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Osadchiy, Maksim and Sidorov, Alexander

Corporate Finance Bank (Russia, Moscow), Novosibirsk State University and Sobolev Institute of Mathematics (Russia, Novosibirsk)

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Maksim Osadchiy[†], Alexander Sidorov[‡]

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Abstract

The paper considers a bank, left to itself, outside of regulation and supervision. The stochastic model allows us to describe the parameters, which create conditions both for the formation of bubbles in the credit market and for the formation of stable banks with self-restrictive behavior that do not require regulatory intervention. The comparative statics of equilibria is studied with respect to the basic parameters of the model, a theoretical assessment is carried out of the probability of bank default based on the values of exogenous factors. Our main task is to evaluate a bank probability of default not by using an econometric empirical approach, but by using microeconomic modeling.

Key words: Banking microeconomics, Credit bubble, Probability of default, Capital adequacy ratio

JEL codes: G21, G28, G32, G33

Introduction

The aim of the work is to study credit bubbles and ability of banks to self-restraint - refusal from unlimited credit expansion without regulatory intervention. The applied task of the study is to create a method for calculating bank ratings based on microeconomic analysis, and not on empirical econometric models. Of course, one cannot finally get rid of empiricism in calculating bank ratings, since many risk factors of bank default cannot be calculated using only microeconomic models, for example, the quality of management and internal control, reputation, etc.

The motivation of the paper is the inclusion in the microeconomic model of the bank of a relatively realistic Vasicek portfolio loss density function, taking into account the correlation of the assets of

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[†]Corporate Finance Bank (Russia, Moscow). Email: OsadchyMS@cfb.ru

[‡]Novosibirsk State University and Sobolev Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences (Russia, Novosibirsk) Email: alex.v.sidorov@gmail.com

borrowers to assess the risk of default of the bank, as well as to study the structure of assets and liabilities of the bank. However, in this paper we will consider the more general form of the portfolio loss density function, not limited to the Vasicek model.

Our work is based on 4 stylized facts related to the field of credit risks:

(1) Many banks tend to blow bubbles - to carry out dangerous credit expansion. They turn into "black holes" - fast-growing banks with unobservable negative capital. This problem is especially acute in developing countries with poor quality of banking regulation and supervision.

(2) If credit expansion covers the entire banking sector, a credit boom begins. If it is not stopped in time, it will inevitably end in a credit crisis. History knows hundreds of examples of such crises (see, e.g., [17, pp. 344-347]). For example, the burst of the US subprime mortgage bubble caused a global crisis.

(3) Credit crises lead to high social costs - unemployment, impoverishment, social instability. To avoid such severe consequences, the regulators limit the expansion of banks by micro-prudential policies, in particular, limiting by CAR (capital adequacy ratio) from below.

(4) Instead of maximally using a capital and increasing risk assets up to the level close to the capital adequacy requirements imposed by the regulator, many banks far exceed the CAR regulatory requirements. CAR has a wide spread [4, Graph I.13]: many banks are undercapitalized, and many banks are overcapitalized.

The key question of the paper: is the bank capable of self-restraint, under certain conditions, without the intervention of the regulator? We will also consider the question: why there is such a large spread of CAR and what explains its magnitude? To do this, consider a hypothetical bank operating without external regulation, and study its behavior. We will build a simple microeconomic model of the bank that takes into account only credit risk.

Our belongs to the class of the Structural models of credit risk. In the framework of our paper a bank is considered as a "Merton firm" with the balance including, on the one hand, the stochastically changing portfolio of assets, on the other hand, the given (i.e., risk-free) liabilities and capital. In turn, the behavior of the assets portfolio is captured by the Vasicek model [19, 20], i.e., as an infinite number of liabilities of the Merton-type firms with the correlating firm's assets. The main result of the Vasicek approach is a derivation of the portfolio's probability density function (PDF) of credit losses, also known as loss distribution of a credit portfolio, from the Brownian motion of the firm's assets. The Vasicek's portfolio credit loss model [19] underlies the Basel Advanced Internal Rating System (AIRB), which was developed to determine regulatory requirements for bank capital, well-known as Basel II and Basel III [3]. Although in reality the credit risk is often accompanied by the risk of outflow of liabilities, in the framework of this paper we do not allow for the liquidity risk, leaving it for a future research.

Similarly to the Merton model, our approach allows to assess the probability of the bank default, which take place when the assets portfolio falls lower than liabilities. However, the main difference of our approach from the Merton model is that we account for the bankruptcy charges. Moreover, the result of our paper shows that this factor plays the crucial role in the banker's decision making. Within the framework of the model, we construct the parameter zones where the bank is capable of self-restraint and where the bubble inflates - unlimited expansion takes place. We will study the structure of these areas. We will also calculate the probability of default of the bank and determine the structure of assets and liabilities of the bank within our model. And we will study the dependence of the probability of default of the bank and its CAR on the parameters.

