Evil deeds in urban economics

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Abstract

The purpose of this note is to update an ancient controversy over the comparison between discrete and continuous agent models of land use and agent location in urban economics. Berliant (1985) shows that that the following statement is self-contradictory: "There is a continuum of agents, each of whom owns or is endowed with a positive Lebesgue measure of land." A corollary follows: "As the number of agents tends to infinity, the set of agents who own a positive Lebesgue measure of land shrinks to zero." The basic question is this: Under what circumstances, if any, can we reconcile the two models? JEL classifications: D51, R13, R14 Keywords: Large urban economies, Continuous and discrete agent models

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You just asked me, when will it end?

Hahahahaha, well let me tell you

Once an evil deed is done, then it never ends

It goes on, and it will go on forever... (Wu Tang Clan, Evil Deeds, *Wu Tang Chamber Music*)

1 Introduction

The purpose of this note is to update an ancient controversy over the comparison between discrete and continuous agent models of land use and agent location in urban economics. Berliant (1985), which is mathematically trivial, shows that the following statement is self-contradictory: "There is a continuum of agents, each of whom owns or is endowed with a parcel of positive Lebesgue measure of land." A corollary follows: "As the number of agents tends to infinity, the set of agents who own a parcel of positive Lebesgue measure of land shrinks to zero." The basic question is this: Under what circumstances, if any, can we reconcile the two models? The literature proceeds with Papageorgiou and Pines (1990), Asami, Fujita and Smith (1991), Berliant (1991), Berliant and ten Raa (1991), Kamecke (1993), and Berliant and Sabarwal (2008). McLean and Meunch (1981) to a certain degree anticipate the controversy. Berliant and Sabarwal (henceforth BS) show that, in the versions of the discrete and continuous models used below, *an empirically relevant comparative static differs in the two types of models.*

In this note, we compare equilibrium price and population densities of the linear (one dimensional) monocentric city models for a continuum of agents and a finite number of agents. The models will feature the same utility functions and endowments for all consumers. The finite model and the continuum model will have the same "number" of consumers. In the finite model this is the actual integer number. In the continuum model, this is interpreted as the (Lebesgue) measure of consumers on the real line. There is an absentee landlord. The city boundary is endogenous, and determined by agricultural land rent.

We note that as population goes to infinity, in both models per capita land consumption tends to zero whereas land prices tend to infinity.

This note is organized as follows. Section 2 introduces and analyzes the continuous agent model. Section 3 does the same for the discrete agent model. Section 4 graphs the equilibrium prices for the two models for comparison. Section 5 presents a regression assuming that the discrete agent model is the true model that is estimated by a continuous agent model. Section 6 compares equilibrium population distributions of the two models. Finally, section 7 provides our conclusions and directions for future research.

2 The Continuous Agent Model

2.1 Notation

Here we introduce a canonical model of urban residential location. Interpretation and explanation can be found in Fujita (1989).

r	Distance from CBD
z	Consumption of composite commodity
s	Land consumption
t	Per kilometer commuting cost
w	Endowment of composite good per capita
$u(s,z)$	Utility function
N	Measure of consumers (exogenous)
$n(r)$	Density of consumers (per kilometer)
$\overline{r}$	City boundary (endogenous)
$\overline{u}$	Utility level (endogenous)
$p(r)$	Per unit rent at distance r from the CBD (endogenous)
1	Agricultural land rent (exogenous)
1	Land supply density (exogenous)

2.2 Equilibrium

Take $u(s,z) = z + \ln(s)$. This is classic, as the resulting rent gradient (derived below) is often used for empirical estimation; see for example Mills (1972, p. 247, Table 1). Quasi-linear utility is useful for solving the finite model explicitly. Typically, we would have to solve the finite model numerically if we don't have quasi-linearity.

