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Abstract

This paper proposes a set of models which can be used to estimate the market risk for a portfolio of crypto-currencies, and simultaneously to estimate also their credit risk using the Zero Price Probability (ZPP) model by Fantazzini et al (2008), which is a methodology to compute the probabilities of default using only market prices. For this purpose, both univariate and multivariate models with different specifications are employed. Two special cases of the ZPP with closed-form formulas in case of normally distributed errors are also developed using recent results from barrier option theory. A backtesting exercise using two datasets of 5 and 15 coins for market risk forecasting and a dataset of 42 coins for credit risk forecasting was performed. The Value-at-Risk and the Expected Shortfall for single coins and for an equally weighted portfolio were calculated and evaluated with several tests. The ZPP approach was used for the estimation of the probability of default/death of the single coins and compared to classical credit scoring models (logit and probit) and to a machine learning algorithm (Random Forest). Our results reveal the superiority of the t-copula/skewed-t GARCH model for market risk, and the ZPP-based models for credit risk.

Keywords: cryptocurrencies, market risk, credit risk, ZPP.

JEL classification: C32, C51, C53, C58, G12, G17, G32, G33.

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1 Introduction

Cryptocurrencies and the cryptomarket have become a popular trend in finance in the last years, as discussed by Antonopoulos (2014), Narayanan et al. (2016), Burniske and Tatar (2017), Fantazzini (2019) and Corbet et al. (2019). Bohr and Bashir (2014) and Yelowitz and Wilson (2015) showed that different people, from crypto and tech enthusiasts to investors, are interested in these new financial tools. Large market capitalizations for the most popular cryptocurrencies, increasing number of crypto funds[1], a growing cumulative number of Initial Coin Offerings (ICOs) and, at the same time, a frequent number of hacks and frauds, make the topic of cryptocurrencies' risk a really urgent problem, see Fantazzini et al. (2016), Fantazzini et al. (2017) and references therein. Hence, the main goal of this paper is to propose a unified framework for the simultaneous modelling of credit and market risk. If we use the formal definition of cryptocurrency proposed by Lansky (2018), then a cryptocurrency can be defined as a system that satisfies these six conditions:

> "*1) The system does not require a central authority, its state is maintained through distributed consensus. 2) The system keeps an overview of cryptocurrency units and their ownership. 3) The system defines whether new cryptocurrency units can be created. If new cryptocurrency units can be created, the system defines the circumstances of their origin and how to determine the ownership of these new units. 4) Ownership of cryptocurrency units can be proved exclusively cryptographically. 5) The system allows transactions to be performed in which ownership of the cryptographic units is changed. A transaction statement can only be issued by an entity proving the current ownership of these units. 6) If two different instructions for changing the ownership of the same cryptographic units are simultaneously entered, the system performs at most one of them.*"
>
> — Lansky (2018, p. 19)

Therefore, a cryptocurrency does not have debt, and it cannot default in a classical sense. However, its price and the investors' demand might drop dramatically because of a revealed scam, hack or other hidden problems that cannot be observed directly from the market data. Because of that, we believe that such risk is not exactly a market one and therefore we propose a definition of credit risk which is somewhat different from the classic one: credit risk for a crypto-coin is its "*death*", a situation when its price drops significantly and a coin becomes illiquid. In this regard, we have to admit that there is not an unambiguous definition of a dead coin, neither in the professional literature, nor in the academic literature. However, it is worth noting that even when considered dead, some coins do still have some negligible daily trading volumes. There are two reasons for this phenomenon: the possibility to recover

[1] `https://cryptofundresearch.com/cryptocurrency-funds-overview-infographic/`

at least a small amount of the initial investment, or to bet on the possible revamp of the dead coin. Differently from stock companies and small and medium enterprises (SMEs), the procedure to revamp a coin is much easier and faster: it only takes a new code or, even easier, an update of the previous code where the major flaws are corrected. This is why the "death" state for a cryptocurrency can be only a temporary situation rather than a permanent one: see, for example, Sid (2018) who discussed the story of *PENG*, which is a coin revamp of an old abandoned project. A brief review of the history and the financial literature devoted to cryptocurrencies is reported in Appendix A.

The first contribution of the paper is a large discussion about credit risk and market risk for cryptocurrencies and how these risks can be defined with these assets. In traditional finance, credit risk is usually defined as the gains and losses on a position (or portfolio) associated with the fulfillment (or not) of contractual obligations, whereas market risk as the gains and losses on the value of a position (or portfolio) due to the changes in market prices (Basel Committee on Banking Supervision (2009) p. 6-7, Hartmann (2010), p. 697). Instead, we argue that credit risk for cryptocurrencies can be defined as the gains and losses on the value of a position of a cryptocurrency that is abandoned and considered dead but which can be potentially revived, while market risk can be described as the gains and losses on the value of a position (or portfolio) of alive cryptocurrencies, due to the movements in market prices in centralized and decentralized exchanges.

The second contribution of the paper is a set of multivariate models which can be used to estimate the market risk for a portfolio of crypto-currencies by using the Value-at-Risk and the Expected Shortfall, and simultaneously to estimate also their credit risk using the Zero Price Probability (ZPP) model by Fantazzini et al. (2008), which is a methodology to compute the probabilities of default using only market prices. Recent papers by Su and Huang (2010), Li et al. (2016) and Dalla Valle et al. (2016) showed the ZPP dominance in terms of default probability estimation with respect to competing models.

The third contribution of this work is the development of closed-form formulas for the ZPP in two special cases, namely the random walk with drift and a GARCH(1,1) model with normal errors, using recent results from barrier option theory by Su and Rieger (2009). Even though crypto-assets are far from being normally distributed, these closed-form formulas can provide a quick estimate of the probability of the coin death and they can give an investor at least a rough idea of the crypto-asset credit risk.

The fourth contribution of the paper is a backtesting exercise using two datasets of 5 and 15 coins for market risk forecasting and a dataset of 42 coins for credit risk forecasting. For the purpose of the multivariate modelling of a portfolio of cryptocurrencies, we employ VAR-DCC and VAR-Copula-GARCH models with different specifications for the error terms. The Value-at-Risk and the Expected Shortfall for the single coins and for an equally weighted portfolio are calculated during a back-testing exercise and then evaluated with several tests. The ZPP approach is used for the estimation of the probability of default/death for the single coins and compared to classical credit scoring models (logit and probit) and to a machine learning algorithm (Random Forest). Our results show that a t-copula/skewed-t GARCH model outperform the competing models for market risk, while ZPP-based models outperform the other models for credit risk.

The paper is organized as follows: the definitions of credit risk and market risk for cryptocurrencies are introducted in Section 2, while the methods proposed for market and credit risk modelling with cryptocurrencies are discussed in Section 3. Section 4 describes the empirical results, while robustness checks are discussed in Section 5. Section 6 briefly concludes.

2 Theoretical framework: Credit and Market risk for cryptocurrencies

The literature review in Appendix A shows that risk management is still in its early stages when cryptocurrencies are of concern. In this regard, there are important differences between cryptocurrency trading and traditional stock and forex trading:

- The *lack of financial oversight for many crypto-based companies and exchanges* means that coins prices can be susceptible to manipulations, pump and dump schemes and market frauds of various types. For example, Gandal et al. (2018) showed that the quick rise in the bitcoin price in 2013 from $150 to more than $1000 was probably due to price manipulation. Griffin and Shams (2018) showed that Tether, a digital currency pegged to US dollars, is likely used to provide price support and manipulate cryptocurrency prices. Moreover, the lack of regulatory oversight is one of the main reasons why there are large and recurring deviations in cryptocurrency prices across exchanges, as shown by Makarov and Schoar (2018)[2].

- *There are no traditional brokers*, and investors can trade cryptocurrencies directly either using centralized exchanges or decentralized exchanges[3].

- Differently from forex and stock markets, *cryptocurrency exchanges do not provide neither cash nor asset insurance* (with some exceptions like Coinbase and Gemini exchanges, which only insure cash deposits): in case of hacking attack, exchanges could lose part or all of investors' cryptocurrencies, and investors would not enjoy any type of government protection. Similar fate would await investors in the case the cryptocurrency turns out to be a scam. Note that if hackers steal the investors'

[2] A table showing the real-time price difference between the last trades across several bitcoin exchanges can be freely accessed at `https://data.bitcoinity.org/markets/arbitrage`

[3] A list of both centralized and decentralized exchanges can be found at `list.wiki/Cryptocurrency_Exchanges`. More information about decentralized exchanges is available at `github.com/distribuyed/index` and references therein.

private keys from the cryptocurrency exchange's wallet, then the investors will permanently lose their assets because cryptocurrency transactions are irreversible (due to the specific nature of a cryptocurrency blockchain): suing the exchange will not help because it will simply declare bankruptcy, leaving the investors with their losses. Therefore, choosing a good exchange is very important when trading cryptocurrencies.

Given this discussion, the separation between market and credit risk becomes even more blurred when dealing with cryptocurrencies than in traditional finance. In the latter case, *credit risk* is broadly defined as the gains and losses on a position or portfolio associated with the fulfillment (or not) of contractual obligations, whereas *market risk* is defined as the gains and losses on the value of a position or portfolio that can take place due to the movements in market prices (such as exchange rates, commodity prices, interest rates, etc.), see Basel Committee on Banking Supervision (2009), Hartmann (2010) and references therein for more details. However, the Basel Committee on Banking Supervision (2009) highlighted that the securitization trend in the last decade "*has diminished the scope for differences in measuring market and credit risk, as securitization transforms the latter into the former*"(Basel Committee on Banking Supervision (2009), p.14). Moreover, market and credit risk are driven by the same economic factors, as a large literature shows –see the special issue on the interaction of market and credit risk in the *Journal of Banking and Finance* in 2010–, so it is difficult to separate these two risks neatly in practical risk measurement and management.

In general, the definition of market risk can be extended rather easily to the case of cryptocurrencies market prices. Instead, credit risk requires a new definition because a cryptocurrency does not pay interest and does not involve the reimbursement of principal amounts, so the previous classical definition cannot be used. To this aim, the concept of a *dead coin* needs to be introduced: in simple terms, a dead coin is a cryptocurrency that does not offer any utility or feature, or it is no more developed and mining ceased, or it was created for hacking or for spreading of malware, or it was purely a scam, or simply a fun/parody project.

Unfortunately, there is not a unique definition of dead coins, neither in the professional literature, nor in the academic literature: in the professional literature, some define dead coins as those whose value drops below 1 cent[4], yet others stress, on top of that, no trading volume, no nodes running, no active community and de-listing from all exchanges[5]. In the academic literature, Feder et al. (2018) proposed the only definition (currently) available: they first define a "candidate peak" as a day in which the 7-day rolling price average is greater than any value 30 days before or after. Moreover, to choose only those peaks with sudden jumps, they define a candidate as a peak only if it is greater than or equal 50% of the

[4]https://www.investopedia.com/terms/d/dead-coin.asp
[5]https://steemit.com/beyondbitcoin/@freshfund/dead-coins-or-dormant-coins

minimum value in the 30 days prior to the candidate peak, and if its value is at least 5% as large as the cryptocurrency's maximum peak. Given these peak data, Feder et al. (2018) consider a coin *abandoned* (= dead), if the daily average volume for a given month is less than or equal to 1% of the peak volume. Besides, if the average daily trading volume for a month following a peak is greater than 10% of the peak value and that currency is currently abandoned, then Feder et al. (2018) change the coin status to *resurrected*.

Despite the differences in these criteria, they have one thing in common: the "death" state is not a permanent state and a dead coin can be revamped/revived, even several times during a relatively short period. For example, Feder et al. (2018) found that 44% of publicly traded coins are subsequently abandoned (18% permanently), while 85% of announced coins fail before being traded publicly. Moreover, they also found that most coins are not traded much, and these coins are more likely to die than their larger counterparts are. Interestingly, they showed that some coins could be revamped up to five times over a 5-year period.

Given this background, *credit risk for cryptocurrencies* can be defined as *the gains and losses on the value of a position of a cryptocurrency that is abandoned and considered dead according to professional and/or academic criteria, but which can be potentially revived and revamped*[6].

Substantially, the differences between credit and market risk in case of cryptocurrencies are of quantitative and temporal nature, not qualitative: if the financial losses and the technical problems can be dealt with the available financial and technical resources, then we have a market event. If the financial losses are too big and the technical problems cannot be solved, then we have a credit event and the cryptocurrency "dies". Moreover, it follows easily that the longer the time horizon, the more probable are large losses and/or technical problems and/or hacking attacks on the cryptocurrency blockchain, so credit risk becomes more important. Note that this latter result is already known in the classical financial literature, given that "*... the ratio of default and (normally distributed) market risk losses is proportional to the square-root of the holding period. Since the ratio goes to 0 as the holding period goes to 0, over short horizons market risk is relatively more important, while over longer horizons losses due to default become more important*"(Basel Committee on Banking Supervision (2009), p.16-17).

Once a credit event takes place, the development of a cryptocurrency stops, its price falls close to zero (or even to zero, if we consider the uninterrupted lack of trading for several days and weeks as evidence of a zero price[7]), but trading may still continue for the reasons discussed in the introduction. At a first glance, this situation may be similar to the so-called "zombie firms": these firms are characterized by

[6] In this paper, we deal only with the credit risk arising from the death of a cryptocurrency. The credit risk due to the possibility that a crypto-exchange is hacked and/or goes bankrupt is examined by Moore and Christin (2013) and Moore et al. (2018).

[7] Crypto-exchanges work 24 hours a day, 365 days a year.

a lack of profitability over an extended period, are rather old and their future expected profitability is rather low. There are no unique criteria to define a zombie firm but, in general, the ability to barely repay the interest on its debts coupled with the inability to repay the principal are common to several criteria proposed in the literature, see Caballero et al. (2008), Adalet et al. (2018) and Banerjee and Hofmann (2018). However, a dead cryptocurrency is a rather different phenomenon from a zombie firm: there is no financial institution providing financial support to continue the project, and the trading of the coin can continue even without any project development. Instead, a zombie firm is kept alive to guarantee that the firm continues its activities, thus saving jobs places.

Given the importance of the zero price barrier to determine the death of a cryptocurrencies, it would be natural to consider reduced-form and hybrid credit risk models in which the asset price jumps to zero at default: see, for example, Linetsky (2006), Carr and Linetsky (2006), Campi et al. (2009), Das and Hanouna (2009), Carr and Wu (2009), Bayraktar and Yang (2011) and Dyrssen et al. (2014). Unfortunately, all these models are based on derivatives data, bond data and/or accounting data, which are not available for cryptocurrencies[8]. This is why a different approach based only on market data is required.

Finally, Table 1 summarizes the main aspects of credit and market risk for cryptocurrencies[9].

	Market risk	**Credit risk**
Definition	Gains and losses on the value of a position or portfolio of ***alive*** cryptocurrencies, that can take place due to the movements in market prices in centralized and decentralized exchanges.	Gains and losses on the value of a position of a cryptocurrency that is ***abandoned and considered dead*** according to professional and/or academic criteria.
Differences from traditional finance	• Lack of financial oversight means that coins prices can be susceptible to manipulations, pump and dump schemes and other market frauds; • Lack of regulatory oversight also explain why prices can differ widely across exchanges; • Cryptocurrency exchanges do not provide neither cash nor asset insurance (but there are exceptions).	• Dead coins can be revamped several times; • Dead coins are very different from "zombie firms"; • Traditional credit risk models cannot be used due to the (current) lack of derivatives data, bond data and/or accounting data.

Table 1: Main aspects of credit and market risk for cryptocurrencies

[8]The CME and the CBOT introduced the first futures on bitcoin in December 2017, whereas options on cryptocurrencies are (currently) traded only on small and illiquid exchanges, with poor or no financial oversight.

[9]Updated lists of dead coins can be found at https://deadcoins.com, www.coinopsy.com/dead-coins. The first site employs a broad definition of dead coins, whereas the second site has stricter selection criteria.

3 Methods: Market and Credit risk models

We first briefly review the Value-at-Risk (VaR) and the Expected Shortfall (ES), as well as a set of statistical tests to measure the goodness-of-fit of these market risk measures from different models after a backtesting exercise[10]. Similarly, we will present several methods to compute the probability of death for a crypto-coin to measure its credit risk, as well as several metrics to evaluate the estimated death probabilities. Two special cases of the ZPP with closed-form formulas in case of normally distributed asset returns will be developed using recent results from barrier option theory. The multivariate time series models used to simultaneously estimate the market and credit risk for a portfolio of crypto-currencies are discussed in Appendix B.

3.1 Market Risk

Market risk arises due to movement in market variables − asset prices, interest rates, FX rates, or in our case − cryptocurrencies prices. In this work, we consider two measures of market risk: the Value-at-Risk (VaR) and the Expected Shortfall (ES).

Value-at-Risk. The Value at Risk at level α (VaR_α) is the minimum loss of the α worst losses an investor can expect to lose over a specific period of time. More formally, if we call $\Delta W(l)$ the change in the portfolio value from time t to $t+l$ and $F_l(x)$ the cumulative distribution function ("cdf") of ΔW_{t+l}, the VaR of a long position over the time horizon l with probability α can be defined as follows:

$$\alpha = Pr[\Delta W(l) \leq VaR_{\alpha,t+l}] = F_l(VaR_{\alpha,t+l}).$$

where VaR is defined as a negative value (loss), even though a part of the financial literature defines it as a positive value, see e.g. Jorion (2006) and Christoffersen (2011). If the cdf is known, then the VaR is its α-quantile times the value of the financial position. The previous definition of the VaR continues to apply to a short position if one uses the distribution of $-\Delta W(l)$, see Jorion (2007) for more details.

If we consider the profit and losses (P&L) for a single coin or for a portolio of crypto-currencies and the normal distribution is used, then the $\mathrm{VaR}_{\alpha,t+l}$ is given by

$$VaR_{\alpha,t+l} = \mu_{t+l} + \sigma_{t+l}\,\Phi_\alpha^{-1}$$

where Φ_α^{-1} is the α-quantile of the standard normal distribution, μ_{t+l} is the forecasted mean of the P&L

[10]Backtesting is the process of assessing the performance of a model, by applying this model to a historical dataset to verify how accurately it would have predicted actual results. See Christoffersen (2011) and McNeil et al. (2015) for a discussion at the textbook level.

distribution at time $t+l$, and σ_{t+l} is the forecasted standard deviation of the P&L distribution at time $t+l$. If the Student's t distribution is used, then the $\mathrm{VaR}_{\alpha,t+l}$ is given by

$$VaR_{\alpha,t+l} = \mu_{t+l} + s_{t+l}\, t^{-1}_{\alpha,\nu}$$

where s_{t+l} is the scale parameter of the Student's t distribution at time $t+l$ given by $s_{t+l} = \sigma_{t+l}\sqrt{(\nu-2)/\nu}$, while $t^{-1}_{\alpha,\nu}$ is the α-quantile of the Student's t distribution with ν degrees of freedom. If copula models are used, simulation methods have to be implemented, see Fantazzini (2008) - section 3.2 for a detailed description, and McNeil et al. (2015) for a general discussion.