Using the model, we study the mechanism of CAR choice. We study also the comparative statics of the banker's decision and of the probability of the bank's default with respect to the model parameters: the interest rates of attraction and allocation of resources, the correlation of borrowers' assets and other factors. There are identified the three areas in the space of exogenous parameters within which the banks are capable of self-restraint, and when they choose the unlimited expansion. This zoning of the model parameters, according to the nature of the equilibrium solution, opens a new way to determine whether the regulator is needed to intervene in banking, what are the boundaries of this intervention and its effectiveness. On the one hand, the concept of laissez-faire can be destructive, since the banking market failure entails far-reaching negative consequences, not so much even for the bank owner as for its many clients. If the bank is large enough, then its default can cause a domino effect. On the other hand, practice shows that the tightening of regulation is faced with the problem of low efficiency of regulatory measures, since banks have ample opportunities to manipulate information, creating the appearance of compliance with regulatory constraints. Another negative effect of over-regulation is a decrease in the efficiency of banks and the economy as a whole. As regulation is tightened, bankers spend too much time and efforts on compliance, instead of doing business. In this regard, it is worth to recall the general economic principle, according to which an economic individual can bypass external constraints, but cannot ignore his/her own incentives.

The classification of the solutions obtained in this paper allows us to identify cases when a state intervention in the bank activity is superfluous, since the decision satisfying the regulator is supported by internal stimuli. If the decision falls into another class, for example, it is characterized by an excessively low CAR, then this intervention is inevitable. Rating agencies use to apply the empirical econometric models to calculate the credit ratings. Our model allows to consider the prospect of calculating bank credit ratings not with empirical econometric models, but with micro-based modeling. Microeconomic models can be useful for evaluating implications of banking regulation.

The paper is organized as follows. Section 1 presents the model of bank that takes into account the partial impairment of bank assets in the event of a bank default, and the conditions for the existence and uniqueness of the equilibria are obtained. The mechanism of the formation of an equilibrium market credit rate in a competitive environment of risk-neutral players is considered. In Section 2, we consider the parametrized classification of the equilibrium states based on CAR and study the comparative statics of equilibrium characteristics both analytically and using computer simulation. Section 3 is devoted to a more visual graphical classification of decisions on the main parameters. The most important result is the determination of compliance with the requirements of Basel III. Finally, Section 4 is devoted to the multi-period extension of this model, in particular, the assessment of the probability of a bank's default in the long term is found. The main results and conclusions of the work are formulated in the Conclusion. A list of notations and abbreviations is placed at the beginning of Appendix.

The related literature

There are two primary types of models that attempt to describe default processes in the credit risk literature: structural and reduced form models. The aim of structural approach is to provide a relationship between default risk and capital structure, unlike the reduced form models, which consider the credit default as exogenous event driven by a stochastic process. Reduced form models do not consider the relation between default and firm value in an explicit manner. Intensity models represent the most extended type of reduced form models. In contrast to structural models, the time of default in intensity models is not determined via the value of the firm, but it is the first jump of an exogenously given jump process. The parameters governing the default hazard rate are inferred from market data. Structural models, pioneered by Black, Scholes [6]and Merton [15], ingeniously employ modern option pricing theory in corporate debt valuation. Merton model was the first structural model and has served as the cornerstone for all other structural models, including ours. A significant extension of Merton was represented by Black and Cox in [5], who managed to relax some of the Merton's assumption.

The next major step in generalizing the structural models was an important contribution of Leland in [13], who explicitly introduced corporate taxes and bankruptcy costs, which may be interpreted as liquidation costs. Thus, he formalized the trade-off framework and provided a way to determine both the optimal default boundary and the value-maximizing optimal capital structure. Since these classical papers the further advances of structural models in various direction. It worth to mention that firms often make their decisions in a principal agent setting, wherein managers, equity holders (borrowers), and creditors may have very different objective functions. The resulting agency problems may have significant implications for the optimal capital structure decisions and optimal contracting decisions, see the papers [9, 10]. The paper [1] develops a structural model of a financial institution that can invest in both liquid and illiquid assets dynamically, maximizing the profit of its shareholders while satisfying some regulatory constraints. It is proved that tightening the liquidity constraint adversely affects their rates of return, while preventing some large losses that occur when the portfolio is very illiquid.

The academic literature on jointly optimal regulation of bank capital and liquidity seems to increase after 2010 by the Basel Committee on Banking Supervision. The paper [2] analyses how banking firms set their capital ratios, that is, the rate of equity capital over assets. In order to study this issue, two theoretical models are developed: the "market" model for banks not affected by capital adequacy regulation, while the second one, the regulatory model, explain the behavior of banks with an optimal market ratio below a legally required regulation.