Consumer optimization problem:

$$\max_{z,s} z + \ln(s)$$

subject to:

$$z + p(r) \cdot s + r \cdot t = w$$

Substituting the budget into the objective function,

$$\max_{s} w - p(r) \cdot s - r \cdot t + \ln(s)$$

$$\begin{aligned}
p(r) \cdot s(r) &= 1 \\
z(r) &= w - p(r) \cdot s(r) - r \cdot t = w - 1 - r \cdot t \\
\ln(s(r)) &= \overline{u} - z(r) = \overline{u} - w + 1 + r \cdot t \\
s(r) &= \exp(\overline{u} - w + 1) \cdot \exp(r \cdot t) \\
p(r) &= n(r) = \frac{1}{s(r)} = \exp(w - 1 - \overline{u}) \cdot \exp(-r \cdot t) \qquad (1)
\end{aligned}$$

Moreover, we know that rent at the urban boundary must be 1:

$$p(\overline{r}) = 1 = \exp(w - 1 - \overline{u}) \cdot \exp(-\overline{r} \cdot t)$$

Hence,

$$\exp(w - 1 - \overline{u}) = \exp(\overline{r} \cdot t)$$

Next,

$$\begin{aligned}
N &= \int_0^{\overline{r}} n(r) dr \\
&= \exp(w - 1 - \overline{u}) \cdot \int_0^{\overline{r}} \exp(-r \cdot t) dr \\
&= \exp(w - 1 - \overline{u}) \cdot \left[-\frac{1}{t} \exp(-r \cdot t) \Big|_0^{\overline{r}} \right] \\
&= \exp(w - 1 - \overline{u}) \cdot \frac{1}{t} \left[1 - \exp(-\overline{r} \cdot t)\right]
\end{aligned}$$

Therefore,

$$\exp(w - 1 - \overline{u}) = \frac{Nt}{1 - \exp(-\overline{r} \cdot t)}$$

and

$$p(r) = \frac{Nt}{1 - \exp(-\overline{r} \cdot t)} \cdot \exp(-r \cdot t)$$

Since $p(\overline{r}) = 1$,

$$\begin{aligned} 1 &= \frac{Nt}{1-\exp(-\overline{r}\cdot t)} \cdot \exp(-\overline{r}\cdot t) \\ 1-\exp(-\overline{r}\cdot t) &= \exp(-\overline{r}\cdot t)Nt \\ 1 &= \exp(-\overline{r}\cdot t)\,[Nt+1] \\ \overline{r} &= \frac{\ln(Nt+1)}{t} \end{aligned}$$

Therefore,

$$p(r) = \exp(-rt)\,[Nt+1]$$

For later use, note that:

$$\ln[p(r)] = \ln(Nt+1) - rt \tag{2}$$

3 The Discrete Agent Model

3.1 An Important Note

There is an apparent discrepancy between the equilibrium land price functions in Berliant and Fujita (1992) (henceforth BF) and Berliant and Sabarwal (2008). In BF, the equations of importance are (4.12) and (4.15), (4.37), (4.39b), (4.41), (4.42), (5.4), and (5.5). In BS, the equation for the price density is given in the short section 2.3, p. 441.

In BF heterogeneity in consumer endowments or wealth, and hence in equilibrium utility levels, is allowed whereas in BS it isn't. In BF, the reason we include the term $f_i^{-1}(x)$ in the price density, and consequently $\sum_{j=2}^{i} \varepsilon_j$ on the expenditure side of the budget constraint is to prevent discrete moves by poorer consumers from the parcels closer to the CBD to parcels farther from the CBD. These are not marginal but discrete moves, so they can't be accounted for using first order conditions. In BS, there is no such issue, since consumers moving farther out will, after satisfying first order conditions for optimization subject to the budget, consume the same bundle as one of their twins, and achieve the same utility level. What happens in BF is that if we used the price density from BS, a poor consumer might find a subset of a wealthier consumer's parcel better. To deter them, we must make prices in outlying parcels higher. This is of particular importance when utility is not quasi-linear.

The bottom line is that both equilibrium price densities are correct. There are multiple equilibrium price densities in this model. The price density in BS is simpler because all consumers are assumed to be identical. In BS, the equilibrium found is best for consumers and worst for the landlord. The equilibria constructed in BF are generally worse than this one for the consumers. Also note that the results in BF do not apply to quasi-linear utility because the boundary condition used in BF (called Assumption 1 there) is not satisfied.

After introducing the notation for the finite model, we shall verify that in the case of log-linear utility used in this note, the equilibrium utility levels achieved by consumers for the price density proposed by BS are indeed identical.

3.2 Notation

We will use analogous notation for the finite model. There are just a few alterations. First, the number of consumers is N, an integer. Second, we index bundles (s_i, z_i) by consumer $i = 1, 2, ..., N$, where consumers (who are *ex ante* identical) are ordered from the CBD outward. Finally, we call the price density for this model $P(r)$ to distinguish it from the continuum of agents model.

