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Abstract

We examine the effects of a fiscal policy which realizes full-employment from a state of under-employment (a sate with deflationary GDP gap). We show that the larger the growth rate of real GDP or the government expenditure by a fiscal policy over the ordinary growth rate is, the smaller the debt-to-GDP ratio at the time when full-employment is realized is, and an aggressive fiscal policy for full-employment can reduce the debt-to-GDP ratio. Therefore, full-employment can be realized by a fiscal policy with smaller debt-to-GDP ratio than before the fiscal policy. An increase in the government expenditure may induce a rise of the interest rate. Since the higher the interest rate is, the larger the debt-to-GDP ratio is, we need an appropriate monetary policy which maintains the low interest rate. Also we show that the condition about propensity to consume for realization of full-employment within one year from under-employment state without increasing the debt-to-GDP ratio before fiscal policy is not demanding. This paper is an attempt of mathematical analysis in a spirit of Modern Monetary Theory.

Key Words: fiscal policy, full-employment, debt-to-GDP ratio.

1 Introduction

Watts and Sharpe (2013) presented a dynamic analysis of debt-to-GDP ratio, and showed that an increase in the government expenditure can reduce the debt-to-GDP ratio. Arranging their model we examine the effects of a fiscal policy which realizes full-employment from a state of under-employment (a state with deflationary GDP gap).

Equations (9) and (A6) in Watts and Sharpe (2013) are confined to a case of one-period. Using a more general model of debt dynamics we consider periods or time required to realize full-employment, and examine the debt-to-GDP ratio at the time when full-employment is realized. The government increases its expenditure to accelerate the economic growth until full-employment is realized. The growth rate of the government expenditure over the ordinary growth rate (the growth rate of the full-employment real GDP) depends on the target growth rate of real GDP over ordinary growth, the share of the government expenditure in real GDP, and the magnitude of multiplier effect. We also show that the condition about propensity to consume for realization of full-employment within one year from under-employment situation without increasing the debt-to-GDP ratio is not demanding.

In the next section we consider a steady state of debt dynamics, and analyze the effects of a fiscal policy to realize full-employment. In Section 3 we present some numerical and graphical simulations based on plausible assumptions of variables.

Let g be the growth rate of the full-employment real GDP, ρ be the extra growth rate of real GDP over g by a fiscal policy (the growth rate of real GDP is $g+\rho$) in a state of under-employment, and γ be the extra growth rate of the government expenditure over g by a fiscal policy (the growth rate of the government expenditure is $g+\gamma$). Under some assumptions about variables we show the following results.

1. The larger the value of ρ is, the faster the full-employment state is realized. (Figure 1)
2. The larger the value of ρ is, the smaller the debt-to-GDP ratio at the time when full-employment is realized is, that is, the more aggressive the fiscal policy is, the smaller the debt-to-GDP ratio at the time when full-employment is realized is. (Figure 3)

The reason for this result is as follows. The smaller the value of ρ is, the long the periods we need to realize full-employment is. On the other hand, as shown in 6 (Proposition 2) below, the share of government expenditure in real GDP at the time when full-employment is realized does not depend on ρ. Thefore, when ρ is small, the accumulated budget deficit including burden of interest is large.

3. When the value of ρ is larger than the critical value, the fiscal policy to realize full-employment reduces the debt-to-GDP ratio. (Figure 4)
4. By a fiscal policy, first the debt-to-GDP ratio decreases, and then it increases. (Figure 5 and 6)
5. Under a situation with some deflationary GDP gap (for example 15%), the condition about propensity to consume for realization of full-employment within one year without increasing the debt-to-GDP ratio is not so demanding (Proposition 1).
6. The share of government expenditure in real GDP at the time when full-employment is realized does not depend on the values of ρ and γ. (Proposition 2)

The main conclusion of this paper is that full-employment can be realized by an aggressive fiscal policy with smaller debt-to-GDP ratio than before the fiscal policy.

An increase in the government expenditure may induce a rise of the interest rate. Since the higher the interest rate is, the larger the debt-to-GDP ratio is (Subsection 3.9), we need an appropriate monetary policy which maintains the low interest rate.

This paper is an attempt of mathematical analysis in a spirit of Modern Monetary Theory.

2 Debt dynamics

According to Watts and Sharpe (2013) we consider a discrete time version of debt dynamics. The variables are as follows. t denotes a period.

c: propensity to consume (including propensity to import), $0 < c < 1$,
τ: tax rate, $0 < \tau < 1$,
$\beta = 1 - c(1-\tau)$,
$Y(0)$: real GDP in period 0,
$Y(t)$: real GDP in period t, $t \geq 0$,
$Y_m(0)$: full-employment real GDP in period 0,
$Y_m(t)$: full-employment real GDP in period t, $t \geq 0$,
$\zeta = \frac{Y_m(0)}{Y(0)}$, $\zeta > 1$,
$\tilde{t}$: the time at which full-employment is realized, $\tilde{t} > 0$,
$G(0)$: government expenditure in period 0,
$G(t)$: government expenditure in period t,
$T(0)$: tax revenue in period 0,
$T(t)$: tax revenue in period t,
$\alpha = \frac{G(0)}{Y(0)}$,
$B(0)$: government budget surplus in period 0,
$B(t)$: government budget surplus in period t,
$b(0) = \frac{B(0)}{Y(0)}$,
$b(t) = \frac{B(t)}{Y(t)}$,
$D(0)$: government debt at the end of period 0,
$D(t)$: government debt at the end of period t,
$d(0) = \frac{D(0)}{Y(0)}$,
$d(t) = \frac{D(t)}{Y(t)}$,
d^*: steady state value of $d(t)$,
g: growth rate of the full-employment real GDP, $g > 0$
ρ: extra growth rate of real GDP by a fiscal policy, $\rho > 0$
γ: extra growth rate of the government expenditure by a fiscal policy, $\gamma > 0$
r: interest rate.

