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# The Evaluation of Model Risk for Probability of Default and Expected Loss

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## The Evaluation of Model Risk for Probability of Default and Expected Loss

### Abstract

The quantification of model risk is still in its infancy. This paper provides an operational quantification of this risk for credit portfolio, when the objective is to approximate the average loss. The methodology is easy to implement and does not require the construction of any worst-case model. The required capital computed to cover for model risk depends on three components, that are an estimated impact of the incorrect model, an evaluated risk of inaccurate estimation of model risk and the prediction error hedge factor. The approach is illustrated by an application to a portfolio of corporate loans segmented by grades.

**Keywords :** Model Risk, Estimation Risk, Specification Risk, Expected Loss, Probability of Default, Required Capital, Prudential Regulation, Difference Estimator.

# 1 Introduction

The prudential regulation for credit portfolios implicitly assumes that models for defining the required capital are well-specified. That implies that this regulation assumes no model risk. The total required capital is then the sum of a regulatory capital<sup>3</sup> and an additional component introduced to capture the estimation risk on the expected loss. The prudential regulation has been aware of the existence of model risk for more than 20 years [see e.g. Derman (1996), Crouhy et al. (1998), Rebonato (2001), OCC (2011), Board of Governors of the Federal Reserve System (2013)]. However the decision to explicitly introduce additional capital for model risk is quite recent. This additional capital, called the Margin of Conservatism (MoC) [EBA (2017)] is often presented as a compensation for an excessively optimistic assessment of the risk by financial institutions, that is a compensation for some underlying bias.

The aim of this paper is to discuss the feasibility of evaluating model risk for credit portfolio in an objective way,<sup>4</sup> that is, based on solid statistical background, not on preferences of supervisors and/or financial institutions. Our starting point is the supervisory decomposition formula of the expected loss (written in a broad manner) as :

$$\text{expected loss} \equiv \text{EL} = \text{EAD} * \text{PD} * (\text{E}) \text{LGD} * \text{CCF}, \quad (1.1)$$

where : EAD is the exposure-at-default, i.e. the face value of the debt at the default event,

PD the probability of default,

(E)LGD the (expected) loss-given-default, that is, one minus the expected ultimate recovery,

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<sup>3</sup>or economic capital for Pillar 2 purpose.

<sup>4</sup>We do not discuss in this note the literature focusing on the effect of model risks on the price of derivative instruments, especially on the misspecification of the copula introduced to jointly price two underlying risks, or on the misspecification on both the historical and risk-neutral models [see e.g. Green, Figlewski (1999), Cont (2004), Detering, Packam (2016)]. The models for pricing financial instruments are usually small scale models whereas the models for retail credits are large scale models.

CCF the credit conversion factor, that is, the fraction of the credit line really used at default.<sup>5</sup>

In truth the literature on the evaluation of model risk in default analysis is particularly sparse. This concerns both the academic and professional literatures, when technical operational advices have to be provided. For instance the most recent documents on margin of conservatism [EBA (2017)] are providing no advice at all and are without any quantitative objective background.

The reason for the lack of academic literature is easily understood if we realize that it is not possible to measure model risk per se. In this respect, the paper by Bertram et al. (2015) is a good illustration of the difficulty. A natural way of measuring model risk is to compare the parametric estimation of the  $PD : p(x; \hat{\theta}_n)$ , where  $\hat{\theta}_n$  denotes the estimate, with a nonparametric estimate  $\hat{p}_n(x)$  of  $p_0(x)$ . But this approach, developed<sup>6</sup> in Bertram et al (2015) when  $X$  contains one variable only, is unfeasible when the number of exogenous variables is large and all cross-effects of variables have to be taken into account. We encounter the curse of dimensionality : the nonparametric approach cannot reasonably be applied and, when it is applied, it will provide poorly accurate results. In other words, deriving a reasonably accurate absolute (functional) measure of model risk is practically unfeasible. The academic literature noted this impossibility; this explains the absence of academic research on a question that is known to be unsolvable.

However, partial answers can be expected :

i) For instance we can expect to compare the model risk of two competing misspecified models (pseudo-models), that is, to measure their relative model risks. The literature on model selection usually assumes that either

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<sup>5</sup>Formula (1.1) is given for an homogenous pool of credit lines. For standard corporate loans, or consumer credits : CCF=1. When different homogenous sets of contracts are aggregated, formula (1.1) is the basic formula before aggregation and is not valid at the aggregate level. In other words formula (1.1) is not satisfying perfect aggregation. However, a kind of aggregate CCF has been defined by the regulator as "conversion factor means the ratio of the currently undrawn amount of commitment that could be drawn and that would be outstanding at default to the currently undrawn amount of the commitment.."

<sup>6</sup>A similar comparison of nonparametric, semi-parametric and parametric approaches is proposed in Danielsson et al. (2018) for the estimation of tail risk on a single asset return. They show that the nonparametric approach can be inappropriate to evaluate the worst scenario for regulatory purpose.

both models are well-specified, in the standard Akaike Information Criterion (AIC) [Akaike (1973), (1974)], or at least one of them is well-specified in the Bayesian Information Criterion (BIC) [Schwarz (1978)]. These simple model choice procedures proposed in the standard softwares are irrelevant, when both pseudo-models are misspecified. Under misspecification, the procedures have to be modified. This has been initially suggested by Takeuchi (1976) leading to the Takeuchi Information Criterion (TIC). A recent complete analysis of model choice under misspecification can be found in Lv, Liu (2014)<sup>7</sup>.

ii) In this article we adopt another approach. We are not attempting to evaluate the model risk per se, but measuring its impact on the approximation of the average loss on the population of interest. We introduce a scalar objective function with a smoothing of the functional measure of model risk, that allows to eliminate both the nonparametric approach and the curse of dimensionality. Indeed a direction of interest is privileged for the application to bank and insurance companies supervision.

The plan of the paper is as follows :

In Section 2, we describe the estimation of the average loss on the population as currently validated by the supervisory authorities, and explain why this approach assumes a well-specified pseudo-model. We also recall how the additional required capital for estimation risk is evaluated.

Section 3 considers misspecified pseudo-models. We first explain how the bias for model risk can be consistently estimated. Next, we show the standard estimate adjusted for bias, leading to a difference estimator, which is an estimation method common in survey sampling theory [see e.g. Cochran (1977), Singh et al. (2013) and the references therein]. Afterwards we evaluate the additional required capital for the estimation error. We show that it has two components, that are a component for compensating the theoretical prediction error and another component for the impact of estimation risk on model risk. We provide in Section 4 an illustration of the approach for a corporate loan portfolio segmented by grades. This allows for comparing in terms of amount of reserves the standard and new methodologies.

We explain in Section 5 how to relax Assumption 1 and extend the quantification of model risk when the true and/or pseudo-model depend on sta-

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<sup>7</sup>The discussion of this literature is out of the scope of the present paper [see Gouriéroux, Monfort (2019,b)]

tionary time varying macro-factors. Section 6 concludes. Technical issues as well as additional illustration are gathered in Appendices.

## 2 The estimated predictor of the average loss and its variability

For expository purpose, we focus first on the main component, i.e. the probability of default, that is, we consider that (E)LGD =1 and CCF=1. Then the decomposition formula simplifies to :

$$EL = EAD * PD. \quad (2.1)$$

We will define more precisely the components in such rather crude decomposition formula to distinguish the theoretical notions from their estimated counterparts, but also to distinguish these notions applied to the sample used for estimation and for the population for which the total loss has to be estimated.

