.2-3) allow take equations on expected transactions and their flows similar to equations on transactions and their flows (17.4; 17.5) as:

$$\frac{\partial}{\partial t}Et_{Qx}(\boldsymbol{k};t,\boldsymbol{z}) + \nabla\cdot\left(Et_{Qx}(\boldsymbol{k};t,\boldsymbol{z})\,\boldsymbol{ve}_{Qx}(\boldsymbol{k};t,\boldsymbol{z})\right) = W_{Qx}(\boldsymbol{k};t,\boldsymbol{z}) \quad (22.6)$$

$$\frac{\partial}{\partial t}\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t,\boldsymbol{z}) + \nabla\cdot\left(\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t,\boldsymbol{z})\,\boldsymbol{ve}_{Qx}(\boldsymbol{k};t,\boldsymbol{z})\right) = \boldsymbol{R}_{Qx}(\boldsymbol{k};t,\boldsymbol{z}) \quad (22.7)$$

Functions W_{Qx}, W_{Qy} and $\boldsymbol{R}_{Qx}$, $\boldsymbol{R}_{Qy}$ in equations (22.6-7) describe action of economic and financial variables, transactions and different expectations, technology, political and other factors that may impact change of expected transactions $Et_{Qx}(\boldsymbol{k};t,z)$ flows $\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t,z)$. Equations on sellers and buyers expected transactions that determine cost $C(t,z)$ (12.10) of transactions (12.8) follow equations similar to (22.6-7).

Similar to (18.-2) equations on $Et_{Qx}(\boldsymbol{k};t)$ (21.5) and flows $\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t)$ (22.4) take form:

$$\frac{d}{dt}Et_{Qx}(\boldsymbol{k};t) = W_{Qx}(\boldsymbol{k};t) = \int d\boldsymbol{z}\, W_{Qx}(\boldsymbol{k};t,\boldsymbol{z}) \quad (22.8)$$

$$\frac{d}{dt}\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t) = \boldsymbol{R}_{Qx}(\boldsymbol{k};t) = \int d\boldsymbol{z}\, \boldsymbol{R}_{Qx}(\boldsymbol{k};t,\boldsymbol{z}) \quad (22.9)$$

Equations (11.2-3) on economic or financial variables $A(t,\boldsymbol{x})$ and their flows $\boldsymbol{P}_A(t,\boldsymbol{x})$, equations (17.4-5) on volume $Q(t,z)$ (12.9) of transactions $BS(t,z)$ (12.8) and transactions flows $\boldsymbol{P}_Q(t,z)$ (15.2-3) and equations (22.6-7) on expected transaction $Et_{Qx}(\boldsymbol{k};t,z)$ and flows $\boldsymbol{Pe}_{Qx}(\boldsymbol{k};t,z)$ complete our approximation of macro financial or economic system based on description of relations between variables, transactions and expectations on economic domain (12.3-4). It is obvious that description of any particular problem requires definition of right hand side factors of equations (11.2-3), (17.4-5), (22.6-7). All specifics and details of economic and financial processes are hidden in and are determined by function $F_A(t,\boldsymbol{x})$ and $\boldsymbol{G}_A(t,\boldsymbol{x})$, $F(t,z)$ and $\boldsymbol{G}(t,z)$, W_{Qx}, W_{Qy} and $\boldsymbol{R}_{Qx}$, $\boldsymbol{R}_{Qy}$. We apply our methods and equations to describe wave propagation of small disturbances of variables (Olkhov, 2016a-2017a), wave propagation of disturbances of transactions (Olkhov, 2018a) and surface waves (Olkhov, 2017b). Our methods permit model business cycles (Olkhov, 2017c; 2019), describe hidden complexities of classical Black-Scholes option pricing model (Olkhov, 2016a;b) and propose Lorentz attractor as possible origin of random behavior of price fluctuations (Olkhov, 2018b). In the next section we describe how perturbations of transactions may define statistics of price disturbances and discuss why it should depend on statistics of volume disturbances.

