Power generation portfolios: A parametric formulation of the efficient frontier

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Abstract

The Portfolio Theory has been extensively used as a planning tool for power generation diversification. However, no one of the existing papers provide a detailed explanation on how the efficient frontier of the Power Generation Portfolio (PGP) is costructed. We provide a parametric formulation of the efficient frontier of PGP of up to 5 technologies. The analysys takes advantages of the fact that the risk of the PGP is a convex function of the shares of the different technologies. The parametric formulation of the efficient frontier of the PGP constitutes a powerfull policy tool for power generation policy-makers.

JEL cassification: D81, G11, Q40, Q49

Keywords: Portfolio, Power Generation, Efficient Frontier, Risk, NPV.

1 Introduction

The Portfolio Theory, developed by Markowitz (1952), has been extensively used to design plans of power generation diversification (See DeLlano-Paz *et al.* (2017) for a review). However, no one of the existing papers provide a detailed explanation on how the efficient frontier of the Power Generation Portfolio (PGP) is constructed. Without any exception, all of them only present a graph depicting the efficient frontier of the corresponding PGP (e.g., Costa *et al.* (2017), Pinheiro Neto *et al.* (2017), Adams and Jamasb (2016), Jain *et al.* (2014), Cunha and Ferreira (2014), Roques *et al.* (2010), Vithayasrichareon *et al.* (2010a), Vithayasrichareon *et al.*(2010b), Roques *et al.* (2008), and Awerbuch and Berger (2003)).

In the present paper we aim to fill this gap in the literature by providing a parametric formulation of the efficient frontier of PGP of up to 5 technologies. Following the existing literature, in present analysis the **efficient frontier** refers to the set of the PGPs that maximize their Expected Net Present Value (ENPV)

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for a given level of risk. This is, the ENPV of an **efficient** PGP can be increased only by increasing its risk (Awerbuch and Berger (2003)). Note that the present analysis could be directly applied to PGP using Levelized Cost of Electricity (LCOE). In such a case, the efficient frontier would be a set of PGPs which can yield the lowest expected energy costs at given, acceptable levels of expected risk (Jansen *et al.* (2006)).

The analysis takes advantages of the fact that the risk of the PGP, given by the Standard Deviation (SD) or the Variance of the NPV, is a convex function of the shares of the different technologies. First, we obtain the shares of the technologies that guarantee the minimum risk of the NPV of the PGP. We then obtain the maximum ENPV of the PGP. Finally, we construct the efficient frontier that corresponds to the parametric equation of the shares of the technologies that link the minimum risk of the NPV of the PGP to its maximum ENPV.

The parametric formulation of the efficient frontier of the PGP allows to tackle the problem of energy generation diversification in an economy. Then, it constitutes a powerful policy tool for power generation policy-makers. Actually, it could be applied to portfolios of assets different than power generation technologies.

The paper also shows that the "portfolio effect" results from the fact that the risk of the PGP is a convex function of the shares of the different technologies.

As this is a methodological paper, instead of focusing the analysis on PGP of a particular economy, we use hypothetical data. This fact allows to show the scope of the methodology and, at the same time, improves exposition simplicity.

The whole analysis relies on the assumption that the covariances of the NPVs of the different technologies is zero. Although this is a strong assumption, it leads to gains in tractability and in the scope of the methodology formulated.

To the best of our knowledge, this is the first effort to provide a detailed methodology to construct, parametrically, the efficient frontier of PGPs.

The paper is organized as follows: In Section 2, we present the preliminaries. Section 3 presents PGP of 2 technologies. Section 4 presents PGP of 3 technologies. PGP of 4 technologies are presented in section 5. PGP of 5 technologies are presented in section 6. Section 7 contains the final remarks and conclusions. The appendix contains the formal proofs.

2 Preliminaries

We apply the Portfolio Theory developed by Markowitz (1952) to find the efficient power generation mix: the ENPV of the generation mix can be increased only by increasing its risk. As usual, risk is measured by the SD or, alternatively, by the variance. Formally, let X_i be the random variable that represents the NPV of technology i. Let Y a random variable describing the NPV of the PGP which is defined as follows:

$$Y = \sum_{i=1}^{n} \alpha_i X_i, \quad \text{with} \quad \sum_{i=1}^{n} \alpha_i = 1. \tag{1}$$

Where $\alpha_i \in [0, 1]$ represent the share of technology i. Following result provides the basic tools for the analysis.

Lemma 1 *Let X_i a random variable that represents the NPV of technology i with mean μ_i and variance σ_i^2. Where $\alpha_i \in [0, 1]$ is the the share of technology $i = 1, 2, \ldots, n$. Let the PGP be represented by the random variable $Y = \sum_{i=1}^{n} \alpha_i X_i$ with $\sum_{i=1}^{n} \alpha_i = 1$. Then*

$$\begin{gathered} E(Y) = \sum_{i=1}^{n} \alpha_i E(X_i), \\ \mu_Y = \sum_{i=1}^{n} \alpha_i \mu_i. \end{gathered} \tag{2}$$

and

$$\begin{gathered} Var(Y) = E[Y - \mu_Y]^2, \\ \sigma_Y^2 = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + \sum\sum_{i<j} \alpha_i \alpha_j \sigma_{i,j}. \end{gathered} \tag{3}$$

where the double summation extends to any values i and j, from 1 to n, such that $i < j$. In addition, $\sigma_{i,j} = E\left[(X_i - \mu_i)\left(X_j - \mu_j\right)\right]$ is the covariance of the NPVs of technologies i and j.

Proof. See pp. 158, Freund *et al.* (2000). ■

First result of Lemma 1 indicates that the ENPV of the PGP is a convex sum of the ENPVs of the different technologies. Following corollary describes such fact.

Corollary 2 *Assume that the ENPV of technology 1 is the greatest while the ENPV of technology n is the lowest. Then, it holds that $\mu_1 \geq \mu_Y \geq \mu_n$ for $\sum_{i=1}^{n} \alpha_i = 1$.*

Second result of Lemma 1 captures the role of the covariances of the NPVs of the different technologies on the risk of the PGP. If the covariances of the NPVs of the different technologies are negative, the risk of the PGP reduces. On the other hand, the risk of the PGP increases when the covariances among the NPVs of the technologies are positive. When some covariances are positive while others are negative, it is difficult to determine the final effect on the risk of the PGP. The existing literature reports that, regardless of the variable used to construct the PGP (NPV, LCOE, capacity factor, or installed capacity), the covariances amongst the different technologies have an absolute value less than one (e.g., Pinheiro Neto *et al.* (2017), Adams and Jamasb (2016), Cunha and Ferreira (2014), Roques *et al.* (2010), and Roques *et al.* (2008)). As the NPVs of the technologies are reported in million of dollars (or pounds) and the shares of the different technologies are less than one, we expect that the term $\sum\sum_{i<j}\alpha_i\alpha_j\sigma_{i,j}$ to be significantly smaller than the term $\sum_{i=1}^{n}\alpha_i^2\sigma_i^2$. Then, for the following analysis we assume that the covariances among the different technologies is zero. this is, $\sigma_{i,j}=0$, for any values i and j , from 1 to n, such that $i<j$. This assumption leads to a lack of precision in calculating the minimum risk of the PGP. However, such loss is compensated by a gain in tractability and by the scope of the methodology formulated.

Worth noting that the assumption that the covariances of the different technologies is zero works well when the PGPs use NPVs or LCOSTs of the different technologies. Nevertheless, such assumption does not seem feasible when the PGPs use capacity factor or installed capacity. In those cases, we can not guarantee that the term $\sum\sum_{i<j}\alpha_i\alpha_j\sigma_{i,j}$ will be significantly smaller than the term $\sum_{i=1}^{n}\alpha_i^2\sigma_i^2$ (e.g., Cunha and Ferreira (2014) and Roques *et al.* (2010)). Then, in such a context, assuming zero covariance amongst the different technologies would lead to meaningful miscalculations of the risk of the NPV of the PGP.

Then, from now on, the risk of the PGP is described by its SD as follows:

$$\sigma_Y=\sqrt{\sum_{i=1}^{n}\alpha_i^2\sigma_i^2} \tag{4}$$

Expression (4) and Corollary 2 provide the tools to construct a parametric formulation of the **efficient frontier** of the PGPs.

As the main contribution of the paper is the parametric formulation of the efficient frontier, instead of focusing the analysis on PGP of a particular economy, we use hypothetical data. This fact allows to show the scope of the methodology, and improves exposition simplicity. We consider five technologies: 1) Hydro Power Plant (Hydro); 2) Wind Power Plant (Wind); 3) Combined Cycle Gas Turbine (CCGT); 4) Advanced Gas-Cooled reactor (Nuclear), and; 5) Integrated Gasification Combined Cycle (Coal). Following table present the statistics of the NPV of the different technologies in USD million.

Statistics	Hydro	Wind	CCGT	Nuclear	Coal
ENPV	500	400	100	-50	-100
St. Dev. NPV	350	450	550	300	400

Table 1: Single technology NPV distribution statistics

Now we have all the building blocks to provide a parametric formulation of the efficient frontier of PGPs. We start with portfolios of two technologies.

3 Portfolios of two technologies

Following result exploits the fact that the SD of the PGP is a convex function of the shares of technologies 1 and 2, (α_1,α_2).

Proposition 3 *From expression (4) the SD of the NPV of the PGP of two technologies is given by* $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2}$. *Assume that the NPV of technology* 2 *is the less risky. For* $\sigma_{1,2} = 0$, $\alpha_i \in [0,1]$ *for* $i = 1,2$, *and* $\alpha_1 + \alpha_2 = 1$ *it holds that*

a) *The risk of the NPV of the PGP, given by* σ_Y, *reaches its global minimum at*

$$\begin{pmatrix} \alpha_1^* \\ \alpha_2^* \end{pmatrix} = \frac{1}{\sigma_1^2+\sigma_2^2}\begin{pmatrix} \sigma_2^2 \\ \sigma_1^2 \end{pmatrix}$$

b) *The minimum risk of the NPV of the PGP is*

$$\sigma_Y^* = \sqrt{\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}} < \sigma_2.$$

Proof. See appendix. ■

3.1 Efficient frontier

Following result provides the parametric formulation of the efficient frontier for PGP of two technologies.

Proposition 4 *Let* $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2}$ *be the SD of the NPV of the PGP. Assume that* $\mu_1 \geq \mu_2$, *then the following holds:*

a) *The* ***efficient frontier*** *corresponds to the following parametric equation of the shares of technologies* 1 *and* 2,

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix},$$

where the parameter α *is such that* $\alpha_1^* \leq \alpha \leq \alpha^{df} \leq 1$. α^{df} *refers to the value of* α *that guarantees certain amount of the ENPV of the PGP.*

b) *The SD in the efficient frontier is given by* $\sigma_Y^* \leq \sigma_y \leq \sigma\left(\alpha^{df}\right)$. *Note that* $\sigma\left(\alpha^{df}\right) \leq \sigma_1$.

c) *The maximum ENPV for every corresponding level of risk is given by* $\mu\left(\alpha_1^*\right) \leq \mu_y \leq \mu\left(\alpha^{df}\right)$. *Note that* $\mu\left(\alpha^{df}\right) \leq \mu_1$.

Proof. See appendix. ■

3.2 Illustrative Portfolios: CCGT-Coal

From Table 1, the corresponding ENPV and variance of the CCGT (cc) and Coal (co) are: $\mu_{cc} = 100$, $\mu_{co} = -100$, $\sigma_{cc}^2 = 302500$, and $\sigma_{co}^2 = 160000$.

1. Following Proposition 3, CCGT corresponds to technology 1 and Coal to technology 2. Then, the shares of the technologies that ensure the minimum risk are

$$(\alpha_{cc}^*, \alpha_{co}^*) = (0.34595, 0.65405)\,.$$

The minimum risk reached by this PGP is $\sigma_Y^* = 323.49$. And the maximum ENPV for such level of risk is $\mu_Y = -30.81$.

2. From Proposition 4, the efficient frontier corresponds to the following parametric equation of the shares of the two technologies

$$\begin{pmatrix} \alpha_{cc} \\ \alpha_{co} \end{pmatrix} = \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix},$$

for $0.34595 \leq \alpha \leq 0.6919$. Note that the PGP of CCGT-Coal reaches the maximum ENPV when $\alpha_{cc} = 1$, and $\mu_Y = \mu_{cc} = 100$. However, devoting a share of 100% to CCGT would be very risky in economic and social terms as the SD of the NPV of CCGT is the greatest, $\sigma_{cc} = 550$. In this case the desicion-maker should take a criteria to define the efficient frontier. We propose the upper limit of the efficient frontier to be $\alpha^{df} = 0.6919$. This fact guarantees that the PGP reaches a risk equal to $\sigma_{co} = 400$, the minimum risk of the two technologies.

