A Short Note on the Infinite Decision Puzzle

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This work draws on the Supertask literature[1] in order to better understand the conceptual and physical possibility of an infinite decision puzzle presented by Barrett and Arntzenius (1999, 2002). The first section presents the puzzle and two possible objections documented in the literature. The next section argues that cardinality and tracking considerations play a key role in understanding the puzzle. The work concludes with a discussion about some implications for the decision theory.

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[1] Benacerraf 1962, Black 1951, Laraudogoitia 1996, 1999 and Silagadze 2005.

1. The Puzzle

Barrett and Arntzenius (1999, 2002) present an economy populated by two players: a Bank and an agent. At time t=0, the first one has an infinite stack of \$1 bills numbered following the natural numbers (.i.e. 1, 2, 3….). Meanwhile the agent has no wealth.

In this context, the bank offers two possible alternative exchange proposals to the agent:

a. Receive the bill that is in the top of stack in any exchange,

or

b. receive the top 3 bills and return to the bank that one holding the small serial number in any exchange.

Assuming that there is only one exchange between the bank and the agent; a rational individual always chooses option (b) netting \$2 (bigger than option (a) netting \$1). Moreover, if the exchange takes place *n* times the payoff of each option are:

$$a.\ \sum_{k=1}^{n} \$1 = \$n \qquad < \qquad b.\ \sum_{k=1}^{n} \$2 = \$2n.$$

Hence, as far as n is bounded, a rational agent always prefers option (b).

1.1. An Infinite Problem

On the contrary, if infinite exchanges are allowed, the setup of the problem and so the decision faced by the agent changes dramatically. Assuming that the time sequence is $t = 0, 1/2, 3/4, 7/8, \ldots$ i.e. $\forall k / k \in \mathbb{N} : t = \frac{2^k - 1}{2^k}$, Now the bank and the agent are involved in a Supertask[2].

In this setup, each sequential task is an exchange of bills between the bank and the agent. Which is the agent's best strategy in this infinite setup? The authors note that if she chooses option *b,* she will end up with nothing at $t=1$: every bill numbered *k* in her hands will be back to the bank at instant $t=1-1/2^K$. On the contrary, if the agent chooses (a), the entire bank stack will end in her hands!

Hence, in the infinite setup, the more profitable strategy is (a).

While in finite time the agent always chooses (b), in an infinite setup it is optimal to choose (a). Is it a paradox? There are two possible objections regarding the existence of this infinite decision puzzle already mentioned in the recent literature.

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[2] See Laraudogoitia (1999) for detailed presentation.

1.2. Two Objections

Machina (2000) argues that because the end state (t=1) is not well defined there is no conceptual possibility for the proposed supertask. However, the supertask actions are taken place for t<1, at t=1 there is no remain actions (Benacerraf 1962). The t-series includes all the instants where actions takes place (t<1). The supertask rules for every t<1, therefore, the state of the world at t=1 could not be a logical consequence of the previous t-series.

Pulier (2000) states that the infinite sum is not well defined, so he argues that the proposed supertask is no possible. Nevertheless, Benacerraf (1962) clearly states that the properties shared by the terms of a succession could not be valid when $n \to \infty$.

Barrett and Arntzenius (2002) defend their position noting that the puzzle proposed has no relation with the summation of the wealth of the agent; on the contrary it relates to the *individual bills* hold by him. They build a problem where one could track the physical trajectory of every bill from the bank to the agent and it return to the bank. The trajectories are always defined and can establish the final situation of the agent at time *t=1*. This is a coherent representation of the problem and has a mathematical structure. So the puzzle remains valid.

The argument presented by Barrett and Arntzenius is an application of the Benacerraf's Critique (1962). He postulates that the arguments about the end state and the properties of the partial sum do not demonstrate the logical impossibility of supertasks. He based his argument on two main ideas (Pérez Laraudogoitia, 2004):

> (a) The state of the world at an instant immediately after the end of a supertask does not necessary follow previous states during the supertask.
> (b) The "finite" properties shared by the terms of a succession could not be valid for the series when the number of terms tends to infinite.

Summing up, the two arguments against the proposed supertask by Machina and Pulier could be refuted using Benacerraf (1962).

2. Cardinality and tracking considerations

Scott and Scott (2005) note that the infinite decision puzzle presented above rest on the mathematical concept of equi-cardinality and the physical possibility of tracking bills. Equi-cardinality between the bills returned to the bank and the natural numbers; and the physical possibility of tracking the bills movement in an Euclidean space following classical mechanics.

2.1. Cardinality

Scott and Scott (2005) postulate that mathematical cardinality and the possibility of tracking the bills anytime are crucial in the correct understanding of the problem. Assuming the same infinite timing as before, one could interpret option (b) as one in where there are two action sequences at any stage:

1. The agent receives from the bank the top 3 bills at any time t. Define A^k as the set of bills hold by the agent at stage *k*.
2. The agent returns the bill holding the small serial number in any exchange. Define B^k as the set of bills hold by the bank at stage *k*.

There is a bijection between these two sets, i.e. $A^K \to B^K / f(k) = 3k$, so they have the same *cardinality*.