5. Asset Pricing and Return

Asset pricing is one of the most important problems of macro finance. We refer (Cochrane and Hansen, 1992; Cochrane and Culp, 2003; Hansen, 2013; Campbell, 2014; Fama, 2014; Cochrane, 2017) as only small part of asset pricing research. Let's study how economic equations on variables, transactions, expectations and their flows can govern asset prices, returns and their fluctuations.

Equations (17.4; 17.5) describe volume $Q(t,z)$ (12.9) of transactions $\boldsymbol{BS}(t,z)$ (12.8) with economic variable E and similar equations model cost $C(t,z)$ (12.10) of transactions $BS(t,z)$. As variable E let's take any particular assets and study how equations (17.4; 17.5) define relations on price and price fluctuations. Let's outline that different aggregations of volume and cost of transactions define different prices. For example relations (12.11) define price of transactions with volume $Q(t,z)$ and cost $C(t,z)$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$, $z=(\boldsymbol{x},\boldsymbol{y})$ and (13.3) define price $p_S(t,\boldsymbol{x})$ of sellers for transactions with variable E from $\boldsymbol{x}$. Transactions of quantity $Q(\boldsymbol{k};t,z)$ (19.5) with cost $C(\boldsymbol{k};t,z)$ (19.6) performed under sellers expectations of type $\boldsymbol{k}=(k,l)$ determine price $p(\boldsymbol{k};t,z)$ of sellers transactions:

$$C(\boldsymbol{k};t,\boldsymbol{z}) = p(\boldsymbol{k};t,\boldsymbol{z})Q(\boldsymbol{k};t,\boldsymbol{z}) \quad ; \quad \boldsymbol{k}=(k,l) \tag{23.1}$$

Relations (13.6) define price $p(t)$ of all transactions with volume $Q(t)$ (13.6) with cost $C(t)$ (13.6) with selected assets E performed in economy at moment t. Price $p(\boldsymbol{k};t)$ of all transactions with volume $Q(\boldsymbol{k};t)$ with cost $C(\boldsymbol{k};t)$ (21.4) with selected assets E performed under expectations $\boldsymbol{k}=(k,l)$ in economy at moment t.

$$C(\boldsymbol{k};t) = p(\boldsymbol{k};t)Q(\boldsymbol{k};t) \quad ; \quad \boldsymbol{k}=(k,l) \tag{23.2}$$

Relations (12.11; 13.3; 13.6; 23.1-2) indicate that price of assets always should be treated in regard to definite aggregation of volume and cost of transactions. Equations on transactions and their flows define equations on prices. As simplest case let's take equations (18.1; 18.2) on volume $Q(t)$ and flow of volume $\boldsymbol{P}_Q(t)$ of transactions

$$\frac{d}{dt}Q(t) = F_Q(t) \quad ; \quad \frac{d}{dt}\boldsymbol{P}_Q(t) = \boldsymbol{G}_Q(t) \tag{23.3}$$

and similar equations on cost $C(t)$ and flow of volume $\boldsymbol{P}_C(t)$ of transactions

$$\frac{d}{dt}C(t) = F_C(t) \quad ; \quad \frac{d}{dt}\boldsymbol{P}_C(t) = \boldsymbol{G}_C(t) \tag{23.4}$$

Equations (23.3-4) define equations on price $p(t)$ (13.6) of all transactions made in economy at moment t with variable E

$$\frac{d}{dt}Q(t) = F_Q(t) \quad ; \quad Q(t)\frac{d}{dt}p(t) + p(t)F_Q(t) = F_C(t) \tag{23.5}$$

$$Q(t)\frac{d}{dt}\boldsymbol{v}_Q(t) + F_Q(t)\boldsymbol{v}_Q(t) = \boldsymbol{G}_Q(t) \;\; ; \;\; Q(t)p(t)\frac{d}{dt}\boldsymbol{v}_C(t) + \boldsymbol{v}_C(t)F_C(t) = \boldsymbol{G}_C(t) \tag{23.6}$$

$$\boldsymbol{P}_Q(t) = Q(t)\boldsymbol{v}_Q(t)\,; \;\; \boldsymbol{P}_C(t) = C(t)\boldsymbol{v}_C(t) = Q(t)p(t)\boldsymbol{v}_C(t) \tag{23.7}$$