Note that in this case we might say that CCGT "**weakly dominates**" the PGP as it has the greatest ENPV which is also relatively risky.[1] On the other hand, a technology "**strongly dominates**" a PGP if it has the greatest ENPV and the lowest risk. Roques *et al.* (2010), Table 4, 2nd scenario, provides a good example where CCGT "strongly dominates" the PGP. However, an economy would face a potential social risk by placing its Power Generation in one single technology. Even in such case, the decision-maker should support Power Generation diversification.

3. The SD in the efficient frontier is given by $323.49 \leq \sigma_y \leq 400$.

4. The maximum ENPV for every corresponding level of risk is given by $-30.81 \leq \mu_y \leq 38.38$.

[1] See Pinheiro Neto *et al.* (2017) for clear example where Hydro "weakly dominates" the PGP.

5. The feasible PGPs of CCGT-Coal are shown in the following figure:

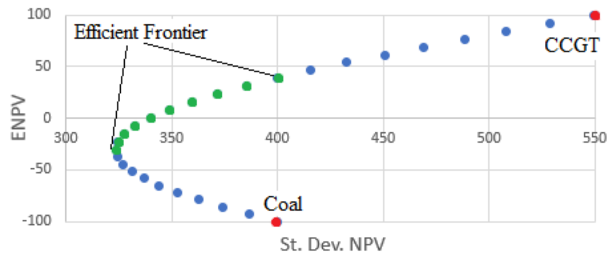
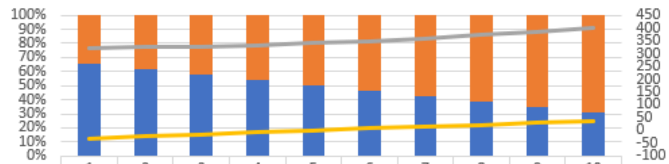


Figure 1: Feasible PGPs of CCGT-Coal

6. The parameters of the efficient frontier are presented in the following figure:



	1	2	3	4	5	6	7	8	9	10
Share CC	0.35	0.38	0.42	0.46	0.50	0.54	0.58	0.62	0.65	0.69
Share Co	0.65	0.62	0.58	0.54	0.50	0.46	0.42	0.38	0.35	0.31
St. Dev.	323.49	324.55	327.69	332.87	339.98	348.90	359.51	371.66	385.21	400.00
ENPV	-30.81	-23.12	-15.43	-7.75	-0.06	7.63	15.32	23.00	30.69	38.38

Figure 2: Efficient Frontier of PGP of CCGT-Coal

4 Portfolios of three technologies

Following result exploits the fact that the SD of the PGP is a convex function of the shares of technologies 1, 2, and 3, $(\alpha_1, \alpha_2, \alpha_3)$.

Proposition 5 *From expression (4) the SD of the NPV of the PGP of three technologies is given by* $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2}$. *Assume that the NPV of technology 2 is the less risky. For* $\sigma_{1,2} = \sigma_{1,3} = \sigma_{2,3} = 0$, $\alpha_i \in [0, 1]$ *for* $i = 1, 2, 3$, *and* $\sum_{i=1}^{3} \alpha_i = 1$ *it holds that*

a) *The risk of the NPV of the of the PGP, given by* σ_Y, *reaches its global minimum at*

$$\begin{pmatrix} \alpha_1^* \\ \alpha_2^* \\ \alpha_3^* \end{pmatrix} = \frac{1}{|A_3|} \begin{pmatrix} \sigma_2^2\sigma_3^2 \\ \sigma_1^2\sigma_3^2 \\ \sigma_1^2\sigma_2^2 \end{pmatrix},$$

where $|A_3| = \sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_2^2\sigma_3^2$.

b) *The minimum risk of the NPV of the PGP is*

$$\sigma_Y^* = \sqrt{\frac{\sigma_1^2\sigma_2^2\sigma_3^2}{|A_3|}} < \sigma_2.$$

Proof. See appendix. ■

4.1 Efficient frontier

Following result provides the parametric formulation of the efficient frontier for PGP of three technologies.

Proposition 6 *Let $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2}$ the SD of the NPV of the PGP. Assume that $\mu_1 \geq \mu_2 \geq \mu_3$, then following holds:*

a) *The **efficient frontier** corresponds to the following parametric equation of the shares of technologies 1, 2, and 3,*

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha^{1+x} \\ \alpha - \alpha^{1+x} \\ 1-\alpha \end{pmatrix},$$

where the parameter α is such that $[\alpha_1^*]^{\frac{1}{1+x}} \leq \alpha \leq \alpha^{df} \leq 1$. Let x be given by $x = \frac{\ln\left[\frac{\alpha_1^*}{\alpha_1^*+\alpha_2^*}\right]}{\ln\left[\alpha_1^*+\alpha_2^*\right]}$. α^{df} refers to the value of α that guarantees certain amount of the ENPV of the PGP.

b) *The SD in the efficient frontier is given by $\sigma_Y^* \leq \sigma_y \leq \sigma\left(\alpha^{df}\right)$. Note that $\sigma\left(\alpha^{df}\right) \leq \sigma_1$.*

c) *The maximum ENPV for every corresponding level of risk is given by $\mu\left([\alpha_1^*]^{\frac{1}{1+x}}\right) \leq \mu_y \leq \mu\left(\alpha^{df}\right)$. Note that $\mu\left(\alpha^{df}\right) \leq \mu_1$.*

Proof. See appendix. ■

4.2 Illustrative Portfolios: CCGT-Nuclear-Coal

From Table 1, the corresponding ENPV and variance of the CCGT (cc), Nuclear (nu) and Coal (co) are: $\mu_{cc} = 100$, $\mu_{nu} = -50$, $\mu_{co} = -100$, $\sigma_{cc}^2 = 302500$, $\sigma_{nu}^2 = 90000$, and $\sigma_{co}^2 = 160000$.

1. Following Proposition 5, CCGT corresponds to technology 1 and Coal to technology 3. Then, the shares of the technologies that ensure the minimum risk are

$$\left(\alpha_{cc}^*, \alpha_{nu}^*, \alpha_{co}^*\right) = (0.15996, 0.53763, 0.30242).$$

The minimum risk reached by this PGP is $\sigma_Y^* = 219.97$. The maximum ENPV for such level of risk is $\mu_Y = -41.128$.

2. From Proposition 6, we obtain $x = \frac{\ln\left[\frac{0.15996}{0.15996+0.53763}\right]}{\ln[0.15996+0.53763]} = 4.089\,4$ and $[\alpha_1^*]^{\frac{1}{1+x}} = [0.15996]^{\frac{1}{5.0894}} = 0.69759$. Then, the efficient frontier corresponds to the following parametric equation of the shares of the three technologies

$$\begin{pmatrix} \alpha_{cc} \\ \alpha_{nu} \\ \alpha_{co} \end{pmatrix} = \begin{pmatrix} \alpha^{5.0894} \\ \alpha - \alpha^{5.0894} \\ 1-\alpha \end{pmatrix},$$

for $0.69759 \leq \alpha \leq 0.93645$. The PGP of CCGT–Nuclear-Coal reaches the maximum ENPV when $\alpha_{cc} = 1$, and $\mu_Y = \mu_{cc} = 100$. In this case, again, CCGT "**weakly dominates**" the PGP as it has the greatest ENPV which is also the most risky, $\sigma_{cc} = 550$. Then, we propose the upper limit of the efficient frontier to be $\alpha^{df} = 0.6919$. This fact guarantees that the PGP reaches a risk equal to $\sigma_{co} = 400$. Although the ENPV of nuclear is the less risky, it is associated to a lower ENPV of the PGP, $\mu_Y = 16.17$. Then, choosing the upper limit of the efficient frontier as $\alpha^{df} = 0.6919$ allows the PGP to reach a greater ENPV, $\mu_y = 54.21$, for a considerable risk.

3. The SD in the efficient frontier is given by $219.97 \leq \sigma_y \leq 400$.

4. The maximum ENPV for every corresponding level of risk is given by $-41.128 \leq \mu_y \leq 54.21$.

5. The feasible PGPs of CCGT-Nuclear-Coal are shown in the following figure:

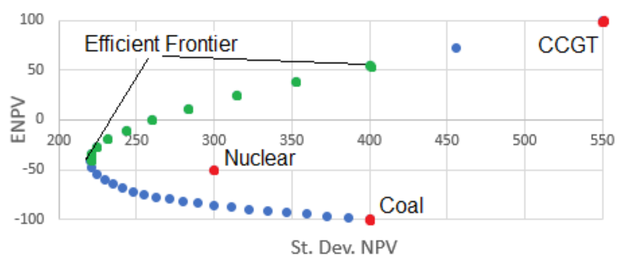
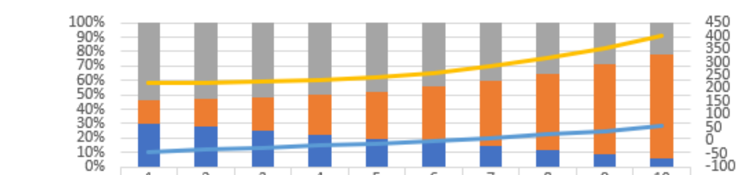


Figure 3: Feasible PGP of CCGT–Nuclear-Coal

6. The parameters of the efficient frontier are presented in figure 4.



	1	2	3	4	5	6	7	8	9	10
Share Nu	0.54	0.53	0.52	0.50	0.47	0.44	0.40	0.35	0.29	0.22
Share CC	0.16	0.19	0.23	0.28	0.33	0.39	0.46	0.53	0.62	0.72
Share Co	0.30	0.28	0.25	0.22	0.20	0.17	0.14	0.12	0.09	0.06
St. Dev.	219.97	221.00	224.62	231.72	243.28	260.32	283.79	314.48	353.06	400.09
ENPV	-41.13	-34.78	-27.62	-19.55	-10.47	-0.27	11.16	23.97	38.27	54.21

Figure 4: Efficient Frontier of PGP of CCGT–Nuclear-Coal

At this stage of the paper we are able to provide a geometric intuition about the parametric formulation of the efficient frontier stated in Proposition 6. We start by plotting the ENPV and the SD of the NPV of the PGP of CCGT-Nuclear-Coal given as follows

$$\sigma_Y = \sqrt{\alpha_{cc}^2 \sigma_{cc}^2 + \alpha_{nu}^2 \sigma_{nu}^2 + (1 - \alpha_{cc} - \alpha_{nu})^2 \sigma_{co}^2},$$
$$\mu_Y = \alpha_{cc}\mu_{cc} + \alpha_{nu}\mu_{nu} + (1 - \alpha_{cc} - \alpha_{nu})\,\mu_{co}.$$

The red surface in Figure 5 corresponds to the risk of the PGP, σ_Y, while the blue plane corresponds to the its ENPV, μ_Y.

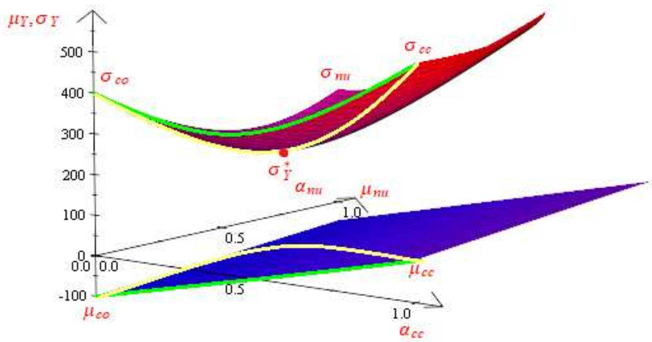


Figure 5: Efficient Frontier of PGP of CCGT–Nuclear-Coal

The green line in the risk of the PGP depicts the risk of the PGP of CCGT-Coal. The green line in the ENPV of the PGP depicts the ENPV the PGP of CCGT-Coal. Then, placing together the corresponding points of the green lines, we obtain the feasible PGP of CCGT-Coal shown in Figure 1.

The yellow line in the risk of the PGP links the risk of Coal, σ_{co}, to the risk of CCGT, σ_{cc}, and the minimum risk of the portfolio, σ_Y^*. The yellow line in the ENPV of the PGP links the ENPV of Coal, μ_{co}, to the ENPV of CCGT, μ_{cc}, and the ENPV corresponding to the minimum risk of the portfolio, $\mu\left([\alpha_1^*]^{\frac{1}{1+x}}\right)$. Then, placing together the corresponding points of the yellow lines, we obtain the feasible PGP of CCGT-Nuclear-Coal shown in Figure 3.