3.3 Equilibrium

With quasi-linear utility, it's easiest to solve for the equilibrium parcels first. The first order conditions for the model with finitely many agents yield:[1]

$$\begin{aligned} \frac{1}{s_i} &= 1 + (N - i) \cdot t \\ s_i &= \frac{1}{1 + (N - i) \cdot t} \end{aligned} \tag{3}$$

The price density $P(r)$ can be defined as follows. For consumer 1, the consumer closest to the CBD with the smallest parcel, the price density is $(N - 1)t + 1$. This applies for r such that $\frac{1}{r} \geq (N - 1)t + 1$ or $0 \leq r \leq \frac{1}{(N-1)t+1}$. Then, $u(s_1, z_1) = w - 1 - \ln(1 + (N - 1) \cdot t)$.

[1] This calculation is obviously not as easy with, say, Cobb-Douglas utility.

For consumer 2, $P(r)$ is defined as follows. For r with $\frac{1}{(N-2)t+1} \geq r \geq \frac{1}{(N-1)t+1}$, $P(r) = \frac{1}{r}$. For r with $\frac{1}{(N-2)t+1} \leq r \leq \frac{1}{(N-1)t+1} + \frac{1}{(N-2)t+1}$, $P(r) = (N-2)t+1$. Then

$$\begin{aligned}
u(s_2, z_2) &= w - \frac{t}{(N-1)t+1} - \ln(1+(N-2)\cdot t) - \int_{\frac{1}{1+(N-1)\cdot t}}^{\frac{1}{1+(N-1)\cdot t}+\frac{1}{1+(N-2)\cdot t}} P(r)dr \\
&= w - \frac{t}{(N-1)t+1} - \ln(1+(N-2)\cdot t) \\
&\quad - \int_{\frac{1}{1+(N-1)\cdot t}}^{\frac{1}{(N-2)t+1}} \frac{1}{r}dr - \int_{\frac{1}{(N-2)t+1}}^{\frac{1}{(N-1)t+1}+\frac{1}{(N-2)t+1}} [(N-2)t+1]\, dr \\
&= w - \frac{t}{(N-1)t+1} - \ln(1+(N-2)\cdot t) \\
&\quad + \ln((N-2)t+1) - \ln(1+(N-1)\cdot t) - \frac{(N-2)t+1}{(N-1)t+1} \\
&= w - 1 - \ln(1+(N-1)\cdot t) \\
&= u(s_1, z_1)
\end{aligned}$$

Thus, the equilibrium utility levels for consumers 1 and 2 are the same. Similar calculations should apply for the other consumers, verifying the claim in Berliant and Sabarwal (2008, p. 441) that this is an equilibrium for the finite model with identical consumers for this particular functional form of utility.[2]

Proceeding to compute the price density for all locations,[3]

$$P(r) = \begin{cases} (N-1)t+1 \text{ for } \frac{1}{(N-1)t+1} \geq r \geq 0 \\ \frac{1}{r-\sum_{j=1}^{i-2}\frac{1}{1+(N-j)\cdot t}} \text{ for } \sum_{j=1}^{i-2}\frac{1}{(N-j)t+1} + \frac{1}{(N-i)t+1} \geq r \geq \sum_{j=1}^{i-1}\frac{1}{(N-j)t+1} \text{ and } i \geq 2 \\ (N-i)t+1 \text{ for } \sum_{j=1}^{i}\frac{1}{(N-j)t+1} \geq r \geq \sum_{j=1}^{i-2}\frac{1}{(N-j)t+1} + \frac{1}{(N-i)t+1} \text{ and } i \geq 2 \end{cases}$$

4 Price Graphs

Note that for the finite model, the price density $P(r)$ must be defined piecewise, segment by segment. This is quite labor intensive.

[2] There are likely other equilibria, with for example a non-constant price density for the first consumer. The one we examine appears to be the equilibrium that maximizes consumer utility and minimizes landlord income among all equilibria.

[3] The slight differences between this expression and the one in BS are a type-o in that paper ($\zeta_s(s - \sum_{k=1}^{n-1} s_k, u)$) and the way we index consumers.

Let's try $N = 10$, $t = 1$ to get things going. The graph of both the discrete agent model land price density, $P(r)$, and the continuous agent model land price density, $p(r)$, are given in Figure 1.