Even simplest form of price equations (23.5-7) demonstrate that right hand factors hide main complexity of price dynamics and price may depend on flows $\boldsymbol{P}_Q(t)$, $\boldsymbol{P}_C(t)$ or velocities $\boldsymbol{v}_Q(t)$, $\boldsymbol{v}_C(t)$ of amount $Q(t)$ and cost $C(t)$ of transactions. Up now we don't know any research on possible impact of flows $\boldsymbol{P}_Q(t)$, $\boldsymbol{P}_C(t)$ and velocities $\boldsymbol{v}_Q(t)$, $\boldsymbol{v}_C(t)$ on price $p(t)$ evolution and fluctuations. Studies of this problem may be very important.

Let's neglect possible impact on flows and show how equations on volume and cost similar to (23.5) may model price fluctuations. Let's study equations similar to (23.5-7) that take into account $Q(\boldsymbol{k};t)$ and cost $C(\boldsymbol{k};t)$ (21.4) that define price $p(\boldsymbol{k};t)$ (23.2) for different expectations $\boldsymbol{k}=(k,l)$. Similar to (23.5-7) equations on $Q(\boldsymbol{k};t)$ and $C(\boldsymbol{k};t)$ (24.1) take form:

$$\frac{d}{dt}Q(\boldsymbol{k};t) = F_Q(\boldsymbol{k};t)\ ;\ \frac{d}{dt}C(\boldsymbol{k};t) = F_C(\boldsymbol{k};t) \tag{24.1}$$

Let's assume that functions $F_Q(\boldsymbol{k};t)$ and $F_C(\boldsymbol{k};t)$ in (24.1) depend on expected transactions $Et_Q(\boldsymbol{k};t)$ (21.5) and $Et_C(\boldsymbol{k};t)$ (21.6) that due to (22.6) and (18.1) follow equations similar to (24.1)

$$\frac{d}{dt}Et_Q(\boldsymbol{k};t) = Fe_Q(\boldsymbol{k};t)\ ;\ \frac{d}{dt}Et_C(\boldsymbol{k};t) = Fe_C(\boldsymbol{k};t) \tag{24.2}$$

Let's assume that functions $Fe_Q(\boldsymbol{k};t)$ and $Fe_C(\boldsymbol{k};t)$ in (24.2) depend on volume $Q(\boldsymbol{k};t)$ and cost $C(\boldsymbol{k};t)$ (24.1). Let's describe small dimensionless perturbations of volumes $Q(\boldsymbol{k};t)$ and cost $C(\boldsymbol{k};t)$ and expected transactions $Et_Q(\boldsymbol{k};t)$ and $Et_C(\boldsymbol{k};t)$:

$$Q(\boldsymbol{k};t) = Q_{0k}\big(1+q(\boldsymbol{k};t)\big)\ ;\ C(\boldsymbol{k};t) = C_{0k}\big(1+c(\boldsymbol{k};t)\big) \tag{24.3}$$

$$\boldsymbol{k} = (k,l);\ k,l = 1,..K$$

$$Et_Q(\boldsymbol{k};t) = Et_{Q0k}\left(1+et_q(\boldsymbol{k};t)\right);\ Et_C(\boldsymbol{k};t) = Et_{C0k}\big(1+et_c(\boldsymbol{k};t)\big) \tag{24.4}$$

and assume that mean values of Q_{0k}, C_{0k}, Et_{Q0k}, Et_{C0k} are slow to compare with small dimensionless disturbances $q(\boldsymbol{k};t)$, $c(\boldsymbol{k};t)$, $et_q(\boldsymbol{k};t)$, $et_c(\boldsymbol{k};t)$. Let's take same assumptions on functions in the right hand side of (24.1; 24.2).

$$F_Q(\boldsymbol{k};t) = F_{Q0k}(1+f_Q(\boldsymbol{k};t))\ ; F_C(\boldsymbol{k};t) = F_{C0k}(1+f_c(\boldsymbol{k};t\ )) \tag{24.5}$$

$$Fe_Q(\boldsymbol{k};t) = Fe_{Q0k}(1+fe_Q(\boldsymbol{k};t))\ ; Fe_{Cl}(\boldsymbol{k};t) = Fe_{C0k}(1+fe_c(\boldsymbol{k};t\ )) \tag{24.6}$$