The fact that $\mu_1 = \mu_{cc} > \mu_{nu} > \mu_{co} = \mu_3$ guarantees that: 1) the efficient frontier of the PGP of CCGT-Nuclear-Coal is a segment of the feasible PGP of CCGT-Nuclear-Coal shown in Figure 3; 2) the efficient frontier does reach the maximum ENPV for a given level of risk, and; 2) the PGP of CCGT-Nuclear-Coal is less risky than any other PGP containing less than three technologies.

5 Portfolios of four technologies

Following result exploits the fact that the SD of the PGP is a convex function of the shares of technologies 1, 2, 3, and 4, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

Proposition 7 *From expression (4) the SD of the NPV of the PGP of four technologies is given by* $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2 + \alpha_4^2\sigma_4^2}$*. Assume that the NPV of technology* 2 *is the less risky. Assume that* $\sigma_{i,j} = 0$*, for any values* i *and* j *, from* 1 *to* 4*, such that* $i < j$*. If* $\alpha_i \in [0,1]$ *for* $i = 1,2,3,4$*, and* $\sum_{i=1}^4 \alpha_i = 1$ *it holds that*

a) *The risk of the NPV of the PGP,* σ_Y*, reaches its global minimum at*

$$\begin{pmatrix} \alpha_1^* \\ \alpha_2^* \\ \alpha_3^* \\ \alpha_4^* \end{pmatrix} = \frac{1}{|A_4|} \begin{pmatrix} \sigma_2^2\sigma_3^2\sigma_4^2 \\ \sigma_1^2\sigma_3^2\sigma_4^2 \\ \sigma_1^2\sigma_2^2\sigma_4^2 \\ \sigma_1^2\sigma_2^2\sigma_3^2 \end{pmatrix}.$$

where $|A_4| = \sigma_1^2\sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_2^2\sigma_4^2 + \sigma_1^2\sigma_3^2\sigma_4^2 + \sigma_2^2\sigma_3^2\sigma_4^2$.

b) *The minimum risk of the NPV of the PGP is*

$$\sigma_Y^* = \sqrt{\frac{\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2}{|A_4|}} < \sigma_2.$$

Proof. See appendix. ■

5.1 Efficient frontier

Following result provides the parametric formulation of the efficient frontier for PGP of four technologies.

Proposition 8 *Let* $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2 + \alpha_4^2\sigma_4^2}$ *the SD of the NPV of the portfolio. Assume that the ENPV of technology* 1 *is the greatest while the ENPV of technology* 4 *is the lowest, then following holds:*

a) *The* ***efficient frontier*** *corresponds to the following parametric equation of the shares of technologies* 1, 2, 3 *and* 4*,*

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \alpha^{1+x_1} \\ \alpha - \alpha^{1+x_1} \\ \alpha^{x_2} - \alpha^{1+x_2} \\ 1 - \alpha - \alpha^{x_2} + \alpha^{1+x_2} \end{pmatrix},$$

where the parameter α *is such that* $[\alpha_1^*]^{\frac{1}{1+x_1}} \leq \alpha \leq \alpha^{df} \leq 1$*. Let* x_1 *and* x_2 *be given by* $x_1 = \frac{\ln\left[\frac{\alpha_1^*}{\alpha_1^*+\alpha_2^*}\right]}{\ln\left[\alpha_1^*+\alpha_2^*\right]}$ *and* $x_2 = \frac{\ln\left[\frac{\alpha_3^*}{\alpha_3^*+\alpha_4^*}\right]}{\ln\left[\alpha_1^*+\alpha_2^*\right]}$*.* α^{df} *refers to the value of* α *that guarantees certain amount of the ENPV of the PGP.*

b) *The SD in the efficient frontier is given by* $\sigma_Y^* \leq \sigma_y \leq \sigma\left(\alpha^{df}\right)$*. Note that* $\sigma\left(\alpha^{df}\right) \leq \sigma_1$*.*

c) *The maximum ENPV for every corresponding level of risk is given by* $\mu\left([\alpha_1^*]^{\frac{1}{1+x_1}}\right) \leq \mu_y \leq \mu\left(\alpha^{df}\right)$*. Note that* $\mu\left(\alpha^{df}\right) \leq \mu_1$*.*

Proof. See appendix. ■

5.2 Illustrative Portfolios: Wind-CCGT-Nuclear-Coal

From Table 1, the corresponding ENPV and variance of the Wind (wd), CCGT (cc), Nuclear (nu) and Coal (co) are: $\mu_{wd} = 400$, $\mu_{cc} = 100$, $\mu_{nu} = -50$, $\mu_{co} = -100$, $\sigma^2_{wd} = 202500$, $\sigma^2_{cc} = 302500$, $\sigma^2_{nu} = 90000$, and $\sigma^2_{co} = 160000$.

1. Following Proposition 7, Wind corresponds to technology 1 and Coal to technology 4. Then, the shares of the four technologies that ensure the minimum risk are

$$(\alpha^*_{wd}, \alpha^*_{cc}, \alpha^*_{nu}, \alpha^*_{co}) = (0.192\,86, 0.129\,11, 0.433\,94, 0.244\,09)\,.$$

   The minimum risk reached by this PGP is $\sigma^*_Y = 197.62$. The maximum ENPV for such level of risk is $\mu_Y = 43.949$.

2. From Proposition 8, we obtain $x_1 = \frac{\ln\left[\frac{0.19286}{0.19286+0.12911}\right]}{\ln[0.19286+0.12911]} = 0.45221$, $x_2 = \frac{\ln\left[\frac{0.43394}{0.43394+0.24409}\right]}{\ln[0.19286+0.12911]} = 0.39379$, and $[\alpha^*_1]^{\frac{1}{1+x_1}} = [0.19286]^{\frac{1}{1.45221}} = 0.32197$. Then, the efficient frontier corresponds to the following parametric equation of the shares of the four technologies

$$\begin{pmatrix} \alpha_{wd} \\ \alpha_{cc} \\ \alpha_{nu} \\ \alpha_{co} \end{pmatrix} = \begin{pmatrix} \alpha^{1.45221} \\ \alpha - \alpha^{1.45221} \\ \alpha^{0.39379} - \alpha^{1.39379} \\ 1 - \alpha - \alpha^{0.39379} + \alpha^{1.39379} \end{pmatrix},$$

   for $0.32197 \leq \alpha \leq 0.732565$. The PGP of Wind-CCGT–Nuclear-Coal reaches the maximum ENPV when $\alpha_{wd} = 1$, and $\mu_Y = \mu_{wd} = 400$. In this case, Wind "**weakly dominates**" the PGP as it has the greatest ENPV which is also relatively risky, $\sigma_{wd} = 450$. We then propose the upper limit of the efficient frontier to be $\alpha^{df} = 0.732565$. This fact guarantees that the PGP reaches a risk equal to $\sigma_{nu} = 300$, the minimum risk of the four technologies.

3. The SD in the efficient frontier is given by $197.\,62 \leq \sigma_y \leq 300$.

4. The maximum ENPV for every corresponding level of risk is given by $43.\,949 \leq \mu_y \leq 249.26$

5. The feasible PGPs of Wind-CCGT-Nuclear-Coal are presented in Figure 6.

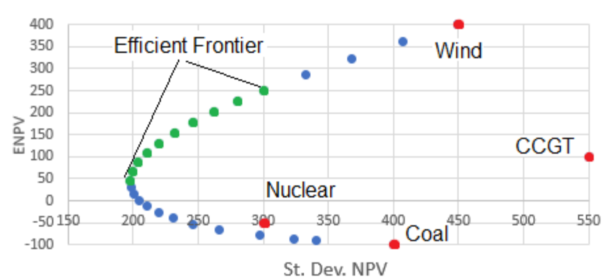
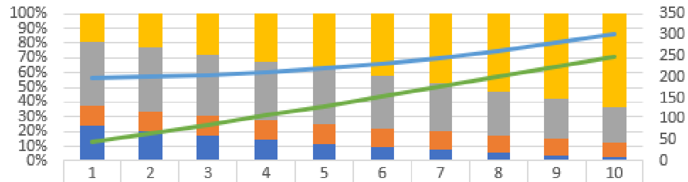


Figure 6: Feasible PGP of Wind-CCGT-Nuclear-Coal

6. Following figure presents the parameters of the efficient frontier.



	1	2	3	4	5	6	7	8	9	10
Share Wd	0.19	0.23	0.28	0.32	0.37	0.42	0.47	0.52	0.58	0.64
Share Nu	0.43	0.43	0.41	0.40	0.38	0.36	0.33	0.30	0.27	0.24
Share CC	0.13	0.13	0.14	0.14	0.13	0.13	0.12	0.12	0.11	0.10
Share Co	0.24	0.21	0.17	0.14	0.12	0.09	0.07	0.06	0.04	0.03
St. Dev.	197.62	199.09	203.37	210.30	219.74	231.56	245.62	261.80	279.96	300.00
ENPV	43.93	64.95	86.46	108.44	130.86	153.72	177.00	200.70	224.79	249.26

Figure 7: Efficient Frontier of PGP of Wind-CCGT-Nuclear-Coal

6 Portfolios of five technologies

Following result exploits the fact that the SD of the PGP is a convex function of the shares of technologies 1, 2, 3, 4 and 5, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$.

Proposition 9 *From expression (4) the SD of the NPV of the PGP of five technologies is given by* $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2 + \alpha_4^2\sigma_4^2 + \alpha_5^2\sigma_5^2}$. *Assume that the NPV of technology 2 is the less risky. Assume that* $\sigma_{i,j} = 0$, *for any values* i *and* j , *from* 1 *to* 5, *such that* $i < j$. *If* $\alpha_i \in [0, 1]$ *for* $i = 1, 2, 3, 4, 5$, *and* $\sum_{i=1}^{5} \alpha_i = 1$ *it holds that*

a) *The risk of the NPV of the PGP,* σ_Y, *reaches its global minimum at*

$$\begin{pmatrix} \alpha_1^* \\ \alpha_2^* \\ \alpha_3^* \\ \alpha_4^* \\ \alpha_5^* \end{pmatrix} = \frac{1}{|A_5|} \begin{pmatrix} \sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2 \\ \sigma_1^2\sigma_3^2\sigma_4^2\sigma_5^2 \\ \sigma_1^2\sigma_2^2\sigma_4^2\sigma_5^2 \\ \sigma_1^2\sigma_2^2\sigma_3^2\sigma_5^2 \\ \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2 \end{pmatrix}.$$

where $|A_5| = \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2 + \sigma_1^2\sigma_2^2\sigma_3^2\sigma_5^2 + \sigma_1^2\sigma_2^2\sigma_4^2\sigma_5^2 + \sigma_1^2\sigma_3^2\sigma_4^2\sigma_5^2 + \sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2$.

b) *The minimum risk of the NPV of the PGP is*

$$\sigma_Y^* = \sqrt{\frac{\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2}{|A_5|}} < \sigma_2.$$

Proof. See appendix. ■

6.1 Efficient frontier

Following result provides the parametric formulation of the efficient frontier for PGP of five technologies.

Proposition 10 *Let* $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2 + \alpha_4^2\sigma_4^2 + \alpha_5^2\sigma_5^2}$ *the SD of the NPV of the portfolio. Assume that the ENPV of technology* 1 *is the greatest while the ENPV of technology* 5 *is the lowest, then following holds:*

a) *The* ***efficient frontier*** *corresponds to following parametric equation of the shares of technologies* 1, 2, 3, 4 *and* 5,

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = \begin{pmatrix} \alpha^{1+x_1+x_2} \\ \alpha^{1+x_2} - \alpha^{1+x_1+x_2} \\ \alpha^{1+x_3} - \alpha^{1+x_2+x_3} \\ \alpha - \alpha^{1+x_2} - \alpha^{1+x_3} + \alpha^{1+x_2+x_3} \\ 1-\alpha \end{pmatrix}.$$

where the parameter α *is such that* $[\alpha_1^*]^{\frac{1}{1+x_1+x_2}} \leq \alpha \leq \alpha^{df} \leq 1$. *Let* x_1, x_2, *and* x_3 *be given by* $x_1 = \frac{\ln\left[\frac{\alpha_1^*}{\alpha_1^*+\alpha_2^*}\right]}{\ln\left[1-\alpha_5^*\right]}$, $x_2 = \frac{\ln\left[\frac{\alpha_1^*+\alpha_2^*}{1-\alpha_5^*}\right]}{\ln\left[1-\alpha_5^*\right]}$, *and* $x_3 = \frac{\ln\left[\frac{\alpha_3^*}{\alpha_3^*+\alpha_4^*}\right]}{\ln\left[1-\alpha_5^*\right]}$. α^{df} *refers to the value of* α *that guarantees certain amount of the ENPV of the PGP.*

b) *The SD in the efficient frontier is given by* $\sigma_Y^* \leq \sigma_y \leq \sigma\left(\alpha^{df}\right)$. *Note that* $\sigma\left(\alpha^{df}\right) \leq \sigma_1$.

c) *The maximum ENPV for every corresponding level of risk is given by* $\mu\left([\alpha_1^*]^{\frac{1}{1+x_1+x_2}}\right) \leq \mu_y \leq \mu\left(\alpha^{df}\right)$. *Note that* $\mu\left(\alpha^{df}\right) \leq \mu_1$.