Backtesting methods for the Value at Risk. For backtesting purposes, it is common practice to split the dataset into two parts (training set and testing set), and to perform a rolling window estimation to compute at each time step the forecasted $VaR_{\alpha,t+l}$ at the desired time horizon l and probability level α. The VaR forecasts and the actual realized series are then compared using the following back-testing techniques:

- Kupiec (1995) 's unconditional coverage test;
- Christoffersen (1998) 's conditional coverage test;
- Loss functions to evaluate VaR forecasts accuracy and to select the best models among a group of competing models using the model confidence set by Hansen, Lunde, and Nason (2011).

The *Kupiec's test* (also known as the *Unconditional Coverage test*) tests the difference between the observed and the expected number of VaR violations. Since the VaR is based on a confidence level α, when we observe T_1 actual exceedances out of T observations (with $T = T_0 + T_1$ and T_0 the number of no VaR violations), we observe an empirical frequency of $\hat{\pi} = T_1/T$ proportion of excessive losses. The Kupiec's test checks whether $\hat{\pi}$ is statistically significantly different from α. It is possible to show that the test of the null hypothesis $H_0 : \hat{\pi} = \alpha$ is given by the following likelihood ratio test statistic, see Kupiec (1995) for details:

$$LR_{uc} = -2\ln\left[(1-\alpha)^{T_0}\alpha^{T_1} \,/\, \{(1-T_1/T)^{T_0}(T_1/T)^{T_1}\}\right] \overset{H_0}{\sim} \chi^2_1$$

Christoffersen (1998) developed a likelihood ratio statistic to test the joint assumption of unconditional coverage and independence of VaR exceedances. The main advantage of this test named the *Conditional Coverage test* is that it can reject a VaR model that forecasts either too many or too few clustered exceedances, while its main limit is the need of (at least) several hundred observations to be accurate.

The test statistic is computed as follows:

$$LR_{cc} = -2\ln\left[(1-\alpha)^{T_0}\alpha^{T_1}\right] + 2\ln\left[(1-\pi_{01})^{T00}\pi_{01}^{T01}(1-\pi_{11})^{T10}\pi_{11}^{T11}\right] \overset{H_0}{\sim} \chi^2_2$$

where T_{ij} is the number of observations with value i followed by j for $i, j = 0, 1$ and

$$\pi_{ij} = \frac{T_{ij}}{\sum_j T_{ij}}$$

are the corresponding probabilities.

Loss functions are useful tools to compare the costs of different admissible choices. We will use the *asymmetric VaR loss function* proposed by Gonzalez-Rivera et al. (2004), which penalizes more heavily the realized losses below the α-th quantile level, that is $y_{t+l} < VaR_{\alpha,t+l}$:

$$l(y_{t+l}, VaR_{\alpha,t+l}) = (\alpha - d^{\alpha}_{t+l})(y_{t+l} - VaR_{\alpha,t+l})$$

where $d^{\alpha}_{t+l} = \mathbf{1}(y_{t+l} < VaR_{\alpha,t+l})$ is the indicator function for the VaR exceedances. This loss function will be used for the *Model Confidence Set* (MCS) by Hansen et al. (2011) to select the best forecasting models among a set of competing models, given a specified confidence level. The MCS approach is an iterative procedure which consists of a test for the null hypothesis of equal predictive ability and an elimination rule: at each iteration, an equivalence test is performed to check if all models in the set of forecasting models $M = M_0$ have an equal forecasting accuracy by testing the following null hypothesis for a given confidence level $1-\beta$:

$$H_{0,M} = E(d_{ij,t}) = 0, \quad \forall i, j \in M, \quad vs \quad H_{A,M} = E(d_{ij,t}) \neq 0$$

where $d_{ij,t} = L_{i,t} - L_{j,t}$ is the sample loss differential between forecasting models i and j and $L_{i,t}$ stands for the loss function of model i at time t. If the null hypothesis cannot be rejected, then $\widehat{M}^*_{1-\beta} = M$. If the null hypothesis is rejected, the elimination rule e_M is used to remove the worst forecasting models from the set M. The procedure is repeated until the null hypothesis cannot be rejected, and the final set of models define the so-called model confidence set $\widehat{M}^*_{1-\beta}$. Hansen et al. (2011) propose different equivalence tests and we discuss here the T-max statistic which we used in the paper. First, the following t-statistics are computed:

$$t_{i\cdot} = \frac{\overline{d}_{i\cdot}}{\widehat{var}(\overline{d}_{i\cdot})}, \quad \text{for} \quad i \in M,$$

where $\overline{d}_{i\cdot} = m^{-1}\sum_{j\in M}\bar{d}_{ij}$ is the simple loss of the i-th model relative to the average losses across models in the set M, and $\overline{d}_{ij} = T^{-1}\sum_{t=1}^{T} d_{ij,t}$ measures the sample loss differential between model i and j. Then, the T-max statistic is computed as $T_{max} = \max_{i\in M}(t_{i\cdot})$. The distribution of this test statistic is non-standard and is estimated using bootstrapping methods, see Hansen et al. (2011) for details. The elimination rule for the T-max statistic is $e_{max,M} = \arg\max_{i\in M}\left(\overline{d}_{i\cdot}/\widehat{var}(\overline{d}_{i\cdot})\right)$.

Expected Shortfall. The Expected Shortfall (ES) measures the average of the worst α losses, where α is a percentile of the P&L distribution. More formally, the expected shortfall at probability level $\alpha \in (0,1)$ is defined as

$$ES_\alpha = \frac{1}{\alpha}\int_0^\alpha F_z^{-1}(X)dz = \frac{1}{\alpha}\int_0^\alpha VaR_z(X)dz$$

where F^{-1} is the inverse function of the cdf of the P&L, which is the Value-at-Risk.

If the normal distribution is used, then the $\mathrm{ES}_{\alpha,t+l}$ is given by,

$$ES_{\alpha,t+l} = \mu_{t+l} + \sigma_{t+l}\,\frac{\phi(\Phi_\alpha^{-1})}{\alpha}$$

where $\phi(\cdot)$ denotes the standard normal density function. Instead, if the Student's t distribution is employed, then the $\mathrm{ES}_{\alpha,t+l}$ is given by

$$ES_{\alpha,t+l} = \mu_{t+l} + s_{t+l}\left(\frac{t_\nu(t_{\alpha,\nu}^{-1})}{\alpha}\left[\frac{\nu + (t_{\alpha,\nu}^{-1})^2}{\nu - 1}\right]\right)$$

where t_ν is the density function of the Student's t distribution with ν degrees of freedom. If copula models are used, similarly to the case of the VaR, simulation methods have to be implemented.

Backtesting methods for the Expected Shortfall. Backtesting the ES is an active field of research since the work of McNeil and Frey (2000), but it has become a very hot topic after Gneiting (2011) showed that ES lacks a mathematical property called elicitability (while VaR does have it), so that it may not be backtested[11]. However, Acerbi and Szekely (2014) showed that the ES can be back-tested and that the property of elicitability is useful only for model selection to choose the best model among a set of competitors. Moreover, Emmer et al. (2015) showed that the ES is elicitable conditionally on the VaR, and that it can be backtested through the approximation of several VaR levels: Kratz et al. (2018) used this latter approach for an empirical application. In this paper, we will use the tests by McNeil and Frey (2000) and by Kratz et al. (2018).

[11]In simple terms, a statistic $\psi(Y)$ of a random variable Y is *elicitable* if it minimizes the expected value of a scoring function S, $\psi(Y) = \arg\min_x E[S(x,Y)]$, where S can be, for the case of the Value-at-Risk, the asymmetric loss function of Gonzalez-Riviera et.al. (2004), while x are the model forecasts. Given a vector of forecasted VaR x_t and a vector of realized P&L y_t, the forecasting model can then be evaluated by minimizing the mean score $\bar{S} = \frac{1}{T}\sum_{t=1}^{T} S(x_t, y_t)$. Elicitability allows for the ranking of the risk models' performance because the scoring function can be used for comparative tests of the models.

McNeil and Frey (2000) proposed a one-sided test based on the *Exceedance Residuals (ER)* that exceed the VaR, that is $ER_{t+l} = (Y_{t+l} - \widehat{ES}_{\alpha,t+l})\mathbf{1}_{Y_{t+l} \leq \widehat{VaR}_{\alpha,t+l}}$, which should form a martingale difference sequence if $\widehat{ES}_{\alpha,t+l}$ and $\widehat{VaR}_{\alpha,t+l}$ are the true ES and VaR, respectively. More specifically, McNeil and Frey (2000) implement the standardized version of this test, where the exceedance residuals are standardized by the forecasted volatility (that is, $ER_{t+l}/\hat{\sigma}_{t+l}$). The null hypothesis is that the expected value of the standardized ER is zero against the alternative that the residuals have mean greater than zero, i.e. the expected shortfall is underestimated,

$$H_0 : \mu = 0, \quad vs \quad H_A : \mu > 0$$

where $\mu = E(ER_{t+l})$. The distribution of this test is computed using the bootstrap method by Efron and Tibshirani (1994) p. 224.

Kratz et al. (2018) proposed a *multilevel VaR backtest* to implicitly backtest the expected shortfall using the initial idea by Emmer et al. (2015), who showed that

$$ES_\alpha \approx \frac{1}{4}[q(\alpha) + q(0.75\alpha + 0.25) + q(0.5\alpha + 0.5) + q(0.25\alpha + 0.75)]$$

where $q(\gamma) = VaR_\gamma$. For example, if $\alpha = 0.05$ then

$$ES_{5\%} \approx \frac{1}{4}[VaR_{5\%} + VaR_{4\%} + VaR_{3\%} + VaR_{2\%}]$$

An estimated ES_α can be considered reliable if the estimates of the four VaR values obtained from the same model are reliable. In the general case, Kratz et al. (2018) consider VaR probability levels $\alpha_1, \ldots, \alpha_N$ defined by

$$\alpha_j = \alpha + \frac{j-1}{N}(1-\alpha), \quad j = 1, \ldots, N,$$

for some starting level α. Kratz et al. (2018) suppose to have a set of $VaR_{\alpha,t}$ and $ES_{\alpha,t}$ forecasts and a series of ex-post losses $\{L_t, t = 1, \ldots, T\}$. If $I_{t,j} = \mathbf{1}_{(L_t > VaR_{\alpha_j,t})}$ is the usual indicator function for a VaR violation at the level α_j and $X_t = \sum_{j=1}^N I_{t,j}$, then the sequence $(X_t)_{t=1,\ldots,T}$ counts the number of VaR levels that were breached. If we define $MN(T, (p_0, \ldots, p_N))$ as the multinomial distribution with T trials, each of which may result in one of $N+1$ outcomes $\{0, 1, \ldots, N\}$ according to probabilities $p_0, \ldots, p_N$ that sum to one, while the observed cell counts are defined by

$$O_j = \sum_{t=1}^T I_{(X_t = j)}, \quad j = 0, 1, \ldots, N$$

then, under the assumptions of unconditional coverage and independence as in Christoffersen (1998), the random vector $(O_0, \ldots, O_N)$ should follow the multinomial distribution $(O_0, \ldots, O_N) \sim MN(T, (\alpha_1 - \alpha_0, \ldots, \alpha_{N+1} - \alpha_N))$. Assuming the multinomial distribution estimated from the data is $MN(T, (\theta_1 - \theta_0, \ldots, \theta_{N+1} - \theta_N))$, Kratz et al. (2018) test the following null and alternative hypotheses:

$$
\begin{aligned}
H_0: &\quad \theta_j = \alpha_j, \quad \text{for} \quad j = 1, \ldots, N \\
H_1: &\quad \theta_j \neq \alpha_j, \quad \text{for at least one } j \in \{1, \ldots, N\}
\end{aligned}
$$

Several test statistics have been proposed in the statistical literature to test the previous hypotheses: Cai and Krishnamoorthy (2006) performed a large simulations study to verify the exact size and power properties of five possible tests, while Kratz et al. (2018) employed three of these five tests. We will employ the *exact method* which is the fifth test statistic reviewed by Cai and Krishnamoorthy (2006), which computes the probability of a given outcome under the null hypothesis using the multinomial probability distribution itself:

$$
P(O_0, O_1, \ldots, O_N) = \frac{T!}{O_0! O_1! \ldots O_N!} (\alpha_1 - \alpha_0)^{O_0} (\alpha_2 - \alpha_1)^{O_1} \ldots (\alpha_{N+1} - \alpha_N)^{O_N}
$$

Cai and Krishnamoorthy (2006) found out that the exact method performs very well, but it can be time-consuming if the number of cells and the sample size T are large. In this latter case, simulation methods can be used to decrease the computational burden[12].

3.2 Credit Risk

In traditional finance, credit risk is the risk of a change in the value of positions associated with an unexpected deterioration in their credit quality, either downgrade or default. In the case of cryptocurrencies, as we previously discussed in section 3, credit risk is the risk of an extreme drop in the coin price following its "death", due to the discovery of a scam or a hacker attack or for other reasons. In this work, we will use the Probability of Default (PD) as the main measure of credit risk. However, when dealing with cryptocurrencies, this probability is usually referred to as the *probability of death of the coin i*, over a period of time $t{+}T$, given that it is alive at the time t:

$$
PD_{i,t} = \mathcal{P}(D^i_{t+T} = 1 | D^i_t = 0)
$$

Several approaches will be considered to estimate this probability using only available data, namely credit scoring models and the Zero Price Probability by Fantazzini, De Giuli, and Maggi (2008). We want to

[12] Several tests reviewed by Cai and Krishnamoorthy (2006) are implemented in the R package `XNomial` available at `cran.r-project.org/web/packages/XNomial` .

emphasize that structural models, like the well known Merton and KMV models, cannot be used because they require accounting data which are not available for cryptocurrencies. Similary to traditional credit risk, we will consider a 1-year time horizon.

Scoring models. Scoring models use statistical techniques to combine different factors into a quantitative score which is used to compute the probability of default, see Mester (1997), Fuertes and Kalotychou (2006), Rodriguez and Rodriguez (2006), Altman and Sabato (2007), Fantazzini and Figini (2008) and Fantazzini and Figini (2009). Credit scoring models can be applied even if ratings and accounting data are not available. Several scoring models typically take the following form:

$$PD_{i,t} = \mathcal{P}(D^i_{t+T} = 1 | D^i_t = 0; \mathbf{X}_{i,t}) = F(\beta' \mathbf{X}_{i,t})$$

In case of a logit model, $F(\beta' \mathbf{X}_{i,t})$ is given by the logistic cdf,

$$F(\beta' \mathbf{X}_{i,t}) = \frac{1}{1 + e^{-(\beta' \mathbf{X}_{i,t})}}$$

while for the probit model $F(\beta' \mathbf{X}_{i,t}) = \Phi(\beta' \mathbf{X}_{i,t})$ where $\Phi(\cdot)$ is a standard normal cdf.

We can also employ machine learning algorithms for solving classification tasks with two classes (dead coin/alive coin): in this work, we will use the *Random Forest* algorithm proposed by Ho (1995) and Breiman (2001). A Random Forest is an ensemble method using a large number of decision trees, where a decision tree is an instrument for decision making with the structure of a tree, i.e. it has branches and leaves. Each branch includes attributes which separate the alternative cases and following the tree through the branches, we end with a (pre-specified) value of an objective function. In case of a classification problem, each leave places an object either in one class or in the competing one. A single decision tree may provide a poor classification and can suffer from over-fitting and model instability: this is why the Random Forest algorithm is commonly used. Such an approach has multiple advantages: it is not sensitive to monotone transformations, it can cope with datasets with a large number of features and classes, and it can be used with continuous variables. See Friedman et al. (2016) and Smith and Koning (2017) for more details about decision trees and random forests.

The Zero Price Probability (ZPP). The Zero Price Probability was firstly introduced in Fantazzini et al. (2008) and relies on the fact that the PD can be estimated by computing the market-implied probability $\mathcal{P}(P_\tau \leq 0)$ with $t < \tau \leq t + T$. Because for a stock price (or a coin price) P_τ is a truncated variable and cannot become less than zero, the Zero-Price Probability is the probability that P_τ goes below the truncation level of zero. Fantazzini et al. (2008) discussed in details why the null price can be used as a default barrier.

A general estimation procedure of the ZPP for multivariate time series is reported below:

1. Consider a generic conditional model for the differences of prices levels $\mathbf{X}_t = \mathbf{P}_t - \mathbf{P}_{t-1}$ without the log-transformation:

$$\mathbf{X}_t = \boldsymbol{\mu}_t + \mathbf{D}_t \mathbf{z}_t \tag{1}$$

where $\boldsymbol{\mu}_t$ is a vector of conditional means, $\mathbf{D}_t = diag(\sigma_{1,t}, \ldots, \sigma_{n,t})$ a diagonal matrix of conditional standard deviations, while $\mathbf{z}_t$ is a vector of standardized errors with a conditional joint distribution given by H_t, which can be any of those discussed in Section 3.1 .

2. Simulate a high number N of price trajectories up to time $t + T$, using the estimated time series model (3) at step 1. We will compute the 1-year ahead probability of death for each coin using $T = 365$ days, given that crypto-assets are traded every day of the week (whereas $T = 250$ would be used in the case of traditional financial assets).

3. The probability of default/death for a crypto-coin i is simply the ratio n/N, where n is the number of times out of N when the simulated price $P^k_{\tau,i}$ touched or crossed the zero barrier along the simulated trajectory:

$$PD_{i,t} = \frac{1}{N} \sum_{k=1}^{N} \mathbf{1}\big\{P^k_{\tau,i} \leq 0 \quad \text{for some } t < \tau \leq t + T\big\}$$

It follows easily from (1) that the models used in section 3.1 for measuring market risk can also be used for measuring credit risk.

A special case: ZPP with normally distributed price differences. The original method proposed in Fantazzini et al. (2008) dealt with general univariate models, where the ZPP can be estimated using only simulation methods. However, if we assume that the price differences are normally distributed and we use some recent theoretical results from barrier option theory, then it is possible to compute the ZPP using a closed-form solution.

Su and Rieger (2009) (Theorem 2.2) showed that if a stock price S_t follows a standard Brownian motion, then the ratio of the probability that the asset hits a barrier $H \in [0, \infty)$ before the maturity T and the probability that the asset is below the barrier at maturity T is equal to 2 :

$$P(\tau \leq T) = 2P(S_T \leq H) = 2\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{H} e^{-\frac{x^2}{2t}} dx$$

where τ is the first t when the underlying asset price touches the barrier H, that is $\min_{0 \leq t \leq T}\{t|S_t \leq H\}$. If the asset follows a geometric Brownian motion with drift, they showed that the ratio of the previous two probabilities is not constant anymore, but for $H \to 0$ it still converges to 2 (see Su and Rieger (2009), Lemma 2.1):

$$\lim_{H \to 0} \frac{P(\tau \leq T)}{P(S_T \leq H)} = 2$$

Using these theoretical results and assuming that the coin price differences $X_t = P_t - P_{t-1}$ are normally distributed, we can find the closed-form solutions for the ZPP in two special cases:

- *Naive constant variance model (NCV).* This model was initially proposed by Li, Yang, and Zou (2016) and it is simply the random walk with drift:

$$X_t = \mu + \varepsilon_t, \quad \varepsilon_t \sim NID(0, \sigma^2)$$

If we use the previous results by Su and Rieger (2009) with $H = 0$ and a geometric Brownian motion with drift, it is straightforward to show that the 1-yead ahead ZPP can be computed at time t as follows:

$$ZPP_{NCV} = P_{NCV}[P_\tau \leq 0] \approx 2\Phi\left(\frac{P_t - \mu T}{\sigma\sqrt{T}}\right), \quad \text{with } t < \tau \leq t+T \tag{2}$$

where P_t is the last price. Note that this formula is an approximation which is valid only in the limit with $\Delta t \to 0$ and $H \to 0$.