The unit of time is a year. We define $D(0)$ and $D(t)$ as the government debts at the ends of period 0 and period t. We assume $g + \rho > r$.

2.1 A steady state

First we examine a steady state of debt dynamics. At the steady state

$$Y(t) = (1+g)^t Y(0),\ G(t) = (1+g)^t G(0),\ T(t) = (1+g)^t T(0).$$

Thus,

$$B(t) = T(t) - G(t) = (1+g)^t B(0).$$

Then, $D(t)$ is calculated as

$$D(t) = (1+r)^t D(0) - \sum_{s=1}^{t} (1+r)^{t-s} B(s)$$

$$= (1+r)^t D(0) - B(0) \sum_{s=1}^{t} (1+r)^{t-s} (1+g)^s$$

$$= (1+r)^t D(0) - (1+r)^t B(0) \sum_{s=1}^{t} \left(\frac{1+g}{1+r}\right)^s$$

$$= (1+r)^t D(0) - (1+r)^t B(0) \frac{\left(\frac{1+g}{1+r}\right)^{t+1} - \left(\frac{1+g}{1+r}\right)}{\left(\frac{1+g}{1+r}\right) - 1}$$

$$= (1+r)^t D(0) - B(0) \frac{(1+g)^{t+1} - (1+g)(1+r)^t}{g-r}.$$

Since $Y(t) = (1+g)^t Y(0)$,

$$\frac{D(t)}{Y(t)} = \left(\frac{1+r}{1+g}\right)^t \frac{D(0)}{Y(0)} - \frac{1+g}{g-r} \frac{B(0)}{Y(0)} \left[1 - \left(\frac{1+r}{1+g}\right)^t\right].$$

Therefore, the debt-to-GDP ratio at time t is obtained as follows.

$$d(t) = \left(\frac{1+r}{1+g}\right)^t d(0) - \frac{1+g}{g-r} b(0) \left[1 - \left(\frac{1+r}{1+g}\right)^t\right].$$

At the steady state

$$d(t) = d(0) = d^*.$$

Then[1],

$$d^* = \frac{1}{1-\left(\frac{1+r}{1+g}\right)^t} \frac{1+g}{r-g} b(0) \left[1 - \left(\frac{1+r}{1+g}\right)^t\right] = \frac{1+g}{r-g} b(0). \qquad (1)$$

2.2 Fiscal policy for full-employment

We assume that there exists a deflationary GDP gap, that is, $Y(0)$ is smaller than the full-employment real GDP, $Y_m(0)$, at time 0. Then, $\zeta > 1$. Since $Y_m(t)$ increases at the rate g,

$$Y_m(t) = (1+g)^t Y_m(0).$$

The government increases the growth rate of its expenditure from g to $g+\gamma$ to increase the growth rate of real GDP from g to $g+\rho$ so as to realize full-employment. Then,

$$Y(t) = (1+g+\rho)^t Y(0).$$

Suppose that at time $\tilde{t}$

$$(1+g+\rho)^{\tilde{t}} Y(0) = (1+g)^{\tilde{t}} Y_m(0),$$

that is, full-employment is realized at $\tilde{t}$. Then, we have

[1] If we define $D(0)$ and $D(t)$ as the government debts at the beginnings of period 0 and period t, then we have

$$d^* = \frac{1}{r-g} b(0).$$

$$\left(\frac{1+g+\rho}{1+g}\right)^{\tilde{t}} = \zeta. \tag{2}$$

$\tilde{t}$ is obtained as follows.

$$\tilde{t} = \frac{\ln\zeta}{\ln\frac{1+g+\rho}{1+g}} = \frac{\ln\zeta}{\ln(1+g+\rho)-\ln(1+g)}. \tag{3}$$

Therefore, the larger the value of ρ is, the smaller the value of $\tilde{t}$ is, that is, the faster the full-employment state is realized. We admit any positive real number for $\tilde{t}$.

If we apply the following approximation of a logarithmic function

$$\ln x = x - 1,$$

to $\ln(1+g+\rho) - \ln(1+g) = \rho$, we have

$$\tilde{t} = \frac{\ln\zeta}{\rho}.$$

Since $G(t)$ increases at the rate $g+\gamma$,

$$G(t) = (1+g+\gamma)^t G(0).$$

We examine the relation between ρ and γ. The increase in real GDP over the ordinary growth is brought by the *multiplier effect* of an increase in the government expenditure over the ordinary growth. Therefore, we have the following relation

$$\frac{1}{\beta}[(1+g+\gamma)^t - (1+g)^t]G(0) = [(1+g+\rho)^t - (1+g)^t]Y(0).$$

This means

$$\frac{1}{\beta}\left[\left(\frac{1+g+\gamma}{1+g}\right)^t - 1\right]G(0) = \left[\left(\frac{1+g+\rho}{1+g}\right)^t - 1\right]Y(0).$$

And so

$$\frac{\alpha}{\beta}\left[\left(\frac{1+g+\gamma}{1+g}\right)^t - 1\right] = \left[\left(\frac{1+g+\rho}{1+g}\right)^t - 1\right].$$

or

$$\left(\frac{1+g+\gamma}{1+g}\right)^t = \frac{\beta}{\alpha}\left[\left(\frac{1+g+\rho}{1+g}\right)^t - 1\right] + 1. \tag{4}$$

Let $t = \tilde{t}$. Then,

$$\left(\frac{1+g+\gamma}{1+g}\right)^{\tilde{t}} = \frac{\beta}{\alpha}\left[\left(\frac{1+g+\rho}{1+g}\right)^{\tilde{t}} - 1\right] + 1 = \frac{\beta}{\alpha}(\zeta - 1) + 1. \tag{5}$$

From this and $\tilde{t} = \frac{\ln\zeta}{\rho}$.