To be able to relate sample and population, we make the following assumption in the beginning of our paper.

### Assumption 1 :

i) The sample is defined by a set of observations  $(y_i, x_i), i = 1, \dots, n$ , where  $y_i$  is the default indicator and  $x_i$  are exogenous variables. Both  $(x_i, y_i), i = 1, \dots, n$  are observed.

ii) The population is defined by a set of variables  $(Y_i, X_i), i = 1, \dots, N$  with the same interpretations as for the sample. The population size  $N$  and the individual exogenous characteristics  $X_i, i = 1, \dots, N$ , are observed,  $Y_i$  is not observed.

iii) The variables  $(y_i, x_i), i = 1, \dots, n, (Y_i, X_i), i = 1, \dots, N$ , are independent identically distributed. The suffix 0 denotes true underlying distribution. For instance  $E_0$  denotes the expectation with respect to the true distribution.

iv) The exposure-at-default  $EAD_i$  is one of the individual exogenous variables :  $EAD_i = a(X_i)$ , say.

The problem is how to approximate the average loss on the population :

$$\widehat{EL}_N = \frac{1}{N} \sum_{i=1}^N (EAD_i * Y_i) = \frac{1}{N} \sum_{i=1}^N [a(X_i) * Y_i], \quad (2.2)$$

from the knowledge of the  $X_i, i = 1, \dots, N$ , and, from a parametric model  $P[Y = 1|X] = p(X; \theta)$ , estimated from the sample. This model being a simplified representation of the reality is likely misspecified, that is, there does not exist a true parameter value  $\theta_0$  such that :

$$p(X; \theta_0) = P_0[Y = 1|X] \equiv p_0(X).$$

In particular we will try to understand the joint roles of specification risk and estimation risk in the error when approximating  $\widehat{EL}_N$ .

The average loss on the population  $\widehat{EL}_N = \frac{1}{N} \sum_{i=1}^N a(X_i)Y_i$ , where  $a(X_i) = EAD_i$ , is usually approximated by :

$$\widehat{EL}_N^* = \frac{1}{N} \sum_{i=1}^N [a(X_i)p(X_i; \hat{\theta}_n)], \quad (2.3)$$

that is, by replacing the unobserved default indicators by the probabilities of default estimated by the (pseudo) model. If the sample size  $n$  is large, the estimator  $\hat{\theta}_n$  is close to its limit  $\theta_0^*$ , called the pseudo-true value of the parameter [Cox (1961)]. Then the approximated average loss is close to :

$$\widehat{EL}_N^* \sim \frac{1}{N} \sum_{i=1}^N [a(X_i)p(X_i; \theta_0^*)]. \quad (2.4)$$

Therefore, for a large sample size  $n$ , the bias of the predictor is :

$$\begin{aligned} B_0 &= E_0(\widehat{EL}_N^* - \widehat{EL}_N) \\ &= E_0\left[\frac{1}{N} \sum_{i=1}^N a(X_i)p(X_i, \theta_0^*) - \frac{1}{N} \sum_{i=1}^N a(X_i)Y_i\right] \\ &\iff B_0 = E_0[a(X)p(X; \theta_0^*) - a(X)Y], \end{aligned} \quad (2.5)$$

where  $E_0$  denotes the expectation with respect to the true distribution.



If the model is well-specified,  $p(X; \theta_0^*) = P_0(Y = 1|X) = E_0(Y|X)$ , and

$$\begin{aligned} B_0 &= E_0[a(X)E_0(Y|X) - a(X)Y] \\ &= E_0[a(X)E_0(Y|X)] - E_0E_0[a(X)Y|X] \\ &= 0, \text{ by the iterated projection theorem.} \end{aligned}$$

In contrast when the (pseudo) model is misspecified, a non zero bias can exist. This bias is a measure of the impact of model risk on the "expected loss".

**Proposition 1 :** The impact of model risk on the expected loss is measured by  $B_0 = E_0[a(X)p(X, \theta_0^*) - a(X)Y]$ .

If the model is well-specified there is no bias. When the model is misspecified, the bias can be equal to zero too in special cases.

**Remark 1 :** The measure  $B_0$  compares the outcomes obtained from the pseudo-model to the true distribution. Since the true distribution cannot be evaluated, a part of the literature on model risk suggests to modify the measure  $B_0$  by introducing a neighbourhood of competitors of  $p(x, \theta)$ , such as a set  $p(x; \theta, \beta)$  of nesting models.<sup>8</sup> Then the new measure arises :

$$\tilde{B} = \max_{\beta, \theta} E_0[a(X)p(X; \theta_0^*) - a(X)p(X; \theta, \beta)],$$

which compares the initial pseudo-model to the "worst" case in the neighbourhood instead of comparing it to the true distribution itself [see e.g. Kerkhof et al. (2002), Danielsson et al. (2016)]. As shown in the next section, it is not necessary to change the true measure  $B_0$  as outlined above, whereas this change is cumbersome and often infeasible. Moreover, as noted in Sibbertsen et al. (2008) : "The set of valid (competing models) to be evaluated has to be massive and it is impossible to set it to sensible limits in practical situations." Typically such analysis are very sensitive to the number and type of "black swans" included in this neighbourhood.

### Example 1: (Pseudo) homogenous segmentation

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<sup>8</sup>This approach usually refers to the robust control and ambiguity aversion literature [see e.g. Hansen, Sargent (2001)].

Let us consider a partition of the domain  $\mathcal{X}$  of possible values of  $X$  into  $J$  segments  $A_j, j = 1, \dots, J$ , and denote the indicator of segment  $j$  by  $Z_j$  :

$$Z_j = 1, \text{ iff } X \in A_j, j = 1, \dots, J.$$

If the pseudo-model assumes homogenous segments, we get :

$$p(X; \theta) = \sum_{j=1}^J \{Z_j PD_j\},$$

with  $\theta = (PD_1, \dots, PD_J)'$ . The unknown  $PD_j$ 's are usually estimated from their sample counterparts :  $\widehat{PD}_j = \sum_{i=1}^n y_i z_{ji} / \sum_{i=1}^n z_{ji}, j = 1, \dots, J$ , and tend to the pseudo-true values :  $PD_{0j}^* = E_0(Y Z_j) / E_0(Z_j) = P_0[Y = 1 | Z_j = 1] = E_0(Y | Z_j = 1), j = 1, \dots, J$ . The theoretical measure  $B_0$  is equal to :

$$\begin{aligned} B_0 &= E_0[a(X) \sum_{j=1}^J Z_j PD_{0j}^* - a(X)Y] \\ &= E_0[a(X)E_0(Y|Z) - a(X)Y] \\ &= E_0 E_0[a(X)E_0(Y|Z) - a(X)Y|Z] \\ &= E_0\{E_0[a(X)|Z]E_0(Y|Z) - a(X)Y|Z\} \\ &= -E_0 Cov_0[a(X), Y|Z]. \end{aligned} \tag{2.6}$$

The bias is the opposite of the average of the within segment covariance between the EAD and the default indicator. Therefore the theoretical measure of the impact of model risk can be of any sign. This bias could be reduced by increasing the number of segments, but with the trade off of an increase of estimation error.