Regulation related to capital requirements is an important issue in the banking sector. One of the indices used to measure how susceptible a bank is to failure, is the capital adequacy ratio (CAR). In general, this index is calculated by dividing a measure of bank capital by an indicator of the level of bank risk. The papers [11] and [16] consider the application of stochastic optimization theory to asset and capital adequacy management in banking and construct continuous-time stochastic models for the dynamics of capital adequacy ratio established by Basel II. This ratio is obtained by dividing the bank's eligible regulatory capital (ERC) by its risk-weighted assets (RWAs) from credit, market and operational risk. The contribution of paper [7] is the construction of a stochastic dynamic model to describe the evolution of bank capital that incorporates capital gains and losses. The gains and losses are represented by loan loss reserves and the unexpected loan losses, respectively. It is studied the optimal capital management problem which maximizes the expectation of bank capital under a risk constraint on the Capital-at-Risk (CaR), where CaR is defined in terms of Value-at-Risk (VaR).

The issue of bank capital adequacy and risk management within a stochastic dynamic setting is studied in the paper [8]. An explicit risk aggregation and capital expression is provided regarding the portfolio choice and capital requirements special context. This leads to a nonlinear stochastic optimal control problem whose solution may be determined by means of dynamic programming algorithm.

Along with the great impact of structural models on the theory of the credit risk and its application, there is reasonable criticism on the prediction power of such models concerning to the pricing of corporative bonds, see, for example, [12]. On the other hand, the structural models are able to predict well the hedge ratios of corporate bonds against the equity of the underlying firm, see [18]. Concluding this short survey of structural models one can say: "All models are wrong, but some are useful."

1 The model

We follow Vasicek approach [20], assuming that the bank provided n loans, and the probability of default of each loan $i \in \{1, \ldots, n\}$ is the same and equals to $PD \in [0, 1]$. It is assumed that the loss given default $LGD = 1$. Random variable ε_i takes two possible values: $\varepsilon_i = 1$, if credit i is defaulted (with probability PD), and $\varepsilon_i = 0$ (with probability $1 - PD$) otherwise. Generally speaking, the random variables ε_i, $i \in \{1, \ldots, n\}$ are not independent. Then a random variable

$$\varepsilon = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i$$

characterizes the share of nonperforming loans, taking values in the interval $[0, 1]$. It is obvious that $\mathbb{E}(\varepsilon) = PD$, while the distribution of ε is not necessary normal due to possible dependence of various ε_i. For now we don't take any specific assumptions on the nature of this dependence, considering the general Cumulative Distribution Function (CDF) $F(z)$ of the random variable ε on $[0, 1]$. Further we put some natural assumption on $F(z)$, which encompass the most important case of the Vasicek distribution of losses.

It is assumed that the bank operates in a perfectly competitive environment, being a price taker. We study a single-period model in which a bank is created at the initial moment $t = 0$ with an initial capital $K_0 > 0$. At the same time the banker chooses the amount $D_0 \geq 0$ of the attracted deposits at the interest rate $R > 0$, and then places the borrowed and its own funds in the uniform loans of the same size at the interest rate $r > R$ before the terminal moment $t = 1$, hence, the loans are equal to $L_0 = K_0 + D_0$. The supply of loans and the demand for deposits are satisfied in full, and interest rates are exogenous parameters, due to the assumption of perfectly competitive environment. At the moment $t = 1$ all loans are repaid, except for those defaulted, hence $L_1 = 0$, and all assets acquire the form of cash M_1. After the deposits are returned and interest is paid at the rate R, the bank's capital becomes equal to $K_1 = M_1 - D_1$.

All balance sheet items, are summarized in Table 1.

Remark 1. This model can be extended to the case of the arbitrary cash $M_0 \geq 0$. However, given the lack of liquidity risk, i.e., the nonzero probability of the premature withdrawal of deposits, easily implies that the optimum choice of the banker will be $M_0 = 0$.

Suppose that the bank's default implies the additional losses, moreover, the banker is "responsible"

Balance sheet items		$t = 0$	$t = 1$
Assets	Cash	$M_0 = 0$	$M_1 = (1 + r)L_0(1 - \varepsilon)$
	Loans	$L_0 = K_0 + D_0$	$L_1 = 0$
Deposits		$D_0 \geq 0$	$D_1 = (1 + R)D_0$
Capital		$K_0 > 0$	$K_1 = M_1 - D_1$

Table 1: Control variables and dependencies between variables

i.e., bears all the costs of bankruptcy. More precisely, if random amount of bank capital at the end of the period

$$\bar{K}_1 = (1 + r)L_0(1 - \varepsilon) - (1 + R)D_0$$

takes positive value, then this amount goes to the banker unchanged. Given $D_0 = L_0 - K_0$, this means that

$$\bar{K}_1 = (r - R - \varepsilon(1 + r))L_0 + (1 + R)K_0.$$

Otherwise, in case of default i.e., $\bar{K}_1 \leq 0$, the bank sells a loan portfolio with a discount $0 \leq d \leq 1$. As result, the terminal capital under default is equal to

$$\hat{K}_1 = \bar{K}_1 - d(1 + r)L_0(1 - \varepsilon).$$

The value $d = 0$ corresponds to the case when there are no additional losses, i.e., $\hat{K}_1 = \bar{K}_1$.