$$\frac{\ln(1+g+\gamma)-\ln(1+g)}{\ln(1+g+\rho)-\ln(1+g)}\ln\zeta = \ln\left[\frac{\beta}{\alpha}(\zeta-1)+1\right].$$

Therefore,

$$\ln(1+g+\gamma) = \frac{\ln(1+g+\rho)-\ln(1+g)}{\ln\zeta}\ln\left[\frac{\beta}{\alpha}(\zeta-1)+1\right] + \ln(1+g). \tag{6}$$

γ is obtained from this equation. This means that the larger the value of ρ is, the larger the value of γ is. Again, if we use approximation of a logarithmic function $\ln x = x - 1$, we obtain

$$g + \gamma = \frac{\rho}{\zeta-1}\frac{\beta}{\alpha}(\zeta - 1) + g.$$

This means

$$\gamma = \rho\frac{\beta}{\alpha}. \tag{7}$$

If $t = 1$, (4) implies (7) without approximation.
$B(t)$ is the sum of the budget surplus growing by g from $B(0)$ and the budget surplus brought by the fiscal policy. It is

$$B(t) = (1+g)^t B(0) + \tau((1+g+\rho)^t - (1+g)^t)Y(0) - ((1+g+\gamma)^t - (1+g)^t)G(0)$$
$$= (1+g)^t B(0) + \tau((1+g+\rho)^t - (1+g)^t)Y(0) - \alpha((1+g+\gamma)^t - (1+g)^t)Y(0).$$

Then, $D(t)$ is written as follows.

$$D(t) = (1+r)^t D(0) - B(0)\sum_{s=1}^{t}(1+r)^{t-s}(1+g)^s$$
$$-\tau Y(0)\sum_{s=1}^{t}(1+r)^{t-s}[(1+g+\rho)^s - (1+g)^s]$$
$$+\alpha Y(0)\sum_{s=1}^{t}(1+r)^{t-s}[(1+g+\gamma)^s - (1+g)^s]$$
$$= (1+r)^t D(0) - (1+r)^t B(0)\sum_{s=1}^{t}\left(\frac{1+g}{1+r}\right)^s$$
$$-(1+r)^t \tau Y(0)\sum_{s=1}^{t}\left[\left(\frac{1+g+\rho}{1+r}\right)^s - \left(\frac{1+g}{1+r}\right)^s\right]$$
$$+(1+r)^t \alpha Y(0)\sum_{s=1}^{t}\left[\left(\frac{1+g+\gamma}{1+r}\right)^s - \left(\frac{1+g}{1+r}\right)^s\right].$$

Since

$$Y(t) = (1+g+\rho)^t Y(0),$$

we get

$$d(t) = \left(\frac{1+r}{1+g+\rho}\right)^t d(0) - \left(\frac{1+r}{1+g+\rho}\right)^t b(0)\sum_{s=1}^{t}\left(\frac{1+g}{1+r}\right)^s$$
$$-\left(\frac{1+r}{1+g+\rho}\right)^t \tau\sum_{s=1}^{t}\left[\left(\frac{1+g+\rho}{1+r}\right)^s - \left(\frac{1+g}{1+r}\right)^s\right]$$
$$+\left(\frac{1+r}{1+g+\rho}\right)^t \alpha\sum_{s=1}^{t}\left[\left(\frac{1+g+\gamma}{1+r}\right)^s - \left(\frac{1+g}{1+r}\right)^s\right]$$
$$= \left(\frac{1+r}{1+g+\rho}\right)^t d(0) - \left(\frac{1+r}{1+g+\rho}\right)^t b(0)\frac{\left(\frac{1+g}{1+r}\right)^{t+1} - \left(\frac{1+g}{1+r}\right)}{\left(\frac{1+g}{1+r}\right) - 1}$$
$$-\left(\frac{1+r}{1+g+\rho}\right)^t \tau\left[\frac{\left(\frac{1+g+\rho}{1+r}\right)^{t+1} - \left(\frac{1+g+\rho}{1+r}\right)}{\left(\frac{1+g+\rho}{1+r}\right) - 1} - \frac{\left(\frac{1+g}{1+r}\right)^{t+1} - \left(\frac{1+g}{1+r}\right)}{\left(\frac{1+g}{1+r}\right) - 1}\right]$$
$$+\left(\frac{1+r}{1+g+\rho}\right)^t \alpha\left[\frac{\left(\frac{1+g+\gamma}{1+r}\right)^{t+1} - \left(\frac{1+g+\gamma}{1+r}\right)}{\left(\frac{1+g+\gamma}{1+r}\right) - 1} - \frac{\left(\frac{1+g}{1+r}\right)^{t+1} - \left(\frac{1+g}{1+r}\right)}{\left(\frac{1+g}{1+r}\right) - 1}\right].$$

Thus,

$$d(t) = \left(\frac{1+r}{1+g+\rho}\right)^t d(0) - \left(\frac{1+r}{1+g+\rho}\right)^t (1+g)b(0)\frac{\left(\frac{1+g}{1+r}\right)^t - 1}{g-r} \tag{8}$$

$$-\left(\tfrac{1+r}{1+g+\rho}\right)^{t}\tau\left[(1+g+\rho)\frac{\left(\frac{1+g+\rho}{1+r}\right)^{t}-1}{g+\rho-r}-(1+g)\frac{\left(\frac{1+g}{1+r}\right)^{t}-1}{g-r}\right]$$

$$+\left(\tfrac{1+r}{1+g+\rho}\right)^{t}\alpha\left[(1+g+\gamma)\frac{\left(\frac{1+g+\gamma}{1+r}\right)^{t}-1}{g+\gamma-r}-(1+g)\frac{\left(\frac{1+g}{1+r}\right)^{t}-1}{g-r}\right].$$