Assuming that the model is well-specified, the approximated average loss  $\widehat{EL}_N^*$  is unbiased<sup>9</sup> of  $\widehat{EL}_N$ , and used to define the so-called economic capital. Then an additional capital is needed to account for the estimation risk, that is for the fact that the pseudo-true value  $\theta_0^*$  is estimated by  $\hat{\theta}_n$  with error. The

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<sup>9</sup>It is also conditionally unbiased, i.e. there is no bias given  $X_1, \dots, X_N$ .

approach is the following : let us assume that the distribution of the estimator is known (or consistently estimated). We deduce from this distribution and the given definition (2.2) of  $\widehat{EL}_N^*$ , the distribution of  $\widehat{EL}_N^*$  (for fixed  $X_i, i = 1, \dots, N$ ). The total required capital can be fixed equal to the value of the 95% quantile, say, of this distribution, i.e. the so-called Value-at-Risk (VaR). When the sample size  $n$  is large and  $\sqrt{n}(\hat{\theta}_n - \theta_0^*)$  is asymptotically normal with mean zero and variance  $\Sigma$ , the computation of the VaR can be obtained by an asymptotic expansion. We have :

$$\begin{aligned}\widehat{EL}_N^* &= \frac{1}{N} \sum_{i=1}^N [a(X_i)p(X_i; \hat{\theta}_n)] \\ &\sim \frac{1}{N} \sum_{i=1}^N a(X_i)p(X_i; \theta_0^*) + \frac{1}{N} \sum_{i=1}^N a(X_i) \frac{\partial p(X_i; \theta_0^*)}{\partial \theta'} (\hat{\theta}_n - \theta_0^*).\end{aligned}$$

If moreover  $N$  is very large, we get :

$$\begin{aligned}\widehat{EL}_N^* &\simeq E_0[a(X)p(X; \theta_0^*)] + E_0 \left[ a(X) \frac{\partial p(X; \theta_0^*)}{\partial \theta'} \right] (\hat{\theta}_n - \theta_0^*) \\ &= E_0[a(X)Y] + E_0 \left[ a(X) \frac{\partial p(X; \theta_0^*)}{\partial \theta'} \right] (\hat{\theta}_n - \theta_0^*) \text{ (since } B_0 = 0).\end{aligned}$$

It follows that :

$$\sqrt{n}[\widehat{EL}_N^* - E_0(a(X)Y)] = E_0 \left[ a(X) \frac{\partial p(X; \theta_0^*)}{\partial \theta'} \right] \sqrt{n}(\hat{\theta}_n - \theta_0^*). \quad (2.7)$$

Therefore :

$$\sqrt{n} \left[ \widehat{EL}_N^* - E_0(a(X)Y) \right] \sim N \left( 0, w^2 = E_0 \left[ a(X) \frac{\partial p(X; \theta_0^*)}{\partial \theta'} \right] \Sigma E_0 \left[ a(X) \frac{\partial p'(X, \theta_0^*)}{\partial \theta} \right] \right), \quad (2.8)$$

and the required capital to be added to  $\widehat{EL}_N^*$  is equal to :

$$RC^+ = \frac{1.64}{\sqrt{n}} \hat{w}_n, \quad (2.9)$$

where  $\hat{w}_n^2$  is a consistent estimator of  $w^2$  and we consider the one sided 95% quantile.  $RC^+$  is introduced for protection against the estimation risk on  $\theta_0^*$ .

This existing approach is misleading for two reasons.

- i) first, it assumes that the model risk can be disregarded (that is,  $B_0 = 0$ ).
- ii) second, it does not account for the theoretical prediction error, that is the impossibility to perfectly predict the indicator  $Y_i$  even if we use all the information in  $X$ , and the true conditional  $PD_0(X)$ . As noted in Chatfield (1993), Section 3, "A pitfall is to think that the quantity required to assess forecast uncertainty is the variance of the forecast rather than the variance of the forecast error".

**Example 1 (continued) :**

Given the segmentation  $Z$ , the approximated average loss is :

$$\begin{aligned} \widehat{EL}_N^* &= \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^J [Z_{ji} a(X_i) \widehat{PD}_j] \\ &= \sum_{j=1}^J \left[ \frac{\sum_{i=1}^N Z_{ji} \sum_{i=1}^N Z_{ji} a(X_i)}{N \sum_{i=1}^N Z_{ji}} \widehat{PD}_j \right] \\ &= \sum_{j=1}^J \left( \frac{N_j}{N} \overline{EAD}_j \widehat{PD}_j \right), \end{aligned}$$

where  $N_j/N$  is the frequency of segment  $j$  and  $\overline{EAD}_j$  the average EAD in this segment.

By applying the  $\delta$ -method to the observed  $\widehat{PD}_j, j = 1, \dots, J$ , for  $n$  large, it is easily shown that the  $\widehat{PD}_j$  are asymptotically independent, normal, with mean  $PD_j^*$ , and a variance that can be estimated by :

$$\widehat{V}(\widehat{PD}_j) \sim \frac{\widehat{PD}_j(1 - \widehat{PD}_j)}{n_j},$$

where  $n_j$  is the size of segment  $j$  in the sample (see Appendix 1). We deduce that :

$$\begin{aligned}
RC^+ &\sim \frac{1.64}{\sqrt{n}} \sqrt{\sum_{j=1}^J \left[ \frac{N_j^2}{N^2} (\overline{EAD}_j)^2 \frac{\widehat{PD}_j(1 - \widehat{PD}_j)}{(n_j/n)} \right]} \\
&\sim \frac{1.64}{\sqrt{n}} \sqrt{\sum_{j=1}^J \left[ \frac{N_j}{N} (\overline{EAD}_j)^2 \widehat{PD}_j(1 - \widehat{PD}_j) \right]},
\end{aligned}$$

if both  $n$  and  $N$  are large.<sup>10</sup>

### 3 The analysis with misspecified models

The aim of this section is to propose a new approach to compute the required capital without assuming that the model PD, i.e. the  $p(X; \theta)$ , is well-specified, and without estimating nonparametrically<sup>11</sup> the true optimal  $PD_0(X) = P_0(Y = 1|X) = E_0(Y|X)$ . The reason is that, for  $N$  large, the

average loss on the population  $\widehat{EL}_N = \frac{1}{N} \sum_{i=1}^N a(X_i)Y_i$  is close to the expected

loss  $EL_0 = E_0[a(X)Y] = E_0[a(X)E_0(Y|X)]$ , and that this expected loss can be estimated directly without using the expression of  $PD_0(X) = E_0(Y|X)$ .

In other words, we are not interested in the conditional  $PD_0(X), \forall X$ , but in a smoothed scalar function of them. Thus, this framework of large number of data (big data) and of high dimensionality (on the  $X$ ), is circumvented since the object of interest has a small dimension, equal to 1 in our example.

First, we explain how to estimate the bias  $B_0$ , and next how to adjust for bias the standard estimator  $\widehat{EL}_N^*$ . Next, we derive the additional required capital from the VaR under misspecification.