- *GARCH(1,1) with normal errors.* If we assume that X_t follows a model with a constant mean and a GARCH(1,1) with normally distributed errors,

$$\begin{aligned} X_t &= \mu + \varepsilon_t \\ \varepsilon_t &= \sigma_t^{1/2}\eta_t, \quad \eta_t \sim NID(0,1) \\ \sigma_t^2 &= \omega + \alpha\varepsilon_t^2 + \beta\sigma_{t-1}^2 \end{aligned}$$

the ZPP can be approximated as follows

$$ZPP_{GARCH11} = P_{GARCH11}[P_\tau \leq 0] \approx 2\Phi\left(\frac{P_t - \mu T}{\sigma_T}\right), \quad \text{with } t < \tau \leq t+T \tag{3}$$

where σ_T^2 is sum of the forecasted variances of all shocks ε_t from $t+1$ till $t+T$, which can be shown to be[13]

[13]Compute the recursive forecasts of the conditional variance from time $t+1$ till time $t+T$; then collect all the common

$$\sigma_T^2 = \omega \cdot [A \times B] + \frac{[1-(\alpha+\beta)^T]}{[1-(\alpha+\beta)]} \cdot \hat{\sigma}^2_{t+1|t} \tag{4}$$

where $\hat{\sigma}^2_{t+1|t}$ can be either the forecasted variance at time $t+1$ conditional on information at time t, or the unconditional variance forecast[14], while A and B are two vectors defined below:

$$\underbrace{A}_{1\times(T-1)} = [(T-1) \quad (T-2) \ \dots \ 2 \quad 1], \qquad \underbrace{B}_{(T-1)\times 1} = \begin{bmatrix} (\alpha+\beta)^0 \\ (\alpha+\beta)^1 \\ \vdots \\ (\alpha+\beta)^{T-3} \\ (\alpha+\beta)^{T-2} \end{bmatrix}$$

There is an increasing empirical literature showing that the statistical distributions of crypto-assets are far from being normally distributed, see e.g. Chu et al. (2015), Osterrieder (2016), Chu et al. (2017), and Osterrieder and Lorenz (2017). Nevertheless, the previous closed-form formulas for the ZPP can provide a quick estimate of the probability of coin death and they can give an investor at least a rough idea of the crypto-asset credit risk. This is why they can be useful instruments for online data providers, who can publish the quotes of traded crypto-assets together with these market-implied PDs[15]. Note that the ZPP can highlight potential fraudulent behavior, like the famous pump-and-dump fraud that involves artificially inflating the price of a stock through various means, to later sell the purchased stock at a higher price to (un-informed) investors who lose all their money. In these cases, the ZPP tends to show very large swings of the estimated PD in short periods of time and any investor should consider such unusual behavior as a potential red flag. See Fantazzini et al. (2008) for some examples with the ZPPs of bankrupt companies which committed different forms of fraud.

Evaluation of prediction quality The main instrument used to check the forecasting performance of a model with binary data is the *confusion matrix* (Kohavi and Provost (1998)). The confusion matrix for a two class classifier is reported in Table 2. In the specific context of our analysis, the entries in the confusion matrix have the following meaning: a is the number of correct predictions that a coin is dead, b is the number of incorrect predictions that a coin is dead, c is the number of incorrect predictions that a coin is alive, while d is the number of correct predictions that a coin is alive.

The confusion matrix is then used to compute the *Area Under the Receiver Operating Characteristic* curve (AUC or AUROC) by Metz and Kronman (1980), Goin (1982) and Hanley and McNeil (1982)

components and use the property of the geometric series. The result will be given by (4).

[14] This second option is preferable in case of risk management.

[15] The ZPP is implemented in the R package `bitcoinFinance` available at `github.com/deanfantazzini/bitcoinFinance`.

Observed/Predicted	*DEAD COIN*	*ALIVE*
DEAD COIN	a	b
ALIVE	c	d

Table 2: Theoretical confusion matrix. Number of: a true positive, b false positive, c false negative, d true negative.

for all forecasting models: the receiver operating characteristic curve is obtained by plotting, for any probability threshold between 0 and 1, the proportion of correctly predicted dead coins $a/(a+b)$ on the y-axis (also known as the sensitivity or hit rate), and the proportion of alive coins predicted as dead coins $c/(c+d)$ (also known as the false positive rate). The AUC lies between zero and one and the closer it is to one the more accurate the forecasting model is, see Sammut and Webb (2011), pp. 869-875, and references therein for more details. We will also compute the model confidence set by Hansen et al. (2011) which was extended by Fantazzini and Maggi (2015) to binary models, to assess the prediction power of our competing models. We will employ the squared loss which is called the *Brier score* when applied on binary outcomes, see Brier (1950).

4 Empirical analysis

We combined a collection of over 1500 coins available from `coinmarketcap.com` with a list of 42 dead coins from `deadcoins.com`, which is a trusted source of information about scams, parody, and other junk coins. Merging both datasets and taking only the coins that are traded from at least the 1st January 2016, we obtained a dataset with 149 alive and 13 dead coins. It is worth noting that dead coins still have some negligible daily trading volumes for the reasons discussed in the Introduction.

The focus of this work is to estimate the market risk and the credit risk for small and medium-sized portfolios of crypto-currencies, so that we considered the following three cases:

- estimation of the market risk measures for a portfolio of the top-5 cryptocurrencies by market capitalization;
- estimation of the market risk measures for a portfolio of 15 coins, consisting of the top-5 coins, 5 random average coins, and 5 junk cryptocurrencies;
- estimation of the credit risk measures for 42 coins, consisting of 29 alive coins and 13 dead coins[16].

[16]We used a sample of 42 coins for credit risk and not a larger sample for two reasons: 1) considering that we used multivariate models for the simultaneous estimation of both market and credit risk, the models discussed in this paper would be unable to estimate the market risk of a portfolio of hundreds (or thousands) of coins. In this case, a completely different set of multivariate models need to be used, the DECO model by Engle and Kelly (2012) being (potentially) one of them. 2) The lack of historical data for the vast majority of dead coins: this problem would make any model suffer from massive selection bias. For these two reasons, the case of large portfolios will be considered in a separate paper.

4.1 Portfolio with top-5 coins

We considered the top-5 coins by market capitalization at the time we started writing this paper in 2018, with daily data ranging between June 2016 and May 2018: BTC (Bitcoin), ETH (Ethereum), XRP (Ripple), LTC (Litecoin) and XLM (Stellar). This period of time witnessed the fast rise of crypto-currencies (with bitcoin reaching almost \$20000), followed by a violent price fall. The descriptive statistics of the daily profits and losses for each coin are presented in Table 3. All coins generally exhibited large skewness and kurtosis, very high coefficients of variations, and unconditional distributions very far from the Gaussian.

	Mean	*Std.Dev.*	*Min*	*Max*	*Skewness*	*Kurtosis*	*Coeff.of Var.*
BTC	9.60	381.65	-2329.30	3608.20	0.97	20.16	39.74
ETH	0.79	27.94	-238.23	151.21	-0.58	15.20	35.27
XRP	0.00	0.07	-0.92	0.78	-0.26	76.95	83.89
XLM	0.00	0.02	-0.17	0.33	4.07	87.40	54.78
LTC	0.16	8.42	-52.53	101.96	3.10	42.05	53.05

Table 3: Descriptive statistics for the daily P&L for each coin (June 2016 - May 2018).

Following Giacomini and Komunjer (2005), Gonzalez-Rivera et al. (2004) and Fantazzini (2009a), we used a rolling forecasting scheme of 522 observations and we left 200 data for the out-of-sample evaluation. We used the first differences of the data without log transformation (that is the P&L) to compute not only their market risk measures, but also their credit risk measures using the Zero Price Probability (ZPP) by Fantazzini et al. (2008). More specifically, we used a VAR(1) model for the conditional means, univariate GARCH(1,1) models for the conditional variances, and the following six multivariate distributions for the standardized errors:

- a DCC(1,1) model with multivariate normal distribution;
- a DCC(1,1) model with multivariate Student's t distribution;
- a normal copula/skewed-t GARCH model with constant correlation matrix;
- a t-copula/skewed-t GARCH model with constant correlation matrix;
- a normal copula/skewed-t GARCH model with a correlation matrix following a DCC(1,1) model;
- a t-copula/skewed-t GARCH model with a correlation matrix following a DCC(1,1) model;

We calculated the 1-day-ahead forecasts for the 1%, 2%, 3%, 4% and 5% VaR levels and for the 5% ES, for each coin and for an equally-weighted portfolio with the five coins. The actual VaR exceedances T_1/T , the p-values of the Kupiec's unconditional coverage test and the p-values of the Christoffersen's conditional coverage test for the VaR forecasts at all quantile levels are reported in Tables 4-6. The asymmetric VaR losses and the forecasting models included in the MCS are reported in Table 7.

	DCC MVN	*DCC MVT*	*Const N-Copula GARCH*	*Const T-Copula GARCH*	*DCC N-Copula GARCH*	*DCC T-Copula GARCH*
			VaR exceedances (T_1/T): 1\% quantile			
BTC	3.52%	1.51%	1.51%	1.01%	1.51%	1.51%
ETH	3.02%	3.52%	3.02%	1.51%	2.51%	3.02%
XRP	1.51%	1.51%	0.50%	0.00%	0.00%	0.00%
XLM	2.01%	1.51%	1.01%	0.50%	1.01%	1.01%
LTC	0.50%	0.00%	0.00%	0.00%	0.00%	0.00%
portfolio	4.02%	2.01%	5.03%	2.51%	2.51%	2.01%
			VaR exceedances (T_1/T): 2\% quantile			
BTC	4.02%	4.52%	3.02%	2.51%	4.02%	4.02%
ETH	4.52%	4.02%	4.02%	3.52%	4.02%	4.02%
XRP	3.52%	4.52%	1.01%	1.01%	1.01%	1.01%
XLM	3.52%	4.02%	1.01%	1.01%	1.01%	1.01%
LTC	1.51%	2.51%	0.50%	0.50%	0.50%	0.50%
portfolio	7.54%	6.53%	6.03%	5.53%	6.03%	6.03%
			VaR exceedances (T_1/T): 3\% quantile			
BTC	5.53%	6.03%	4.52%	4.02%	4.52%	4.52%
ETH	5.53%	6.03%	6.03%	4.52%	6.03%	6.03%
XRP	4.02%	6.03%	1.51%	2.01%	2.01%	2.01%
XLM	3.52%	6.03%	2.51%	1.01%	3.02%	3.02%
LTC	3.02%	5.03%	1.01%	1.01%	1.51%	1.01%
portfolio	8.54%	8.54%	9.55%	10.05%	8.54%	8.54%
			VaR exceedances (T_1/T): 4\% quantile			
BTC	6.03%	6.53%	5.03%	5.03%	6.03%	5.53%
ETH	7.04%	7.54%	8.04%	7.54%	8.54%	8.54%
XRP	4.02%	8.54%	2.51%	2.51%	2.51%	2.51%
XLM	3.52%	6.03%	3.02%	3.02%	3.02%	3.02%
LTC	4.52%	10.05%	2.01%	1.51%	2.01%	2.01%
portfolio	9.55%	11.06%	10.55%	10.05%	10.05%	9.55%
			VaR exceedances (T_1/T): 5\% quantile			
BTC	6.53%	7.54%	6.53%	6.03%	6.53%	6.53%
ETH	8.04%	8.04%	8.54%	8.54%	9.05%	9.05%
XRP	5.03%	10.55%	3.02%	4.02%	3.52%	3.52%
XLM	4.52%	7.54%	4.52%	3.52%	4.02%	4.02%
LTC	5.53%	12.56%	2.51%	2.01%	3.02%	2.51%
portfolio	11.06%	12.56%	12.06%	11.06%	10.55%	11.06%

Table 4: VaR exceedances T_1/T for each coin and for the equally-weighted portfolio.

	DCC MVN	*DCC MVT*	*Const N-Copula GARCH*	*Const T-Copula GARCH*	*DCC N-Copula GARCH*	*DCC T-Copula GARCH*
			Kupiec's test p-value: VaR 1\%			
BTC	**0.01**	0.50	0.50	0.99	0.50	0.50
ETH	**0.02**	**0.01**	**0.02**	0.50	0.07	**0.02**
XRP	0.50	0.50	0.44	0.05	0.05	0.05
XLM	0.21	0.50	0.99	0.44	0.99	0.99
LTC	0.44	0.05	0.05	0.05	0.05	0.05
portfolio	**0.00**	0.21	**0.00**	0.07	0.07	0.21
			Kupiec's test p-value: VaR 2\%			
BTC	0.07	**0.03**	0.34	0.62	0.07	0.07
ETH	**0.03**	0.07	0.07	0.17	0.07	0.07
XRP	0.17	**0.03**	0.27	0.27	0.27	0.27
XLM	0.17	**0.07**	0.27	0.27	0.27	0.27
LTC	0.60	0.62	0.07	0.07	0.07	0.07
portfolio	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
			Kupiec's test p-value: VaR 3\%			
BTC	0.06	**0.03**	0.24	0.42	0.24	0.24
ETH	0.06	**0.03**	**0.03**	0.24	**0.03**	**0.03**
XRP	0.42	**0.03**	0.17	0.38	0.38	0.38
XLM	0.68	**0.03**	0.68	0.06	0.99	0.99
LTC	0.99	0.13	0.06	0.06	0.17	0.06
portfolio	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
			Kupiec's test p-value: VaR 4\%			
BTC	0.17	0.09	0.48	0.48	0.17	0.30
ETH	0.05	**0.02**	**0.01**	**0.02**	**0.00**	**0.00**
XRP	0.99	**0.00**	0.25	0.25	0.25	0.25
XLM	0.72	0.17	0.46	0.46	0.46	0.46
LTC	0.71	**0.00**	0.11	0.11	0.11	0.11
portfolio	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
			Kupiec's test p-value: VaR 5\%			
BTC	0.34	0.13	0.34	0.52	0.34	0.34
ETH	0.07	0.07	0.04	**0.04**	**0.02**	**0.02**
XRP	0.99	**0.00**	0.17	0.51	0.31	0.31
XLM	0.75	0.13	0.75	0.31	0.51	0.51
LTC	0.74	**0.00**	0.08	**0.03**	0.17	0.08
portfolio	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**

Table 5: Kupiec's tests for each coin and for the equally-weighted portfolio.

	DCC MVN	*DCC MVT*	*Const N-Copula GARCH*	*Const T-Copula GARCH*	*DCC N-Copula GARCH*	*DCC T-Copula GARCH*
			Christoffersen's test p-value: VaR 1\%			
BTC	0.02	0.76	0.76	0.98	0.76	0.76
ETH	0.03	**0.01**	**0.03**	0.76	0.05	**0.03**
XRP	0.76	0.76	0.73	0.14	0.14	0.14
XLM	0.42	0.76	0.98	0.73	0.98	0.98
LTC	0.73	0.14	0.14	0.14	0.14	0.14
portfolio	**0.00**	0.42	**0.00**	0.17	0.17	0.42
			Christoffersen's test p-value: VaR 2\%			
BTC	0.14	0.06	0.53	0.78	0.14	0.14
ETH	0.01	0.12	0.12	0.18	0.12	0.12
XRP	0.30	0.06	0.53	0.53	0.53	0.53
XLM	0.30	0.14	0.53	0.53	0.53	0.53
LTC	0.83	0.78	0.20	0.20	0.20	0.20
portfolio	0.00	0.00	0.00	0.01	0.00	0.00
			Christoffersen's test p-value: VaR 3\%			
BTC	0.05	**0.04**	0.33	0.52	0.33	0.33
ETH	0.05	**0.04**	0.08	0.36	0.08	0.08
XRP	0.52	**0.04**	0.38	0.63	0.63	0.63
XLM	0.71	0.08	0.81	0.16	0.83	0.83
LTC	0.83	0.18	0.16	0.16	0.38	0.16
portfolio	0.00	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
			Christoffersen's test p-value: VaR 4\%			
BTC	0.16	0.13	0.63	0.63	0.16	0.52
ETH	0.09	0.05	**0.03**	0.07	0.02	0.02
XRP	0.71	**0.01**	0.46	0.46	0.46	0.46
XLM	0.73	0.37	0.63	0.63	0.63	0.63
LTC	0.61	**0.00**	0.26	0.26	0.26	0.26
portfolio	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
			Christoffersen's test p-value: VaR 5\%			
BTC	0.33	0.22	0.33	0.33	0.33	0.33
ETH	0.16	0.16	0.10	0.10	0.06	0.06
XRP	0.59	**0.00**	0.32	0.58	0.46	0.46
XLM	0.62	0.31	0.62	0.46	0.58	0.58
LTC	0.49	**0.00**	0.18	0.08	0.32	0.18
portfolio	**0.00**	**0.00**	**0.00**	**0.00**	**0.01**	**0.00**

Table 6: Christoffersen's tests for each coin and for the equally-weighted portfolio.

	DCC MVN	*DCC MVT*	*Const N-Copula GARCH*	*Const T-Copula GARCH*	*DCC N-Copula GARCH*	*DCC T-Copula GARCH*	
			Asymmetric VaR Loss and inclusion in the MCS: 1\% quantile				
BTC	0.004	0.004	0.005	0.005	0.004		0.005
ETH	0.005	0.004	0.005	0.005	0.005		0.005
XRP	0.008	0.008	0.007	0.007	0.007		0.007
XLM	0.005	0.007	0.007		0.007		0.007
LTC	0.004	0.004					
portfolio	0.003	0.003	0.003	0.003	0.003		0.003
			Asymmetric VaR Loss and inclusion in the MCS: 2\% quantile				
BTC	0.008	0.008	0.008	0.008			0.008
ETH	0.009	0.009	0.009	0.009	0.009		0.009
XRP	0.012	0.014	0.013	0.013	0.013		0.013
XLM	0.010	0.012	0.012	0.012			0.012
LTC	0.008	0.008					
portfolio	0.005	0.005	0.005	0.005	0.005		0.004
			Asymmetric VaR Loss and inclusion in the MCS: 3\% quantile				
BTC	0.011	0.012	0.012	0.012	0.012		0.012
ETH	0.012	0.012	0.012	0.012	0.012		0.012
XRP	0.016	0.018	0.017	0.017	0.017		0.017
XLM	0.014	0.016	0.016	0.016	0.016		0.016
LTC	0.011	0.011					
portfolio	0.006	0.006	0.006	0.006	0.006		0.006
			Asymmetric VaR Loss and inclusion in the MCS: 4\% quantile				
BTC	0.014	0.014	0.015	0.015	0.015		0.015
ETH	0.015	0.015	0.015	0.015	0.015		0.015
XRP	0.020	0.021	0.021	0.021	0.021		0.021
XLM	0.018	0.019	0.020				
LTC	0.014	0.015					
portfolio	0.007	0.008	0.008	0.008	0.008		0.008
			Asymmetric VaR Loss and inclusion in the MCS: 5\% quantile				
BTC	0.017	0.017	0.017	0.017	0.017		0.017
ETH	0.017	0.018	0.018	0.018	0.018		0.018
XRP	0.023	0.024	0.024	0.024	0.024		0.024
XLM	0.021	0.022					
LTC	0.016	0.019					
portfolio	0.009	0.009	0.009	0.009	0.009		0.009

Table 7: Asymmetric VaR Loss and MCS: an empty cell means the model is not included into the MCS.