Proof. See appendix. ■

6.2 Illustrative Portfolios: Hydro-Wind-CCGT-Nuclear-Coal

From Table 1, the corresponding ENPV and variance of the Hydro (hy), Wind (wd), CCGT (cc), Nuclear (nu) and Coal (co) are: $\mu_{hy} = 500$, $\mu_{wd} = 400$, $\mu_{cc} = 100$, $\mu_{nu} = -50$, $\mu_{co} = -100$, $\sigma_{hy}^2 = 122500$, $\sigma_{wd}^2 = 202500$, $\sigma_{cc}^2 = 302500$, $\sigma_{nu}^2 = 90\,000$, and $\sigma_{co}^2 = 160000$.

1. To apply Proposition 9, consider that Hydro corresponds to technology 1 and Coal to technology 5. Then, the shares of the technologies that ensure the minimum risk are

   $$(\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*, \alpha_5^*) = (0.24174, 0.14624, 0.09\,7896, 0.32904, 0.18508).$$

   The minimum risk of this PGP is given by $\sigma_Y^* = 172.09$. And the maximum ENPV for such level of risk is $\mu_Y = 154.20$.

2. From Proposition 10, we obtain $x_1 = \frac{\ln\left[\frac{0.24174}{0.24174+0.14624}\right]}{\ln[1-0.18508]} = 2.3115$, $x_2 = \frac{\ln\left[\frac{0.24174+0.14624}{1-0.18508}\right]}{\ln[1-0.18508]} = 3.6261$, $x_3 = \frac{\ln\left[\frac{0.097896}{0.097896+0.32904}\right]}{\ln[1-0.18508]} = 7.1958$, and $[\alpha_1^*]^{\frac{1}{1+x_1+x_2}} =$

$[0.241\,74]^{\frac{1}{6.9376}} = 0.814\,92$. Then, the efficient frontier corresponds to the following parametric equation of the shares of the three technologies

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = \begin{pmatrix} \alpha^{6.9376} \\ \alpha^{4.6261} - \alpha^{6.9376} \\ \alpha^{8.1958} - \alpha^{11.822} \\ \alpha - \alpha^{4.6261} - \alpha^{8.1958} + \alpha^{11.822} \\ 1 - \alpha \end{pmatrix},$$

for $0.814\,92 \leq \alpha \leq 0.97647$. The PGP of Hydro-Wind-CCGT–Nuclear-Coal reaches the maximum ENPV when $\alpha_{hy} = 1$, and $\mu_Y = \mu_{hy} = 5000$. In this case, Hydro "**weakly dominates**" the PGP. We propose the upper limit of the efficient frontier to be $\alpha^{df} = 0.97647$ to guarantees that the PGP reaches a risk equal to $\sigma_{nu} = 300$, the minimum risk of the five technologies.

3. The SD in the efficient frontier is given by $172.09 \leq \sigma_y \leq 300$.

4. The maximum ENPV for every corresponding level of risk is given by $154.20 \leq \mu_y \leq 446.87$

5. Following figure presents the feasible PGPs of Hydro-Wind-CCGT-Nuclear-Coal

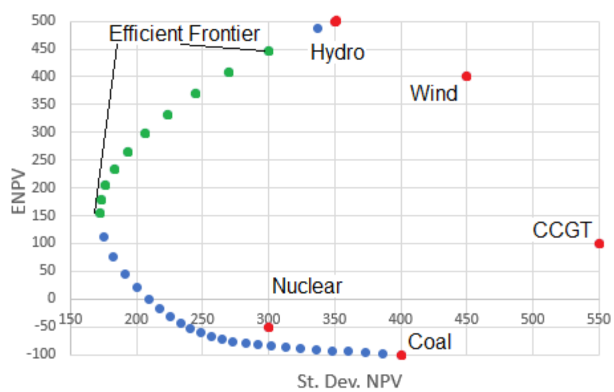
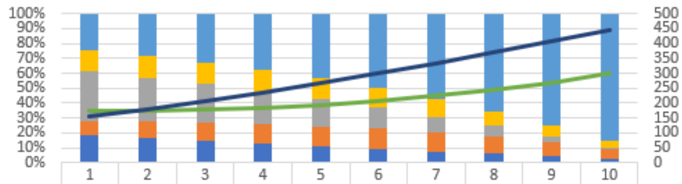


Figure 8: Feasible PGP: Hydro-Wind-CCGT–Nuclear-Coal

6. The parameters of the efficient frontier are presented in figure 9.



	1	2	3	4	5	6	7	8	9	10
Share Hy	0.24	0.28	0.33	0.38	0.43	0.50	0.57	0.65	0.75	0.85
Share Wd	0.15	0.15	0.15	0.14	0.14	0.13	0.12	0.10	0.08	0.05
Share Nu	0.33	0.30	0.26	0.22	0.18	0.14	0.10	0.07	0.04	0.01
Share CC	0.10	0.11	0.12	0.13	0.13	0.13	0.13	0.12	0.10	0.07
Share Co	0.19	0.17	0.15	0.13	0.11	0.10	0.08	0.06	0.04	0.02
St. Dev.	172.09	173.18	176.78	183.32	193.13	206.46	223.46	244.38	269.63	300.01
ENPV	154.20	179.11	205.96	234.78	265.58	298.34	332.99	369.42	407.47	446.87

Figure 9: Efficient Frontier of PGP of Hydro-Wind-CCGT–Nuclear-Coal

7 Final remarks and conclusions

Present paper tackle the problem of energy generation diversification by providing a parametric formulation of the efficient frontier of PGP for up to 5 technologies. Then, the parametric formulation of PGP constitutes a powerful policy tool for power generation policy-makers. Actually, it could be applied to portfolios of assets different than power generation technologies.

The paper also shows, implicitly, the source of what is called the "portfolio effect": risk reduction attained through diversification. The portfolio effect results from the fact that the risk of the PGP is a convex function of the shares of the different technologies. Part b) of Propositions 3, 5, 7, and 9 guarantee the existence of the portfolio effect.

From the structure of the paper, it is straight forward to extend the methodology to obtain the shares of technologies to guarantee the minimum risk of PGP of more than 5 technologies. The reader only have to follow the sequence depicted by Propositions 3, 5, 7, and 9. However, the parametric formulation of the efficient frontier of PGPs of more than 5 technologies should be obtained doing the corresponding mathematical proofs. They could be done by following the proof of Propositions 4, 6, 8, and 10.

The complete analysis relies on the assumption that the covariances of the NPV amongst the different technologies is zero. Depending on computational availability, future research could be extended to verify the actual effect of the correlation of the NPVs on the minimum risk of the portfolio.

References

Adams, R., Jamasb, T., 2016. Optimal Power Generation Portfolios: An Application to the UK, Cambridge Working Papers in Economics 1646 / Energy Policy Research Group (EPRG), Working Paper 1620, August, Faculty of Economics, University of Cambridge.

Awerbuch, S., Berger, M., 2003. Applying portfolio Theory to EU Electricity Planning and Policy-Making. IEA/EET Working Paper. IEA. Paris

Costa, O. L.V., Ribeiro, C. O., Rego, E. E., Stern, J. M., Parente, V., Kileber, S., 2017. Robust portfolio optimization for electricity planning: An application based on the Brazilian electricity mix. Energy Economics 64: 158–169.

Cunha, J., Ferreira, P., 2014. Designing electricity generation portfolios using the mean-variance approach, Int. J. Sustain. Energy Plan. Manag., 4: 17–30.

DeLlano-Paz, F., Calvo-Silvosa, A., Antelo, S. I., Soares, I., 2017. Energy planning and modern portfolio theory: A review. Renewable and Sustainable Energy Review 77: 636–651.

Freund, J. E., Miller, I, Miller, M., 2000. Estadística matemática con aplicaciones. Pearson educación.

Jain, S., Roelofs, F., Oosterlee, C. W., 2014. Decision-support tool for assessing future nuclear reactor generation portfolios. Energy Economics 44: 99-112.

Jansen, J.C., Beurskens, L.W.M., Tilburg, X.V., 2006. Application of portfolio analysis to the Dutch generating mix. ECN report C-05-100. Energy research council of Netherlands.

Markowitz H. M., 1952. Portfolio Selection, Journal of Finance, Vol. 7, pp 77-91.

Pinheiro Neto, D., Domingues, E. G., Coimbra, A. P., de Almeida, A. T., Alves, A. J., Calixto W. P., (2017), "Portfolio optimization of renewable energy assets: hydro, wind, and photovoltaic energy in the regulated market in Brazil," Energy Economics, vol. 64, pp. 238–250, 2017.

Roques, F. A., Hiroux, C., Saguan, M., 2010. Optimal wind power deployment in Europe — A portfolio approach, Energy Policy 38, pp. 3245-3256.

Roques, F. A., Newbery, D. M., Nuttall, W. J., 2008. Fuel mix diversification incentives in liberalized electricity markets: A Mean–Variance Portfolio theory approach, Energy Economics, Volume 30, Issue 4: 1831-1849.

Vithayasrichareon, P., MacGill, I.F., Wen, F.S., 2010a. Electricity generation portfolio evaluation for highly uncertain and carbon constrained future electricity industries. IEEE Power and Energy Society General Meeting.

Vithayasrichareon, P., MacGill, I.F., Wen, F., 2010b. Electricity Generation Portfolio Analysis for Coal, Gas and Nuclear Plant under Future Uncertainties. 4th IASTED Asian Conference on Power and Energy Systems

Appendix

Proof of Proposition 3. The SD of the PGP of two technologies is given by $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2}$. Assume that he NPV of technology 2 is the less risky. For $\sigma_{1,2} = 0$, $\alpha_i \in [0,1]$ for $i = 1,2$, and $\alpha_1 + \alpha_2 = 1$. For tractability, most of the proof uses the variance of the PGP instead of its SD.

Proof of a) We need to find the shares of technologies 1 and 2, given by (α_1, α_2), that guarantees the minimum risk (variance) of the NPV of the PGP. For tractability, we start by assuming that $\alpha_2 = 1 - \alpha_1$. Then, the variance of the NPV of the PGP is given by $\sigma_Y^2 = \alpha_1^2\sigma_1^2 + (1-\alpha_1)^2\sigma_2^2$. First, we find the critical point. The First Order Conditions (FOC) are:

$$\frac{\partial\sigma_Y^2}{\partial\alpha_1} = 2\alpha_1\sigma_1^2 + 2(1-\alpha_1)(-1)\sigma_2^2 = 0, \tag{5}$$

From expression (5) we have

$$\alpha_1\sigma_1^2 + \alpha_1\sigma_2^2 = \sigma_2^2, \tag{6}$$

which leads to

$$\alpha_1^* = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}, \tag{7}$$

Then, $\alpha_2^* = 1 - \alpha_1^* = \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}$. The critical point of the variance of the NPV of the PGP is

$$(\alpha_1^*, \alpha_2^*) = \frac{1}{\sigma_1^2+\sigma_2^2}\left(\sigma_2^2, \sigma_1^2\right), \tag{8}$$

To verify that the variance of the NPV of the PGP, σ_Y^2, has a minimum at the critical point (α_1^*, α_2^*) we need the Second Order Conditions (SOC):

$$\frac{\partial^2\sigma_Y^2}{\partial\alpha_1^2} = 2\left[\sigma_1^2 + \sigma_2^2\right] > 0.$$

Then, the variance σ_Y^2 has a minimum at point (α_1^*, α_2^*).