#### 3.1 Estimation of the bias

Whereas the defaults are not yet observed on the population of interest, they have been observed on the sample  $(y_i, x_i), i = 1, \dots, n$ . Therefore, by Assumption 1, a consistent estimator of  $B_0$  is :

<sup>10</sup>Note that  $N_j/N \sim n_j/n$  by Assumption 1.

<sup>11</sup>Indeed the number of exogenous variables  $X$  is often very large (high dimensionality) and the kernel or neural network estimators of  $PD_0(X)$  are both unreliable and non robust.

$$\hat{B}_n = \frac{1}{n} \sum_{i=1}^n a(x_i)p(x_i; \hat{\theta}_n) - \frac{1}{n} \sum_{i=1}^n a(x_i)y_i. \quad (3.1)$$

Thus, contrary to a common belief, we do not have to know the true distribution, or an accurate model approximating the truth, to approximate consistently the effect of model risk on the average loss on the population.

### 3.2 The adjusted estimated predictor

The approximation of the bias in (3.1) can be used to transform the standard estimated predictor  $\widehat{EL}_N^*$  into the difference estimator :

$$\widetilde{EL}_N^* = \widehat{EL}_N^* - \hat{B}_n = \frac{1}{N} \sum_{i=1}^N a(X_i)p(X_i; \hat{\theta}_n) - \frac{1}{n} \sum_{i=1}^n a(x_i)p(x_i; \hat{\theta}_n) + \frac{1}{n} \sum_{i=1}^n a(x_i)y_i, \quad (3.2)$$

and use this bias adjusted version of the estimator to define the "economic capital".

**Remark 2 :** An alternative for the bias adjustment is the ratio estimator [Cochran (1977)] :  $\widetilde{\widetilde{EL}}_N^* = \widehat{EL}_N^* \sum_{i=1}^n a(x_i)y_i / [\sum_{i=1}^n a(x_i)p(x_i; \hat{\theta}_n)]$ . This alternative is not considered here.

### 3.3 The asymptotic expansion

Let us now consider the difference between  $\widetilde{EL}_N^*$  and the average loss  $\widehat{EL}_N$  on the population of interest. We have, for large sample size  $n$  :

$$\begin{aligned}
& \widetilde{EL}_N^* - \widehat{EL}_N \\
&= \frac{1}{N} \sum_{i=1}^N a(X_i)p(X_i; \hat{\theta}_n) - \frac{1}{n} \sum_{i=1}^n a(x_i)p(x_i; \hat{\theta}_n) + \frac{1}{n} \sum_{i=1}^n a(x_i)y_i - \frac{1}{N} \sum_{i=1}^N a(X_i)Y_i \\
&\sim \frac{1}{N} \sum_{i=1}^N a(X_i)p(X_i; \theta_0^*) + \frac{1}{N} \sum_{i=1}^N a(X_i) \frac{\partial p(X_i; \theta_0^*)}{\partial \theta'} (\hat{\theta}_n - \theta_0^*) \\
&\quad - \frac{1}{n} \sum_{i=1}^n a(x_i)p(x_i; \theta_0^*) - \frac{1}{n} \sum_{i=1}^n a(x_i) \frac{\partial p(x_i; \theta_0^*)}{\partial \theta'} (\hat{\theta}_n - \theta_0^*) \\
&\quad + \frac{1}{n} \sum_{i=1}^n a(x_i)y_i - \frac{1}{N} \sum_{i=1}^N a(X_i)Y_i.
\end{aligned}$$

Let us also assume  $N$  large of the same order as  $n$  :  $\frac{N}{n} \sim \mu$ . We get :

$$\begin{aligned}
& \sqrt{N}(\widetilde{EL}_N^* - \widehat{EL}_N) \\
&\sim \frac{1}{\sqrt{N}} \sum_{i=1}^N [a(X_i)p(X_i, \theta_0^*) - E_0[a(X)p(X; \theta_0^*)]] - \frac{1}{\sqrt{N}} \sum_{i=1}^N [a(X_i)Y_i - E_0(a(X)Y)] \\
&\quad - \sqrt{\frac{N}{n}} \frac{1}{\sqrt{n}} \sum_{i=1}^n [a(x_i)p(x_i; \theta_0^*) - E_0[a(X)p(X; \theta_0^*)]] + \sqrt{\frac{N}{n}} \frac{1}{\sqrt{n}} \sum_{i=1}^n [a(x_i)y_i - E_0(a(X)Y)],
\end{aligned}$$

since the sum of the two other terms equivalent to  $\sqrt{\frac{N}{n}} E_0[a(X) \delta \frac{p(X; \theta_0^*)}{\delta \theta'}] \sqrt{n}(\hat{\theta}_n - \theta_0^*)$  with opposite signs become negligible.

As the data in the sample are independent of the data in the population, we get the asymptotic distribution of the approximation error :

$$\sqrt{N}(\widetilde{EL}_N^* - \widehat{EL}_N) \approx N[0, (1 + \mu)V_0[a(X)p(X; \theta_0^*) - a(X)Y]]. \quad (3.3)$$

This limiting distribution depends on the distribution of  $\hat{\theta}_n$  through the pseudo-true value only.

Then, the additional required capital becomes :

$$RC^+ = \frac{1.64}{\sqrt{N}} \sqrt{1 + \mu} \sqrt{\hat{V}_n [a(x_i)p(x_i; \hat{\theta}_n) - a(x_i)y_i]}, \quad (3.4)$$

where  $\hat{V}_n$  denotes the empirical variance computed on the sample.

To summarize the (estimated) total required capital is :

$$RC = \widehat{EL}_N^* - \hat{B}_n + \frac{1.64}{\sqrt{N}} \sqrt{1 + \mu} \sqrt{\hat{V}_n [a(x_i)p(x_i; \hat{\theta}_n) - a(x_i)y_i]}. \quad (3.5)$$

It includes 4 components :

- i)  $\widehat{EL}_N^*$  is the standard economic capital assuming well-specified model.
- ii)  $\hat{B}_n$  is the estimation of (the impact of) model risk.
- iii)  $\text{In } (RC^+)^2 : \frac{(1.64)^2}{N} \hat{V}_n$  is the evaluation of the prediction error.
- iv)  $\text{In } (RC^+)^2 : \frac{(1.64)^2}{n} \hat{V}_n$  is the evaluation of the estimation risk on model risk due to the replacement of  $B_0$  by  $\hat{B}_n$ .

The comparison between the existing formulas (2.1), (2.6), (2.7), and the new formula (3.5) shows clearly changes in the computation of required capital when the model risk has been taken into account. The decomposition shows that it is difficult to separate the model risk, defined as the risk of using an incorrect model (specification risk), from the estimation risk. This explains the most recent definition by the OCC (2011), where the model risk is related to losses from the use of an uncorrect model and from the uncertainty present in the estimation procedure. What really matters is the relevance of the final result derived from the estimated pseudo-model. This is a global result and, to adjust for specification and estimation risks, it is not necessary to disentangle the different sources of errors, such as missing variables in the PD, choice of an ad-hoc segmentation, or of an estimation method, which is not consistent (even if the pseudo-model were well-specified)... In fact just the result matters. This also explains why the approach is still valid for an automatic prediction algorithm as long as the calibrated parameters of



the algorithm are the same in the estimation/validation sample and in the population.