Table 4-7 show that the t-copula with skewed-t GARCH marginals can be a good compromise for precise VaR estimates across different quantile levels, particularly for the most extreme quantiles (1% and 2%) which are the most important for regulatory purposes, see BIS (2013) and BIS (2016). This evidence is consistent with other results reported in the financial literature using copulas, see Cherubini et al. (2004), Fantazzini (2008), Fantazzini (2009a), Patton (2009), Weiss (2011), Weiss (2013), Patton (2013) and McNeil et al. (2015). The worse VaR results for central quantiles (4% and 5%) were expected, due to the computational problems associated with the estimation of GARCH models with small samples ($\leq$ 500 observations), see Fantazzini (2009a) for more details. Unfortunately, increasing the estimation window is not advisable when using crypto-assets, due to the potential presence of structural breaks caused by changes in local regulations, the arrival of new investors, hacking attacks and massive improvements in mining hardware, see e.g. Bouri et al. (2016), Fantazzini et al. (2016), Fantazzini et al. (2017) and Mensi et al. (2018). This is why we employed a rolling estimation window of 522 observations, which is similar to the window size suggested by Hwang and Valls Pereira (2006), who investigated the small sample properties of the maximum likelihood estimates of ARCH and GARCH models. The asymmetric VaR losses of the competing models are rather close and almost all models are included into the MCS (for both the single coins and the portfolio case). This latter result was expected because Fantazzini (2009a) showed that, when small samples are considered and the data are skewed and leptokurtic, the biases in the GARCH parameters are so large that they deliver conservative VaR estimates, even with a simple multivariate normal distribution.

The p-values of the exceedance residuals test by McNeil and Frey (2000) and of the multilevel VaR backtest with $N = 4$ levels by Kratz et al. (2018) for the forecasted ES 5% are reported in Table 8.

	DCC MVN	*DCC MVT*	*Const N-Copula GARCH*	*Const T-Copula GARCH*	*DCC N-Copula GARCH*	*DCC T-Copula GARCH*
ES 5% test 1: Exceedance Residuals test by McNeil and Frey (2000)						
BTC	1.00	1.00	0.78	0.75	0.66	0.67
ETH	0.85	1.00	0.49	0.70	0.52	0.52
XRP	0.28	1.00	0.66	0.91	0.83	0.81
XLM	**0.00**	0.99	0.87	0.93	0.86	0.85
LTC	0.96	1.00	0.99	0.98	1.00	0.99
portfolio	0.25	1.00	0.12	0.21	0.15	0.44
ES 5% test 2: multilevel VaR backtest by Kratz et al. (2018)						
BTC	0.09	**0.01**	0.51	0.81	0.17	0.22
ETH	**0.01**	**0.01**	**0.00**	**0.02**	**0.00**	**0.00**
XRP	0.68	**0.00**	0.22	0.51	0.42	0.42
XLM	0.70	**0.03**	0.76	0.21	0.73	0.73
LTC	0.97	**0.00**	**0.02**	**0.00**	0.08	**0.02**
portfolio	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**

Table 8: ER test by McNeil and Frey (2000) and multilevel VaR backtest by Kratz et al. (2018).

The ER test by McNeil and Frey (2000) does not reject the null hypothesis for almost any model, except for a normal DCC model. Instead, the multilevel VaR backtest by Kratz et al. (2018) is much more selective, and it rejects the null hypothesis of a correctly specified multinomial distribution for both

DCC models and (to a lesser extent) copula models. These results confirm the evidence reported in table 4 that DCC models work poorly with extreme quantiles, while copula models have problems with central quantiles.

4.2 Portfolio with 15 coins

We considered the following 15 coins with daily data ranging between June 2016 and May 2018: BTC (Bitcoin), ETH (Ethereum), XRP (Ripple), LTC (Litecoin), XLM (Stellar), XCP (Counterparty), SHIFT (Shift), RVR (RevolutionVR), THC (HempCoin), CLAM (Clams), GAME (GameCredits), MINT (MintCoin), ABY (ArtByte), GLD (GoldCoin), EFL (e-Gulden). The actual VaR exceedances T_1/T and the p-values of the Kupiec's unconditional coverage test are reported in Table 9, while the p-values of the Christoffersen's conditional coverage test, the asymmetric VaR losses and the forecasting models included in the MCS are reported in Table 10. The p-values of the exceedance residuals test by McNeil and Frey (2000) and of the multilevel VaR backtest with $N = 4$ levels by Kratz et al. (2018) for the forecasted ES 5% are reported in Table 11.

The results with 15 coins are rather similar to the previous analysis with 5 coins: copula-GARCH models pass the vast majority of Kupiec and Christoffersen tests (whereas DCC models do not) and are usually more precise for extreme quantiles, particularly the t-copula/skewed-t GARCH model. However, the asymmetric VaR losses are not very different and in several cases, all models are included in the MCS. The ER test by McNeil and Frey (2000) rejects the null hypothesis for very few models, while the multilevel VaR backtest by Kratz et al. (2018) again rejects the null hypothesis much more for DCC models than for copula models. In general, DCC models seem to provide worse model fits of the multivariate distribution tails compared to copula models, thus confirming recent simulation studies by Muller and Righi (2018).

4.3 Credit risk for 42 coins

We computed the probability of death/default for a set of 42 coins reported in Table 12 , using the methods described in section 3.2.2 .

We split our dataset into a training set containing all coins up to May 2017, and a validation (out-of-time) set ranging from June 2017 till May 2018. Note that the coins which were considered dead in May 2017, were also included in the validation set because they kept on trading and they could be potentially revived, see the previous discussion in Section 3. We estimated all models using the training subset and then we computed the 1-year-ahead forecasts for the probability of death for each coin: these forecasts were used with the validation dataset to compute the AUC scores, Brier scores and the MCS.

	DCC MVN	*DCC MVT*	*Const N-Copula GARCH*	*Const T-Copula GARCH*	*DCC N-Copula GARCH*	*DCC T-Copula GARCH*	*DCC MVN*	*DCC MVT*	*Const N-Copula GARCH*	*Const T-Copula GARCH*	*DCC N-Copula GARCH*	*DCC T-Copula GARCH*
	VaR exceedances (T_1/T): 1\% quantile						**Kupiec's test p-value: VaR 1\%**					
BTC	4.02%	1.01%	2.01%	0.50%	1.01%	1.01%	**0.00**	0.99	0.21	0.44	0.99	0.99
ETH	4.02%	2.51%	3.02%	1.01%	1.51%	1.51%	**0.00**	0.07	**0.02**	0.99	0.50	0.50
XRP	4.52%	2.01%	1.51%	0.50%	0.50%	0.50%	**0.00**	0.21	0.50	0.44	0.44	0.44
XLM	5.03%	3.02%	3.52%	1.51%	2.51%	2.51%	**0.00**	**0.02**	**0.01**	0.50	0.07	0.07
LTC	2.51%	0.50%	1.01%	0.00%	0.00%	0.00%	0.07	0.44	0.99	0.05	0.05	0.05
XCP	4.02%	3.02%	3.02%	3.52%	3.52%	3.52%	**0.00**	**0.02**	**0.02**	0.01	**0.01**	**0.01**
SHIFT	5.03%	3.02%	2.51%	1.51%	1.51%	1.51%	**0.00**	**0.02**	0.07	0.50	0.50	0.50
RVR	4.52%	4.02%	2.51%	1.51%	2.01%	2.01%	**0.00**	**0.00**	0.07	0.50	0.21	0.21
THC	3.02%	3.02%	1.01%	0.50%	0.50%	0.50%	**0.02**	**0.02**	0.99	0.44	0.44	0.44
CLAM	3.02%	1.01%	1.51%	0.50%	0.50%	0.50%	**0.02**	0.99	0.50	0.44	0.44	0.44
GAME	2.01%	3.02%	2.01%	1.01%	1.01%	1.01%	0.07	**0.02**	0.21	0.99	0.99	0.99
MINT	4.02%	1.51%	1.51%	1.01%	1.01%	1.01%	**0.00**	0.50	0.50	0.99	0.99	0.99
ABY	4.52%	3.02%	1.51%	1.51%	1.51%	1.51%	**0.00**	**0.02**	0.50	0.50	0.50	0.50
GLD	2.01%	1.51%	1.51%	0.50%	0.50%	0.50%	0.21	0.50	0.50	0.44	0.44	0.44
EFL	2.51%	2.01%	1.01%	0.00%	0.00%	0.00%	0.07	0.21	0.99	0.05	0.05	0.05
portfolio	7.04%	6.03%	7.04%	4.52%	5.03%	5.03%	**0.00**	**0.00**	0.00	**0.00**	**0.00**	**0.00**
	VaR exceedances (T_1/T): 2\% quantile						**Kupiec's test p-value: VaR 2\%**					
BTC	7.04%	4.02%	3.02%	1.01%	1.51%	1.51%	**0.00**	0.07	0.34	0.27	0.60	0.60
ETH	5.53%	3.52%	4.52%	1.51%	4.02%	4.02%	**0.00**	0.17	**0.03**	0.60	0.07	0.07
XRP	5.03%	4.52%	3.52%	1.51%	2.01%	2.01%	**0.01**	**0.03**	0.17	0.60	0.99	0.99
XLM	6.53%	6.03%	4.02%	4.02%	4.02%	4.02%	**0.00**	**0.00**	0.07	0.07	0.07	0.07
LTC	3.52%	4.02%	1.01%	0.50%	0.50%	0.50%	0.17	0.07	0.27	0.07	0.07	0.07
XCP	5.53%	3.52%	4.02%	6.03%	6.03%	6.03%	**0.00**	0.17	0.07	**0.00**	**0.00**	**0.00**
SHIFT	5.03%	4.02%	4.52%	3.52%	3.52%	3.52%	**0.01**	0.07	**0.03**	0.17	0.17	0.17
RVR	5.03%	5.03%	4.02%	3.02%	3.02%	3.02%	**0.01**	**0.01**	0.07	0.34	0.34	0.34
THC	3.52%	3.02%	2.51%	0.50%	1.51%	1.51%	0.17	0.34	0.62	0.07	0.60	0.60
CLAM	3.02%	1.51%	2.01%	1.01%	1.01%	1.01%	0.34	0.60	0.99	0.27	0.27	0.27
GAME	4.52%	4.52%	3.02%	2.51%	3.02%	3.02%	**0.01**	**0.03**	0.34	0.62	0.34	0.34
MINT	5.53%	4.52%	3.02%	2.51%	2.51%	2.51%	**0.00**	**0.03**	0.34	0.62	0.62	0.62
ABY	5.53%	5.53%	4.02%	4.52%	4.02%	4.52%	**0.00**	**0.00**	0.07	**0.03**	0.07	**0.03**
GLD	2.51%	3.02%	2.01%	0.50%	0.50%	0.50%	0.62	0.34	0.99	0.07	0.07	0.07
EFL	2.51%	4.52%	1.51%	0.00%	0.00%	0.00%	0.62	**0.03**	0.60	**0.00**	**0.00**	**0.00**
portfolio	8.04%	8.04%	8.04%	7.04%	7.04%	7.04%	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
	VaR exceedances (T_1/T): 3\% quantile						**Kupiec's test p-value: VaR 3\%**					
BTC	8.04%	5.03%	5.03%	3.02%	5.03%	5.03%	**0.00**	0.13	0.13	0.99	0.13	0.13
ETH	6.03%	5.53%	5.03%	4.02%	4.52%	4.52%	0.03	0.06	0.13	0.42	0.24	0.24
XRP	5.03%	6.03%	6.03%	3.02%	3.02%	3.02%	0.13	**0.03**	**0.03**	0.99	0.99	0.99
XLM	7.54%	6.53%	5.03%	4.02%	4.52%	4.52%	**0.00**	**0.01**	0.13	0.42	0.24	0.24
LTC	5.03%	7.04%	2.01%	1.01%	1.51%	2.01%	0.13	**0.00**	0.38	0.06	0.17	0.38
XCP	6.53%	5.53%	6.53%	6.53%	7.04%	7.04%	**0.01**	0.06	**0.01**	**0.01**	**0.00**	**0.00**
SHIFT	6.03%	6.53%	5.53%	4.52%	4.52%	4.52%	**0.03**	**0.01**	0.06	0.24	0.24	0.24
RVR	5.53%	6.03%	5.03%	5.03%	5.03%	5.03%	0.06	**0.03**	0.13	0.13	0.13	0.13
THC	3.52%	6.03%	3.02%	2.01%	2.01%	2.01%	0.68	**0.03**	0.99	0.38	0.38	0.38
CLAM	4.02%	2.51%	2.01%	1.51%	1.51%	1.51%	0.42	0.68	0.38	0.17	0.17	0.17
GAME	5.53%	5.53%	4.02%	4.52%	5.03%	5.03%	**0.03**	0.06	0.42	0.24	0.13	0.13
MINT	5.53%	6.03%	4.02%	4.52%	4.52%	4.52%	0.06	**0.03**	0.42	0.24	0.24	0.24
ABY	6.53%	6.53%	8.04%	6.03%	6.03%	6.03%	**0.01**	**0.01**	**0.00**	**0.03**	**0.03**	**0.03**
GLD	3.02%	3.52%	2.51%	2.01%	1.51%	1.51%	0.99	0.68	0.68	0.38	0.17	0.17
EFL	2.51%	5.53%	2.01%	0.50%	0.50%	0.50%	0.68	0.06	0.38	**0.01**	**0.01**	**0.01**
portfolio	9.55%	10.05%	10.55%	7.04%	7.54%	7.54%	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
	VaR exceedances (T_1/T): 4\% quantile						**Kupiec's test p-value: VaR 4\%**					
BTC	9.05%	7.04%	6.03%	6.03%	7.04%	7.04%	**0.00**	0.05	0.17	0.17	0.05	0.05
ETH	7.04%	6.53%	6.03%	4.52%	5.03%	5.53%	0.05	0.09	0.17	0.71	0.48	0.30
XRP	5.53%	8.54%	6.53%	3.52%	4.02%	4.02%	0.30	**0.00**	0.09	0.72	0.99	0.99
XLM	7.54%	7.54%	5.53%	4.52%	4.52%	5.03%	**0.02**	**0.02**	0.30	0.71	0.71	0.48
LTC	5.03%	8.04%	2.01%	2.01%	2.51%	3.02%	0.48	**0.01**	0.11	0.11	0.25	0.46
XCP	6.53%	6.03%	8.04%	7.54%	7.54%	8.04%	0.09	0.17	**0.01**	**0.02**	**0.02**	**0.01**
SHIFT	6.53%	7.04%	6.53%	5.53%	5.53%	5.53%	0.09	0.05	0.09	0.30	0.30	0.30
RVR	6.53%	8.54%	5.53%	5.03%	5.53%	5.53%	0.09	**0.00**	0.30	0.48	0.30	0.30
THC	3.52%	7.04%	3.02%	2.51%	2.51%	2.51%	0.72	0.05	0.46	0.25	0.25	0.25
CLAM	5.53%	5.03%	3.02%	2.01%	2.01%	2.51%	0.30	0.48	0.46	0.11	0.11	0.25
GAME	6.03%	7.04%	5.53%	5.53%	6.03%	6.03%	0.09	0.05	0.30	0.30	0.17	0.17
MINT	7.04%	6.53%	4.52%	4.52%	4.52%	4.52%	0.05	0.09	0.71	0.71	0.71	0.71
ABY	7.04%	10.05%	8.04%	7.54%	7.54%	7.54%	0.05	**0.00**	0.01	0.02	**0.02**	**0.02**
GLD	4.02%	6.53%	3.52%	3.02%	2.51%	3.02%	0.99	0.09	0.72	0.46	0.25	0.46
EFL	3.02%	6.53%	2.51%	1.01%	1.01%	1.51%	0.46	0.09	0.25	**0.01**	**0.01**	0.04
portfolio	11.06%	12.56%	10.55%	9.55%	9.55%	9.55%	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
	VaR exceedances (T_1/T): 5\% quantile						**Kupiec's test p-value: VaR 5\%**					
BTC	9.05%	9.05%	7.04%	7.04%	7.54%	7.54%	**0.02**	0.02	0.21	0.21	0.13	0.13
ETH	8.04%	7.54%	7.04%	4.52%	6.53%	6.53%	0.07	0.13	0.21	0.75	0.34	0.34
XRP	8.04%	10.05%	8.04%	4.02%	4.02%	4.02%	0.07	**0.00**	0.07	0.51	0.51	0.51
XLM	7.54%	8.04%	7.54%	5.03%	5.53%	5.53%	0.13	0.07	0.13	0.99	0.74	0.74
LTC	6.53%	10.55%	4.02%	2.51%	4.02%	4.52%	0.34	**0.00**	0.51	0.08	0.51	0.75
XCP	7.54%	8.04%	8.54%	8.54%	8.54%	8.54%	0.13	0.07	0.04	0.04	**0.04**	0.04
SHIFT	7.04%	9.05%	7.54%	6.03%	6.03%	6.03%	0.21	**0.02**	0.13	0.52	0.52	0.52
RVR	6.53%	9.55%	7.04%	5.53%	6.03%	6.03%	0.34	**0.01**	0.21	0.74	0.52	0.52
THC	4.52%	8.54%	3.52%	2.51%	2.51%	2.51%	0.75	**0.04**	0.31	0.08	0.08	0.08
CLAM	6.03%	7.04%	3.52%	3.02%	2.51%	3.02%	0.52	0.21	0.31	0.17	0.08	0.17
GAME	6.03%	7.54%	6.03%	6.03%	6.03%	6.03%	0.34	0.13	0.52	0.52	0.52	0.52
MINT	7.54%	8.04%	4.52%	5.03%	5.03%	5.03%	0.13	0.07	0.75	0.99	0.99	0.99
ABY	7.04%	11.06%	9.55%	7.54%	7.54%	7.54%	0.21	**0.00**	**0.01**	0.13	0.13	0.13
GLD	4.02%	8.54%	5.53%	4.52%	4.02%	4.02%	0.51	**0.04**	0.74	0.75	0.51	0.51
EFL	3.52%	8.54%	3.02%	1.51%	1.51%	1.51%	0.31	**0.04**	0.17	**0.01**	**0.01**	**0.01**
portfolio	12.06%	13.57%	12.06%	10.05%	10.05%	10.05%	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**

Table 9: VaR exceedances T_1/T (left) and Kupiec's tests (right) for each coin and for the equally-weighted portfolio. P-values smaller than 0.05 are in bold font.