Let $t=\tilde{t}$. Then,

$$d(\tilde{t})=\left(\tfrac{1+r}{1+g+\rho}\right)^{\tilde{t}}d(0)-\left(\tfrac{1+r}{1+g+\rho}\right)^{\tilde{t}}(1+g)b(0)\frac{\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1}{g-r} \tag{9}$$

$$-\left(\tfrac{1+r}{1+g+\rho}\right)^{\tilde{t}}\tau\left[(1+g+\rho)\frac{\left(\frac{1+g+\rho}{1+r}\right)^{\tilde{t}}-1}{g+\rho-r}-(1+g)\frac{\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1}{g-r}\right]$$

$$+\left(\tfrac{1+r}{1+g+\rho}\right)^{\tilde{t}}\alpha\left[(1+g+\gamma)\frac{\left(\frac{1+g+\gamma}{1+r}\right)^{\tilde{t}}-1}{g+\gamma-r}-(1+g)\frac{\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1}{g-r}\right].$$

From this

$$d(\tilde{t})-d(0)=\left[\left(\tfrac{1+r}{1+g+\rho}\right)^{\tilde{t}}-1\right]d(0)-\left(\tfrac{1+r}{1+g+\rho}\right)^{\tilde{t}}(1+g)b(0)\frac{\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1}{g-r} \tag{10}$$

$$-\left(\tfrac{1+r}{1+g+\rho}\right)^{\tilde{t}}\tau\left[(1+g+\rho)\frac{\left(\frac{1+g+\rho}{1+r}\right)^{\tilde{t}}-1}{g+\rho-r}-(1+g)\frac{\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1}{g-r}\right]$$

$$+\left(\tfrac{1+r}{1+g+\rho}\right)^{\tilde{t}}\alpha\left[(1+g+\gamma)\frac{\left(\frac{1+g+\gamma}{1+r}\right)^{\tilde{t}}-1}{g+\gamma-r}-(1+g)\frac{\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1}{g-r}\right].$$

Because $\left(\frac{1+r}{1+g+\rho}\right)^{\tilde{t}}-1<0$ by $g+\rho>r$, (10) is decreasing with respect to $d(0)$. γ is obtained from (6), and $\tilde{t}$ is obtained from (3).

If we assume, from (1)

$$d(0)=\frac{1+g}{r-g}b(0),$$

then

$$d(\tilde{t})-d(0)\big|_{d(0)=\frac{1+g}{r-g}b(0)}$$

$$=\left[\left(\frac{1+r}{1+g+\rho}\right)^{\tilde{t}}-1\right]d(0)+\left(\frac{1+r}{1+g+\rho}\right)^{\tilde{t}}d(0)\left[\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1\right]$$

$$-\left(\tfrac{1+r}{1+g+\rho}\right)^{\tilde{t}}\tau\left[(1+g+\rho)\frac{\left(\frac{1+g+\rho}{1+r}\right)^{t}-1}{g+\rho-r}-(1+g)\frac{\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1}{g-r}\right]$$

$$+\left(\tfrac{1+r}{1+g+\rho}\right)^{\tilde{t}}\alpha\left[(1+g+\gamma)\frac{\left(\frac{1+g+\gamma}{1+r}\right)^{t}-1}{g+\gamma-r}-(1+g)\frac{\left(\frac{1+g}{1+r}\right)^{\tilde{t}}-1}{g-r}\right]$$

$$= \left[\left(\frac{1+g}{1+g+\rho}\right)^{\tilde{t}} - 1\right] d(0)$$

$$- \left(\frac{1+r}{1+g+\rho}\right)^{\tilde{t}} \tau \left[(1+g+\rho)\frac{\left(\frac{1+g+\rho}{1+r}\right)^{t} - 1}{g+\rho-r} - (1+g)\frac{\left(\frac{1+g}{1+r}\right)^{\tilde{t}} - 1}{g-r}\right]$$

$$+ \left(\frac{1+r}{1+g+\rho}\right)^{\tilde{t}} \alpha \left[(1+g+\gamma)\frac{\left(\frac{1+g+\gamma}{1+r}\right)^{t} - 1}{g+\gamma-r} - (1+g)\frac{\left(\frac{1+g}{1+r}\right)^{\tilde{t}} - 1}{g-r}\right].$$

When $t = 1$, with (7)

$$d(1) - d(0)|_{d(0)=\frac{1+g}{r-g}b(0)} = \left[\left(\frac{1+g}{1+g+\rho}\right) - 1\right] d(0) \tag{11}$$

$$- \left(\frac{1+r}{1+g+\rho}\right) \tau \left[(1+g+\rho)\frac{\left(\frac{1+g+\rho}{1+r}\right) - 1}{g+\rho-r} - (1+g)\frac{\left(\frac{1+g}{1+r}\right) - 1}{g-r}\right]$$

$$+ \left(\frac{1+r}{1+g+\rho}\right) \alpha \left[(1+g+\gamma)\frac{\left(\frac{1+g+\gamma}{1+r}\right) - 1}{g+\gamma-r} - (1+g)\frac{\left(\frac{1+g}{1+r}\right) - 1}{g-r}\right]$$

$$= \frac{1}{1+g+\rho}(-\rho d(0) - \rho\tau + \alpha\gamma) = \frac{\rho}{1+g+\rho}(-d(0) - \tau + \beta).$$

For (11) to be negative it is necessary and sufficient that

$$\beta < d(0) + \tau.$$

This is the result in the appendix of Watts and Sharpe (2013) .