Notice that when more consumers are added by increasing N, due to the quasi-linear utility function, the $P(r)$ function is only modified from $N = 1$ leftward. That is, $P(r)$ is the same for $i = 2, 3, ..., 10$. What this means is that the graph is extended to the left of $r = 0$, but remains almost the same as the graph below for $r > 0$. The only difference is in the first consumer, who is special.

$$p(r) = \begin{cases} \exp(-r) \cdot 11 & 0 \leq r \leq \ln(11) \\ 1 & \ln(11) \leq r \end{cases}$$

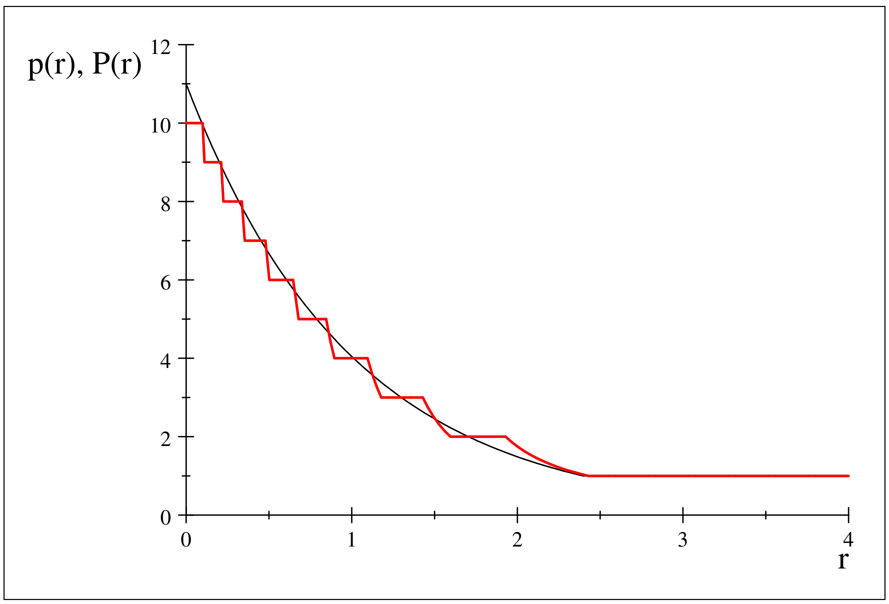

Figure 1: Price densities: p(r) for continuum model (black), P(r) for finite model (red); $t = 1$

Now let's try $t = 10$. The land price densities are given in Figure 2.

$$p(r) = \begin{cases} \exp(-r \cdot 10) \cdot 101 & 0 \leq r \leq \ln(101)/10 \\ 1 & \ln(101)/10 \leq r \end{cases}$$

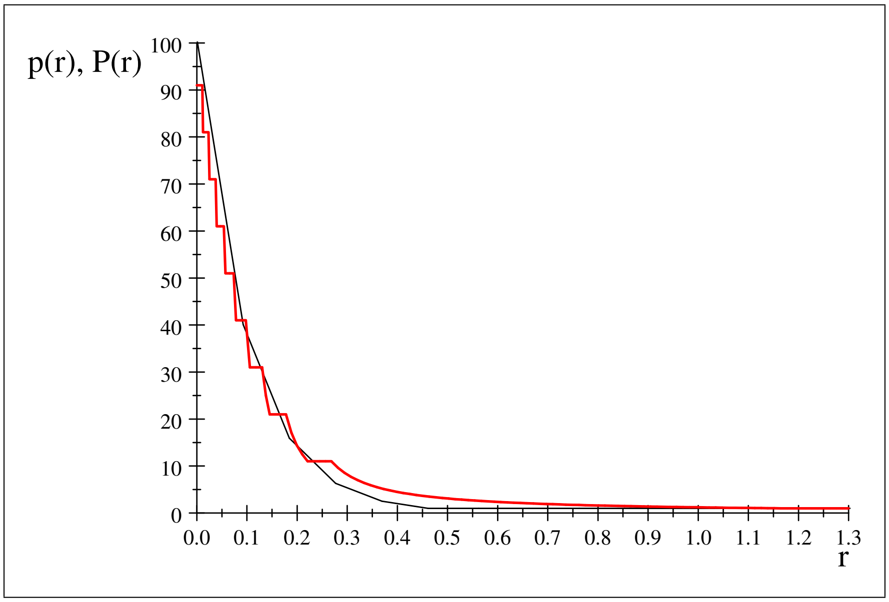

Figure 2: Price densities: p(r) for continuum model (black), P(r) for finite model (red); $t = 10$