	DCC MVN	*DCC MVT*	*Const N-Copula GARCH*	*Const T-Copula GARCH*	*DCC N-Copula GARCH*	*DCC T-Copula GARCH*	*DCC MVN*	*DCC MVT*	*Const N-Copula GARCH*	*Const T-Copula GARCH*	*DCC N-Copula GARCH*	*DCC T-Copula GARCH*
	Christoffersen's test p-value: VaR 1\%						**Asymmetric VaR 1% Loss and MCS**					
BTC	**0.00**	0.98	0.42	0.73	0.98	0.98	0.010	0.006	0.008	0.007	0.006	0.006
ETH	**0.00**	0.17	0.06	0.98	0.76	0.76	0.010	0.006	0.007	0.007	0.007	0.007
XRP	**0.00**	0.42	0.76	0.73	0.73	0.73	0.006	0.006	0.007	0.006	0.006	0.006
XLM	**0.00**	0.06	**0.02**	0.76	0.17	0.17	0.009	0.008	0.009	0.007	0.007	0.007
LTC	0.17	0.73	0.98	0.14	0.14	0.14	0.010	0.005	0.008	0.007	0.007	0.006
XCP	**0.00**	0.06	0.06	**0.02**	**0.02**	**0.02**	0.013	0.013	0.010	0.014	0.014	0.014
SHIFT	**0.00**	0.06	0.17	0.76	0.76	0.76	0.007	0.007	0.007		0.007	0.007
RVR	**0.00**	0.00	0.17	0.76	0.42	0.42	0.015	0.010	0.010	0.009	0.009	0.009
THC	0.06	0.06	0.98	0.73	0.73	0.73	0.008	0.009	0.008	0.009		0.009
CLAM	0.06	0.98	0.76	0.73	0.73	0.73	0.015	0.014		0.014	0.014	0.014
GAME	0.17	0.06	0.42	0.98	0.98	0.98	0.012	0.008	0.009	0.008	0.008	0.008
MINT	**0.00**	0.76	0.76	0.98	0.98	0.98	0.014	0.014	0.015	0.015	0.015	0.015
ABY	**0.00**	0.06	0.76	0.76	0.76	0.76	0.012	0.007	0.009	0.008	0.008	0.008
GLD	0.42	0.76	0.76	0.73	0.73	0.73	0.015	0.014		0.014	0.014	0.014
EFL	0.17	0.42	0.98	0.14	0.14	0.14	0.016	0.009	0.014	0.015	0.014	0.014
portfolio	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**		0.005		0.005	0.005	0.005
	Christoffersen's test p-value: VaR 2\%						**Asymmetric VaR 2% Loss and MCS**					
BTC	**0.00**	0.14	0.53	0.53	0.83	0.83	0.015	0.012	0.013	0.011	0.011	0.011
ETH	**0.01**	0.18	0.06	0.83	0.14	0.14	0.015	0.010	0.012	0.012	0.012	0.012
XRP	**0.02**	0.06	0.30	0.83	0.92	0.92	0.012	0.012	0.012	0.010	0.011	0.010
XLM	**0.00**	**0.00**	0.14	0.14	0.14	0.14	0.013	0.013		0.012	0.012	0.012
LTC	0.30	0.14	0.53	0.20	0.20	0.20	0.014	0.009	0.011	0.011	0.011	0.010
XCP	**0.01**	0.30	0.12	**0.00**	**0.00**	**0.00**	0.020	0.019	0.017	0.022	0.023	0.023
SHIFT	**0.02**	0.14	0.06	0.30	0.30	0.30	0.013	0.013	0.014	0.013	0.014	0.013
RVR	**0.03**	**0.03**	0.14	0.53	0.53	0.53	0.020	0.018	0.018	0.017	0.018	0.018
THC	0.30	0.53	0.78	0.20	0.83	0.83	0.012	0.013	0.015			
CLAM	0.53	0.83	0.92	0.53	0.53	0.53	0.025	0.025	0.030	0.024	0.024	0.023
GAME	**0.02**	0.06	0.53	0.78	0.53	0.53	0.018	0.016	0.014	0.014	0.014	0.014
MINT	**0.01**	0.06	0.53	0.78	0.78	0.78	0.024	0.024	0.025	0.025	0.025	0.026
ABY	**0.01**	**0.01**	0.14	0.06	0.14	0.06	0.019	0.014	0.014	0.016	0.017	0.017
GLD	0.78	0.53	0.92	0.20	0.20	0.20	0.022	0.020		0.020	0.021	0.020
EFL	0.78	0.06	0.83	**0.02**	**0.02**	**0.02**	0.023	0.018	0.022	0.023	0.023	0.022
portfolio	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	0.009	0.008	0.009	0.007	0.008	0.007
	Christoffersen's test p-value: VaR 3\%						**Asymmetric VaR 3% Loss and MCS**					
BTC	**0.00**	0.07	0.25	0.83	0.07	0.07	0.017			0.016	0.016	0.016
ETH	0.08	0.05	0.25	0.43	0.36	0.36	0.019	0.015	0.016	0.016	0.017	0.016
XRP	0.18	**0.04**	**0.04**	0.83	0.83	0.83	0.016	0.017	0.016	0.015	0.015	0.015
XLM	**0.01**	**0.04**	0.18	0.52	0.33	0.33	0.017	0.017		0.017	0.017	0.017
LTC	0.18	**0.02**	0.63	0.16	0.38	0.63	0.017	0.014	0.015	0.015	0.014	0.014
XCP	**0.04**	0.15	**0.02**	**0.04**	**0.01**	**0.01**	0.026	0.025	0.023	0.029	0.029	0.029
SHIFT	**0.04**	**0.02**	0.09	0.33	0.33	0.33	0.018	0.018	0.019	0.018	0.019	0.018
RVR	0.15	**0.04**	0.18	0.25	0.25	0.25	0.025	0.024	0.024	0.023	0.024	0.023
THC	0.71	0.08	0.83	0.63	0.63	0.63	0.016	0.018				
CLAM	0.52	0.81	0.63	0.38	0.38	0.38	0.033	0.035	0.040	0.032	0.032	0.032
GAME	0.08	0.09	0.52	0.33	0.18	0.18	0.023	0.021	0.018	0.020	0.021	0.020
MINT	0.15	**0.04**	0.52	0.33	0.33	0.33	0.033	0.032	0.034	0.035	0.035	0.035
ABY	**0.02**	**0.02**	**0.00**	**0.04**	**0.04**	**0.04**	0.033	0.032	0.034	0.035	0.035	0.035
GLD	0.83	0.44	0.81	0.63	0.38	0.38	0.028	0.025		0.026	0.026	0.026
EFL	0.81	0.09	0.63	0.04	**0.04**	**0.04**	0.029	0.025		0.031	0.030	0.029
portfolio	**0.00**	**0.00**	**0.00**	**0.02**	**0.00**	**0.00**	0.010	0.010	0.010	0.009	0.009	0.009
	Christoffersen's test p-value: VaR 4\%						**Asymmetric VaR 4% Loss and MCS**					
BTC	**0.00**	**0.03**	0.16	0.16	**0.03**	**0.03**	0.025	0.022	0.023	0.020	0.021	0.021
ETH	0.09	**0.03**	**0.03**	0.66	0.17	0.18	0.022	0.019	0.020	0.020	0.020	0.020
XRP	0.30	**0.02**	0.10	0.73	0.71	0.71	0.020	0.021	0.020	0.019	0.020	0.019
XLM	0.07	0.07	0.30	0.61	0.61	0.46	0.021	0.020		0.021	0.021	0.020
LTC	0.46	0.04	0.26	0.26	0.46	0.63	0.021	0.019	0.018	0.018	0.017	0.017
XCP	0.24	0.16	**0.03**	0.05	0.05	**0.03**	0.031	0.030	0.029	0.034	0.035	0.034
SHIFT	0.10	0.05	0.10	0.30	0.30	0.30	0.022	0.022	0.023	0.023	0.023	0.023
RVR	0.24	**0.02**	0.30	0.63	0.52	0.52	0.030	0.028	0.028	0.028	0.028	0.028
THC	0.73	0.14	0.63	0.46	0.46	0.46	0.019	0.021				
CLAM	0.30	0.46	0.63	0.26	0.26	0.46	0.041	0.043	0.048	0.041	0.041	0.040
GAME	0.24	0.14	0.52	0.52	0.37	0.37	0.027	0.026	0.022	0.026	0.026	0.025
MINT	0.14	0.10	0.61	0.61	0.61	0.61	0.040	0.040	0.041	0.043		
ABY	0.05	**0.00**	0.04	**0.02**	**0.02**	**0.02**	0.029	0.025	0.028	0.030	0.031	0.031
GLD	0.71	0.13	0.45	0.63	0.46	0.63	0.033	0.030		0.031	0.031	0.031
EFL	0.63	0.24	0.46	0.04	0.04	0.12	0.035	0.031		0.037	0.036	0.036
portfolio	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	0.012	0.012	0.012	0.011	0.011	0.011
	Christoffersen's test p-value: VaR 5\%						**Asymmetric VaR 5% Loss and MCS**					
BTC	**0.03**	**0.00**	0.09	0.09	0.08	0.08	0.029	0.026	0.027	0.025	0.026	0.025
ETH	0.16	0.08	0.09	0.67	0.33	0.33	0.026	0.023	0.024	0.024	0.024	0.024
XRP	0.07	**0.01**	0.05	0.58	0.58	0.58	0.023	0.025	0.024	0.023	0.023	0.023
XLM	0.31	0.18	0.31	0.59	0.49	0.49	0.024	0.023		0.024	0.024	0.024
LTC	0.63	**0.01**	0.58	0.18	0.58	0.62	0.023	0.022	0.020	0.021	0.020	0.020
XCP	0.22	0.16	0.10	0.10	0.10	0.10	0.036	0.034	0.034	0.039	0.039	0.039
SHIFT	0.16	**0.01**	0.31	0.37	0.37	0.37	0.025	0.025	0.027	0.026	0.027	0.026
RVR	0.63	**0.02**	0.29	0.84	0.77	0.77	0.033	0.033	0.033	0.032	0.032	0.032
THC	0.62	0.10	0.46	0.18	0.18	0.18	0.022	0.024				
CLAM	0.37	0.16	0.46	0.32	0.18	0.32	0.048	0.051	0.055	0.048	0.048	0.047
GAME	0.63	0.31	0.77	0.77	0.77	0.77	0.031	0.029	0.026	0.030	0.030	0.030
MINT	0.31	0.05	0.62	0.59	0.59	0.59	0.047	0.046	0.048			
ABY	0.16	**0.00**	**0.02**	0.09	0.09	0.09	0.033	0.030	0.032	0.035	0.036	0.036
GLD	0.58	0.10	0.84	0.62	0.58	0.58	0.037	0.035		0.036	0.037	0.036
EFL	0.46	0.10	0.32	**0.03**	**0.03**	**0.03**	0.039	0.036			0.043	
portfolio	**0.00**	**0.00**	**0.00**	**0.01**	**0.01**	**0.01**	0.013	0.013	0.014	0.012	0.012	0.012

Table 10: Christoffersen's tests, asymmetric VaR Loss and MCS (an empty cell means the model is not included) for each coin and for the equally-weighted portfolio.

	DCC MVN	*DCC MVT*	*Const N-Copula GARCH*	*Const T-Copula GARCH*	*DCC N-Copula GARCH*	*DCC T-Copula GARCH*	*DCC MVN*	*DCC MVT*	*Const N-Copula GARCH*	*Const T-Copula GARCH*	*DCC N-Copula GARCH*	*DCC T-Copula GARCH*
	ES 5\% test 1: Exceedance Residuals test						**ES 5\% test 2: multilevel VaR backtest**					
BTC	0.99	1.00	0.76	1.00	1.00	1.00	**0.00**	**0.01**	0.21	0.36	0.10	0.10
ETH	0.97	0.99	0.70	0.32	0.49	0.46	**0.00**	0.06	0.06	0.90	0.26	0.22
XRP	0.67	0.96	0.91	0.98	0.96	0.96	**0.02**	**0.00**	**0.03**	0.94	0.98	0.98
XLM	**0.01**	1.00	0.04	0.11	0.14	0.12	**0.00**	**0.00**	0.10	0.47	0.37	0.34
LTC	0.26	1.00	0.85	0.99	1.00	1.00	0.29	**0.00**	0.36	**0.00**	**0.03**	0.07
XCP	0.96	1.00	0.54	0.06	**0.04**	**0.03**	**0.00**	0.06	**0.00**	**0.00**	**0.00**	**0.00**
SHIFT	0.63	1.00	0.35	0.07	0.10	0.09	**0.01**	**0.00**	**0.02**	0.39	0.39	0.39
RVR	0.49	0.99	0.34	0.17	0.12	0.10	**0.02**	**0.00**	0.13	0.50	0.39	0.39
THC	0.10	0.99	0.12	0.25	0.20	0.20	0.71	**0.02**	0.79	0.10	0.28	0.28
CLAM	0.90	0.97	0.15	0.94	0.93	0.96	0.62	0.66	0.71	0.15	0.10	0.22
GAME	0.87	0.98	0.47	0.33	0.23	0.22	**0.01**	**0.02**	0.62	0.61	0.32	0.32
MINT	0.95	1.00	0.52	0.63	0.65	0.79	**0.00**	**0.01**	0.82	0.82	0.82	0.82
ABY	0.90	1.00	0.86	0.39	0.42	0.29	**0.00**	**0.00**	**0.00**	**0.01**	**0.01**	**0.01**
GLD	0.87	1.00	0.78	0.91	0.86	0.85	0.96	0.11	0.98	0.38	0.19	0.24
EFL	0.94	1.00	0.93	0.97	0.98	0.97	0.76	**0.01**	0.42	**0.00**	**0.00**	**0.00**
portfolio	0.54	0.99	**0.00**	0.05	**0.02**	**0.02**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**

Table 11: ER test by McNeil and Frey (2000) and multilevel VaR backtest by Kratz et al. (2018).

Ticker	*Name*	*Ticker*	*Name*	*Ticker*	*Name*
BTC	Bitcoin	SC	Siacoin	MUE	MonetaryUnit
ETH	Ethereum	GAME	GameCredits	RADS	Radium
XRP	XRP	DMD	Diamond	OMNI	Omni
XLM	Stellar	THC	HempCoin	MINT	MintCoin
LTC	Litecoin	BLK	BlackCoin	ABY	ArtByte
USDT	Tether	AMP	Synereo	GLD	GoldCoin
XMR	Monero	CLAM	Clams	HUC	HunterCoin
DASH	Dash	PND	Pandacoin	EFL	e-Gulden
XEM	NEM	GRC	GridCoin	EGC	EverGreenCoin
DOGE	Dogecoin	POT	PotCoin	ORB	Orbitcoin
LSK	Lisk	IOC	I/O Coin	WDC	WorldCoin
BTS	BitShares	XMY	Myriad	GUN	Guncoin
DCR	Decred	FLO	FLO	TTC	TittieCoin
DGB	DigiByte	XST	Stealth	QTL	Quatloo

Table 12: Set of coins used for credit risk measurement

The last month trading volume, the one-year trading volume and the average yearly search volume index as provided by Google Trends were used as regressors for the logit, probit and random forest models[17]. For computing the ZPP, we employed the random walk model with the closed-formula (2), the univariate GARCH(1,1) model with normal errors and the closed-formula (3), univariate GARCH(1,1) models with student's t and skewed-t errors, DCC(1,1) models with standardized errors following either a multivariate normal or a multivariate t distribution, and Copula-GARCH models with skewed-t marginals and either a normal copula or a t-copula (both with constant and dynamic parameters). The AUC scores, the Brier score and the models included in the MCS, for both the training and the validation datasets, are reported in Table 13.

Classical credit scoring models performed better in the training sample, whereas the models perfor-

[17]The *Search Volume Index* by Google Trends computes how many searches have been done for a keyword or a topic on Google over a specific period of time and a specific region. See `https://support.google.com/trends/?hl=en` for more details.

Model	*AUC (training)*	*AUC (validation)*	*Brier score/MCS (training)*	*Brier score/MCS (validation)*
Logit	0.91	0.91	0.097	0.125
Probit	0.91	0.90	0.097	0.124
Random forest	0.80	0.90	0.154	0.103
ZPP RW	0.75	0.81	0.180	0.300
ZPP GARCH n	0.40	0.52		
ZPP GARCH t	0.56	0.63	0.187	0.301
ZPP GARCH sk.t	0.59	0.66	0.186	0.303
ZPP DCC n	0.60	0.76	0.168	0.277
ZPP DCC t	0.48	0.66	0.143	0.233
ZPP N.Copula sk.t-G.	0.56	0.73	0.230	0.262
ZPP T.Copula sk.t-G.	0.58	0.70		0.297
ZPP N.Copula DCC sk.t-G	0.59	0.72	0.246	0.280
ZPP T.Copula DCC sk.t-G	0.57	0.74	0.200	0.228

Table 13: AUC, Brier scores and MCS (an empty cell means the model is not included).

mances are much closer when the validation/out-of-time sample is considered. In the latter case, the ZPPs computed with multivariate models improved their performances in term of AUC, but they are still worse then credit scoring models. However, almost all models are included in the MCS, thus showing that there is insufficient information in the data to separate good and bad models: this outcome was expected due to the small sample size involved. Interestingly, the ZPP computed with the simple random walk with drift performed remarkably well, thus confirming the empirical evidence by Li et al. (2016). This result and the improved performance of the ZPP computed with multivariate models can probably be explained using again the simulation results in Fantazzini (2009a), who showed that the biased parameters of a multivariate distribution computed with a small sample of skewed and leptokurtic data can deliver conservative VaR estimates in the left tail of the distribution: considering that the left tail of the P&L distribution is the key element to compute the ZPP, a positively biased probability may have helped these models to forecast the riskiest coins. Another reason which could have contributed to the better performance of all multivariate models in the validation sample is the different coins' dependence structure in the training and validation time samples. Figure 1 reports the heat maps of the correlation matrices for all coins in the training sample (left) and the validation sample (right): it is evident that the second sample witnessed a much stronger dependence among coins than the first sample, and we would have a similar picture if we used measures of rank correlation like the Kendall' tau and the Spearman's rho (not reported). For example, the average correlation between coins in the training sample was 0.15, whereas it was 0.32 in the validation sample. This stronger dependence may explain why multivariate models performed better than the univariate models thanks to a larger information set.