Thus let's take mean values of Q_{0k}, C_{0l}, Et_{Q0k}, Et_{C0l} and F_{Q0k}, F_{C0k}, Et_{Q0k}, Et_{C0k} as constants and equations on disturbances take form:

$$Q_{0k}\ \frac{d}{dt}q(\boldsymbol{k};t) = F_{Q0k}f_q(\boldsymbol{k};t)\ ;\ C_{0l}\frac{d}{dt}c(\boldsymbol{k};t) = F_{Q0k}f_c(\boldsymbol{k};t) \tag{25.1}$$

$$Et_{Q0k}\ \frac{d}{dt}et_q(\boldsymbol{k};t) = Fe_{Q0k}fe_q(\boldsymbol{k};t)\ ;\ Et_{C0l}\ \frac{d}{dt}et_c(\boldsymbol{k};t) = Fe_{C0l}fe_c(\boldsymbol{k};t) \tag{25.2}$$

To consider (25.1; 25.2) as self-consistent equations let's take that disturbances $f_q(\boldsymbol{k};t)$ and $f_c(\boldsymbol{k};t)$ in (25.1) depend on disturbances $et_q(\boldsymbol{k};t)$ and $et_c(\boldsymbol{k};t)$ and $fe_q(\boldsymbol{k};t)$ and $fe_c(\boldsymbol{k};t)$ in (25.2) depend on disturbances of $q(\boldsymbol{k};t)$ and $c(\boldsymbol{k};t)$.

$$f_q(\boldsymbol{k};t) = a_{qk}et_q(\boldsymbol{k};t)\ \ ;\ \ \ f_c(\boldsymbol{k};t) = a_{ck}et_c(\boldsymbol{k};t) \tag{26.1}$$

$$fe_q(\boldsymbol{k};t) = be_{qk}q(\boldsymbol{k};t)\ \ ;\ \ \ fe_c(\boldsymbol{k};t) = be_{ck}c(\boldsymbol{k};t) \tag{26.2}$$

Due to (26.1; 26.2) equations (25.1-2) take form:

$$Q_{0k}\ \frac{d}{dt}q(\boldsymbol{k};t) = a_{qk}F_{Q0k}et_q(\boldsymbol{k};t)\ \ ;\ C_{0k}\frac{d}{dt}c(\boldsymbol{k};t) = a_{ck}F_{Q0k}\ et_c(\boldsymbol{k};t) \tag{26.3}$$

$$Et_{Q0k}\ \frac{d}{dt}et_q(\boldsymbol{k};t) = be_{qk}Fe_{Q0k}q(\boldsymbol{k};t)\ \ ;\ Et_{C0k}\frac{d}{dt}et_c(\boldsymbol{k};t) = be_{ck}Fe_{C0k}\ c(\boldsymbol{k};t) \tag{26.4}$$

For

$$\omega_{qk}^2 = -a_{qk} be_{qk} \frac{F_{Q0k}}{Q_{0k}} \frac{Fe_{Q0k}}{Et_{Q0k}} > 0 \;\; ; \;\; \omega_{ck}^2 = -\, a_{ck} be_{ck} \frac{F_{Q0k}}{C_{0k}} \frac{Fe_{C0k}}{Et_{C0k}} > 0 \qquad (27.1)$$

equations on disturbances *q(**k**;t), c(**k**;t), et_q(**k**;t), et_c(**k**;t)* take form of harmonic oscillators:

$$\left(\frac{d^2}{dt^2} + \omega_{qk}^2\right) q(\boldsymbol{k};t) = 0 \;\; ; \;\; \left(\frac{d^2}{dt^2} + \omega_{ck}^2\right) c(\boldsymbol{k};t) = 0 \qquad (27.2)$$

$$\boldsymbol{k} = (k,l);\; k,l = 1,..K$$

$$\left(\frac{d^2}{dt^2} + \omega_{qk}^2\right) et_q(\boldsymbol{k};t) = 0 \;\; ; \;\; \left(\frac{d^2}{dt^2} + \omega_{ck}^2\right) et_c(\boldsymbol{k};t) = 0 \qquad (27.3)$$