5 Regression

Since the model with a continuum of agents is often used empirically, here we analyze the contrast between the two models. If the continuum model makes sense only when there is a finite model nearby, we perform the following exercise. Suppose that the true model is the finite one. We generate data using the finite model equilibrium. Use the model with 10 consumers as illustrated in Figure 1. For each consumer i, calculate the front location distance to their equilibrium parcel, called r_i. Then compute the average price on a parcel

$$p_i = \frac{\int_{r_i}^{r_i+s_i} P(r)dr}{s_i}$$

There will be 10 data points. Then we run the regression implied for the continuum equilibrium price gradient, (2). That is,

$$\ln(p_i) = \ln(Nt + 1) - r_i \cdot t + \varepsilon_i$$

If the continuum model is a good approximation to the finite model, the constant in this regression should be $\ln(11) = 2.3979$, and the coefficient on r_i should be -1.

Front Location	Average Price	ln(Average Price)
0	10	2.3026
.1	9.0482	2.2026
.21111	8.0534	2.0861
.33611	7.0597	1.9544
.47897	6.0678	1.803
.64563	5.0783	1.625
.84563	4.0926	1.4092
1.0956	3.113	1.1356
1.429	2.1443	.76281
1.929	1.1931	.17655

In fact, the constant is 2.321991 with standard error .007885, whereas the regression coefficient on r_i is -1.09772 with standard error .008557. The regression estimate of the finite model using the continuum model is thus:

$$p(r) = \exp(-r \cdot 1.09772) \cdot 10.196$$

Next, in Figure 3 we reproduce Figure 1 with the regression estimate:

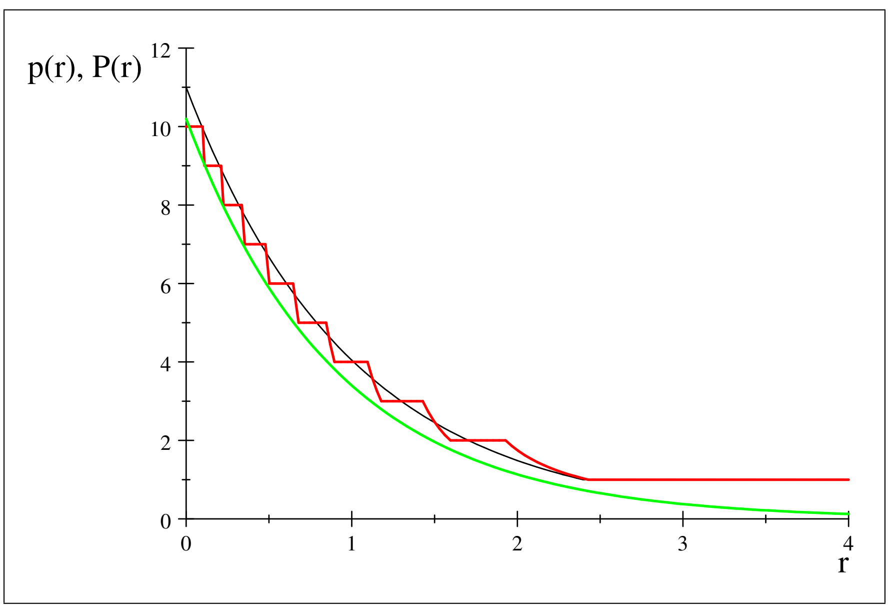

Figure 3: Price densities: p(r) for continuum model (black), P(r) for finite model (red), regression estimate of finite model (green); $t = 1$

With this specification, namely using the front location for distance from the CBD and the average price for the unit price of the parcel, and presuming that the finite model is the true model, in other words it generates the data, we have the following conclusions from the regression. The unit commuting cost is overestimated relative to both the finite model and the continuum model. The intercept (unit price of land at the CBD) is underestimated. Therefore, if the continuum model is used empirically, it will generate biased estimates.

An alternative would be to generate locations randomly and use the best fit of the continuum model to the finite model. But this makes little sense, since one is not likely to observe prices at different points of the same parcel. Moreover, sampling will be more frequent for larger parcels.