Proof of b) Then, the minimum value of the variance, σ_Y^{*2}, of the NPV of the PGP is given by

$$\sigma_Y^{*2} = \frac{1}{\left[\sigma_1^2+\sigma_2^2\right]^2}\left[\left(\sigma_2^2\right)^2\sigma_1^2 + \left(\sigma_1^2\right)^2\sigma_2^2\right],$$

$$\sigma_Y^{*2} = \sigma_Y^2 = \frac{\sigma_1^2\sigma_2^2}{\left[\sigma_1^2+\sigma_2^2\right]^2}\left[\left[\sigma_2^2 + \sigma_1^2\right]\right],$$

$$\sigma_Y^{*2} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} = \alpha_2^*\sigma_2^2 < \sigma_2^2,$$

Then

$$\sigma_Y^* < \sigma_2. \tag{9}$$

The NPV of the PGP is less risky than the NPV of the less risky technology. ■

Proof of Proposition 4. Let $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2}$ the SD of the NPV of the PGP. From Proposition 3 we know that the risk of the NPV of the PGP, σ_Y, reaches its global minimum at point (α_1^*, α_2^*). Assume that $\mu_1 \geq \mu_2$.To obtain the parametric formulation of the efficient frontier, we write the variance of the portfolio as follows:

$$\sigma_Y^2 = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2, \tag{10}$$

for $\alpha \in [0,1]$. Note that when $\alpha = 1$, then $\sigma_Y^2 = \sigma_1^2$, the variance of the NPV of the PGP equals the variance of technology 1. This scenario ensures that technology 1, which has the greatest ENPV, receives a share of 100%. On the other hand, when $\alpha = 0$, then $\sigma_Y^2 = \sigma_2^2$, the variance of the NPV of the PGP equals the variance of technology 2. The latter implies that technology 2, which has the lower ENPV, receives a share of 100%. Then, this way of expressing the variance of the NPV of the PGP allows to have portfolios assigning a share of 100% to the technologies with the greater and lower ENPV. To be sure that expression (10) allows to reach the point (α_1^*, α_2^*) where σ_Y reaches its global minimum, it should hold that

$$\alpha_1^* = \alpha, \tag{11}$$

$$\alpha_2^* = 1 - \alpha, \tag{12}$$

Expressions (11) and (12) lead to the fact that the shares of technologies 1 and 2 in this PGPs are given by the following expressions

$$\alpha_1 = \alpha, \tag{13}$$

$$\alpha_2 = 1 - \alpha, \tag{14}$$

From expressions (13) and (11), **the PGP with lowest risk (variance or SD) is given when**

$$\alpha = \alpha_1^*. \tag{15}$$

Now we need to find the PGP with the greatest ENPV. The ENPV of the PGP is given by:

$$\mu_Y = \alpha_1\mu_1 + \alpha_2\mu_2, \tag{16}$$

substituting expressions (13) and (14) into expression (16) leads to

$$\mu_Y = \alpha\mu_1 + (1-\alpha)\mu_2,$$

it is straight forward to obtain that

$$\tfrac{d\mu_Y}{d\alpha} = \mu_1 - \mu_2 > 0,$$

because of the assumption that $\mu_1 \geq \mu_2$. Then, the **PGP reaches its maximum ENPV** when $\alpha = 1$, and $\mu_Y = \mu_1$ and $\sigma_Y^2 = \sigma_1^2$. However, there could be an alternative criteria to choose the maximum ENPV of the PGP. For example, if the NPV of technology 2 is the less risky, then, the criteria could be to choose α^{df} such that $\sigma_Y^2\left(\alpha^{df}\right)=\sigma_2^2$. In this case $\alpha^{df}<1$. Then, the PGP **with maximum ENPV** given when $\alpha=\alpha^{df}$. Then, **the efficient frontier** is given by expressions (13) and (14) for $\alpha_1^*\leq\alpha\leq\alpha^{df}$. As a consequence, the SD in the efficient frontier is given by $\sigma_Y^*\leq\sigma_y\leq\sigma\left(\alpha^{df}\right)$ while the maximum ENPV for every corresponding level of risk is given by $\mu\left(\alpha_1^*\right)\leq\mu_y\leq\mu\left(\alpha^{df}\right)$. Note that $\mu\left(\alpha^{df}\right)\leq\mu_1$ and $\sigma\left(\alpha^{df}\right)\leq\sigma_1$. ■

Proof of Proposition 5. The SD of the of the NPV of the PGP of three technologies is given by $\sigma_Y=\sqrt{\alpha_1^2\sigma_1^2+\alpha_2^2\sigma_2^2+\alpha_3^2\sigma_3^2}$. Assume that the NPV of technology 2 is the less risky. For $\sigma_{1,2}=\sigma_{1,3}=\sigma_{2,3}=0$, $\alpha_i\in[0,1]$ for $i=1,2,3$, and $\sum_{i=1}^3\alpha_i=1$. For tractability, most of the proof uses the variance of the PGP instead of its SD.

Proof of a) We need to find the shares of technologies 1, 2, and 3, given by $(\alpha_1,\alpha_2,\alpha_3)$ that ensures the minimum risk (variance) of the NPV of the PGP. For tractability, we start by assuming that $\alpha_3=1-\alpha_1-\alpha_2$. Then, the variance of the NPV of the PGP is given by $\sigma_Y^2=\alpha_1^2\sigma_1^2+\alpha_2^2\sigma_2^2+\left(1-\alpha_1-\alpha_2\right)^2\sigma_3^2$. First, we find the critical point. The FOC are:

$$\frac{\partial\sigma_Y^2}{\partial\alpha_1}=2\alpha_1\sigma_1^2+2\left(1-\alpha_1-\alpha_2\right)(-1)\,\sigma_3^2=0, \tag{17}$$

$$\frac{\partial\sigma_Y^2}{\partial\alpha_2}=2\alpha_2\sigma_2^2+2\left(1-\alpha_1-\alpha_2\right)(-1)\,\sigma_3^2=0, \tag{18}$$

from expression (17) we have

$$\alpha_1\left[\sigma_1^2+\sigma_3^2\right]+\alpha_2\sigma_3^2=\sigma_3^2, \tag{19}$$

from expression (18) we have

$$\alpha_1\sigma_3^2+\alpha_2\left[\sigma_2^2+\sigma_3^2\right]=\sigma_3^2. \tag{20}$$

Expressions (19) and (20) lead to the following system of equations

$$\begin{bmatrix}\sigma_1^2+\sigma_3^2 & \sigma_3^2\\ \sigma_3^2 & \sigma_2^2+\sigma_3^2\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\begin{bmatrix}\sigma_3^2\\ \sigma_3^2\end{bmatrix}, \tag{21}$$

Calculating the inverse of matrix $A_3=\begin{bmatrix}\sigma_1^2+\sigma_3^2 & \sigma_3^2\\ \sigma_3^2 & \sigma_2^2+\sigma_3^2\end{bmatrix}$ we end up with

$$\begin{bmatrix}\alpha_1^*\\ \alpha_2^*\end{bmatrix}=\frac{1}{|A_3|}\begin{bmatrix}\sigma_2^2+\sigma_3^2 & -\sigma_3^2\\ -\sigma_3^2 & \sigma_1^2+\sigma_3^2\end{bmatrix}\begin{bmatrix}\sigma_3^2\\ \sigma_3^2\end{bmatrix}, \tag{22}$$

where $|A_3|=\sigma_1^2\sigma_2^2+\sigma_1^2\sigma_3^2+\sigma_2^2\sigma_3^2$. Leading to the result

$$\begin{bmatrix}\alpha_1^*\\ \alpha_2^*\end{bmatrix}=\frac{1}{|A_3|}\begin{bmatrix}\sigma_2^2\sigma_3^2\\ \sigma_1^2\sigma_3^2\end{bmatrix}, \tag{23}$$

Then, $\alpha_3^*=1-\alpha_1^*-\alpha_2^*=\frac{\sigma_1^2\sigma_2^2}{|A_3|}$. The critical point of the variance of the NPV of the PGP is

$$\left(\alpha_1^*,\alpha_2^*,\alpha_3^*\right)=\frac{1}{|A_3|}\left(\sigma_2^2\sigma_3^2,\sigma_1^2\sigma_3^2,\sigma_1^2\sigma_2^2\right). \tag{24}$$

To verify that the variance of the NPV of the PGP, σ_Y^2, has a minimum at point $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$ we need the SOC. The Hessian matrix is as follows:

$$H = 2 \begin{bmatrix} \sigma_1^2 + \sigma_3^2 & \sigma_3^2 \\ \sigma_3^2 & \sigma_2^2 + \sigma_3^2 \end{bmatrix},$$

Following the criteria of the leading principal minors of the Hessian matrix, we have

$$H_1 = 2\left(\sigma_1^2 + \sigma_3^2\right) > 0.$$

$$H_2 = 2\left|A_3\right| = 2\left(\sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_2^2\sigma_3^2\right) > 0.$$

The two leading principal minors of the Hessian matrix are positive for any $(\alpha_1, \alpha_2, \alpha_3)$. Then, the variance of the NPV of the PGP is a convex function of the shares of the three technologies, $(\alpha_1, \alpha_2, \alpha_3)$. As a consequence, the variance of the NPV of the PGP, σ_Y^2, has a global minimum at point $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$, given by expression (24).

Proof of b) Then, the minimum value of the variance of the NPV of the PGP is

$$\sigma_Y^{*2} = \tfrac{1}{[A_3]^2}\left[\left(\sigma_2^2\sigma_3^2\right)^2\sigma_1^2 + \left(\sigma_1^2\sigma_3^2\right)^2\sigma_2^2 + \left(\sigma_1^2\sigma_2^2\right)^2\sigma_3^2\right],$$

$$\sigma_Y^{*2} = \tfrac{\sigma_1^2\sigma_2^2\sigma_3^2}{[A_3]^2}\left[\sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_2^2\sigma_3^2\right],$$

$$\sigma_Y^{*2} = \tfrac{\sigma_1^2\sigma_2^2\sigma_3^2}{A_3} = \alpha_2^*\sigma_2^2 < \sigma_2^2,$$

Then

$$\sigma_Y^* < \sigma_2. \tag{25}$$

The NPV of the PGP is less risky than the NPV of the less risky technology. ■

Proof of Proposition 6. Let $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2}$ the SD of the NPV. From Proposition 5 we know that the risk of the NPV of the PGP, σ_Y, reaches its global minimum at point $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$. Assume that $\mu_1 \geq \mu_2 \geq \mu_3$.To obtain the parametric formulation of the efficient frontier, we write the variance of the portfolio as follows:

$$\begin{aligned} \sigma_Y^2 &= \alpha^2\left(\beta^2\sigma_1^2 + (1-\beta)^2\sigma_2^2\right) + (1-\alpha)^2\sigma_3^2, \\ \sigma_Y^2 &= \alpha^2\beta^2\sigma_1^2 + \alpha^2(1-\beta)^2\sigma_2^2 + (1-\alpha)^2\sigma_3^2, \end{aligned} \tag{26}$$

for $\alpha, \beta \in [0,1]$. Note that when $\alpha = \beta = 1$, then $\sigma_Y^2 = \sigma_1^2$, the variance of the portfolio equals the variance of technology 1. This fact implies that technology 1, which has the greatest ENPV, receives a share of 100%. On the other hand, when $\alpha = 0$, then $\sigma_Y^2 = \sigma_3^2$, the variance of the portfolio equals the variance of technology 3. Then, technology 3, which has the lower ENPV, receives a share of 100%. Then, this formulation of the variance of the NPV of the PGP allows to have portfolios that assign a share of 100% to the technologies with the greatest and lowest ENPV. To be sure that expression (26) allows to reach the point $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$, where σ_Y reaches its global minimum, it should hold that

$$\alpha_1^* = \alpha\beta, \tag{27}$$

$$\alpha_2^* = \alpha\,(1-\beta)\,, \tag{28}$$

$$\alpha_3^* = (1-\alpha)\,, \tag{29}$$

from expression (27)

$$\alpha = \tfrac{\alpha_1^*}{\beta} \tag{30}$$

substituting expression (30) into expression (28) leads to

$$\beta = \tfrac{\alpha_1^*}{\alpha_1^*+\alpha_2^*}, \tag{31}$$

substituting expression (31) into expression (30) leads to

$$\alpha = \alpha_1^* + \alpha_2^*. \tag{32}$$

Assume that

$$\beta = \beta\,(\alpha) = \alpha^x \tag{33}$$

to ensure that $\beta \in [0,1]$ for $\alpha \in [0,1]$. Then, from expression (31) and (32) we have

$$\tfrac{\alpha_1^*}{\alpha_1^*+\alpha_2^*} = (\alpha_1^* + \alpha_2^*)^x\,.$$

which leads to

$$x = \frac{\ln\left[\frac{\alpha_1^*}{\alpha_1^*+\alpha_2^*}\right]}{\ln\left[\alpha_1^*+\alpha_2^*\right]}. \tag{34}$$

Substituting expression (33) into expressions (27), (28), and (29) leads to the fact that the shares of technologies $1, 2$ and 3 in this portfolio are given by the following expressions

$$\alpha_1 = \alpha\beta = \alpha^{1+x}, \tag{35}$$

$$\alpha_2 = \alpha\,(1-\beta) = \alpha - \alpha^{1+x}, \tag{36}$$

$$\alpha_3 = 1 - \alpha. \tag{37}$$

From expressions (27) and (35), **the PGP with lowest risk (variance or SD) is given when**

$$\alpha = [\alpha_1^*]^{\frac{1}{1+x}}\,. \tag{38}$$

Now we need to fond the PGP with the corresponding greatest ENPV. The ENPV of the PGP is given by

$$\mu_Y = \alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3, \tag{39}$$

substituting expressions (35), (36), and (37) into expression (39) leads to

$$\mu_Y = \alpha^{1+x}\mu_1 + \left(\alpha - \alpha^{1+x}\right)\mu_2 + (1-\alpha)\,\mu_3,$$