Simple solutions of (27.2) for dimensionless disturbances *q(**k**;t)* and *c(**k**;t)*

$$q(\boldsymbol{k};t) = g_{qk} sin\omega_{qk} t + \; d_{qk} cos\omega_{qk} t \quad g_{qk}, d_{qk} \ll 1 \qquad (27.4)$$

$$c(\boldsymbol{k};t) = g_{cl} sin\omega_{ck} t + d_{cl} cos\omega_{ck} t \;\; ; \;\; g_{ck}, d_{ck} \ll 1 \qquad (27.5)$$

Relations (27.4-5) describe simple harmonic fluctuations of disturbances *q(**k**;t)* of volume *Q(**k**;t)* and disturbance *c(**k**;t)* of cost *C(**k**;t)* of transactions under expectations ***k*** =(*k,l*).

Now let's study disturbances of cost *C(t)*, volume *Q(t)* and price *p(t)* for (13.6) as:

$$Q(t) = \; \sum_{k,l} Q_{0kl}\big(1 + q(\boldsymbol{k};t)\big) = Q_0 \sum_{k,l} \lambda_{kl}\big(1 + q(\boldsymbol{k};t)\big) \qquad (28.1)$$

$$C(t) = \; \sum_{k,l} C_{0kl}\big(1 + c(\boldsymbol{k};t)\big) = C_0 \sum_{k,l} \mu_{kl}\big(1 + c(\boldsymbol{k};t)\big) \qquad (28.2)$$

Relations (28.1) describe impact of dimensionless disturbances *q(**k**;t)* on volume *Q(t)* and (28.2) describe impact of dimensionless disturbances *c(**k**;t)* on cost *C(t),* ***k****=(k,l)*:

$$Q_0 = \sum_{k,l} Q_{0kl} \;\; ; \;\; \lambda_{kl} = \frac{Q_{0kl}}{Q_0} \;\; ; \; C_0 = \sum_{k,l} C_{0kl} \;\; ; \;\; \mu_{kl} = \frac{C_{0kl}}{C_0} \qquad (28.3)$$

$$\sum \lambda_{kl} = \; \sum \mu_{kl} = 1 \;\; ; \qquad (28.4)$$

Relations (13.6) define price *p(t)* for transactions with volume *Q(t)* and cost *C(t)*:

$$p(t) = \frac{C(t)}{Q(t)} = \frac{\sum_{k,l} C(k,l;t)}{\sum_{k,l} Q(k,l;t)} \quad ; \quad p_0 = \frac{C_0}{Q_0} = \frac{\sum_{k,l} C_{0kl}}{\sum_{k,l} Q_{0kl}} \qquad (28.5)$$

In linear approximation by disturbances *q(k,l;t)* and *c(k,l;t)* price *p(t)* (28.5) take form:

$$p(t) = \frac{C(t)}{Q(t)} = \frac{C_0 \sum_{k,l} \mu_{kl}\big(1 + c(k,l;t)\big)}{Q_0 \sum_{k,l} \lambda_{kl}\big(1 + q(k,l;t)\big)} = p_0 \left[1 + \sum\nolimits_{k,l} \mu_{kl} c(k,l;t) - \sum\nolimits_{k,l} \lambda_{kl} q(k,l;t)\right]$$

$$p(t) = p_0[1 + \pi(t)] = p_0\big[1 + \sum_{k,l}(\mu_{kl} c(k,l;t) - \lambda_{kl} q(k,l;t))\big] \qquad (28.6)$$

Dimensionless fluctuations of price *π(t)* (28.6) equals weighted sum of disturbances *q(k,l;t)* and *c(k,l;t)* as (28.7):

$$\pi(t) = \sum_{k,l} \mu_{kl}\, c(k,l;t) - \lambda_{kl} q(k,l;t) \qquad (28.7)$$

Now let's take (23.2; 24.3) and present *π(t)* in other form:

$$C(k,l;t) = C_{0kl}[1 + c(k,l;t)] = p_{0kl}[1 + \pi(k,l;t)] Q_{0kl}[1 + q(k,l;t)] \qquad (29.1)$$