On the other hand, if $\tilde{t} = 1$ in (10), also with (7) we have

$$d(1) - d(0) = \left[\left(\frac{1+r}{1+g+\rho}\right) - 1\right] d(0) - \left(\frac{1+g}{1+g+\rho}\right) b(0) + \left(\frac{1}{1+g+\rho}\right)(\beta - \tau)\rho$$

$$= \frac{1}{1+g+\rho}[(r-g)d(0) - (1+g)b(0) + \rho(-d(0) + \beta - \tau)]. \tag{12}$$

From (3) if $\tilde{t} = 1$,

$$\rho = (1+g)(\zeta - 1).$$

Therefore, for (12) to be negative it is necessary and sufficient that

$$\beta < d(0) + \tau - \frac{(r-g)d(0) - (1+g)b(0)}{(1+g)(\zeta-1)}. \tag{13}$$

Note that $\beta = 1 - c(1-\tau)$. This is a condition for realizing full-employment within one year. When $\tau = 0.25$, $\alpha = 0.28$, $g = 0.025$, $r = 0.015$, $b(0) = -0.015$ and $\zeta = 1.15$, (13) means

$$c > \frac{781}{1845} \approx 0.4233. \tag{14}$$

This is not so demanding condition. It means that when deflationary GDP gap is 15%, we can realize full-employment within one year without increasing the debt-to-GDP ratio. Summarizing the result,

Proposition 1

The condition for propensity to consume for realizing full-employment within one year without increasing the debt-to-GDP ratio before fiscal policy is not demanding.

$\alpha = \frac{G(0)}{Y(0)}$ is the share of the government expenditure in real GDP at period 0. GDP grows at the rate $g+\rho$, on the other hand the government expenditure grows at the rate $g+\gamma$, and $\gamma > \rho$. The larger the values of ρ and γ are, the smaller the number of periods necessary for realization of full-employment is. The value of α at $\tilde{t}$ is denoted by

$$\alpha(\tilde{t}) = \frac{G(\tilde{t})}{Y(\tilde{t})} = \left(\frac{1+g+\gamma}{1+g+\rho}\right)^{\tilde{t}} \alpha = \left(\frac{1+g}{1+g+\rho}\right)^{\tilde{t}} \left(\frac{1+g+\gamma}{1+g}\right)^{\tilde{t}} \alpha.$$

From (2) and (5), we get

$$\alpha(\tilde{t}) = \frac{1}{\zeta}\left[\frac{\beta}{\alpha}(\zeta - 1) + 1\right]\alpha.$$

This is constant, that is, it does not depend on ρ and γ. We have shown the following result.

Proposition 2 *The share of government expenditure in real GDP at the time when full-employment is realized does not depend on the values of ρ and γ.*

3 Graphical simulations

We present some simulation results. Assume the following values for the variables.

$c = 0.55,\ \tau = 0.25,\ \alpha = 0.28,\ g = 0.025,\ r = 0.015,\ b(0) = -0.015$ and $\zeta = 1.15$.

We do not assume that $d(0)$ and $b(0)$ have steady state values described in (1). We assume that g and r are constant, and $g > r$[2]. However, in Subsection 3.9 we examine a case where $r > g$. Also in Subsection 3.10 we examine a case where $d(0)$ and $b(0)$ have steady state values.

3.1 Relation between $\boldsymbol{\rho}$ and $\tilde{\boldsymbol{t}}$

In addition to the above assumptions we assume $d(0) = 0.5$. Figure 1 represents the relation between ρ and $\tilde{t}$. As (3) suggests, the larger the value of ρ is, the smaller the value of $\tilde{t}$ is, that is, the faster the full-employment state is realized. Therefore, the more aggressive the fiscal policy is, the faster the full-employment state is realized. For example, when $\rho = 0.05$, $\tilde{t} \approx 3$, when $\rho = 0.1$, $\tilde{t} \approx 1.5$.

[2] In Mitchell, Wray and Watts (2019) (pp. 357-358) it is stated that when $g > r$, there exists a stable steady state value of the debt-to-GDP ratio.

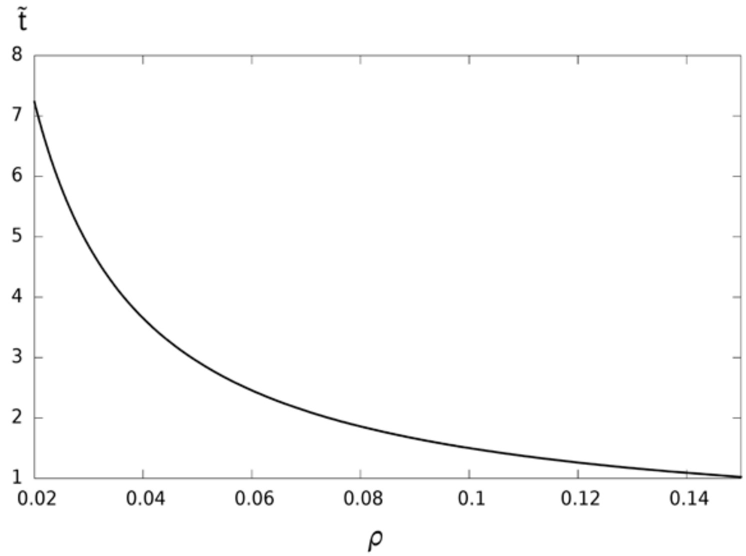

Figure 1: The relation between ρ and $\tilde{t}$

3.2 Relation between $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$

Again we assume $d(0) = 0.5$. Figure 2 represents the relation between the value of ρ and the value of γ according to (6). As it suggests, the larger the value of ρ is, the larger the value of γ is. For example, when $\rho = 0.05$, $\gamma \approx 0.1$, when $\rho = 0.1$, $\gamma \approx 0.2$.