The formula for required capital is derived for a given pseudo-model. When there are several competing pseudo-models  $p_k(x; \theta_k), k = 1, \dots, K$ , say, we get different levels of required capital  $RC_k, k = 1, \dots, K$ . In a prudential perspective, it has been proposed to take the worst model in order to fix the level of  $RC$ , i.e.  $RC_w = \max_k RC_k$ , [see e.g. <sup>12</sup> Kerkhof et al. (2002), Danielsson et al. (2016), Muller, Righi (2018)]. Such an approach can be defended if the set of pseudo-models is stable in time, since  $RC_w$  significantly depends on this set. Why do we have to keep such a worst model in the set? Simply since the objective can vary. Instead of approximating

$$\frac{1}{N} \sum_{i=1}^N EAD_i Y_i = \frac{1}{N} \sum_{i=1}^N [a(X_i) Y_i],$$

may be we will have also to approximate a weighted quantity of the type :  $\frac{1}{N} \sum_{i=1}^N [\Pi(X_i) EAD_i Y_i] \equiv \frac{1}{N} \sum_{i=1}^N \tilde{a}(X_i) Y_i$

corresponding to another structure of clientele, or to the introduction of another regulatory component  $LGD_i$  or  $CCF_i$ . Then the worst pseudo-model for  $a(X)$  might become the best model for  $\tilde{a}(X)$ . This means that formula (3.5) could be inverted in order to find the "weights"  $a(X)$  for which the required capital is the largest, or the smallest.

**Remark 3 :** From this operational viewpoint the set of pseudo-models is fixed. Therefore, we do not have to reestimate a combination of pseudo-models that are appropriate for each given criterion. If the pseudo-models are the models proposed by different financial institutions, the regulatory agency cannot combine such models without providing to all institutions the information on the models of their competitors. If several pseudo-models belong to a same institution and were constructed for various purposes, it can be costly to reestimate an approximate combination each time the supervisor suggests a different objective function [see Gouieroux, Monfort (2019), a, for the governance of a set of pseudo-models].

**Remark 4 :** The estimated asymptotic Gaussian VaR used in the formula

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<sup>12</sup>However these papers focus on the effect of model risk on risk measures such as VaR and expected shortfall, whereas we highlight in our paper the importance of adjusting for bias, that is the main impact of model risk when estimating an average loss.

(3.5) [as the one in formula (2.9)] also introduces an additional estimation risk that could be evaluated [see e.g. Hansen (2006), Lawrenz (2006), Alexander, Sarabia (2012), Gouriéroux, Zakoian (2013)]. The study of this additional risk is out of the scope of the present paper.

**Remark 5 :** To avoid a voluntary overfitting or readjustment of the estimate  $\hat{\theta}_n$  in order to get  $\hat{B}_n = 0$ , that is a regulatory arbitrage, it is necessary to distinguish a sample to be used for estimation providing  $\hat{\theta}_n$  from another independent one  $(x_i, y_i), i = 1, \dots, n$  to be used for validation, i.e. for computing <sup>13</sup> $\hat{B}_n$ . The estimator  $\hat{\theta}_n$  is now independent of the validation sample, but the expression (3.5) of the required capital is still valid.

**Remark 6 :** The result is easily extended to a nonrepresentative sample, such as a sample drawn from a population similar to the population  $(X_i, Y_i), i = 1, \dots, n$  with unequal weights. Then, the adjustment for bias  $\hat{B}_n$  and the estimated variance  $\hat{V}_n$  have to be weighted. For expository purpose, we focus first on the main component, i.e. the probability of default, that is, we consider that (E)LGD=1 and CCF=1. Then the decomposition formula is still valid with weighted means and weights equal to the inverse of inclusion probabilities [see Horwitz, Thompson (1952), Cochran (1977) and Appendix 2]. Contrary to a common belief [see e.g. EBA (2017), p15], the use of non representative sample improve the accuracy of standard estimators, if these estimators are preliminary adjusted for non representativeness.

**Example 1** (continued).

For a segmentation  $Z$ , we have :

$$\hat{B}_n = \sum_{j=1}^J \frac{n_j}{n} \overline{ead}_j \widehat{PD}_j - \sum_{j=1}^J \frac{n_j}{n} \overline{(ead * y)}_j,$$

where  $\overline{ead}_j$  (resp.  $\overline{(ead * y)}_j$ ) is the averaged  $EAD$  (resp.  $EAD * Y$ ) on segment  $j$  of the sample. The expression of  $\hat{B}_n$  involves the sample counterpart of the within segment averaged autocovariances between  $EAD$  and  $Y$ .

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<sup>13</sup>In learning theory, the first sample is called the sample of learning, whereas the validation sample is called the test sample. We avoid this terminology since the validation sample is not used for testing, but for learning the bias magnitude.

The term of variance appearing in the additional  $RC^+$  becomes :

$$\begin{aligned}
& V_0[a(X)p(X; \theta_0^*) - a(X)Y] \\
= & V_0\left[\sum_{j=1}^J (Z_j a(X) PD_{0j}^* - Z_j a(X) Y)\right] \\
= & E_0 \left\{ \left[ \sum_{j=1}^J (Z_j a(X) PD_{0j}^* - Z_j a(X) Y) \right]^2 \right\} - (E_0 [\sum_{j=1}^J (Z_j a(X) PD_{0j}^* - Z_j a(X) Y)])^2 \\
= & E_0 \left\{ \sum_{j=1}^J Z_j a(X)^2 [PD_{0j}^* - Y]^2 \right\} - B_0^2 \\
= & E_0 \left[ \sum_{j=1}^J Z_j a(X)^2 [Y(1 - 2PD_{0j}^*) + PD_{0j}^{*2}] \right] - B_0^2.
\end{aligned}$$

It is consistently estimated by :

$$\sum_{j=1}^J \frac{n_j}{n} \left\{ \overline{[(ead)^2 y]_j} (1 - 2\widehat{PD}_j) + \overline{[(ead)^2]_j} \widehat{PD}_j^2 \right\} - \widehat{B}_n^2,$$

where  $\overline{[(ead)^2 y]_j}$  [resp.  $\overline{[(ead)^2]_j}$ ] is the mean of  $(EAD)^2 \cdot y$  (resp.  $(EAD)^2$ ) computed on segment  $j$ .

## 4 An illustration

To illustrate the implementation of the approach introduced in the previous sections to compute and decompose the required capital, we consider a representative subportfolio of corporate loans of a European bank. We focus on small corporates. We assume that the default is the default of the corporate, that will imply the joint default of all its loans. Then the EAD is the sum of the EAD of its loans. The portfolio includes about 40 000 firms (and much more loans) at the date the required capital is calculated, i.e. the end of 2016. The prediction of the loss concerns the next year 2017.

Following the current supervision (see the structure of the guidelines by EBA (2017)), there is a first phase of model development, when the (pseudo) models are estimated. There can exist different pseudo-models since the loans can be granted by different entities of the bank, in particular if the bank has a structure decentralized by regions, or industrial sector, each regional/sectorial entity with possibly a specific scoring model. To introduce a minimal coherency between these scoring systems, a common rating scale, called master scale is introduced. This master scale has 13 grades (whereas the minimum number of grades of a master scale demanded by the supervision is 7). Following the current regulation they include grades as the standard  $A^+, A, \dots, C^+, C, C^-$ , and several additional grades for the treatment of firms entering in (resp. being in, exiting from) a very risky situation. This master scale defines a segmentation that is frequently used when the models are validated and the required capital computed. Thus we will apply the pseudo-model of Example 1 to the (pseudo) homogenous segmentation corresponding to this master scale.