From (28.6-7) and (29.1) in linear approximation by *c(k,l;t), π(k,l;t)* and *q(k,l;t)* obtain:

$$C_{0kl} = p_{0kl} Q_{0kl} \quad ; \quad c(k,l;t) = \pi(k,l;t) + q(k,l;t) \qquad (29.2)$$

Let's substitute (29.2) into (28.7):

$$\pi(t) = \sum_{k,l} \mu_{kl}\pi(k,l;t) + \sum_{k,l}(\mu_{kl} - \lambda_{kl})\, q(k,l;t) \tag{29.3}$$

Relations (29.3) describe price perturbations $\pi(t)$ as weighted sum of partial price disturbances $\pi(k,l;t)$ and volume disturbances $q(k,l;t)$. Thus statistics of price disturbances $\pi(t)$ is defined by statistics of partial price disturbances $\pi(k,l;t)$ and statistics of volume disturbances $q_k(k,l;t)$.

Relations between disturbances of quantity and cost of transactions on one hand and disturbances of expectations that approve these transactions may be the source of random evolution. Random behavior of disturbances of quantity and cost of transactions induce random motion of price disturbances $\pi(t)$ and in (Olkhov, 2018b) we present model of Lorentz attractor as a possible factor that cause random behavior of price.

Return perturbations. Price disturbances (29.3) cause perturbations of return $r(t,d)$:

$$r(t,d) = \frac{p(t)}{p(t-d)} - 1 \tag{30.1}$$

Let's introduce partial returns $r(\boldsymbol{k};t,d)$ for price $p(\boldsymbol{k};t)$ (23.2) and "returns" $w(\boldsymbol{k};t,d)$ for volumes $Q(\boldsymbol{k};t)$ (24.3), $\boldsymbol{k}=(k,l)$:

$$r(\boldsymbol{k};t,d) = r(k,l;t,d) = \frac{p(k,l;t)}{p(k,l;t-d)} - 1 \ ; \ w(\boldsymbol{k};t,d) = w(k,l;t,d) = \frac{Q(k,l;t)}{Q(k,l;t-d)} - 1 \tag{30.2}$$

Let's assume for simplicity that mean price p_{0kl} and Q_{0kl} (29.2) are constant during time term d and present (30.1, 30.2) as

$$r(t,d) = \frac{\pi(t)-\pi(t-d)}{1+\pi(t-d)} \ ; \ w(k,l;t,d) = \frac{q(k,l;t)-q(k,l;t-d)}{1+q(k,l;t-d)} \tag{30.3}$$

$$r(t,d) = \sum \mu_{kl}\frac{1+\pi(k,l;t-d)}{1+\pi(t-d)} r(k,l;t,d) + \sum(\mu_{kl} - \lambda_{kl})\frac{1+q(k,l;t-d)}{1+\pi(t-d)} w(k,l;t,d) \tag{30.4}$$

Let's define

$$\varepsilon_{kl}(t-d) = \mu_{kl}\frac{1+\pi(k,l;t-d)}{1+\pi(t-d)} \ ; \ \eta_{kl}(t-d) = (\mu_{kl} - \lambda_{kl})\frac{1+q(k,l;t-d)}{1+\pi(t-d)} \tag{30.5}$$

$$\sum_{k,l}[\varepsilon_{kl}(t-d) + \eta_{kl}(t-d)] = 1 \tag{30.6}$$

$$r(t,d) = \sum_{k,l} \varepsilon_{kl}(t-d) r(k,l;t,d) + \sum_{k,l} \eta_{kl}(t-d) w(k,l;t,d) \tag{30.7}$$