It is straight forward to obtain that

$$\frac{d\mu_Y}{d\alpha} = [1+x]\,\alpha^x\,[\mu_1 - \mu_2] + \mu_2 - \mu_3 > 0,$$

because of the assumption that $\mu_1 \geq \mu_2 \geq \mu_3$. Then, the **PGP reaches its maximum ENPV** when $\alpha = 1$, and $\mu_Y = \mu_1$ and $\sigma_Y^2 = \sigma_1^2$. However, there could be an alternative criteria to choose the maximum ENPV of the PGP. For example, if the NPV of technology 2 is the less risky, then, the criteria could be to choose α^{df} such that $\sigma_Y^2\left(\alpha^{df}\right) = \sigma_2^2$. In this case $\alpha^{df} < 1$. Then, the **PGP with maximum ENPV** is given when $\alpha = \alpha^{df}$. Then, **the efficient frontier** is given by expressions (35), (36), and (37) for $[\alpha_1^*]^{\frac{1}{1+x}} \leq \alpha \leq \alpha^{df}$. As a consequence, the SD in the efficient frontier is given by $\sigma_Y^* \leq \sigma_y \leq \sigma\left(\alpha^{df}\right)$ while the maximum ENPV for every corresponding level of risk is given by $\mu\left([\alpha_1^*]^{\frac{1}{1+x}}\right) \leq \mu_y \leq \mu\left(\alpha^{df}\right)$. Note that $\mu\left(\alpha^{df}\right) \leq \mu_1$ and $\sigma\left(\alpha^{df}\right) \leq \sigma_1$. ■

Proof of Proposition 7. The SD of the NPV of the PGP of four technologies is given by $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2 + \alpha_4^2\sigma_4^2}$. Assume that the NPV of technology 2 is the less risky. If $\sigma_{i,j} = 0$, for any values i and j , from 1 to 4, such that $i < j$. If $\alpha_i \in [0,1]$ for $i = 1,2,3,4$, and $\sum_{i=1}^{4}\alpha_i = 1$. For tractability, most of the proof uses the variance of the PGP instead of its SD.

Proof of a) We need to find the shares of technologies 1, 2, 3, and 4, given by $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, that ensures the minimum risk (variance) of the NPV of the PGP. For tractability, we start by assuming that $\alpha_4 = 1-\alpha_1-\alpha_2-\alpha_3$. Then, the variance of the NPV of the PGP is given by $\sigma_Y^2 = \alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2 + (1-\alpha_1-\alpha_2-\alpha_3)^2\sigma_4^2$. First, we find the critical point. The FOC are:

$$\frac{\partial\sigma_Y^2}{\partial\alpha_1} = 2\alpha_1\sigma_1^2 + 2\,(1-\alpha_1-\alpha_2-\alpha_3)\,(-1)\,\sigma_4^2 = 0, \tag{40}$$

$$\frac{\partial\sigma_Y^2}{\partial\alpha_2} = 2\alpha_2\sigma_2^2 + 2\,(1-\alpha_1-\alpha_2-\alpha_3)\,(-1)\,\sigma_4^2 = 0, \tag{41}$$

$$\frac{\partial\sigma_Y^2}{\partial\alpha_3} = 2\alpha_3\sigma_3^2 + 2\,(1-\alpha_1-\alpha_2-\alpha_3)\,(-1)\,\sigma_4^2 = 0, \tag{42}$$

from expression (40) we have

$$\alpha_1\left[\sigma_1^2+\sigma_4^2\right] + \alpha_2\sigma_4^2 + \alpha_3\sigma_4^2 = \sigma_4^2, \tag{43}$$

from expression (41) we have

$$\alpha_1\sigma_4^2 + \alpha_2\left[\sigma_2^2+\sigma_4^2\right] + \alpha_3\sigma_4^2 = \sigma_4^2, \tag{44}$$

from expression (42) we have

$$\alpha_1\sigma_4^2 + \alpha_2\sigma_4^2 + \alpha_3\left[\sigma_3^2+\sigma_4^2\right] = \sigma_4^2. \tag{45}$$

Expressions (43), (44), and (45) lead to the following system of equations

$$\begin{bmatrix} \sigma_1^2+\sigma_4^2 & \sigma_4^2 & \sigma_4^2 \\ \sigma_4^2 & \sigma_2^2+\sigma_4^2 & \sigma_4^2 \\ \sigma_4^2 & \sigma_4^2 & \sigma_3^2+\sigma_4^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \sigma_4^2 \\ \sigma_4^2 \\ \sigma_4^2 \end{bmatrix}, \tag{46}$$

Calculating the inverse of matrix $A_4 = \begin{bmatrix} \sigma_1^2+\sigma_4^2 & \sigma_4^2 & \sigma_4^2 \\ \sigma_4^2 & \sigma_2^2+\sigma_4^2 & \sigma_4^2 \\ \sigma_4^2 & \sigma_4^2 & \sigma_3^2+\sigma_4^2 \end{bmatrix}$ we end up with

$$\begin{bmatrix} \alpha_1^* \\ \alpha_2^* \\ \alpha_3^* \end{bmatrix} = \tfrac{1}{|A_4|} \begin{bmatrix} \sigma_2^2\sigma_3^2+\sigma_2^2\sigma_4^2+\sigma_3^2\sigma_4^2 & -\sigma_3^2\sigma_4^2 & -\sigma_2^2\sigma_4^2 \\ -\sigma_3^2\sigma_4^2 & \sigma_1^2\sigma_3^2+\sigma_1^2\sigma_4^2+\sigma_3^2\sigma_4^2 & -\sigma_1^2\sigma_4^2 \\ -\sigma_2^2\sigma_4^2 & -\sigma_1^2\sigma_4^2 & \sigma_1^2\sigma_2^2+\sigma_1^2\sigma_4^2+\sigma_2^2\sigma_4^2 \end{bmatrix} \begin{bmatrix} \sigma_4^2 \\ \sigma_4^2 \\ \sigma_4^2 \end{bmatrix}, \tag{47}$$

where $|A_4| = \sigma_1^2\sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_2^2\sigma_4^2 + \sigma_1^2\sigma_3^2\sigma_4^2 + \sigma_2^2\sigma_3^2\sigma_4^2$. The solution is the system of equations is

$$\begin{bmatrix} \alpha_1^* \\ \alpha_2^* \\ \alpha_3^* \end{bmatrix} = \frac{1}{|A_4|} \begin{bmatrix} \sigma_2^2\sigma_3^2\sigma_4^2 \\ \sigma_1^2\sigma_3^2\sigma_4^2 \\ \sigma_1^2\sigma_2^2\sigma_4^2 \end{bmatrix}, \tag{48}$$

Then, $\alpha_4^* = 1-\alpha_1^*-\alpha_2^*-\alpha_3^* = \frac{\sigma_1^2\sigma_2^2\sigma_3^2}{|A_4|}$. The critical point of the variance of the NPV of the PGP is

$$(\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*) = \tfrac{1}{|A_4|}\left(\sigma_2^2\sigma_3^2\sigma_4^2, \sigma_1^2\sigma_3^2\sigma_4^2, \sigma_1^2\sigma_2^2\sigma_4^2, \sigma_1^2\sigma_2^2\sigma_3^2\right). \tag{49}$$

To verify that the variance of the NPV of the PGP, σ_Y^2, has a minimum at point $(\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*)$ we need the SOC. The Hessian matrix is as follows:

$$H = 2\begin{bmatrix} \sigma_1^2+\sigma_4^2 & \sigma_4^2 & \sigma_4^2 \\ \sigma_4^2 & \sigma_2^2+\sigma_4^2 & \sigma_4^2 \\ \sigma_4^2 & \sigma_4^2 & \sigma_3^2+\sigma_4^2 \end{bmatrix}, \tag{50}$$

Following the criteria of the leading principal minors of the Hessian matrix, we have

$$H_1 = 2\left(\sigma_1^2+\sigma_4^2\right) > 0,$$

$$H_2 = 2\begin{vmatrix} \sigma_1^2+\sigma_4^2 & \sigma_4^2 \\ \sigma_4^2 & \sigma_2^2+\sigma_4^2 \end{vmatrix} = 2\left(\sigma_1^2\sigma_2^2+\sigma_1^2\sigma_4^2+\sigma_2^2\sigma_4^2\right) > 0,$$

$$H_3 = 2\,|A_4| > 0.$$

The three leading principal minors of the Hessian matrix are positive for any $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Then, the variance of the NPV of the PGP is a convex function of the shares of the four, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. As a consequence, the variance of the NPV of the PGP, σ_Y^2, has a global minimum at point $(\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*)$, given by expression (49).

Proof of b) Then, the minimum value of the variance of the NPV of the portfolio is

$$\sigma_Y^{*2} = \tfrac{1}{[|A_4|]^2}\left[\left(\sigma_2^2\sigma_3^2\sigma_4^2\right)^2\sigma_1^2 + \left(\sigma_1^2\sigma_3^2\sigma_4^2\right)^2\sigma_2^2 + \left(\sigma_1^2\sigma_2^2\sigma_4^2\right)^2\sigma_3^2 + \left(\sigma_1^2\sigma_2^2\sigma_3^2\right)^2\sigma_4^2\right],$$

$$\sigma_Y^{*2} = \sigma_Y^2 = \tfrac{\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2}{[|A_4|]^2}\left[\sigma_2^2\sigma_3^2\sigma_4^2 + \sigma_1^2\sigma_3^2\sigma_4^2 + \sigma_1^2\sigma_2^2\sigma_4^2 + \sigma_1^2\sigma_2^2\sigma_3^2\right],$$

$$\sigma_Y^{*2} = \frac{\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2}{|A_4|} = \alpha_2^*\sigma_2^2 < \sigma_2^2,$$

Then

$$\sigma_Y^* < \sigma_2. \tag{51}$$

The NPV of the portfolio is less risky than the NPV of the less risky technology. ■

Proof of Proposition 8. Let $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2 + \alpha_4^2\sigma_4^2}$ the SD of the NPV. From Proposition 7 we know that σ_Y reaches its global minimum at point $(\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*)$. Assume that the ENPV of technology 1 is the greatest while the ENPV of technology 4 is the lowest. To obtain the parametric formulation of the efficient frontier, we write the variance of the portfolio as follows:

$$\begin{aligned} \sigma_Y^2 &= \alpha^2\left[\beta^2\sigma_1^2 + (1-\beta)^2\sigma_2^2\right] + (1-\alpha)^2\left[\gamma^2\sigma_3^2 + (1-\gamma)^2\sigma_4^2\right], \\ \sigma_Y^2 &= \alpha^2\beta^2\sigma_1^2 + \alpha^2(1-\beta)^2\sigma_2^2 + (1-\alpha)^2\gamma^2\sigma_3^2 + (1-\alpha)^2(1-\gamma)^2\sigma_4^2 \end{aligned} \tag{52}$$

for $\alpha, \beta, \gamma \in [0,1]$. Note that when $\alpha = \beta = 1$, then $\sigma_Y^2 = \sigma_1^2$, the variance of the portfolio equals the variance of technology 1. This fact implies that technology 1, which has the greatest ENPV, receives a share of 100%. On the other hand, when $\alpha = \gamma = 0$, then $\sigma_Y^2 = \sigma_4^2$, the variance of the portfolio equals the variance of technology 4. Then, technology 4, which has the lowest ENPV, receives share of 100%. Then, this formulation of the variance of the NPV of the PGP allows to have portfolios that assign a share of 100% to the technologies with the greatest and lowest ENPV. To be sure that expression (52) allows to reach the point $(\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*)$ where σ_Y reaches its global minimum, it should holds that

$$\alpha_1^* = \alpha\beta, \tag{53}$$

$$\alpha_2^* = \alpha(1-\beta), \tag{54}$$

$$\alpha_3^* = (1-\alpha)\gamma, \tag{55}$$

$$\alpha_4^* = (1-\alpha)(1-\gamma), \tag{56}$$

from expression (53)

$$\alpha = \frac{\alpha_1^*}{\beta}, \tag{57}$$

substituting expression (57) into expression (54) leads to

$$\beta = \frac{\alpha_1^*}{\alpha_1^* + \alpha_2^*}, \tag{58}$$

substituting expression (58) into expression (57) leads to

$$\alpha = \alpha_1^* + \alpha_2^*. \tag{59}$$

From expression (55)

$$1 - \alpha = \frac{\alpha_3^*}{\gamma} \tag{60}$$

substituting expression (60) into expression (56) leads to

$$\gamma = \frac{\alpha_3^*}{\alpha_3^*+\alpha_4^*}. \tag{61}$$

Assume that

$$\beta = \beta(\alpha) = \alpha^{x_1} \tag{62}$$

$$\gamma = \gamma(\alpha) = \alpha^{x_2} \tag{63}$$

to ensure that $\beta \in [0,1]$ and $\gamma \in [0,1]$ for $\alpha \in [0,1]$. Then, substituting expressions (58) and (59) into expression (62) he have

$$\frac{\alpha_1^*}{\alpha_1^*+\alpha_2^*} = (\alpha_1^* + \alpha_2^*)^{x_1}.$$

which leads to

$$x_1 = \frac{\ln\left[\frac{\alpha_1^*}{\alpha_1^*+\alpha_2^*}\right]}{\ln\left[\alpha_1^*+\alpha_2^*\right]}. \tag{64}$$