Finally, we remark that the AUCs estimated with an out-of-time validation dataset can be higher than those estimated with an in-sample dataset at a previous time, because new future data may well

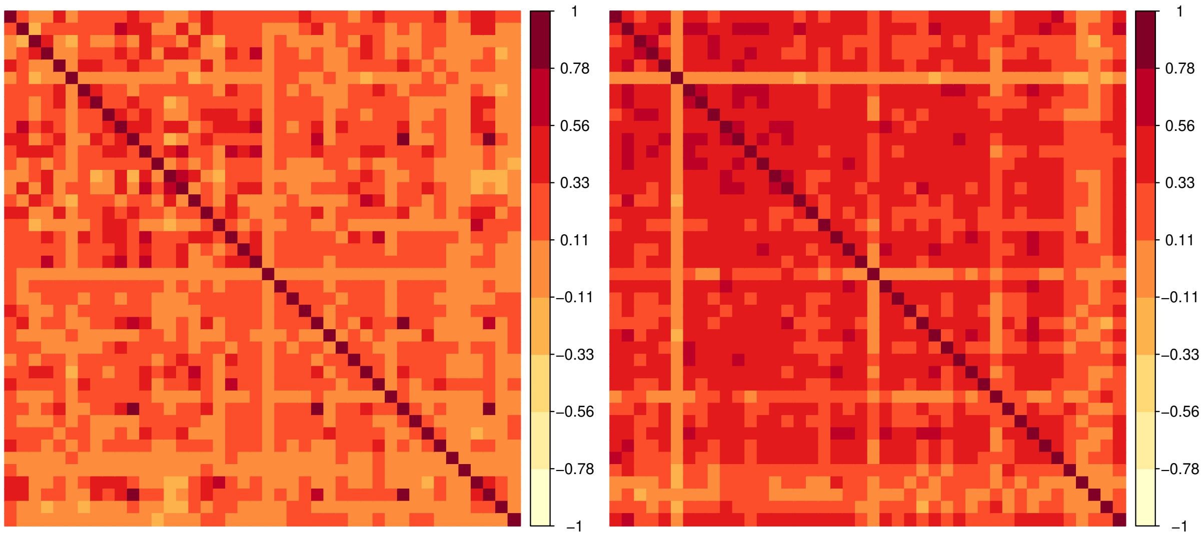

Figure 1: Heat maps of the correlation matrices in the training sample (June 2016 - May 2017, left) and the validation sample (June 2017 - May 2018, right)

come from a different distribution, see Figure 1 for an explicative example. Moreover, an estimated AUC may widely differ from the true metric in a small sample. In this regard, Hanczar et al. (2010) performed an extensive simulation study with several models and cross-section training and test datasets, and they showed that "*(i) for small samples the root mean square differences of the estimated and true metrics are considerable; (ii) even for large samples, there is only weak correlation between the true and estimated metrics; and (iii) generally, there is weak regression of the true metric on the estimated metric*" (Hanczar et al. (2010), p. 822). Interestingly, they found that with $N = 50$ the estimated AUCs may differ from the true AUC with amounts up to ± 0.2: given that we worked with temporal datasets and parameter-intensive models, this range may be even higher. Besides, this simulation evidence also helps to explain why the MCS included almost all models. All metrics computed by Hanczar et al. (2010) with small simulated samples showed very large variances and any model selection based on these metrics would be difficult: hence the inability of the MCS approach to distinguish between good and bad models.

5 Robustness checks

We wanted to verify that our previous results hold also with different portfolios and different models. Therefore, we performed a series of robustness checks, considering a capitalization-weighted portfolio instead of an equally-weighted portfolio, and we computed the market risk measures assuming the cryptocurrencies P&L to be comonotonic.

5.1 Capitalization-weighted portfolios

The equally-weighted portfolio is the main benchmark of the financial industry and literature, see e.g. DeMiguel et al. (2007) and references therein. Moreover, Hu et al. (2018) and Brauneis and Mestel (2019) analyzed cryptocurrency-portfolios and showed that the simple 1/N portfolio outperforms all competing portfolio strategies. However, most investors in crypto-currencies use capitalization-weighted portfolios, where individual assets are weighted according to their total market capitalization see, for example, the monthly trading volume rankings published at `coinmarketcap.com/currencies/volume/monthly`. Moreover, capitalization-weighted portfolios are important benchmarks in the financial literature, see Sharpe (1992), Black and Litterman (1992) and Reilly and Brown (2002). Therefore, we performed the previous back-testing analysis with capitalization-weighted portfolios considering 5 and 15 coins, respectively. The actual VaR exceedances T_1/T, the p-values of the Kupiec's unconditional coverage test, the p-values of the Christoffersen's conditional coverage test, the p-values of the exceedance residuals test by McNeil and Frey (2000) and of the multilevel VaR backtest by Kratz et al. (2018) for the forecasted 5%ES are reported in Table 14.

	DCC MVN	*DCC MVT*	*Const N-Copula GARCH*	*Const T-Copula GARCH*	*DCC N-Copula GARCH*	*DCC T-Copula GARCH*	*DCC MVN*	*DCC MVT*	*Const N-Copula GARCH*	*Const T-Copula GARCH*	*DCC N-Copula GARCH*	*DCC T-Copula GARCH*
	VaR exceedances (T_1/T): 1\% quantile						**Kupiec's test p-value: VaR 1\%**					
5 coins	3.52%	1.51%	3.52%	2.51%	2.51%	2.01%	**0.01**	0.50	0.01	0.07	0.07	0.21
15 coins	6.53%	4.52%	6.53%	3.52%	4.52%	4.02%	**0.00**	**0.00**	**0.00**	**0.01**	**0.00**	**0.00**
	VaR exceedances (T_1/T): 2\% quantile						**Kupiec's test p-value: VaR 2\%**					
5 coins	5.53%	5.53%	5.53%	5.03%	5.53%	5.03%	**0.00**	**0.00**	**0.00**	**0.01**	**0.00**	**0.01**
15 coins	7.04%	6.53%	7.04%	5.53%	5.53%	5.53%	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
	VaR exceedances (T_1/T): 3\% quantile						**Kupiec's test p-value: VaR 3\%**					
5 coins	6.53%	7.04%	8.54%	6.53%	6.53%	6.53%	**0.01**	**0.00**	**0.00**	**0.01**	**0.01**	**0.01**
15 coins	8.04%	8.54%	7.54%	6.03%	6.53%	6.53%	**0.00**	**0.00**	**0.00**	**0.03**	**0.01**	**0.01**
	VaR exceedances (T_1/T): 4\% quantile						**Kupiec's test p-value: VaR 4\%**					
5 coins	8.04%	9.55%	9.55%	9.05%	8.04%	8.54%	**0.01**	**0.00**	**0.00**	**0.00**	**0.01**	**0.00**
15 coins	8.54%	11.56%	9.05%	7.04%	6.53%	6.53%	**0.00**	**0.00**	**0.00**	0.05	0.09	**0.09**
	VaR exceedances (T_1/T): 5\% quantile						**Kupiec's test p-value: VaR 5\%**					
5 coins	9.05%	12.06%	9.55%	9.55%	10.55%	10.55%	**0.02**	**0.00**	**0.01**	**0.01**	**0.00**	**0.00**
15 coins	10.05%	12.06%	10.55%	9.05%	9.05%	9.05%	**0.00**	**0.00**	**0.00**	**0.02**	**0.02**	**0.02**
	Christoffersen's test p-value: VaR 1\%						**ES 5\% test 1: Exceedance Residuals test**					
5 coins	**0.02**	0.76	0.02	0.17	0.17	0.42	0.45	0.99	0.21	0.50	0.54	0.68
15 coins	**0.00**	**0.00**	**0.00**	**0.02**	**0.00**	**0.00**	0.74	0.99	0.06	0.38	0.23	0.26
	Christoffersen's test p-value: VaR 2\%						**ES 5\% test 2: multilevel VaR backtest**					
5 coins	0.01	0.01	0.01	0.02	0.01	0.02	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
15 coins	**0.00**	**0.00**	**0.00**	**0.01**	**0.01**	**0.01**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
	Christoffersen's test p-value: VaR 3\%											
5 coins	**0.02**	**0.01**	**0.00**	**0.02**	**0.02**	**0.02**						
15 coins	**0.00**	**0.00**	**0.01**	0.08	**0.04**	**0.04**						
	Christoffersen's test p-value: VaR 4\%											
5 coins	**0.03**	**0.00**	**0.00**	**0.00**	**0.01**	**0.01**						
15 coins	**0.02**	**0.00**	**0.00**	0.14	0.24	0.24						
	Christoffersen's test p-value: VaR 5\%											
5 coins	0.06	**0.00**	**0.02**	**0.02**	**0.01**	**0.01**						
15 coins	**0.02**	**0.00**	**0.00**	0.06	0.06	0.06						

Table 14: Capitalization-weighted portfolios risk measures: Kupiec's unconditional coverage test, Christoffersen's conditional coverage test, ER test by McNeil and Frey (2000), multilevel VaR back-test by Kratz et al. (2018).

The risk estimates are slightly more precise than the corresponding estimates with equally-weighted portfolios, probably due to the less extreme distributions of top cryptocurrencies compared to coins with small capitalizations. Nevertheless, the null hypotheses of the Kupiec's unconditional coverage test, Christoffersen's conditional coverage test, and the multilevel VaR backtest are again rejected for most portfolio risk estimates, while the ER test by McNeil and Frey (2000) does not highlight any particular misspecification for the ES 5%. T-copula based models often provide the most precise estimates and, in general, copula models tend to fare better than DCC models with large portfolios than with small portfolios.

5.2 Comonotonic assets

A set of random variables is comonotonic if they are perfectly positively dependent, which means that they are almost surely strictly increasing functions of a single random variable, see McNeil et al. (2015) for a detailed discussion at the textbook level. In this case, the Value-at-Risk and the expected shortfall are additive for a sum of comonotonic random variables, see Dhaene et al. (2006) for a proof. Therefore, portfolio risk measures can be computed using only the marginal models without the need to estimate the dependence structure (which is represented by the so-called *comonotonicity copula*).

The actual VaR exceedances T_1/T, the p-values of the Kupiec's unconditional coverage test, the p-values of the Christoffersen's conditional coverage test, the p-values of the exceedance residuals test by McNeil and Frey (2000) and of the multilevel VaR backtest by Kratz et al. (2018) for the forecasted 5%ES in case of an equally-weighted portfolio are reported in Table 15.

	Marginal models			***Marginal models***			***Marginal models***			***Marginal models***		
	Normal GARCH	*Student GARCH*	*Skewed-t GARCH*	*Normal GARCH*	*Student GARCH*	*Skewed-t GARCH*	*Normal GARCH*	*Student GARCH*	*Skewed-t GARCH*	*Normal GARCH*	*Student GARCH*	*Skewed-t GARCH*
	VaR exceedances (T_1/T): 1%			**Kupiec's test p.: VaR 1%**			**C. test p-value: VaR 1%**			**ES 5%: E.R. test**		
5 coins	1.01%	1.01%	0.00%	0.99	0.99	0.05	0.98	0.98	0.14	0.85	0.98	0.99
15 coins	1.51%	0.00%	1.01%	0.50	0.05	0.99	0.76	0.14	0.98	0.62	1.00	0.83
	VaR exceedances (T_1/T): 2%			**Kupiec's test p.: VaR 2%**			**C. test p-value: VaR 2%**			**ES 5%: m.VaR backtest**		
5 coins	2.01%	1.51%	1.01%	0.99	0.60	0.27	0.92	0.83	0.53	0.96	0.79	0.12
15 coins	1.51%	1.01%	1.01%	0.60	0.27	0.27	0.83	0.53	0.53	0.49	0.75	0.12
	VaR exceedances (T_1/T): 3%			**Kupiec's test p.: VaR 3%**			**C. test p-value: VaR 3%**					
5 coins	3.02%	4.02%	1.01%	0.99	0.42	0.06	0.83	0.52	0.16			
15 coins	2.51%	2.01%	1.01%	0.68	0.38	0.06	0.81	0.63	0.16			
	VaR exceedances (T_1/T): 4%			**Kupiec's test p.: VaR 4%**			**C. test p-value: VaR 4%**					
5 coins	3.02%	5.03%	2.01%	0.46	0.48	0.11	0.63	0.63	0.26			
15 coins	2.51%	4.02%	2.01%	0.25	0.99	0.11	0.46	0.71	0.26			
	VaR exceedances (T_1/T): 5%			**Kupiec's test p.: VaR 5%**			**C. test p-value: VaR 5%**					
5 coins	4.52%	6.03%	3.02%	0.75	0.52	0.17	0.67	0.33	0.32			
15 coins	3.02%	4.52%	3.52%	0.17	0.75	0.31	0.32	0.62	0.46			

Table 15: Equally-weighted portfolios risk measures with comonotonic assets: Kupiec's unconditional coverage test, Christoffersen's conditional coverage test, ER test by McNeil and Frey (2000), multilevel VaR backtest by Kratz et al. (2018).

All estimated portfolio risk measures pass the backtesting specification tests, with the student's t GARCH marginals providing the most accurate measures, whereas the other marginal models are slightly more conservative. These results will definitely not surprise traders and financial professionals dealing with crypto-currencies, who several times repeated that the crypto-market is still a "one-man show" driven by the bitcoin price, see e.g. Mitsuru et al. (2014), Bitconnect (2017) and DeMichele (2018).

6 Conclusions

Credit and market risks for cryptocurrencies are more interlinked than for traditional assets, and their differences are of quantitative and temporal nature, not qualitative. Credit risk for cryptocurrencies can be defined as the gains and losses on the value of a position of a cryptocurrency that is abandoned and considered dead but which can be potentially revived, while market risk can be described as the gains and losses on the value of a position (or portfolio) of alive cryptocurrencies, due to the movements in market prices in centralized and decentralized exchanges.

This paper proposed a set of models to estimate simultaneously both the market risk and the credit risk for a portfolio of crypto-currencies. Moreover, two closed-form formulas for the ZPP model in case of normally distributed errors were also developed using recent results from barrier option theory. These formulas can be useful to get a rough idea of the crypto-asset credit risk and may be interesting to online data providers, who can publish the quotes of crypto-assets together with their market-implied credit risk measures.

A backtesting exercise for market risk modelling using two datasets of 5 and 15 coins was performed. Our results showed that the t-copula with skewed-t GARCH marginals can be a good compromise for precise VaR estimates across different quantile levels, particularly the most extreme quantiles (1% and 2%) which are the most important for regulatory purposes. However, the asymmetric VaR losses of the competing models were rather close and all models were included in the MCS for almost all coins and for the portfolio case: this result was expected because Fantazzini (2009a) showed that, when small samples are considered and the data are skewed and leptokurtic, the biases in the GARCH parameters are so large that they can deliver conservative VaR estimates, even with a simple multivariate normal distribution. The backtesting of the expected shortfall estimates showed that DCC models often underestimated the true ES, whereas t-copula/skewed-t GARCH models were generally fine.

A backtesting exercise for credit risk modelling was performed using a dataset of 42 coins. The empirical analysis showed that classical credit scoring models performed better in the training sample, whereas the models' performances were much closer in the validation sample, with the simple ZPP computed using a random walk with drift performing remarkably well. In general, all multivariate models performed much better in the validation sample than in the training sample, thanks to the much stronger dependence shown by coins in the validation sample.

Finally, we performed a set of robustness checks to verify that our results also hold under different forecasting setups. We found out that risk estimates using capitalization-weighted portfolios are slightly more precise than those with equally-weighted portfolios, probably due to the less extreme distributions of top cryptocurrencies compared to coins with small capitalizations. Moreover, market risk measures computed assuming the cryptocurrencies P&L to be comonotonic passed all the backtesting specification tests, thus confirming financial professional literature showing that crypto-currencies are mainly driven by the bitcoin price.

The extreme volatility, skewness, kurtosis, and dependence of cryptocurrencies due to the frequent presence of structural breaks and price manipulations pose a serious challenge to any multivariate risk model. Skewed-t marginals and copulas allowing for dynamic dependence are a good starting point, but the length of the estimation window plays also an important role due to multiple breaks of different nature. In this regard, a rolling estimation window of approximately 500 observations can be considered a good compromise, considering the numerical properties of GARCH models discussed by Hwang and Valls Pereira (2006), together with the simulation evidence reported by Pesaran and Timmermann (2007), who showed that in a regression with multiple breaks the optimal window for estimation includes all of the observations after the last break, plus a limited number of observations before this break. This setup is viable with portfolios of small sizes, but it is hardly feasible with portfolios of medium and large sizes, due to a large number of parameters involved. Moreover, the strong dependence of cryptocurrencies from the bitcoin price becomes much more complex and unstable to model with larger portfolios. In such a situation, a numerically efficient solution is to assume the assets to be comonotonic.

The number of coins that we used for backtesting is rather low, and this can be a potential limitation of our analysis. However, we want to remark that we considered only small and medium-sized portfolios to avoid additional model complexity and due to the lack of historical data for the vast majority of dead coins. The last issue is certainly a major stumbling block for fully developing models for credit risk measurement and management with crypto-assets, because the lack of these data would make any model suffer from massive selection bias: for example, in the middle of 2017, there were more than 1500 traded alive coins and almost 800 dead coins, but historical market data were available only for a few dozens dead coins. The retrieval and the analysis of such data are left as an avenue for future research. Another potential future development is the analysis of the interactions effects between market risk and credit risk for cryptocurrencies and how they change over time.

References

Acerbi, Carlo, and Balazs Szekely. 2014. "Back-Testing Expected Shortfall." *Risk* 27: 76–81.

Adalet McGowan, Muge, Dan Andrews, and Valentine Millot. 2018. "The Walking Dead? Zombie Firms and Productivity Performance in Oecd Countries." *Economic Policy* 33 (96): 685–736.

Aielli, Gian Piero. 2013. "Dynamic Conditional Correlation: On Properties and Estimation." *Journal of Business & Economic Statistics* 31 (3): 282–99.

Altman, Edward I, and Gabriele Sabato. 2007. "Modelling Credit Risk for Smes: Evidence from the Us Market." *Abacus* 43 (3): 332–57.

Antonopoulos, Andreas M. 2014. *Mastering Bitcoin: Unlocking Digital Cryptocurrencies.* " O'Reilly Media, Inc."

Back, Adam. 2002. "Hashcash-a Denial of Service Counter-Measure."

Balcilar, Mehmet, Elie Bouri, Rangan Gupta, and David Roubaud. 2017. "Can Volume Predict Bitcoin Returns and Volatility? A Quantiles-Based Approach." *Economic Modelling* 64: 74–81.

Bali, Turan G, and Hao Zhou. 2016. "Risk, Uncertainty, and Expected Returns." *Journal of Financial and Quantitative Analysis* 51 (3): 707–35.

Banerjee, Ryan, and Boris Hofmann. 2018. "The Rise of Zombie Firms: Causes and Consequences." *BIS Quarterly Review, September*, 67–78.

Basel Committee on Banking Supervision. 2009. "Findings on the Interaction of Market and Credit Risk." Bank for International Settlements, Working paper n. 16, May.

Bauwens, Luc, and Sebastien Laurent. 2005. "A New Class of Multivariate Skew Densities, with Application to Generalized Autoregressive Conditional Heteroscedasticity Models." *Journal of Business & Economic Statistics* 23 (3): 346–54.

Bauwens, Luc, Christian M Hafner, and Sebastien Laurent. 2012. *Handbook of Volatility Models and Their Applications.* Vol. 3. Wiley.

Bauwens, Luc, Sebastien Laurent, and Jeroen VK Rombouts. 2006. "Multivariate Garch Models: A Survey." *Journal of Applied Econometrics* 21 (1): 79–109.

Bayraktar, Erhan, and Bo Yang. 2011. "A Unified Framework for Pricing Credit and Equity Derivatives." *Mathematical Finance* 21 (3): 493–517.

BIS. 2013. "Fundamental Review of the Trading Book: A Revised Market Risk Framework." *Basel Committee on Banking Supervision.*

———. 2016. "Minimum Capital Requirements for Market Risk - Publication No. 352." *Basel*

Committee on Banking Supervision.

Bitconnect. 2017. “How Is the Price of Cryptocurrency Defined?” Bitconnect editorial board.

Black, Fischer, and Robert Litterman. 1992. “Global Portfolio Optimization.” *Financial Analysts Journal* 48 (5): 28–43.

Bohr, Jeremiah, and Masooda Bashir. 2014. “Who Uses Bitcoin? An Exploration of the Bitcoin Community.” In *2014 Twelfth Annual International Conference on Privacy, Security and Trust*, 94–101. IEEE.

Bouoiyour, Jamal, and Refk Selmi. 2015. “What Does Bitcoin Look Like?” *Annals of Economics & Finance* 16 (2).

Bouoiyour, Jamal, Refk Selmi, and Aviral Kumar Tiwari. 2015. “Is Bitcoin Business Income or Speculative Foolery? New Ideas Through an Improved Frequency Domain Analysis.” *Annals of Financial Economics* 10 (01): 1550002.