Relations (30.6-7) describe return (30.1) as sum of partial returns and volume "returns" $w(k,l;t,d)$ (30.2-3). Sum for coefficients μ_{kl} and μ_{kl}-λ_{kl} for price $p(t)$ (28.6), $\pi(t)$ (29.3) and $\varepsilon_{kl}(t)$ and $\eta_{kl}(t)$ for return $r(t,d)$ (30.1) equals unit but (29.3) and (30.7) can't be treated as averaging procedure as some coefficients μ_{kl}-λ_{kl} and $\eta_{kl}(t)$ should be negative. If mean price (29.2) $p_{0kl}=p_0$ for all pairs of expectations (k,l) then from (28.6, 28.7) obtain

$$p_{0kl} = p_0 = const \ \rightarrow \ \lambda_{kl} = \mu_{kl} \ ; \ \eta_{kl}(t) = 0 \text{ for all } k,l \tag{30.8}$$

and relations (29.3; 30.7) take simple form

$$\pi(t) = \sum_{k,l} \mu_{kl}\, \pi(k,l;t) \tag{30.9}$$

$$r(t,d) = \sum_{k,l} \mu_{kl} \frac{1+\pi(k,l;t-d)}{1+\pi(t-d)} r(k,l;t,d) = \sum_{k,l} \mu_{kl} \frac{\pi(k,l;t)-\pi(k,l;t-d)}{1+\pi(t-d)} \tag{30.10}$$

Thus assumption (30.8) on prices (29.2) for all pairs of expectations (k,l) cause representation (30.9, 30.10) of price disturbances $\pi(t)$ as sum of partial price disturbances $\pi(k,l;t)$ weighted by μ_{kl} (28.3) for different pairs of expectations (k,l). If coefficients μ_{kl} (28.3) are random then their statistics impact statistic properties of price disturbances $\pi(t)$. If (30.8) is not valid then price disturbances $\pi(t)$ should take form (29.3) and depend on partial disturbances $\pi(k,l;t)$, volume perturbations $q(k,l;t)$ and statistics of λ_{kl} and μ_{kl} (28.3). Assumption (30.8) cause returns as (30.10), otherwise returns take (30.7). Actually expectations are key factors for market competition and different expectations (k,l) should cause different mean partial prices p_{0kl}. That produce complex representation of price (29.3) and return (30.7) disturbances as well as impact volatility and statistic distributions of price and return disturbances.

6. Conclusions

Development of methods of economic and financial theory is an endless problem. Above we present only beginnings of the theory framework. Let's resume main issues of our approach. We model macroeconomic system by three elements – variables, transactions and expectations of economic agents. We distribute agents by their risk ratings as coordinates on economic domain and describe macro variables, transactions and expectations as density functions of risks. We regard risks as main drivers of economic and financial evolution and consider economic activity of agents as the main source of risks. Any financial or economic activity is related with risks and we propose that risk-free models have nothing common with reality. We show that changes of risk ratings of agents due to any reasons induce economic flows of variables, transactions and expectations and these flows produce significant impact on evolution of macroeconomic system. Flows of variables, transactions and expectations double number of properties that define evolution of macro economy.

Different variables like demand and supply have different flows and that cause permanent perturbations of supply and demand. That makes existence of any market supply-demand equilibrium very doubtful. Current discussion on "Rebuilding macroeconomic theory" (Vines and Wills, 2018a) doesn't study this important issue and we assume that further research on applicability of GE concept to economic modeling is required.

Financial variables as functions of risks $\boldsymbol{x}$ describe state of macroeconomic system. Transactions as functions of risks $z=(\boldsymbol{x},\boldsymbol{y})$ describe dynamics of economic variables and thus describe evolution of macro economy. Expectations are most "etheric" economic substance that impact agents to perform transactions. Numerous expectations are determined by variables, transactions, expectations of other agents, by economic regulation, technology or climate forecasts and etc. These factors project impact of environment on economic and financial processes and add most complexity to macro modeling. Different expectations cause different impact on various transactions. Economic or financial value of expectations, their importance and influence on macroeconomic evolution should be measured proportionally to amount of transactions performed under these expectations. That makes description of macroeconomic dynamics absolutely exciting problem.

We apply our methods to description of asset pricing and return fluctuations. Statistical properties of asset pricing and return fluctuations should be studied with respect to description of corresponding transactions. That conclusion redirects studies of price statistics to studies of relations between transactions and expectations.

Many problems should be studied further. Econometric problems and observation of economic and financial variables, transactions and expectations of agents and agents risk assessment are among the central. We hope that our methods may help for better description of economic and financial processes.

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