2.2. Physical possibility

However, this is only a necessary condition for two infinite sets to be equal; having the same cardinality does not mean “identical”.

Assuming that the previous justification convinces the reader about the conceptual possibility of the proposed supertask, let’s now analyse its potential physical possibility.

Following Grünbaum (1970) and Pérez Laraudogoitia (2004), a supertask is kinematically possible if the following two conditions are satisfied:

1. All the bills travel at bounded speed
2. $\forall t : 0 \le t < 1$, the position of all the moving bills approach a defined limit.

Richard Sainsbury (1988) suggests that the continuity principle could solve some infinite puzzles (e.g. supertasks). The principle states that *the trajectories of the moving bodies are continuous lines.* Formally, for every body *i*:

$$\forall t : \lim_{u \to t} \overline{X}_i(u) = \overline{X}_i(t)$$

where $\overline{X}_i$ is the position of *i*.

Please note that there is no final exchange in our experiment. Regarding to physical world assumed here, the bills are bodies in Euclidean space. Moreover, the position is a function continuous in time. Formally, the space is a plane *(x,y)* and the position is defined by a function $f(x, y, t)$ continuous in *x, y* and *t*.

The second condition for equality in an infinite framework proposed by Scott and Scott (2005) is that it must be possible to *track* each bill movement from A to B (i.e. the existence of a physical trajectory). Formally, this is done by noting that every bill with serial number “k” will be returned to the bank at $t = 1 - \frac{1}{2^k}$.

Hence, at the end of the supertask the two sets are equals (A=B), and if the agent chooses option (b) he will end with no bills.

In a different physical setup, tracking could not be possible. For example, in a world that follows quantum mechanics laws, it is no possible to track each bill movement. One could only talk about fall or increase of the number of bills, with no individualisation. In this physical world, only cardinality apply so the puzzle remains because it is possible to have infinitely bills at t=1.

3. Implications for the decision theory

Barrett and Arntzenius (2002) describe a similar puzzle where the wealth of the agent is finite. They assume that each additional bill has positive but decreasing marginal utility, so the total wealth in any case is a finite amount. This example clearly shows that the puzzle is not destroy by supporting the idea that there is no infinite wealth in the real world, on the contrary, it is based on infinite but identifiable trajectories of transaction of goods. On the other hand, the authors accept that the puzzle requires an agent confronting an infinite sequence of decisions.

What about the energy needed for moving the bills? In the original version the authors assume infinite. However, the also describes an alternative puzzle where the total energy is upper bounded: each bill, in any subsequent transaction is left closer to the frontier between the bank and the agent, so the total energy used remains finite. This example assumes that there is no minimum energy required for a single transaction. (Barrett and Arntzenius 2002)

Conclusion

Barrett and Arntzenius (1999, 2002) proposed a puzzle where one-step rationality contradicts the global rationality. Cardinality and tractability considerations give a proper account of the puzzle. Moreover, in the context of business decision problems (problems that include physical bills), there is no possibility of separate the logical proposal from the physical possibility of the supertask (Scott and Scott 2005).

References

Allis, V. and Koetsier, T., 1991, 'On Some Paradoxes of the Infinite', British Journal for the Philosophy of Science, 42, pp. 187-194

Allis, V. and Koetsier, T., 1995, 'On Some Paradoxes of the Infinite II', British Journal for the Philosophy of Science, 46, pp. 235-247

Barrett, J. A. and Arntzenius, F. (1999) An Infinite Decision Puzzle. *Theory and Decision,* 46 (1)**,** 101-103.

Barrett, J. A. and Arntzenius, F. (2002) Why the Infinite Decision Puzzle Is Puzzling. *Theory and Decision,* 52 (2)**,** 139-147.

Benacerraf, P. (1962) Tasks, Super-Tasks, and the Modern Eleatics. *The Journal of Philosophy,* 59 (24)**,** 765-784.

Black, M. (1951) 'Achilles and the Tortoise'. *Analysis,* 11 (5)**,** 91-101.

Earman, J., and Norton, J. (1996) 'Infinite Pains: The Trouble with Supertasks', in Benacerraf and His Critics, A. Morton and S. Stich (eds.), Oxford: Blackwell, pp. 231-261.

Grünbaum, A. (1970) 'Modern Science and Zeno's Paradoxes of Motion', in Salmon [1970], pp. 200-250

Pérez Laraudogoitia, J. (1996) A Beautiful Supertask. *Mind,* 105 (417)**,** 81-83.

Pérez Laraudogoitia, J. (2004) "Supertasks", *The Stanford Encyclopedia of Philosophy* (Winter 2004 Edition), Edward N. Zalta (ed.), URL = http://plato.stanford.edu/archives/win2004/entries/spacetime-supertasks/>.

Sainsbury, R. (1995) *Paradoxes*, Cambridge: Cambridge University Press.2nd edition.

Scott, M. and Scott, A. (2005) Infinite Exchange Problems. Theory and Decision, 57 (4), 397-406.

Silagadze, Z. K. (2005) Zeno Meets Modern Science. *Acta Physica Polonica B,* 36 (10)**,** 2887.