Bouri, Elie, Georges Azzi, and Anne Haubo Dyhrberg. 2017. “On the Return-Volatility Relationship in the Bitcoin Market Around the Price Crash of 2013.” *Economics: The Open-Access, Open-Assessment E-Journal* 11 (2): 1–16.

Bouri, Elie, Luis A Gil-Alana, Rangan Gupta, and David Roubaud. 2016. “Modelling Long Memory Volatility in the Bitcoin Market: Evidence of Persistence and Structural Breaks.” *International Journal of Finance & Economics*, 24(1): 412-426 .

Brandvold, Morten, Peter Molnar, Kristian Vagstad, and Ole Christian Andreas Valstad. 2015. “Price Discovery on Bitcoin Exchanges.” *Journal of International Financial Markets, Institutions and Money* 36: 18–35.

Brauneis, Alexander, and Roland Mestel. 2019. “Cryptocurrency-Portfolios in a Mean-Variance Framework.” *Finance Research Letters* forthcoming.

Breiman, Leo. 2001. “Random Forests.” *Machine Learning* 45 (1): 5–32.

Brier, Glenn W. 1950. “Verification of Forecasts Expressed in Terms of Probability.” *Monthly Weather Review* 78 (1): 1–3.

Buchholz, Martis, Jess Delaney, Joseph Warren, and Jeff Parker. 2012. “Bits and Bets, Information, Price Volatility, and Demand for Bitcoin.” *Economics* 312.

Burniske, Chris, and Jack Tatar. 2017. *Cryptoassets:The Innovative Investors Guide to Bitcoin and Beyond*. McGraw Hill Professional.

Caballero, Ricardo J, Takeo Hoshi, and Anil K Kashyap. 2008. “Zombie Lending and Depressed Restructuring in Japan.” *American Economic Review* 98 (5): 1943–77.

Cai, Yong, and K Krishnamoorthy. 2006. “Exact Size and Power Properties of Five Tests for

Multinomial Proportions." *Communications in Statistics-Simulation and Computation* 35 (1): 149–60.

Campi, Luciano, Simon Polbennikov, and Alessandro Sbuelz. 2009. "Systematic Equity-Based Credit Risk: A Cev Model with Jump to Default." *Journal of Economic Dynamics and Control* 33 (1): 93–108.

Caporin, Massimiliano, and Michael McAleer. 2013. "Ten Things You Should Know About the Dynamic Conditional Correlation Representation." *Econometrics* 1 (1): 115–26.

———. 2014. "Robust Ranking of Multivariate Garch Models by Problem Dimension." *Computational Statistics & Data Analysis* 76: 172–85.

Carr, Peter, and Vadim Linetsky. 2006. "A Jump to Default Extended Cev Model: An Application of Bessel Processes." *Finance and Stochastics* 10 (3): 303–30.

Carr, Peter, and Liuren Wu. 2009. "Stock Options and Credit Default Swaps: A Joint Framework for Valuation and Estimation." *Journal of Financial Econometrics* 8 (4): 409–49.

Catania, Leopoldo, Stefano Grassi, and Francesco Ravazzolo. 2018. "Predicting the Volatility of Cryptocurrency Time–Series." In *Mathematical and Statistical Methods for Actuarial Sciences and Finance*.

Chan, Stephen, Jeffrey Chu, Saralees Nadarajah, and Joerg Osterrieder. 2017. "A Statistical Analysis of Cryptocurrencies." *Journal of Risk and Financial Management* 10 (2): 12.

Chaum, David. 1983. "Blind Signatures for Untraceable Payments." In *Advances in Cryptology*, 199–203.

Chaum, David, and Stefan Brands. 1997. "'Minting'electronic Cash." *IEEE Spectrum* 34 (2). IEEE: 30–34.

Cheah, Eng-Tuck, and John Fry. 2015. "Speculative Bubbles in Bitcoin Markets? An Empirical Investigation into the Fundamental Value of Bitcoin." *Economics Letters* 130: 32–36.

Cherubini, Umberto, Elisa Luciano, and Walter Vecchiato. 2004. *Copula Methods in Finance*. John Wiley & Sons.

Christoffersen, Peter. 2011. *Elements of Financial Risk Management*. Academic Press.

Christoffersen, Peter F. 1998. "Evaluating Interval Forecasts." *International Economic Review*, 841–862.

Chu, Jeffrey, Stephen Chan, Saralees Nadarajah, and Joerg Osterrieder. 2017. "GARCH Modelling of Cryptocurrencies." *Journal of Risk and Financial Management* 10 (4): 17.

Chu, Jeffrey, Saralees Nadarajah, and Stephen Chan. 2015. "Statistical Analysis of the Exchange Rate of Bitcoin." *PloS One* 10 (7): e0133678.

Ciaian, Pavel, Miroslava Rajcaniova, and d'Artis Kancs. 2016. "The Digital Agenda of Virtual Currencies: Can Bitcoin Become a Global Currency?" *Information Systems and E-Business Management*

14 (4): 883–919.

Corbet, Shaen, Brian Lucey, and Larisa Yarovaya. 2018. "Datestamping the Bitcoin and Ethereum Bubbles." *Finance Research Letters* 26: 81–88.

Corbet, Shaen, Brian Lucey, Andrew Urquhart, and Larisa Yarovaya. 2019. "Cryptocurrencies as a Financial Asset: A Systematic Analysis." *International Review of Financial Analysis* forthcoming.

Dalla Valle, Luciana, Maria Elena De Giuli, Claudia Tarantola, and Claudio Manelli. 2016. "Default Probability Estimation via Pair Copula Constructions." *European Journal of Operational Research* 249 (1): 298–311.

Das, Sanjiv R, and Paul Hanouna. 2009. "Implied Recovery." *Journal of Economic Dynamics and Control* 33 (11): 1837–57.

DeMichele, Thomas. 2018. "Why Do Altcoin Prices Often Follow Bitcoin Price?" Cryptocurrency facts.

DeMiguel, Victor, Lorenzo Garlappi, and Raman Uppal. 2007. "Optimal Versus Naive Diversification: How Inefficient Is the 1/N Portfolio Strategy?" *The Review of Financial Studies* 22 (5): 1915–53.

Dhaene, Jan, Steven Vanduffel, Marc J Goovaerts, Rob Kaas, Qihe Tang, and David Vyncke. 2006. "Risk Measures and Comonotonicity: A Review." *Stochastic Models* 22 (4): 573–606.

Dyhrberg, Anne Haubo. 2016a. "Bitcoin, Gold and the Dollar–A Garch Volatility Analysis." *Finance Research Letters* 16: 85–92.

———. 2016b. "Hedging Capabilities of Bitcoin. Is It the Virtual Gold?" *Finance Research Letters* 16: 139–44.

Dyrssen, Hannah, Erik Ekstrom, and Johan Tysk. 2014. "Pricing Equations in Jump-to-Default Models." *International Journal of Theoretical and Applied Finance* 17 (03): 1450019.

Efron, Bradley, and Robert J Tibshirani. 1994. *An Introduction to the Bootstrap*. CRC press.

Emmer, S, M Kratz, and D Tasche. 2015. "What Is the Best Risk Measure in Practice?" *Journal of Risk* 18: 31–60.

Engle, Robert. 2002. "Dynamic Conditional Correlation: A Simple Class of Multivariate Generalized Autoregressive Conditional Heteroskedasticity Models." *Journal of Business & Economic Statistics* 20 (3): 339–50.

Engle, Robert F, and Kevin Sheppard. 2001. "Theoretical and Empirical Properties of Dynamic Conditional Correlation Multivariate Garch." National Bureau of Economic Research.

Fantazzini, Dean. 2008. "Dynamic Copula Modelling for Value at Risk." *Frontiers in Finance and Economics* 5 (2): 72–108.

———. 2009a. "The Effects of Misspecified Marginals and Copulas on Computing the Value at Risk:

A Monte Carlo Study." *Computational Statistics & Data Analysis* 53 (6): 2168–88.

———. 2009b. "Value at Risk for High-Dimensional Portfolios: A Dynamic Grouped T-Copula Approach." In *The Var Implementation Handbook*, edited by Greg Gregoriou, 253–82. McGraw-Hill.

Fantazzini, Dean. 2019. *Quantitative Finance with R and Cryptocurrencies.* Amazon KDP, ISBN-13: 978-1090685315.

Fantazzini, Dean, and Silvia Figini. 2008. "Default Forecasting for Small-Medium Enterprises: Does Heterogeneity Matter?" *International Journal of Risk Assessment and Management* 11 (1-2): 138–63.

———. 2009. "Random Survival Forests Models for Sme Credit Risk Measurement." *Methodology and Computing in Applied Probability* 11 (1): 29–45.

Fantazzini, Dean, and Mario Maggi. 2015. "Proposed Coal Power Plants and Coal-to-Liquids Plants in the Us: Which Ones Survive and Why?" *Energy Strategy Reviews* 7: 9–17.

Fantazzini, Dean, Maria Elena De Giuli, and Mario Maggi. 2008. "A New Approach for Firm Value and Default Probability Estimation Beyond Merton Models." *Computational Economics* 31 (2): 161–80.

Fantazzini, Dean, Erik Nigmatullin, Vera Sukhanovskaya, and Sergey Ivliev. 2016. "Everything You Always Wanted to Know About Bitcoin Modelling but Were Afraid to Ask. Part 1." *Applied Econometrics* 44: 5–24.

———. 2017. "Everything You Always Wanted to Know About Bitcoin Modelling but Were Afraid to Ask. Part 2." *Applied Econometrics* 45: 5–28.

Feder, Amir, Neil Gandal, JT Hamrick, Tyler Moore, and Marie Vasek. 2018. "The Rise and Fall of Cryptocurrencies." In *Proc. of the Workshop on the Economics of Information Security (Weis).*

Fernandez, Carmen, and Mark FJ Steel. 1998. "On Bayesian Modeling of Fat Tails and Skewness." *Journal of the American Statistical Association* 93 (441) Group: 359–71.

Friedman, Jerome, Trevor Hastie, and Robert Tibshirani. 2016. *The Elements of Statistical Learning.* Vol. 1.

Fuertes, Ana-Maria, and Elena Kalotychou. 2006. "Early Warning Systems for Sovereign Debt Crises: The Role of Heterogeneity." *Computational Statistics & Data Analysis* 51 (2): 1420–41.

Gandal, Neil, JT Hamrick, Tyler Moore, and Tali Oberman. 2018. "Price Manipulation in the Bitcoin Ecosystem." *Journal of Monetary Economics* 95: 86–96.

Garcia, David, and Frank Schweitzer. 2015. "Social Signals and Algorithmic Trading of Bitcoin." *Royal Society Open Science* 2 (9): 150288.

Garcia, David, Claudio J Tessone, Pavlin Mavrodiev, and Nicolas Perony. 2014. "The Digital Traces of Bubbles: Feedback Cycles Between Socio-Economic Signals in the Bitcoin Economy." *Journal of the Royal Society Interface* 11 (99): 20140623.

Gerlach, Jan-Christian, Guilherme Demos, and Didier Sornette. 2018. "Dissection of Bitcoin's Multiscale Bubble History from January 2012 to February 2018." *arXiv Preprint arXiv:1804.06261*.

Giacomini, Raffaella, and Ivana Komunjer. 2005. "Evaluation and Combination of Conditional Quantile Forecasts." *Journal of Business & Economic Statistics* 23 (4): 416–31.

Gkillas, Konstantinos, and Paraskevi Katsiampa. 2018. "An Application of Extreme Value Theory to Cryptocurrencies." *Economics Letters* 164: 109–11.

Glaser, Florian, Kai Zimmermann, Martin Haferkorn, Moritz Weber, and Michael Siering. 2014. "Bitcoin-Asset or Currency? Revealing Users' Hidden Intentions."

Gneiting, Tilmann. 2011. "Making and Evaluating Point Forecasts." *Journal of the American Statistical Association* 106 (494): 746–62.

Goin, James E. 1982. "ROC Curve Estimation and Hypothesis Testing: Applications to Breast Cancer Detection." *Pattern Recognition* 15 (3): 263–69.

Gonzalez-Rivera, Gloria, Tae-Hwy Lee, and Santosh Mishra. 2004. "Forecasting Volatility: A Reality Check Based on Option Pricing, Utility Function, Value-at-Risk, and Predictive Likelihood." *International Journal of Forecasting* 20 (4): 629–45.

Griffin, John M, and Amin Shams. 2018. "Is Bitcoin Really Un-Tethered?" University of Texas at Austin.

Hanczar, Blaise, Jianping Hua, Chao Sima, John Weinstein, Michael Bittner, and Edward R Dougherty. 2010. "Small-Sample Precision of Roc-Related Estimates." *Bioinformatics* 26 (6): 822–30.

Hanley, James A, and Barbara J McNeil. 1982. "The Meaning and Use of the Area Under a Receiver Operating Characteristic (Roc) Curve." *Radiology* 143 (1): 29–36.

Hansen, Bruce E. 1994. "Autoregressive Conditional Density Estimation." *International Economic Review*. JSTOR, 705–30.

Hansen, Peter R, Asger Lunde, and James M Nason. 2011. "The Model Confidence Set." *Econometrica* 79 (2): 453–97.

Hartmann, Philipp. 2010. "Interaction of Market and Credit Risk." *Journal of Banking and Finance* 4 (34): 697–702.

Hayes, Adam. 2015. "A Cost of Production Model for Bitcoin." New School for Social Research, Department of Economics.

———. 2017. "Cryptocurrency Value Formation: An Empirical Study Leading to a Cost of Production Model for Valuing Bitcoin." *Telematics and Informatics* 34 (7): 1308–21.

Ho, Tin Kam. 1995. "Random Decision Forests." In *Document Analysis and Recognition, 1995., Proceedings of the Third International Conference on*, 1:278–82. IEEE.

Hu, Albert, Christine A Parlour, and Uday Rajan. 2018. "Cryptocurrencies: Stylized Facts on a

New Investible Instrument.” Haas School of Business, UC Berkeley.

Hwang, Soosung, and Pedro L Valls Pereira. 2006. “Small Sample Properties of Garch Estimates and Persistence.” *The European Journal of Finance* 12 (6-7): 473–94.

Joe, Harry. 1997. *Multivariate Models and Multivariate Dependence Concepts.* CRC Press.

Jorion, Philippe. 2006. “Value at Risk: The New Benchmark for Managing Financial Risk.” *NY: McGraw-Hill Professional.*

———. 2007. *Financial Risk Manager Handbook.* Vol. 406. Wiley.

Katsiampa, Paraskevi. 2017. “Volatility Estimation for Bitcoin: A Comparison of Garch Models.” *Economics Letters* 158: 3–6.

Kohavi, R, and F Provost. 1998. “Glossary of Terms.” *Machine Learning* 30: 271–74.

Kostovetsky, Leonard, and Hugo Benedetti. 2018. “Digital Tulips? Returns to Investors in Initial Coin Offerings.” Boston College working paper.

Kratz, Marie, Yen H Lok, and Alexander J McNeil. 2018. “Multinomial Var Backtests: A Simple Implicit Approach to Backtesting Expected Shortfall.” *Journal of Banking & Finance* 88: 393–407.

Kristoufek, Ladislav. 2013. “Can Google Trends Search Queries Contribute to Risk Diversification?” *Scientific Reports* 3. Nature Publishing Group.

Kupiec, Paul. 1995. “Techniques for Verifying the Accuracy of Risk Measurement Models.” *The Journal of Derivatives*, 3(2).

Lansky, Jan. 2018. “Possible State Approaches to Cryptocurrencies.” *Journal of Systems Integration* 9 (1): 19–31.

Li, Lili, Jun Yang, and Xin Zou. 2016. “A Study of Credit Risk of Chinese Listed Companies: ZPP Versus Kmv.” *Applied Economics* 48 (29): 2697–2710.

Linetsky, Vadim. 2006. “Pricing Equity Derivatives Subject to Bankruptcy.” *Mathematical Finance* 16 (2): 255–82.

Liu, Ruiping, Zhichao Shao, Guodong Wei, and Wei Wang. 2017. “GARCH Model with Fat-Tailed Distributions and Bitcoin Exchange Rate Returns.” *Journal of Accounting, Business and Finance Research* 1 (1): 71–75.

MacDonell, Alec. 2014. “Popping the Bitcoin Bubble: An Application of Log-Periodic Power Law Modeling to Digital Currency.” *University of Notre Dame Working Paper.*

McNeil, Alexander J, and Rudiger Frey. 2000. “Estimation of Tail-Related Risk Measures for Heteroscedastic Financial Time Series: An Extreme Value Approach.” *Journal of Empirical Finance* 7 (3-4). Elsevier: 271–300.

McNeil, Alexander J, Rudiger Frey, and Paul Embrechts. 2015. *Quantitative Risk Management:*

Concepts, Techniques and Tools. Princeton U.P.

Mensi, Walid, Khamis Hamed Al-Yahyaee, and Sang Hoon Kang. 2018. "Structural Breaks and Double Long Memory of Cryptocurrency Prices: A Comparative Analysis from Bitcoin and Ethereum." *Finance Research Letters*.

Mester, Loretta J. 1997. "What's the Point of Credit Scoring?" *Business Review* 3 (Sep/Oct): 3–16.

Metz, Charles E, and Helen B Kronman. 1980. "Statistical Significance Tests for Binormal Roc Curves." *Journal of Mathematical Psychology* 22 (3): 218–43.

Mitsuru, I, Y Kitamura, and M Tsutomu. 2014. "Is Bitcoin the Only Cryptocurrency in the Town?" Discussion Paper Series A No.602, Institute of Economic Research Hitotsubashi University, Tokyo.

Moore, Tyler, and Nicolas Christin. 2013. "Beware the Middleman: Empirical Analysis of Bitcoin-Exchange Risk." In *International Conference on Financial Cryptography and Data Security*, 25–33. Springer.

Moore, Tyler, Nicolas Christin, and Janos Szurdi. 2018. "Revisiting the Risks of Bitcoin Currency Exchange Closure." *ACM Transactions on Internet Technology* 18.

Muller, Fernanda Maria, and Marcelo Brutti Righi. 2018. "Numerical Comparison of Multivariate Models to Forecasting Risk Measures." *Risk Management* 20 (1): 29–50.

Naimy, Viviane Y, and Marianne R Hayek. 2018. "Modelling and Predicting the Bitcoin Volatility Using Garch Models." *International Journal of Mathematical Modelling and Numerical Optimisation* 8 (3): 197–215.

Nakamoto, Satoshi. 2008. "Bitcoin: A Peer-to-Peer Electronic Cash System."

Narayanan, Arvind, Joseph Bonneau, Edward Felten, Andrew Miller, and Steven Goldfeder. 2016. *Bitcoin and Cryptocurrency Technologies: A Comprehensive Introduction*. Princeton University Press.

Nelsen, Roger B. 1999. *An Introduction to Copulas*. Vol. 139. Springer-Verlag, New York.

Osterrieder, Joerg. 2016. "The Statistics of Bitcoin and Cryptocurrencies." Advanced Risk; Portfolio Management Paper.

Osterrieder, Joerg, and Julian Lorenz. 2017. "A Statistical Risk Assessment of Bitcoin and Its Extreme Tail Behavior." *Annals of Financial Economics* 12 (01): 1750003.