The misspecifications are at two levels : when the pseudo-models are developed, but also when the master scale is introduced. Indeed it is implicitly assumed that the cross segments : master scale x firm's size are homogenous. To give some insight about the possible within heterogeneity, we provide in Figure 1, the distributions of the exposure at risk for the master grade segments denoted 1, 2, 3, 4, 5, say. These grades are ordered and have been selected in order to get grades similar to investment grades (1 and 2) and to speculative grades (3,4,5).

[Insert Figure 1 : Distribution of EAD within Segments]

For the first grades, we observe several modes in the distribution of the exposures, that disappear for speculative grades. The exposure is not very concentrated within a grade. If these segments were homogenous, we will have no bias from formula (2.6). The large heterogeneity in Figure 1 will likely imply a non zero bias to be adjusted for.

Similarly there is also some heterogeneity for the other segmentation variable : small firm, medium size firms, large size firms..., that is considered. Indeed this depends significantly on the definition of the size. In our case, the small firms have an annual turnover smaller or equal to 7.5 million €. The EAD is related to the size of the firm measured by the turnover, but the relationship is not strict as seen in Figure 2.

[Insert Figure 2 : Plot of EAD vs Size]

In fact the total exposure of a firm is the sum of its EAD in the different banks, where it has debt. By considering a single bank we have a partial view about this total exposure<sup>14</sup>. We also observe in Figure 2 extreme exposures, especially when they are compared to the turnover. These extreme values can have a significant impact on the evaluation of the required capital.

Let us now compare the two ways of computing the required capital, with and without accounting for model risk. The approaches require sufficiently large sample/population sizes and not too low default probability. For grade 1,2 the sample sizes are 38, and 150, but no default has been observed on the past. The analysis of the risk in these grades require other techniques [see e.g. Pluto, Tasche (2005), Kiefer (2009), EBA (2015), (2018), Castor et al. (2017)] and is out of the scope of our paper. We focus below on the other grades 3, 4, 5. We provide in Table 1 characteristics of the population of interest corresponding to year 2017 and the validation sample corresponding to year 2016. The pseudo-model has been estimated on several years before 2015 to cover a cycle as usually demanded by EBA supervision.<sup>15</sup> Then in Table 2 we give the average exposure and the different components of the required capital with a ELGD fixed to the standard value of 45%.

**Table 1 : Sample and Population**

grade	Population size (2017)	validation sample size (2016)	average EAD population
3	1240	1105	1184993 €
4	3588	3107	722266 €
5	7393	6725	566962 €

**Table 2 : Decomposition of the Required Capital**

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<sup>14</sup>The total exposure of the bank for the small firms is about 25 billion €.

<sup>15</sup>At lest 5 years of observations are required by the supervision for model estimation.

grade	standard EL	additional RC well-specified	adjustment for model risk	estimation risk + prediction error misspecified
3	279 €	252 €	-198€	41 €
4	269 €	114 €	-50 €	486 €
5	590 €	111 €	-433 €	203 €

These decompositions are given by firm, that is for all contracts of a defaulted firm. They are given in level, but can also be considered in proportion of the exposure. For instance the standard EL represents 0.025%, 0.037%, 0.124% of the EAD, respectively, that corresponds to PD equal to 0.055%, 0.082%, 0.255%, respectively. As expected they increase with the grade. Under well specification the additional RC (3rd column of Table 2) is mainly impacted by the size of the validation sample.

Let us now discuss the decomposition under misspecification. We see that the bias adjustment is negative. This is a consequence of probabilities of default decreasing over the period 2007-2017. The parameters of the pseudo-model estimated on 2010-2015 overestimate the resulting PD's and this overestimation is adjusted for when comparing to the validation sample. Note that the effect of the misspecification is prudent for all grades, but is significantly different for grade 4. This is due to a special default in the validation sample with an extreme EAD, following the death of the manager of the firm. This shows that this segment is less homogenous with respect to the EAD compared to the grades 3 and 5, and that the sample size is likely too small. We provide in Appendix 3, the results for year 2018 with the enlarged validation sample covering years 2016/2017.

The possibility of negative adjustment is explicitly mentioned in the guideline of the ECB (2017), 4.4.2, 26 : "In order to overcome estimation errors in PD and LGD estimates... institutions should apply adequate methodologies for correcting the identified errors ("appropriate adjustment")... This adjustment can have both positive and negative effect on the risk parameter". Our evaluation of the estimation risk on this bias adjustment corresponds to ECB (2017), 4.4.2, 27 : "Where such appropriate adjustments are used, institutions should apply a MoC (Margin of Conservatism) to account for the additional estimation error associated with these adjustments..."

There is clearly a need for bias adjustment. Depending on the grade 4, 5, 6, but also on the period of interest (here 2017) and on the validation sample (here 2016) (see also Appendix 3), the bias can be positive, or negative,

the total required capital being of course always positive. Moreover, for the small probabilities of default of our applications and the rather large EAD heterogeneity a minimal validation sample size of about  $n = 6000$  is preferable (see Appendix 3). Of course, this problem does not appear for consumer loans, credit cards, mortgages where the number of loans is of several hundred thousands or millions.

## 5 Models with macro-factors

### 5.1 The assumptions

The results of Sections 2 and 3 have been derived under Assumption 1 of i.i.d. observation. If this assumption is not satisfied, another model risk is created that has not been taken into account yet. Let us briefly explain how to extend the approach when the true and/or the pseudo-models depend on time, for instance, due to the presence of (stationary) macro risk factors. Assumption 1 has to be weakened, and the time effect needs to be explicitly introduced. The new set of assumptions is the following :

**Assumption 2 :**

- i) There exist three types of variables :  $X_i, Y_i$  and macrofactors  $F_t$ .
- ii) The pseudo-model is parametric  $p(X_i, F_t; \theta)$  in order to approximate at date  $t$  the probability of default at  $t + 1$ .
- iii) This pseudo-model is estimated on a set of observations corresponding to dates before date 1. The estimator of  $\theta$  is  $\hat{\theta}_n$ .
- iv) There exist  $T$  validation samples :

$$(x_{it}, y_{it}), i = 1, \dots, n_t, t = 1, \dots, T.$$

- v) The population of interest is the population at date  $T + 1$  :

$$(X_{i,T+1}, Y_{i,T+1}), i = 1, \dots, N_{T+1}.$$

- vi) Conditional on the process of macrofactors, the data  $(x_{it}, y_{it}), i = 1, \dots, n_t, t = 1, \dots, T + 1$  are independent.
- vii) The conditional distributions of  $(x_{it}, y_{it})$  given  $F_t = f, F_{t-1} = f_{-1}$  are identical.
- viii) The process  $(F_t)$  is strictly stationary.

Some macro-risk factors are observed at date  $t$  by the statistician and included as explanatory variables in the pseudo-model. Other risk factors can be i) observable, but not taken into account in the pseudo-model, or ii) are unobservable and implicitly stochastic. These unobserved stochastic factors create cross-sectional as well as serial dependence in the data, not captured in the misspecified model.