Now, substituting expressions (59) and (61) into expression (63) he have

$$\frac{\alpha_3^*}{\alpha_3^*+\alpha_4^*} = (\alpha_1^* + \alpha_2^*)^{x_2}.$$

which leads to

$$x_2 = \frac{\ln\left[\frac{\alpha_3^*}{\alpha_3^*+\alpha_4^*}\right]}{\ln\left[\alpha_1^*+\alpha_2^*\right]}. \tag{65}$$

Substituting expression (62) and (63)into expressions (53), (54), (55), and (56) leads to the fact that the share of technologies $1, 2, 3$ and 4 in this portfolio is given by the following expressions

$$\alpha_1 = \alpha\beta = \alpha^{x_1+1}, \tag{66}$$

$$\alpha_2 = \alpha(1-\beta) = \alpha - \alpha^{x_1+1}, \tag{67}$$

$$\alpha_3 = \alpha^{x_2} - \alpha^{x_2+1}. \tag{68}$$

$$\alpha_4 = 1 - \alpha - \alpha^{x_2} + \alpha^{x_2+1}. \tag{69}$$

From expressions (53) and (66), **the PGP with lowest risk (variance or SD) is given when**

$$\alpha = [\alpha_1^*]^{\frac{1}{1+x_1}}. \tag{70}$$

Now we need to find the portfolio with the corresponding greatest ENPV. The ENPV of the PGP is given by

$$\mu_Y = \alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3 + \alpha_4\mu_4, \tag{71}$$

substituting expressions (66), (67), (68), and (69) into expression (71) leads to

$$\mu_Y = \alpha^{1+x_1}\mu_1 + \left(\alpha - \alpha^{1+x_1}\right)\mu_2 + \left(\alpha^{x_2} - \alpha^{1+x_2}\right)\mu_3 + \left(1 - \alpha - \alpha^{x_2} + \alpha^{1+x_2}\right)\mu_4,$$

It is straight forward to obtain that

$$\frac{d\mu_Y}{d\alpha} = (1+x_1)\,\alpha^{x_1}\left[\mu_1-\mu_2\right]+\left[x_2\alpha^{x_2-1}-(1+x_2)\,\alpha^{x_2}\right]\left[\mu_3-\mu_4\right]+\left[\mu_2-\mu_4\right]>0,$$

because of the assumption that the ENPV of technology 1 is the greatest while the NPV of technology 4 is the lowest. Then, **the portfolio reaches its maximum ENPV when** $\alpha = 1$, and $\mu_Y = \mu_1$ and $\sigma_Y^2 = \sigma_1^2$. However, there could be an alternative criteria to choose the maximum ENPV. For example, if the NPV of technology 2 is the less risky, then, the criteria could be to choose α^{df} such that $\sigma_Y^2\left(\alpha^{df}\right) = \sigma_2^2$. Then, the **PGP with maximum ENPV** is given when $\alpha = \alpha^{df}$. Then, **the efficient frontier**is given by expressions (66), (67), (68), and (69) for $[\alpha_1^*]^{\frac{1}{1+x_1}} \leq \alpha \leq \alpha^{df}$. As a consequence, the SD in the efficient frontier is given by $\sigma_Y^* \leq \sigma_y \leq \sigma\left(\alpha^{df}\right)$ while the maximum ENPV for every corresponding level of risk is given by $\mu\left([\alpha_1^*]^{\frac{1}{1+x_1}}\right) \leq \mu_y \leq \mu\left(\alpha^{df}\right)$. Note that $\mu\left(\alpha^{df}\right) \leq \mu_1$ and $\sigma\left(\alpha^{df}\right) \leq \sigma_1$. ■

Proof of Proposition 9. The SD of the NPV of the PGP of five technologies is given by $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2+\alpha_2^2\sigma_2^2+\alpha_3^2\sigma_3^2+\alpha_4^2\sigma_4^2+\alpha_5^2\sigma_5^2}$. Assume that he NPV of technology 2 is the less risky. If $\sigma_{i,j} = 0$, for any values i and j , from 1 to 5, such that $i < j$. If $\alpha_i \in [0,1]$ for $i = 1,2,3,4,5$, and $\sum_{i=1}^{5}\alpha_i = 1$. For tractability, most of the proof uses the variance of the PGP instead of its SD.

Proof of a) We need to find the shares of technologies 1, 2, 3, 4, and 5, given by $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, that ensures the minimum risk (variance) of the NPV of the PGP. For tractability, we start by assuming that $\alpha_5 = 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4$. Then, the variance of NPV of the PGP is given by $\sigma_Y^2 = \alpha_1^2\sigma_1^2+\alpha_2^2\sigma_2^2+\alpha_3^2\sigma_3^2+\alpha_4^2\sigma_4^2+(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)^2\sigma_5^2$. First, we find the critical point. The FOC are:

$$\frac{\partial\sigma_Y^2}{\partial\alpha_1} = 2\alpha_1\sigma_1^2+2\,(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\,(-1)\,\sigma_5^2 = 0, \tag{72}$$

$$\frac{\partial\sigma_Y^2}{\partial\alpha_2} = 2\alpha_2\sigma_2^2+2\,(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\,(-1)\,\sigma_5^2 = 0, \tag{73}$$

$$\frac{\partial\sigma_Y^2}{\partial\alpha_3} = 2\alpha_3\sigma_3^2+2\,(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\,(-1)\,\sigma_5^2 = 0, \tag{74}$$

$$\frac{\partial\sigma_Y^2}{\partial\alpha_4} = 2\alpha_3\sigma_3^2+2\,(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\,(-1)\,\sigma_5^2 = 0, \tag{75}$$

from expression (72) we have

$$\alpha_1\left[\sigma_1^2+\sigma_5^2\right]+\alpha_2\sigma_5^2+\alpha_3\sigma_5^2+\alpha_4\sigma_5^2 = \sigma_5^2, \tag{76}$$

from expression (73) we have

$$\alpha_1\sigma_5^2+\alpha_2\left[\sigma_2^2+\sigma_5^2\right]+\alpha_3\sigma_5^2+\alpha_4\sigma_5^2 = \sigma_5^2. \tag{77}$$

from expression (74) we have

$$\alpha_1\sigma_5^2+\alpha_2\sigma_5^2+\alpha_3\left[\sigma_3^2+\sigma_5^2\right]+\alpha_4\sigma_5^2 = \sigma_5^2. \tag{78}$$

from expression (75) we have

$$\alpha_1\sigma_5^2 + \alpha_2\sigma_5^2 + +\alpha_3\sigma_5^2 + \alpha_4\left[\sigma_4^2 + \sigma_5^2\right] = \sigma_5^2. \tag{79}$$

Expressions (76), (77), (78), and (79) lead to the following system of equations

$$\begin{bmatrix} \sigma_1^2+\sigma_5^2 & \sigma_5^2 & \sigma_5^2 & \sigma_5^2 \\ \sigma_5^2 & \sigma_2^2+\sigma_5^2 & \sigma_5^2 & \sigma_5^2 \\ \sigma_5^2 & \sigma_5^2 & \sigma_3^2+\sigma_5^2 & \sigma_5^2 \\ \sigma_5^2 & \sigma_5^2 & \sigma_5^2 & \sigma_4^2+\sigma_5^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \sigma_5^2 \\ \sigma_5^2 \\ \sigma_5^2 \\ \sigma_5^2 \end{bmatrix}, \tag{80}$$

Calculating the inverse of matrix $A_5 = \begin{bmatrix} \sigma_1^2+\sigma_5^2 & \sigma_5^2 & \sigma_5^2 & \sigma_5^2 \\ \sigma_5^2 & \sigma_2^2+\sigma_5^2 & \sigma_5^2 & \sigma_5^2 \\ \sigma_5^2 & \sigma_5^2 & \sigma_3^2+\sigma_5^2 & \sigma_5^2 \\ \sigma_5^2 & \sigma_5^2 & \sigma_5^2 & \sigma_4^2+\sigma_5^2 \end{bmatrix}$
we end up with

$$\begin{bmatrix} \alpha_1^* \\ \alpha_2^* \\ \alpha_3^* \\ \alpha_4^* \end{bmatrix} = \tfrac{1}{|A_5|} \begin{bmatrix} \begin{bmatrix} \sigma_2^2\sigma_3^2\sigma_4^2+ \\ \sigma_2^2\sigma_3^2\sigma_5^2+ \\ \sigma_2^2\sigma_4^2\sigma_5^2+ \\ \sigma_3^2\sigma_4^2\sigma_5^2 \end{bmatrix} & -\sigma_3^2\sigma_4^2\sigma_5^2 & -\sigma_2^2\sigma_4^2\sigma_5^2 & -\sigma_2^2\sigma_3^2\sigma_5^2 \\ -\sigma_3^2\sigma_4^2\sigma_5^2 & \begin{bmatrix} \sigma_1^2\sigma_3^2\sigma_4^2+ \\ \sigma_1^2\sigma_3^2\sigma_5^2+ \\ \sigma_1^2\sigma_4^2\sigma_5^2+ \\ \sigma_3^2\sigma_4^2\sigma_5^2 \end{bmatrix} & -\sigma_1^2\sigma_4^2\sigma_5^2 & -\sigma_1^2\sigma_3^2\sigma_5^2 \\ -\sigma_2^2\sigma_4^2\sigma_5^2 & -\sigma_1^2\sigma_4^2\sigma_5^2 & \begin{bmatrix} \sigma_1^2\sigma_2^2\sigma_4^2+ \\ \sigma_1^2\sigma_2^2\sigma_5^2+ \\ \sigma_1^2\sigma_4^2\sigma_5^2+ \\ \sigma_2^2\sigma_4^2\sigma_5^2 \end{bmatrix} & -\sigma_1^2\sigma_2^2\sigma_5^2 \\ -\sigma_2^2\sigma_3^2\sigma_5^2 & -\sigma_1^2\sigma_3^2\sigma_5^2 & -\sigma_1^2\sigma_2^2\sigma_5^2 & \begin{bmatrix} \sigma_1^2\sigma_2^2\sigma_3^2+ \\ \sigma_1^2\sigma_2^2\sigma_5^2+ \\ \sigma_1^2\sigma_3^2\sigma_5^2+ \\ \sigma_2^2\sigma_3^2\sigma_5^2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \sigma_5^2 \\ \sigma_5^2 \\ \sigma_5^2 \\ \sigma_5^2 \end{bmatrix} \tag{81}$$

where $|A_5| = \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2 + \sigma_1^2\sigma_2^2\sigma_3^2\sigma_5^2 + \sigma_1^2\sigma_2^2\sigma_4^2\sigma_5^2 + \sigma_1^2\sigma_3^2\sigma_4^2\sigma_5^2 + \sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2$. The solution is the system of equations is

$$\begin{bmatrix} \alpha_1^* \\ \alpha_2^* \\ \alpha_3^* \\ \alpha_4^* \end{bmatrix} = \frac{1}{|A_5|} \begin{bmatrix} \sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2 \\ \sigma_1^2\sigma_3^2\sigma_4^2\sigma_5^2 \\ \sigma_1^2\sigma_2^2\sigma_4^2\sigma_5^2 \\ \sigma_1^2\sigma_2^2\sigma_3^2\sigma_5^2 \end{bmatrix}, \tag{82}$$

Then, $\alpha_4^* = 1 - \alpha_1^* - \alpha_2^* - \alpha_3^* - \alpha_4^* = \frac{\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2}{|A_5|}$. The critical point of the variance of the NPV of the PGP is

$$\begin{pmatrix} \alpha_1^* \\ \alpha_2^* \\ \alpha_3^* \\ \alpha_4^* \\ \alpha_5^* \end{pmatrix} = \frac{1}{|A_5|} \begin{pmatrix} \sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2 \\ \sigma_1^2\sigma_3^2\sigma_4^2\sigma_5^2 \\ \sigma_1^2\sigma_2^2\sigma_4^2\sigma_5^2 \\ \sigma_1^2\sigma_2^2\sigma_3^2\sigma_5^2 \\ \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2 \end{pmatrix}. \tag{83}$$