Patton, Andrew. 2013. "Copula Methods for Forecasting Multivariate Time Series." In *Handbook of Economic Forecasting*, 2:899–960. Elsevier.

Patton, Andrew J. 2006a. "Estimation of Multivariate Models for Time Series of Possibly Different Lengths." *Journal of Applied Econometrics* 21 (2): 147–73.

———. 2006b. "Modelling Asymmetric Exchange Rate Dependence." *International Economic Review* 47 (2): 527–56.

———. 2009. "Copula–based Models for Financial Time Series." In *Handbook of Financial Time*

Series, 767–85.

Pesaran, M Hashem, and Allan Timmermann. 2007. "Selection of Estimation Window in the Presence of Breaks." *Journal of Econometrics* 137 (1): 134–61.

Pichl, Lukas, and Taisei Kaizoji. 2017. "Volatility Analysis of Bitcoin." *Quantitative Finance and Economics* 1: 474–85.

Reilly, Frank K, and Keith C Brown. 2002. *Investment Analysis and Portfolio Management.* Cengage Learning.

Rodriguez, Arnulfo, and Pedro N Rodriguez. 2006. "Understanding and Predicting Sovereign Debt Rescheduling: A Comparison of the Areas Under Receiver Operating Characteristic Curves." *Journal of Forecasting* 25 (7): 459–79.

Sammut, Claude, and Geoffrey I Webb. 2011. *Encyclopedia of Machine Learning* Science & Business Media.

Satchell, Stephen, and John Knight. 2011. *Forecasting Volatility in the Financial Markets.*

Sharpe, William F. 1992. "Asset Allocation: Management Style and Performance Measurement." *Journal of Portfolio Management* 18 (2): 7–19.

Sid. 2018. "How Peng Coin Will Surge 8-12x These Coming Weeks." Medium.

Sklar, M. 1959. "Fonctions de Repartition an Dimensions et Leurs Marges." *Publ. Inst. Statist. Univ. Paris* 8: 229–31.

Smith, Chris, and Mark Koning. 2017. *Decision Trees and Random Forests: A Visual Introduction for Beginners.* Independently published.

Stavroyiannis, Stavros. 2018. "Value-at-Risk and Related Measures for the Bitcoin." *The Journal of Risk Finance* 19 (2). Emerald Publishing Limited: 127–36.

Su, En-Der, and Shih-Ming Huang. 2010. "Comparing Firm Failure Predictions Between Logit, Kmv, and Zpp Models: Evidence from Taiwan's Electronics Industry." *Asia-Pacific Financial Markets* 17 (3). Springer: 209–39.

Su, Lujing, and Marc O Rieger. 2009. "How Likely Is It to Hit a Barrier? Theoretical and Empirical Estimates." *Technical Report, Working Paper No. 594, National Centre of Competence in Research, Financial Valuation and Risk Management.*

Thies, Sven, and Peter Molnar. 2018. "Bayesian Change Point Analysis of Bitcoin Returns." *Finance Research Letters* 27: 223–27.

Trucios, Carlos. 2018. "Forecasting Bitcoin Risk Measures: A Robust Approach."

Tse, Yiu K, and Albert K C Tsui. 2002. "A Multivariate Generalized Autoregressive Conditional Heteroscedasticity Model with Time-Varying Correlations." *Journal of Business & Economic Statistics*

20 (3): 351–62.

Weiss, Gregor NF. 2013. "Copula-Garch Versus Dynamic Conditional Correlation: An Empirical Study on Var and Es Forecasting Accuracy." *Review of Quantitative Finance and Accounting* 41 (2): 179–202.

Weiss, Gregor NF. 2011. "Are Copula-Gof-Tests of Any Practical Use? Empirical Evidence for Stocks, Commodities and Fx Futures." *The Quarterly Review of Economics and Finance* 51 (2): 173–88.

Woo, D., I. Gordon, and V. Iaralov. 2013. "Bitcoin: A First Assessment." *FX and Rates* December 2013. Bank of America Merrill Lynch.

Yelowitz, Aaron, and Matthew Wilson. 2015. "Characteristics of Bitcoin Users: An Analysis of Google Search Data." *Applied Economics Letters* 22 (13): 1030–6.

Appendix A: a review of the history and the financial literature devoted to cryptocurrencies

Brief historical overview: Bitcoin & sons

Bitcoin was not the first cryptocurrency proposed in the IT literature, see e.g. the works by Chaum (1983), Chaum and Brands (1997) and Back (2002). The idea of Bitcoin was presented in a paper published in 2008 by a group of anonymous authors under the pseudonym of Satoshi Nakamoto (Nakamoto (2008)). Since then, Bitcoin has become the most popular online decentralized currency, which allows users to make transactions without any third party by using a peer-to-peer protocol. The system is based on cryptography algorithms which provide security for all operations, see Antonopoulos (2014), Narayanan et al. (2016) for more details. The Bitcoin market capitalization was close to \$ 70 bn at the end of 2018 and represented almost 50% of all cryptocurrencies total capitalization. The price for a single bitcoin decreased to approximately \$ 4000, after reaching a top of almost \$ 20000 at the end of 2017. Most cryptocurrencies are *hard forks* of the Bitcoin protocol, that is they introduced new rules to create blocks which are not considered valid by the older (Bitcoin) software. A description of the history of bitcoin (and cryptocurrencies in general) can be found in Burniske and Tatar (2017).

A common way to raise funds to create a new coin, app, or service with cryptocurrencies is to launch an *Initial Coin Offering* (ICO). An ICO is a type of crowdfunding, where funds are collected by selling a fixed number of new coins to investors. It is similar to an Initial Public Offering (IPO) of a company, but there are some important differences: ICOs may fall outside current regulations and can be prone to scams and securities law violations. Moreover, holding coins not necessarily gives dividends, but services and goods manufactured by the company. According to the CoinDesk State of Blockchain Q1 2018[18], the number of ICOs is still increasing (number of ICOs in 2017 was twice larger than in 2016) and its funding attained \$ 12 bn. Only in the first quarter of 2018, there were 202 ICOs with \$ 6.3 bn of total funding. However, investing in cryptocurrencies can be an extremely risky process and the final return depends on the strategy adopted: Kostovetsky and Benedetti (2018) shows that approximately 56 percent of crypto startups that raise money through ICOs die within four months of their initial coin offerings. However, they also show that the representative ICO investor earns 82% and the safest strategy is to acquire coins in an ICO and then sell them on the first day -if individual investors can participate in ICOs-, or to sell them in the first six months, otherwise.

[18] `https://www.coindesk.com/research/state-blockchain-q1-2018`

Literature review

The Bitcoin phenomenon has attracted significant attention in the academic literature with regard to its fundamental value (Woo, Gordon, and Iaralov (2013), Garcia et al. (2014), Hayes (2015), Hayes (2017)), price dynamics (Buchholz et al. (2012), Kristoufek (2013), Garcia et al. (2014), Garcia and Schweitzer (2015), Glaser et al. (2014), Bouoiyour and Selmi (2015), Bouoiyour, Selmi, and Tiwari (2015), Ciaian, Rajcaniova, and Kancs (2016), Bouri, Azzi, and Dyhrberg (2017)), bubble modelling (MacDonell (2014), Cheah and Fry (2015), Gerlach, Demos, and Sornette (2018)), price discovery (Brandvold et al. (2015)), and more recently about univariate volatility modelling (Dyhrberg (2016a), Dyhrberg (2016b)), Balcilar et al. (2017), Chu et al. (2017), Katsiampa (2017), Liu et al. (2017), Pichl and Kaizoji (2017), Naimy and Hayek (2018) and Catania, Grassi, and Ravazzolo (2018). See Fantazzini et al. (2016) and Fantazzini et al. (2017) for a large survey of the econometric and mathematical tools which have been proposed so far to model the bitcoin price and several related issues, highlighting advantages and limits.

With regards to risk management for cryptocurrencies, the number of works is much more limited: Chu et al. (2017) compared twelve GARCH models with seven popular cryptocurrencies, and their fits were assessed in terms of five in-sample criteria and out-of-sample Value at Risk performances. Chan et al. (2017) analyzed the (unconditional) statistical properties of seven cryptocurrencies, while Osterrieder and Lorenz (2017) examined the tail behavior of bitcoin returns using extreme value distributions but no backtesting was performed. Stavroyiannis (2018) examined a set of market risk measures for the BTC and compared these measures with the SP500 index, the Brent crude oil spot price and the gold spot price by performing a backtesting analysis. Gkillas and Katsiampa (2018) studied the tail behavior of the returns of five major cryptocurrencies by using again extreme value analysis and computing the Value-at-Risk and Expected Shortfall, but no backtesting analysis was implemented. Trucios (2018) compared the one-step-ahead volatility forecast of Bitcoin using several GARCH-type models and also evaluated the performance of several procedures when estimating the Value-at-Risk. As it is possible to notice, all these studies dealt only with univariate models, focused almost exclusively on the Value-at-Risk and volatility forecasting, while only three works performed backtesting analysis.

In general, standard models for market risk tend to work poorly with cryptocurrencies due to the frequent presence of structural breaks, see Bouri et al. (2016), Fantazzini et al. (2016), Fantazzini et al. (2017), Mensi, Al-Yahyaee, and Kang (2018) and Thies and Molnar (2018). To make matters worse, price manipulations and market frauds caused by the lack of financial oversight (Gandal et al. (2018), Griffin and Shams (2018)) and the fact that cryptocurrencies are still mainly used for speculative purposes, make financial bubbles a recurring phenomenon, see Corbet, Lucey, and Yarovaya (2018), Cheah and Fry (2015) and Gerlach, Demos, and Sornette (2018). A potential solution could be to use complex model specifications able to accommodate structural breaks and extreme price volatility, but this would come at the cost of lower computational tractability and potential model over-fitting. Moreover, this solution would become quickly unfeasible in the multivariate case, due to the well-known curse of dimensionality. Finally, we remark that credit risk modelling and the implications of having invested in dead coins have not been considered so far.

Appendix B: Multivariate time series models

We employed two multivariate models for simultaneously computing the market and credit risk measures of a portfolio of cryptocurrencies: the VAR-DCC and the VAR-Copula-GARCH models. The DCC model was originally proposed by Engle (2002) and Tse and Tsui (2002), while copula-GARCH models were discussed in Cherubini, Luciano, and Vecchiato (2004), A. J. Patton (2006a), A. J. Patton (2006b), Fantazzini (2008) and Fantazzini (2009b). We provide below a brief review of these two approaches, while we refer the interested reader to Bauwens, Hafner, and Laurent (2012) for a more detailed treatment at the textbook level.

These two approaches share similar building blocks for the conditional mean and the conditional variance: for the mean, a Vector Auto-Regression model (VAR) is used, while for the variance a set of GARCH models was employed. Let $\mathbf{Y}_t$ be a vector stochastic process of dimension $n \times 1$, then a conditional model for $\mathbf{Y}_t$ can be expressed as follows:

$$\mathbf{Y}_t = \boldsymbol{\mu}_t + \mathbf{D}_t \mathbf{z}_t$$

where $\boldsymbol{\mu}_t$ is a vector of conditional means, $\mathbf{D}_t = diag(\sigma_{1,t}, \ldots, \sigma_{n,t})$ a diagonal matrix of conditional standard deviations, while $\mathbf{z}_t$ is a vector of standardized errors with a conditional multivariate distribution $H_t(z_{1,t}, \ldots, z_{n,t}; \boldsymbol{\theta})$ and parameter vector $\boldsymbol{\theta}$.

The conditional means are modelled with a VAR(p) model,

$$\boldsymbol{\mu}_t = \mathbf{a}_0 + \sum_{m=1}^{p} \mathbf{A}_m \mathbf{Y}_{t-m}$$

while the conditional variances with GARCH(p,q) models,

$$\sigma_{i,t}^2 = \omega_i + \sum_{m=1}^{p} \alpha_{i,m} (\sigma_{i,t-m} z_{i,t-m})^2 + \sum_{k=1}^{q} \beta_{i,k} \sigma_{i,t-k}^2$$

Other univariate GARCH models, like the Exponential-GARCH, the Threshold-GARCH, etc. can be used. Where the VAR-DCC and the VAR-Copula-GARCH models differ is how they specify the conditional joint distribution H_t.

Dynamic Conditional Correlation (DCC) models

The DCC model by Engle (2002) assumes that H_t is a multivariate Normal or Student's t distribution with correlation matrix $\mathbf{R}_t$,

$$\mathbf{R}_t = (diag\mathbf{Q}_t)^{-1/2}\mathbf{Q}_t(diag\mathbf{Q}_t)^{-1/2} \tag{5}$$

where the $n \times n$ symmetric positive definite matrix $\mathbf{Q}_t$ is given by:

$$\mathbf{Q}_t = \left(1 - \sum_{l=1}^{L}\theta_{1,l} - \sum_{s=1}^{S}\theta_{2,s}\right)\bar{\mathbf{Q}} + \sum_{l=1}^{L}\theta_{1,l}\mathbf{z}_{t-l}\mathbf{z}'_{t-l} + \sum_{s=1}^{S}\theta_{2,s}\mathbf{Q}_{t-s}$$

where $\bar{\mathbf{Q}}$ is the $n \times n$ unconditional variance/covariance matrix of $\mathbf{z}_t$, $\theta_{1,l}(\geq 0)$ and $\theta_{2,s}(\geq 0)$ are scalar parameters satisfying $\sum_{l=1}^{L}\theta_{1,l} + \sum_{s=1}^{S}\theta_{2,s} < 1$ to have $\mathbf{Q}_t > 0$ and $\mathbf{R}_t > 0$. $\mathbf{Q}_t$ is the covariance matrix of $\mathbf{z}_t$, so that $q_{ii,t}$ is not equal to 1 by construction and it is subsequently transformed into a correlation matrix by (5).

In the multivariate normal case, the total likelihood can be decomposed in two parts, so that the models for the conditional means and variances can be estimated in a first stage, while the parameters of the conditional correlation are estimated in a second stage using the parameters from the first step. Instead, in case of a multivariate student's t distribution, the estimation should be performed either in one step (so that the shape parameter is jointly estimated for all volatility models), or in two stages: in this latter case, the first step is based on a Gaussian quasi-maximum likelihood (QML) estimator for the conditional means and variances, while the shape parameter is estimated in a second step, as suggested by Bauwens and Laurent (2005).

The DCC model is one of the most used models in applied finance, thanks to its computational flexibility that allows this model to be computed also with portfolios consisting of up to 100 assets (Engle and Sheppard (2001), Bauwens, Hafner, and Laurent (2012)). Despite known problems (Aielli (2013), Caporin and McAleer (2013)), it still remains an important benchmark when multivariate volatility modelling is of concern, see Bauwens, Laurent, and Rombouts (2006), Satchell and Knight (2011), Caporin and McAleer (2014) and Bali and Zhou (2016).

Copula-VAR-GARCH models

Copula theory provides an easy way to deal with the (usually) complex multivariate modelling. Using the so-called Sklar (1959) theorem, a joint distribution can be factored into the marginals and a dependence function called a *copula*. The joint distribution H_t can be expressed as follows:

$$\mathbf{z}_t \sim H_t(z_{1,t}, \ldots, z_{n,t}; \boldsymbol{\theta}) \equiv C_t(F_{1,t}(z_1; \delta_1), \ldots, F_{n,t}(z_{n,t}; \delta_n); \boldsymbol{\gamma}) \tag{6}$$

which means that the joint distribution H_t of a vector of standardized errors $\mathbf{z}_t$ is the copula $C_t(\cdot; \boldsymbol{\gamma})$ of the cumulative distribution functions of the innovation marginals $F_{1,t}(z_1; \delta_1), \ldots, F_{n,t}(z_{n,t}, ; \delta_n)$, where $\boldsymbol{\gamma}, \delta_1, \ldots, \delta_n$ are the copula and marginal parameters, respectively. For more details about copulas, we refer the interested reader to the textbooks by Joe (1997) and Nelsen (1999), while Cherubini, Luciano, and Vecchiato (2004) provide a detailed discussion of copula techniques for financial applications. It follows from (6) that the log-likelihood function for the joint conditional distribution $H_t(\cdot, \boldsymbol{\theta})$ is given by

$$l(\boldsymbol{\theta}) = \sum_{t=1}^{T} \log \left(c\big(F_{1,t}(z_1; \delta_1), \ldots, F_{n,t}(z_{n,t}; \delta_n); \boldsymbol{\gamma}\big) \right) + \sum_{t=1}^{T} \sum_{i=1}^{n} \log f_i(z_{i,t}; \delta_i)$$

where c is the copula density function, whereas f_i are the marginal densities. The maximization of the previous log-likelihood with respect to the parameters $(\boldsymbol{\gamma}, \delta_1, \ldots, \delta_n)$ can be made in 1 step, or in several steps by partitioning of the parameter vector into separate parameters for each margin and the parameters for the copula. This multi-step procedure is known as the method of Inference Functions for Margins (IFM), see Joe (1997) for details.

It is rather straightforward to show that the previous DCC model can be represented as a special case within a more general copula framework,

$$\begin{aligned} \mathbf{Y}_t &= \boldsymbol{\mu}_t + \mathbf{D}_t \mathbf{z}_t, \quad \text{where} \quad \mathbf{z}_t \sim H_t(z_{1,t}, \ldots, z_{n,t}; \boldsymbol{\theta}), \quad \text{and} \\ \mathbf{z}_t &\sim H_t \equiv C_t^{Normal}(F_{1,t}^{Normal}(z_1; \delta_1), \ldots, F_{n,t}^{Normal}(z_{n,t}; \delta_n); \mathbf{R}_t) \end{aligned}$$

see e.g. A. J. Patton (2006a), A. J. Patton (2006b), Fantazzini (2008) and Fantazzini (2009b) for more details.

The copula approach allows us to consider more general cases than a multivariate normal DCC model. For example, if we consider skewed-t distributions for the marginals, like those proposed by Hansen (1994) or Fernandez and Steel (1998), then a multivariate model allowing for marginal skewness, kurtosis and normal dependence can be expressed as follows:

$$\begin{aligned} \mathbf{Y}_t &= \boldsymbol{\mu}_t + \mathbf{D}_t \mathbf{z}_t \\ \mathbf{z}_t &\sim H_t \equiv C_t^{Normal}(F_{1,t}^{Skewed-t}(z_{1,t}; \delta_1), \ldots, F_{n,t}^{Skewed-t}(z_{n,t}; \delta_n); \mathbf{R}_t) \end{aligned}$$

where $F_{i,t}^{Skewed-t}$ is the cumulative distribution function of the marginal Skewed-t, and $\mathbf{R}_t$ can be made constant or time-varying, as in the constant conditional correlation (CCC) model or in the DCC model, respectively.

If we suppose that our assets have symmetric tail dependence, we can use a Student's t copula, instead,

$$
\begin{aligned}
\mathbf{Y}_t &= \boldsymbol{\mu}_t + \mathbf{D}_t \mathbf{z}_t \\
\mathbf{z}_t &\sim H_t \equiv C_t^{Student'sT}(F_{1,t}^{Skewed-t}(z_1;\delta_1), \ldots, F_{n,t}^{Skewed-t}(z_{n,t};\delta_n); \mathbf{R}_t, \nu)
\end{aligned}
$$

where ν denotes the degrees of freedom of the t-copula.