6 Population Graph

We will graph the cumulative population at distance r from the CBD. For the continuous population model, with our functional form, we can explicitly compute from (1) its integral, namely the cumulative population distribution in the case $N = 10$, $t = 1$:

$$\int_0^{r'} n(r)dr = 11 - 11\exp(-r')$$

For the discrete model, we cumulate agents at the back ends of their parcels. For this purpose, we utilize (3), and call the cumulative $N(r)$. Figure 4 illustrates the cumulative population as a function of distance from the CBD.

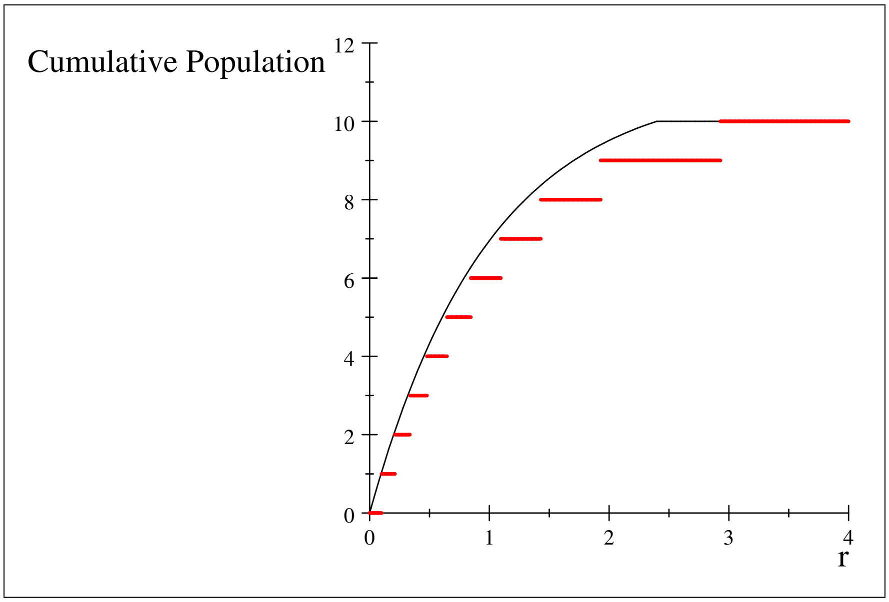

Figure 4: Cumulative population: $\int$n(r)dr for continuum model (black), $N(r)$ for finite model (red); $t = 1$

It is obvious that deviations between the models get larger farther from the CBD.

7 Conclusions

There are several implications of this exercise:

1. The overall fit is not bad, but there are a few very important caveats.

2. It's not clear which parameters are crucial for this exercise. We have examined only $t = 1$ and $t = 10$, using log quasi-linear utility. It might be interesting to multiply $\ln(s)$ in the utility function by a constant.

3. It would be of interest to use Cobb-Douglas to introduce income effects, but this is hard computationally. The level of utility and land consumption are more difficult to solve in the finite model. We could perhaps use another quasi-linear utility, but the character of the solutions will likely not change.

4. It's not clear that increasing N would do much in this context of quasi-linear utility. For the finite model, things would appear to be more or less the same, with more consumer land consumption added below location 0. For

the continuum model, only the constant attached to the price changes. **This suggests that the approximation will not get better or worse as N increases, in the sense that per-consumer error will be constant.**

5. But it is of interest to explore the comparison between the two models as N increases from 10 to 20 to 100. **An important issue is if we make N very large, then per capita land consumption will necessarily be close to 0, whereas prices will be very large. In other words, can N be made large enough so that the models are similar, but so that land consumption is not ridiculously small?** This seems to depend on the choice of units.

6. How should the model be calibrated, namely how should parameters be chosen? That will also depend on the choice of units.

7. As is obvious from the figures, if we use the *same parameters* for both the continuum and discrete models, the continuum model has higher equilibrium prices closer to the CBD, whereas the finite model has equilibrium prices higher farther from the CBD. Since the discrete model is not self-contradictory, we can infer a bias in results derived from the continuum model, such as comparative statics.

8. Consider the finite model to be the truth, and the continuum model to be the approximation. Then, **there appears to be a systematic bias in the continuum model estimate of the finite model price.** Specifically, *empirical estimates of the rent gradient from the continuum model are biased upward* relative to the value inserted into the finite model. Can this be proved generally?

9. How should the bias be corrected?

10. The cumulative population distributions for the two models also diverge as the distance from the CBD increases.

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