To verify that the variance of the NPV of the PGP, σ_Y^2, has a minimum at point $(\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*, , \alpha_5^*)$ we need the SOC. The Hessian matrix is as follows:

$$H = 2 \begin{bmatrix} \sigma_1^2 + \sigma_5^2 & \sigma_5^2 & \sigma_5^2 & \sigma_5^2 \\ \sigma_5^2 & \sigma_2^2 + \sigma_5^2 & \sigma_5^2 & \sigma_5^2 \\ \sigma_5^2 & \sigma_5^2 & \sigma_3^2 + \sigma_5^2 & \sigma_5^2 \\ \sigma_5^2 & \sigma_5^2 & \sigma_5^2 & \sigma_4^2 + \sigma_5^2 \end{bmatrix}, \tag{84}$$

Following the criteria of the leading principal minors of the Hessian matrix, we have

$$H_1 = 2\left(\sigma_1^2 + \sigma_5^2\right) > 0,$$

$$H_2 = 2 \begin{vmatrix} \sigma_1^2 + \sigma_5^2 & \sigma_5^2 \\ \sigma_5^2 & \sigma_2^2 + \sigma_5^2 \end{vmatrix} = 2\left(\sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_5^2 + \sigma_2^2\sigma_5^2\right) > 0,$$

$$H_3 = 2 \begin{vmatrix} \sigma_1^2 + \sigma_5^2 & \sigma_5^2 & \sigma_5^2 \\ \sigma_5^2 & \sigma_2^2 + \sigma_5^2 & \sigma_5^2 \\ \sigma_5^2 & \sigma_5^2 & \sigma_3^2 + \sigma_5^2 \end{vmatrix} = 2\left(\sigma_1^2\sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_2^2\sigma_5^2 + \sigma_1^2\sigma_3^2\sigma_5^2 + \sigma_2^2\sigma_3^2\sigma_5^2\right) > 0,$$

$$H_4 = 2\left|A_5\right| > 0.$$

The four leading principal minors of the Hessian matrix are positive for any $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$.Then, the variance of the NPV of the PGP is a convex function of the shares of the five technologies $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$. As a consequence, the variance of the NPV of the PGP, σ_Y^2, has a global minimum at point $(\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*, \alpha_5^*)$, given by expression (83).

Proof of b) Then, the minimum value of the variance of the NPV of the portfolio is

$$\sigma_Y^{*2} = \tfrac{1}{[|A_5|]^2}[\left(\sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2\right)^2\sigma_1^2 + \left(\sigma_1^2\sigma_3^2\sigma_4^2\sigma_5^2\right)^2\sigma_2^2 + \\ \left(\sigma_1^2\sigma_2^2\sigma_4^2\sigma_5^2\right)^2\sigma_3^2 + \left(\sigma_1^2\sigma_2^2\sigma_3^2\sigma_5^2\right)^2\sigma_4^2 + \left(\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2\right)^2\sigma_5^2]$$

$$\sigma_Y^{*2} = \tfrac{\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2}{[|A_5|]^2}\left[\sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2 + \sigma_1^2\sigma_3^2\sigma_4^2\sigma_5^2 + \sigma_1^2\sigma_2^2\sigma_4^2\sigma_5^2 + \sigma_1^2\sigma_2^2\sigma_3^2\sigma_5^2 + \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2\right],$$

$$\sigma_Y^{*2} = \tfrac{\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2}{|A_5|} = \alpha_2^*\sigma_2^2 < \sigma_2^2,$$

Then

$$\sigma_Y^* < \sigma_2. \tag{85}$$

The NPV of the portfolio is less risky than the NPV of the less risky technology.
■

Proof of Proposition 10. Let $\sigma_Y = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2 + \alpha_4^2\sigma_4^2 + \alpha_5^2\sigma_5^2}$ the SD of the NPV. From Proposition 9 we know that σ_Y reaches its global minimum at point $(\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*, \alpha_5^*)$. Assume that the ENPV of technology 1 is the greatest while the ENPV of technology 5 is the lowest. To obtain the parametric formulation of the efficient frontier, we write the variance of the portfolio as follows:

$$\sigma_Y^2 = \alpha^2\left(\eta^2\left[\beta^2\sigma_1^2 + (1-\beta)^2\sigma_2^2\right] + (1-\eta)^2\left[\gamma^2\sigma_3^2 + (1-\gamma)^2\sigma_4^2\right]\right) + (1-\alpha)^2\sigma_5^2, \\ \sigma_Y^2 = \alpha^2\eta^2\beta^2\sigma_1^2 + \alpha^2\eta^2(1-\beta)^2\sigma_2^2 + \alpha^2(1-\eta)^2\gamma^2\sigma_3^2 + \alpha^2(1-\eta)^2(1-\gamma)^2\sigma_4^2 + (1-\alpha)^2\sigma_5^2 \tag{86}$$

for $\alpha, \eta, \beta, \gamma \in [0,1]$. Note that when $\alpha = \beta = \eta = 1$, then $\sigma_Y^2 = \sigma_1^2$, the variance of the portfolio equals the variance of technology 1. This fact implies that technology 1, which has the greatest ENPV, receives a share of 100%. On the other hand, when $\alpha = 0$, then $\sigma_Y^2 = \sigma_5^2$, the variance of the portfolio equals the variance of technology 5. Then, technology 5, which has the lower ENPV, receives a share of 100%. Then, this formulation of the variance of the NPV of the PGP allows to have portfolios that assign a share of 100% to the technologies with the greatest and lowest ENPV. To be sure that expression (86) allows to reach the point $(\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*, \alpha_5^*)$ where σ_Y reaches its global minimum, it should hold that

$$\alpha_1^* = \alpha\eta\beta, \tag{87}$$

$$\alpha_2^* = \alpha\eta\left(1-\beta\right), \tag{88}$$

$$\alpha_3^* = \alpha\left(1-\eta\right)\gamma, \tag{89}$$

$$\alpha_4^* = \alpha\left(1-\eta\right)\left(1-\gamma\right), \tag{90}$$

$$\alpha_5^* = \left(1-\alpha\right), \tag{91}$$

from expression (87)

$$\alpha\eta = \tfrac{\alpha_1^*}{\beta} \tag{92}$$

substituting expression (92) into expression (88) leads to

$$\beta = \tfrac{\alpha_1^*}{\alpha_1^*+\alpha_2^*}, \tag{93}$$

substituting expression (93) into expression (92) leads to

$$\alpha\eta = \alpha_1^* + \alpha_2^*. \tag{94}$$

From expression (91)

$$\alpha = 1 - \alpha_5^*, \tag{95}$$

substituting expression (95) into expression (94) leads to

$$\eta = \tfrac{\alpha_1^*+\alpha_2^*}{1-\alpha_5^*}. \tag{96}$$

From expression (89)

$$\alpha\left(1-\eta\right) = \tfrac{\alpha_3^*}{\gamma}, \tag{97}$$

substituting expression (97) into expression (90) leads to

$$\gamma = \tfrac{\alpha_3^*}{\alpha_3^*+\alpha_4^*}, \tag{98}$$

Assume that

$$\beta = \beta\left(\alpha\right) = \alpha^{x_1} \tag{99}$$

$$\eta = \eta\left(\alpha\right) = \alpha^{x_2} \tag{100}$$

$$\gamma = \gamma(\alpha) = \alpha^{x_3} \tag{101}$$

to ensure that $\beta \in [0,1]$, $\eta \in [0,1]$ and $\gamma \in [0,1]$ for $\alpha \in [0,1]$. Then, substituting expressions (93) and (95) into expression (99) he have

$$\frac{\alpha_1^*}{\alpha_1^*+\alpha_2^*} = (1-\alpha_5^*)^{x_1}.$$

which leads to

$$x_1 = \frac{\ln\left[\frac{\alpha_1^*}{\alpha_1^*+\alpha_2^*}\right]}{\ln\left[1-\alpha_5^*\right]}. \tag{102}$$

Substituting expressions (95) and (98) into expression (100) he have

$$\frac{\alpha_1^*+\alpha_2^*}{1-\alpha_5^*} = (1-\alpha_5^*)^{x_2}.$$

which leads to

$$x_2 = \frac{\ln\left[\frac{\alpha_1^*+\alpha_2^*}{1-\alpha_5^*}\right]}{\ln\left[1-\alpha_5^*\right]}. \tag{103}$$

Substituting expressions (95) and (96) into expression (101) he have

$$\frac{\alpha_3^*}{\alpha_3^*+\alpha_4^*} = (1-\alpha_5^*)^{x_3}.$$

which leads to

$$x_3 = \frac{\ln\left[\frac{\alpha_3^*}{\alpha_3^*+\alpha_4^*}\right]}{\ln\left[1-\alpha_5^*\right]}. \tag{104}$$

substituting expression (99), (100) and (101) into expressions (87), (88), (89), (90), and (91) leads to the fact that the share of technologies $1, 2, 3, 4$ and 5 in this portfolio is given by the following expressions,

$$\alpha_1 = \alpha\eta\beta = \alpha^{1+x_1+x_2}, \tag{105}$$

$$\alpha_2 = \alpha\eta(1-\beta) = \alpha^{1+x_2} - \alpha^{1+x_1+x_2}, \tag{106}$$

$$\alpha_3 = \alpha(1-\eta)\gamma = \alpha^{1+x_3} - \alpha^{1+x_2+x_3}, \tag{107}$$

$$\alpha_4 = \alpha(1-\eta)(1-\gamma) = \alpha - \alpha^{1+x_2} - \alpha^{1+x_3} + \alpha^{1+x_2+x_3}, \tag{108}$$

$$\alpha_5 = 1-\alpha. \tag{109}$$

From expressions (87) and (105), the **PGP with lowest risk (variance or SD) is given when**

$$\alpha = \left[\alpha_1^*\right]^{\frac{1}{1+x_1+x_2}}. \tag{110}$$

Now we need to find the portfolio with the corresponding greatest ENPV. The ENPV of the PGP is given by

$$\mu_Y = \alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3 + \alpha_4\mu_4 + \alpha_5\mu_5, \tag{111}$$

substituting expressions (105), (106), (107), (108), and (109) into expression (111) leads to

$$\mu_Y = \left(\alpha^{1+x_1+x_2}\right)\mu_1 + \left(\alpha^{1+x_2} - \alpha^{1+x_1+x_2}\right)\mu_2 + \\ \left(\alpha^{1+x_3} - \alpha^{1+x_2+x_3}\right)\mu_3 + \left(\alpha - \alpha^{1+x_2} - \alpha^{1+x_3} + \alpha^{1+x_2+x_3}\right)\mu_4 + (1-\alpha)\mu_5,$$

It is straight forward to obtain that

$$\frac{d\mu_Y}{d\alpha} = (1+x_1+x_2)\,\alpha^{x_1+x_2}\left[\mu_1 - \mu_2\right] + (1+x_2)\,\alpha^{x_2}\left[\mu_2 - \mu_4\right] + \\ \left[(1+x_3)\,\alpha^{x_3} - (1+x_2+x_3)\,\alpha^{x_2+x_3}\right]\left[\mu_3 - \mu_4\right] + \left[\mu_4 - \mu_5\right] > 0,$$

because of the assumption that the ENPV of technology 1 is the greatest while the NPV of technology 5 is the lowest. Then, **the portfolio reaches its maximum ENPV when** $\alpha = 1$, and $\mu_Y = \mu_1$ and $\sigma_Y^2 = \sigma_1^2$. However, there could be an alternative criteria to choose the maximum ENPV. For example, if the NPV of technology 2 is the less risky, then, the criteria could be to choose α^{df} such that $\sigma_Y^2\left(\alpha^{df}\right) = \sigma_2^2$. Then, the **PGP with maximum ENPV** is given when $\alpha = \alpha^{df}$. Then, **the efficient frontier** is given by expressions (105), (106), (107), (108), and (109) for $\left[\alpha_1^*\right]^{\frac{1}{1+x_1+x_2}} \leq \alpha \leq \alpha^{df}$. As a consequence, the SD in the efficient frontier is given by $\sigma_Y^* \leq \sigma_y \leq \sigma\left(\alpha^{df}\right)$ while the maximum ENPV for every corresponding level of risk is given by $\mu\left(\left[\alpha_1^*\right]^{\frac{1}{1+x_1+x_2}}\right) \leq \mu_y \leq \mu\left(\alpha^{df}\right)$. Note that $\mu\left(\alpha^{df}\right) \leq \mu_1$ and $\sigma\left(\alpha^{df}\right) \leq \sigma_1$. ■