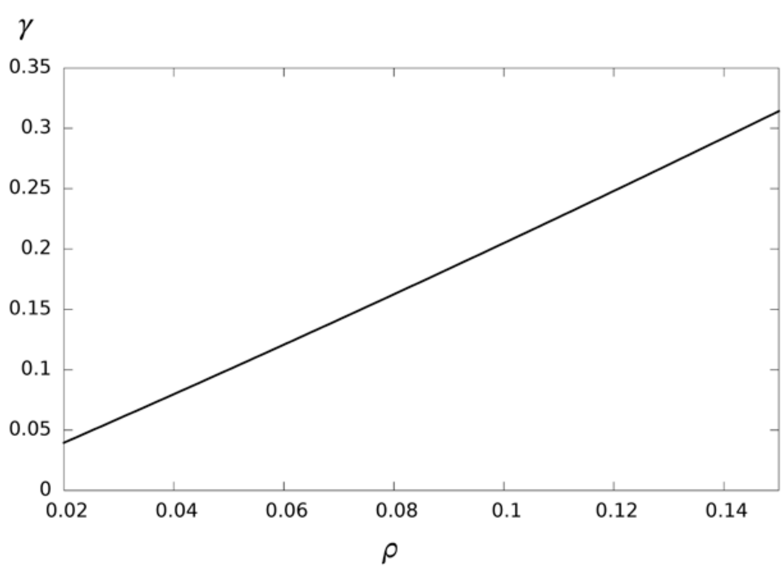

Figure 2: The relation between ρ and γ

3.3 Relation between $\boldsymbol{\rho}$ and $\boldsymbol{d(\tilde{t})}$

We assume $d(0) = 0.5$. Figure 3 represents the relation between ρ and $d(\tilde{t})$ according to (9). The larger the value of ρ is, the smaller the value of $d(\tilde{t})$ is, that is, the smaller the debt-to-GDP ratio at the time when full-employment is realized is.

As we said in the introduction, the reason for this result is as follows. The smaller the value of ρ is, the longer the periods we need to realize full-employment is. On the other hand, as shown in Proposition 2, the share of government expenditure in real GDP at the time when full-employment is realized does not depend on ρ. Thefore, when ρ is small, the accumulated budget deficit including burden of interest is large.

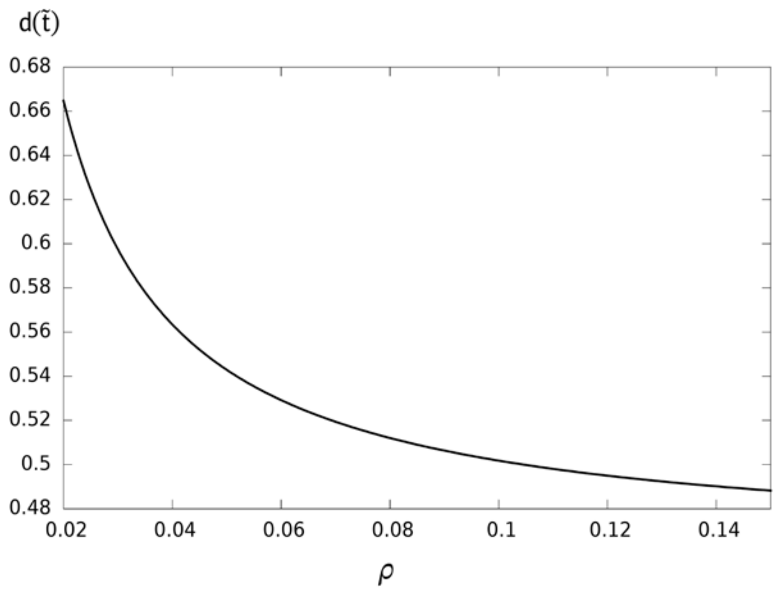

Figure 3: The relation between ρ and $d(\tilde{t})$

3.4 Relation between $\boldsymbol{\rho}$ and $\boldsymbol{d(\tilde{t}) - d(0)}$

We assume $d(0) = 0.5$. Figure 4 represents the relation between ρ and $d(\tilde{t}) - d(0)$, which is the difference between the debt-to-GDP ratio at $\tilde{t}$ and that at $t = 0$, according to (10). The larger the value of ρ is, the smaller the value of $d(\tilde{t}) - d(0)$ is. If ρ is larger than about 0.1, the debt-to-GDP ratio at $t = \tilde{t}$ is smaller than that at $t = 0$, that is, the aggressive fiscal policy to realize full-employment reduces the debt-to-GDP ratio.

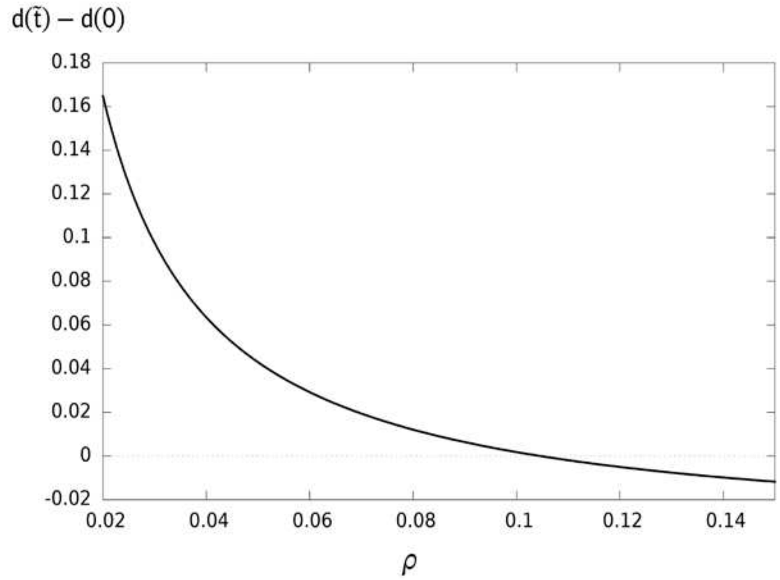

Figure 4: The relation between ρ and $d(\tilde{t}) - d(0)$

3.5 Relation between $\boldsymbol{t}$ and $\boldsymbol{d(t)}$

We assume $d(0) = 0.5$ and $\rho = 0.12$. Figure 5 represents the relation between the time (t) and the value of $d(t)$ according to (8) for $0 < t \le \tilde{t}$. First $d(t)$ decreases, then it increases.