## 5.2 The impact of model risk and the difference estimator

The average loss of interest :

$$\widehat{EL}_{N,T+1} = \frac{1}{N_{T+1}} \sum_{i=1}^{N_{T+1}} [a(X_{i,T+1})Y_{i,T+1}], \quad (5.1)$$

is usually approximated by :

$$\widehat{EL}_{N,T+1}^* = \frac{1}{N_{T+1}} \sum_{i=1}^{N_{T+1}} [a(X_{i,T+1})p(X_{i,T+1}, F_T, \hat{\theta}_n)]. \quad (5.2)$$

The asymptotic bias due to model risk is :

$$\begin{aligned} B_0 &= E_0(\widehat{EL}_{N,T+1}^* - \widehat{EL}_{N,T+1}) \\ &\simeq E_0[a(X_{T+1})p(X_{T+1}, F_T; \theta_0^*) - a(X_{T+1})Y_{T+1}], \end{aligned} \quad (5.3)$$

where the expectation is with respect to the true distribution of  $X_{T+1}, Y_{T+1}$  and  $F_T$ .

By Assumption 2 vi), vii), viii), the analogues computed on the validation samples have the same asymptotic theoretical mean. Therefore,

$$\hat{B}_{n,t} = \frac{1}{n_t} \sum_{i=1}^{n_t} [a(x_{i,t})p(x_{i,t}; F_{t-1}, \hat{\theta}_n)] - \frac{1}{n_t} \sum_{i=1}^{n_t} [a(x_{i,t})y_{i,t}], \quad (5.4)$$

is such that :

$$E_0 \hat{B}_{n,t} \simeq B_0, t = 1, \dots, T, \quad (5.5)$$

if the validation sample sizes are sufficiently large.



Therefore, we can adjust the standard estimator (5.2) for the misspecification bias by considering the difference estimator :

$$\widetilde{EL}_{N,T+1}^* = \widehat{EL}_{N,T+1}^* - \frac{1}{T} \sum_{t=1}^T \hat{B}_{n,t}. \quad (5.6)$$

In other words, the standard approximation is adjusted for bias by taking into account the prediction errors obtained from the T validation samples. In practice these errors can all be computed at time T. In this respect, it is necessary to record and update the set of validation samples in a separate database, and to know precisely the specification  $p(x, F; \hat{\theta}_n)$  of the pseudo-model.

The above methodology rationalizes on more objective backgrounds the multiplicative factor 3 applied to market risk in order to take model risk into account, as proposed by the Basle Committee in the past [ see Stahl (1997)] Ex-post this crude multiplicative factor has shown to be compatible with the performances of the VaR through the crisis [Jaschke et al. (2007)]. However, our approach shows that the adjustment cannot be set fixed. It depends on the objective function, i.e. a weighted or unweighted measure of average loss, and of the risk of the pseudo-model.

### 5.3 Additional required capital

The formula of additional required capital needs to be modified to take into account the variability of macro-factors and the difficulty of predicting them accurately. More precisely, we can expand the quantity :

$$\sqrt{N_{T+1}} \left[ \widetilde{EL}_{N,T+1}^* - \widehat{EL}_{N,T+1}^* + \frac{1}{T} \sum_{t=1}^T \hat{B}_{n,t} \right], \quad (5.7)$$

when the sizes  $n_t, t = 1, \dots, T, N_{t+1}$  are large with  $N_{T+1}/n_t \sim \mu_t$ , to see that this quantity is approximately Gaussian for given factor path. Given a factor path  $(F_t)$ , the conditional mean is :

$$m(F_{T+1}, F_T) + \frac{1}{T} \sum_{t=1}^T m(F_t, F_{t-1}), \quad (5.8)$$

where  $m(F_t, F_{t-1}) = E_0(\hat{B}_{n,t}|F_t, F_{t-1})$ , and the conditional variance is :

$$\sigma^2(F_{T+1}, F_T) + \sum_{t=1}^T \mu_t \sigma^2(F_t, F_{t-1}), \quad (5.9)$$

with  $\sigma^2(F_t, F_{t-1}) = V_0(\hat{B}_{n,t} | F_t, F_{t-1})$ .

The difficulty is now to reintegrate with respect to the underlying factor path. This demand to account for all potential macro-factors, not only the small number introduced in the pseudo-model, and also to know the factor dynamics. We encounter again the curse of dimensionality for the macro-variable now, and the difficulty of modelling accurately the joint dynamics of these  $F'_t$ 's.

There exists an alternative, if the number  $T$  of validation samples is sufficiently large. This number  $T$  might be increased in practice by passing to a monthly frequency of observations instead of an annual frequency. Indeed the monthly data are usually available for individual loans and neglecting these data implies a loss of information. For expository purpose let us assume  $\mu_t = 1, t = 1, \dots, T$ . Then the observed  $\hat{B}_{n,t}, t = 1, \dots, T$  are observations of a single stationary time series and the direct analysis of this serie may provide a (conditional) VaR on the approximation error.

Thus an alternative computation of the required capital is :

$$\text{Total RC} = \widetilde{EL}_{N,T+1}^* + \frac{1}{T} \sum_{t=1}^T \hat{B}_{nt}, \quad (5.10)$$

defining the economic capital as :

$$EC = \widetilde{EL}_{N,T+1}^* + \frac{1}{T} \sum_{t=1}^T \hat{B}_{nt}, \quad (5.11)$$

and the additional RC as :

$$RC^+ = VaR_{T+1}(\hat{B}_n) - \frac{1}{T} \sum_{t=1}^T \hat{B}_{nt}. \quad (5.12)$$

## 6 Concluding Remarks

The aim of this paper is to clarify what is feasible and what is infeasible for an operational and objective quantification of model risk in the framework of

the Capital Requirement Regulation and IRB approach. Let us summarize the main messages of this note.

- i) The model risk per se cannot be evaluated since the reality is unknown and cannot be approximated accurately and in a robust way.
- ii) The impact of model risk on the average loss on a population can be evaluated, if there are no macro risk factors. This leads to a decomposition of required capital with a component to estimate model risk, another one to account for the estimation risk impact of model risk, and a last component for compensating the theoretical prediction errors.
- iii) When the probability of default depends on stationary macro-factors, it is possible to adjust for the bias due to model risk. However, the associated estimation risk is more difficult to derive analytically.
- iv) A solution for evaluating this estimation risk effect on model risk exists, if there is a series of validation samples. Then, the curse of dimensionality due to the large number of potential macro-risk factors is circumvented, and the analysis has only to consider a single time series of prediction errors. Of course the validation samples have to be managed by the external auditor/supervisor and not by the statistician estimating the pseudo-model to avoid overfitting.
- v) If there is an unpredictable change of regime in the macro-risk factors, the validation samples are no longer informative and the objective quantification of the impact of model risk is infeasible.
- vi) Finally the approach is mainly considering the standard risks, not the extreme tails risks. This is compatible with the regulation in which the additional reserves for tail risks are computed through the stress tests.

The technique proposed in this note shows the importance of the validation samples and of estimation methods based on difference estimators, that are standard in survey sample theory, but not well known in financial statistics or financial econometrics. This requires large cross-sectional dimensions and for instance is not valid for evaluating model risk on a single asset and its derivatives [see e.g. Hull, Suo (2002), Cont (2004), Detering, Packam (2016) for some attempts of measuring model risk].