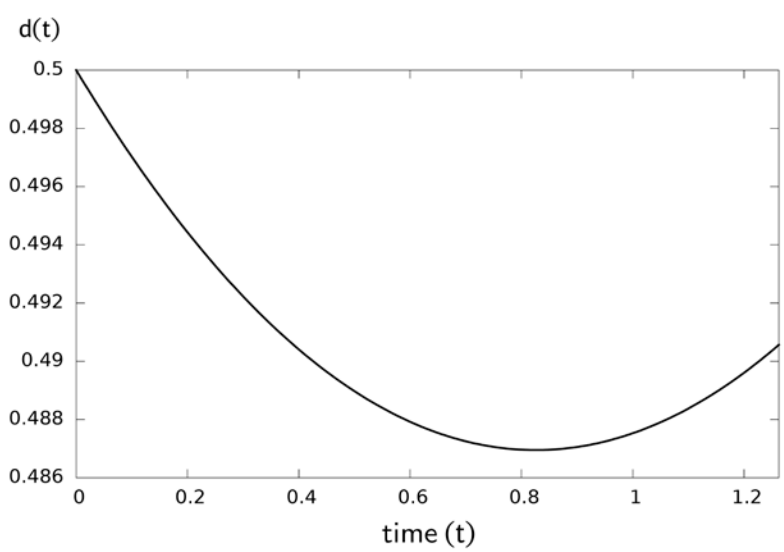

Figure 5: The relation between the time and $d(t)$

3.6 *Relation between* $\boldsymbol{t}$ *and* $\boldsymbol{d(t) - d(0)}$

Again we assume $d(0) = 0.5$ and $\rho = 0.12$. Figure 6 represents the relation between the time (t) and the value of $d(t) - d(0)$. First $d(t) - d(0)$ decreases, then it increases.

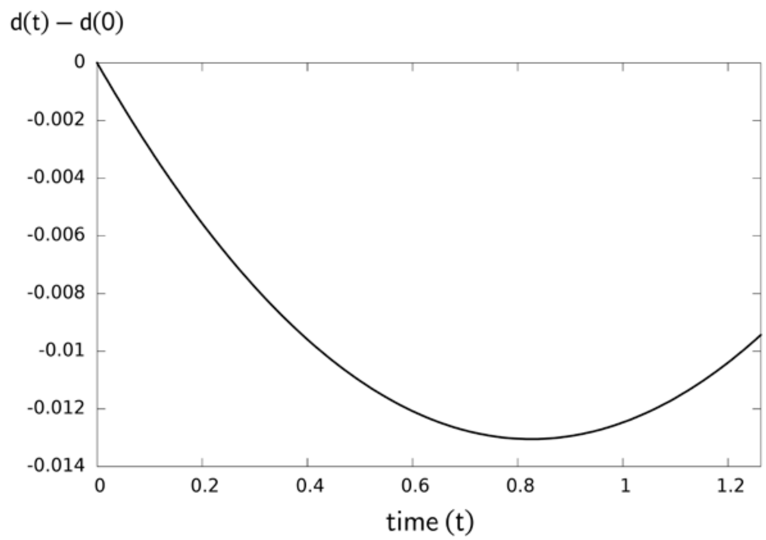

Figure 6: The relation between the time and $d(t) - d(0)$

3.7 *Relation between* $\boldsymbol{d(0)}$ *and* $\boldsymbol{d(\tilde{t})}$

We assume $\rho = 0.12$. Figure 7 represents the relation between the value of $d(0)$ and the value of $d(\tilde{t})$ according to (9). It is a straight line whose slope is smaller then one.

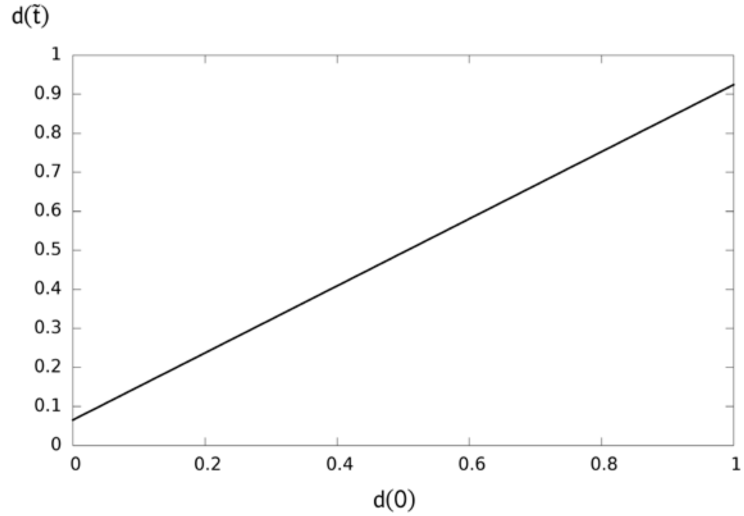

Figure 7: The relation between $d(0)$ and $d(\tilde{t})$

3.8 *Relation between* $\boldsymbol{d(0)}$ *and* $\boldsymbol{d(\tilde{t}) - d(0)}$

Again we assume $\rho = 0.12$. Figure 8 represents the relation between the value of $d(0)$ and the value of $d(\tilde{t}) - d(0)$ according to (10). It is a straight line whose slope is negative.

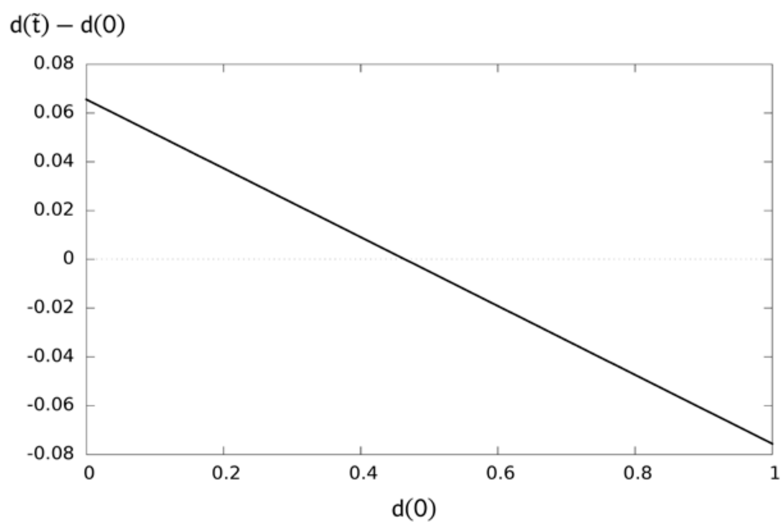

Figure 8: The relation between $d(0)$ and $d(\tilde{t}) - d(0)$

3.9 *Relation between* $\boldsymbol{\rho}$ *and* $\boldsymbol{d(\tilde{t}) - d(0)}$ *with low and high interest rates*

We assume $r = 0.035$. The values of other variables are the same as those in the previous cases. In Figure 9 we compare the relation between ρ and $d(\tilde{t})-d(0)$ in a case of low interest rate and that in a case of high interest rate.

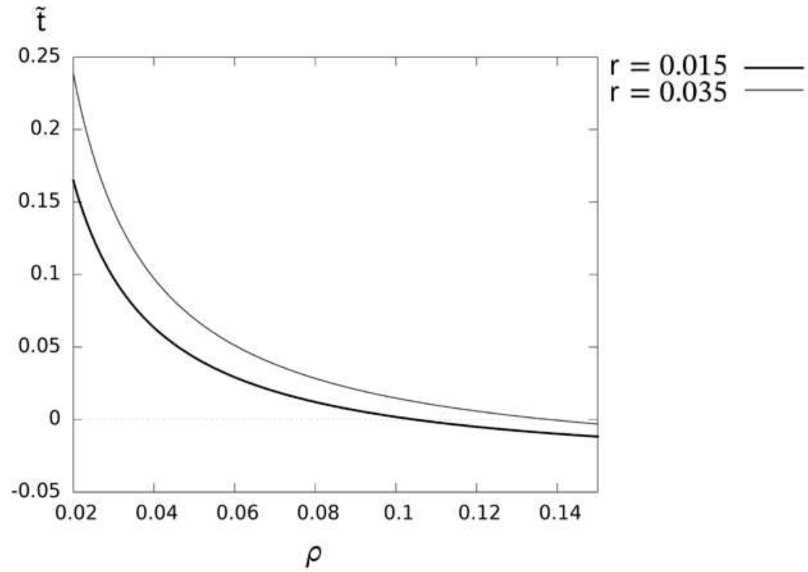

Figure 9: The relation between ρ and $d(\tilde{t})-d(0)$ with low and high interest rates

With higher interest rate the debt-to-GDP ratio at the time when full-employment is realized is less likely smaller than that at period 0.

3.10 Relation between $\boldsymbol{\rho}$ *and* $\boldsymbol{d(\tilde{t})-d(0)}$ *when* $\boldsymbol{b(0)=\frac{r-g}{1+g}d(0)}$ *and* $\boldsymbol{b(0)=-0.015}$

We assume that $b(0)$ and $d(0)$ have steady state values, that is, $b(0)=\frac{r-g}{1+g}d(0)$. In Figure 10 we compare the relation between ρ and $d(\tilde{t})-d(0)$ in a case of $b(0)=-0.015$ and that when $b(0)=\frac{r-g}{1+g}d(0)$.

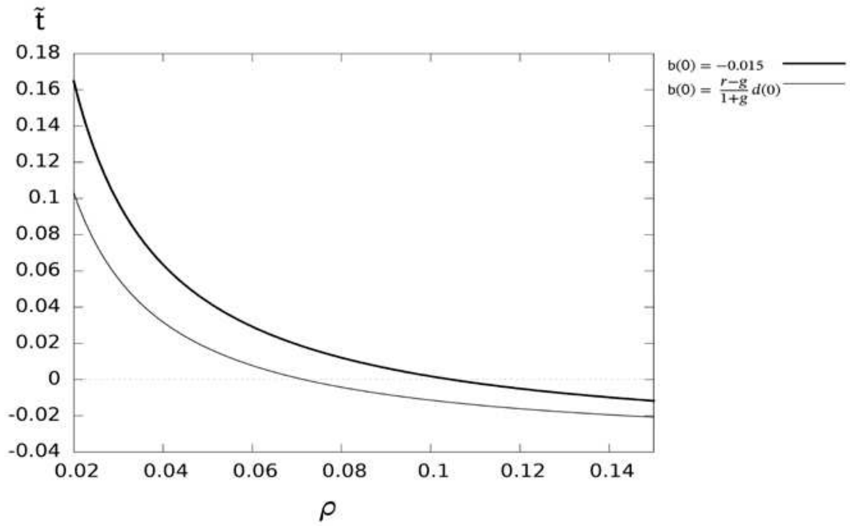

Figure 10: The relation between ρ and $d(\tilde{t}) - d(0)$ when $b(0) = \frac{r-g}{1+g}d(0)$ and $b(0) = -0.015$

If $b(0)$ and $d(0)$ have steady state values, the debt-to-GDP ratio at the time when full-employment is realized is more likely smaller than that at period 0 than the case where $b(0) = -0.015 < \frac{r-g}{1+g}d(0)$.

4 Concluding Remark

We admit that $\tilde{t}$ has any positive real number. However, our model of debt dynamics is a discrete time model. We want to study effects of a fiscal policy to realize full-employment by a more general continuous time version of debt dynamics.

Acknowledgment

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