This technique can be applied to other objective functions, such as the measures of the performance of a pseudo-model to compute the Value-at-Risk on a large portfolio of financial assets, and to the governance of a set of pseudo-models [see Gouriéroux, Monfort (2019, a)].

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## Appendix 1

### Asymptotic Distribution of $\widehat{PD}$

We show in this appendix that the standard asymptotic distribution of  $\widehat{PD}$  assuming homogeneity within segment is still valid if there exists within segment heterogeneity. Let us consider a given segment characterized by the indicator  $Z$ . We have :

$$\widehat{PD} = \frac{\sum_{i=1}^n y_i z_i}{\sum_{i=1}^n z_i}.$$

Note that the size of the segment, that is the term in the denominator, is stochastic. This explains why standard formulas do not necessarily apply.

Under the assumption of i.i.d. observations  $(z_i, y_i), i = 1, \dots, n$ , we get :

$$\frac{1}{\sqrt{n}} \begin{bmatrix} \sum_{i=1}^n [y_i z_i - E_0(YZ)] \\ \sum_{i=1}^n (z_i - E_0 Z) \end{bmatrix} \sim N(0, \Sigma),$$

where  $\Sigma = \begin{pmatrix} V_0(YZ) & cov_0(YZ, Z) \\ \bullet & V_0 Z \end{pmatrix} = \begin{pmatrix} E_0(YZ)[1 - E_0(YZ)] & E_0(YZ)[1 - E_0 Z] \\ \bullet & E_0 Z(1 - E_0 Z) \end{pmatrix}$   
 since both  $Y$  and  $Z$  are dummy variables.

Then, by the  $\delta$ -method, we deduce :

$$\sqrt{n}(\widehat{PD} - PD) \sim N[0, w^2],$$

where :



$$\begin{aligned}
w^2 &= \left[ \frac{1}{E_0 Z}, -\frac{E_0(YZ)}{E_0(Z)^2} \right] \Sigma \left[ \frac{1}{E_0 Z}, -\frac{E_0(YZ)}{(E_0 Z)^2} \right]' \\
&= \frac{1}{E_0(Z)} \left( \frac{E_0(YZ)}{E_0(Z)} \right) \left( 1 - \frac{E_0(YZ)}{E_0(Z)} \right) \\
&= \frac{1}{E_0 Z} PD(1 - PD).
\end{aligned}$$

The expression of the estimated variance follows.

## Appendix 2

### Non-Representative Sample

Let us now consider the case of a non-representative sample. This situation can be described as follows :

i) We consider another population  $(X_i^*, Y_i^*), i = 1, \dots, N^*$ , independent of the population of interest  $(X_i, Y_i), i = 1, \dots, N$ , such that the data  $(X_i, Y_i), (X_i^*, Y_i^*)$  are i.i.d..

ii) Then a sample  $(x_i, y_i), i = 1, \dots, n$  is drawn independently in  $(X_i^*, Y_i^*), i = 1, \dots, N^*$ , with weights  $a^*(X_i^*)$ , say, such that  $\sum_{i=1}^{N^*} a^*(X_i^*) = 1$ . If  $a^*(X_i^*) = \frac{1}{N^*}$ , the sample is representative, it is not representative, otherwise.

The samples used for estimation are frequently non representative in order to overweight the individuals with risky exogenous characteristics. This practice allows for more accurate estimators, if these estimators are first adjusted for the non-representativeness of data.

In our framework this adjustment is introduced when estimating  $B_0$  and  $V_0[a(X)p(X; \theta_0^*) - a(X)Y]$ , from the sample.

For instance the expression (3.1) of  $\hat{B}_n$  has to be replaced by

$$\hat{B}_n = \frac{N^*}{n} \sum_{i=1}^n \left[ \frac{a(x_i)}{a^*(x_i)} p(x_i; \hat{\theta}_n) \right] - \frac{N^*}{n} \sum_{i=1}^n \left[ \frac{a(x_i)}{a^*(x_i)} p(x_i; \hat{\theta}_n) \right],$$

and similarly the associated change of probability has also to be applied when estimating the variance term.

### Appendix 3

#### Enlarged Validation Sample

We provide below the decomposition of the required capital for grades 3, 4, 5, when the population of interest is 2018 and the validation sample covers the years 2016 and 2017. The new tables are given below :

**Table 1' :Sample and Population**

grade	Population size (2018)	validation sample size (2016 – 2017)	average EAD population
3	1443	2345	1104265 €
4	3790	6695	704563 €
5	7899	14118	589639 €

**Table 2' :Decomposition of the Required Capital**

grade	standard EL	adjustment for model risk	estimation risk + prediction error misspecified
3	352 €	-241 €	60 €
4	323 €	-90 €	326 €
5	657 €	-435 €	197 €

Therefore the total required capital per firm is 169 €, 559 € and 419 € for grades 3, 4, 5, respectively.

Figure 1: Distribution of EAD within Segments

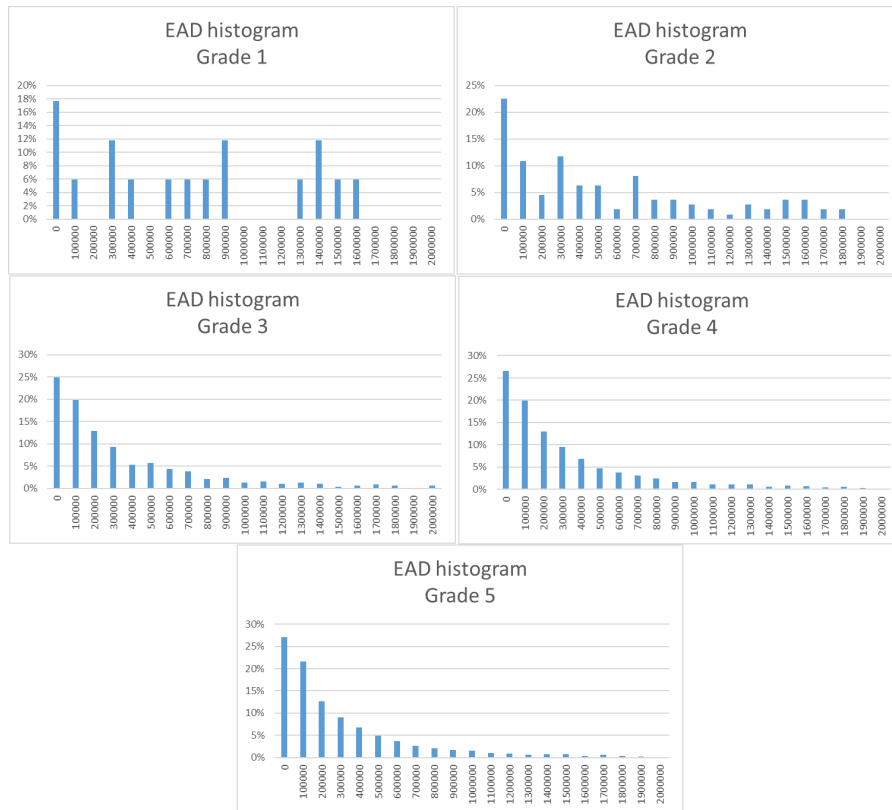


Figure 2: Plot of EAD vs Size