Note that the condition of the bank default

$$\bar{K}_1 \leq 0 \iff \varepsilon \geq \frac{r - R}{1 + r} + \frac{1 + R}{1 + r}\frac{K_0}{L_0}. \tag{1.1}$$

In other words, the default takes place if and only if the losses exceed the certain threshold, which depends on the Capital Adequacy Ratio (CAR)

$$k(L_0) = \frac{K_0}{L_0}.$$

Here we assume that the risk weight of loans is equal to 1.

To save space, let's introduce the following notification

$$\hat{\mathcal{E}} = \frac{r - R}{1 + r},$$

which implies

$$\frac{1 + R}{1 + r} = 1 - \hat{\mathcal{E}}, \; \frac{\tilde{r} - R}{1 + r} = \hat{\mathcal{E}} - PD,$$

and let

$$\mathcal{E}(k) = \hat{\mathcal{E}} + (1 - \hat{\mathcal{E}})k.$$

Due to (1.1) $\mathcal{E}(k(L_0))$ may be interpreted as a threshold value of losses, which triggers the bank default. The value $\hat{\mathcal{E}} = \mathcal{E}(0)$ may be also interpreted as the limit loss threshold when the loan portfolio increases unrestrictedly, because $\lim\limits_{L_0 \to \infty} k(L_0) = 0$.

Thus, the general definition of the terminal capital is

$$K_1 = \begin{cases} \bar{K}_1, & \text{if } \varepsilon < \mathcal{E}(k(L_0)) \\ \bar{K}_1 - d(1+r)L_0(1-\varepsilon), & \text{otherwise.} \end{cases}$$

The banker's problem is to maximize the expected value of the terminal capital $\mathbb{E}(K_1)$ under condition $D_0 \geq 0 \iff L_0 \geq K_0$. Given

$$\mathbb{E}(\bar{K}_1) = (r - R - PD(1+r))L_0 + (1+R)K_0 = (1+r)\left[(\hat{\mathcal{E}} - PD)L_0 + (1-\hat{\mathcal{E}})K_0\right],$$

we obtain that the expected terminal capital is equal to

$$\mathbb{E}(K_1) = (1+r)\left[(\hat{\mathcal{E}} - PD)L_0 + (1-\hat{\mathcal{E}})K_0 - dL_0 \int_{\mathcal{E}(k(L_0))}^{1} (1-z)f(z)dz\right].$$

Given $f(z) = F'(z)$ is a PDF of the random losses ε, we may interpret the function

$$\mathrm{ret}(z) \equiv (1-z)f(z)$$

as a weighted share of the returned loans and let

$$\mathrm{Ret}(x) = \int_x^1 \mathrm{ret}(z)\mathrm{d}z.$$

The function $\mathrm{Ret}(x)$ is obviously decreasing, because $\mathrm{Ret}'(x) = -\mathrm{ret}(x)$, and satisfies $\mathrm{Ret}(0) = 1 - PD$, $\mathrm{Ret}(1) = 0$. Therefore, the expected terminal capital $\mathbb{E}(K_1) = (1+r)U(L_0)$, where

$$U(L_0) = (1-\hat{\mathcal{E}})K_0 + (\hat{\mathcal{E}} - PD)L_0 - d \cdot L_0 \mathrm{Ret}(\mathcal{E}(k(L_0))), \tag{1.2}$$

is the reduced objective function.

Now the original banker's problem is equivalent to

$$\max U(L_0) \text{ s.t. } L_0 \geq K_0.$$

Theorem 1. *Let $\tilde{r} > R$ and the weighted share of returns $ret(z)$ is strictly decreasing on the interval $PD < z < 1$, then $U''(L_0) < 0$ for all $L_0 \geq K_0$. Moreover, if inequality*

$$d > \frac{\hat{\mathcal{E}} - PD}{Ret(\hat{\mathcal{E}})} \tag{1.3}$$

holds, then there exists unique solution of the banker's problem

$$L_0^* = \frac{K_0}{k^*}, \; D_0^* = \frac{1-k^*}{k^*} K_0 \tag{1.4}$$

where $k^* \in (0,1)$ is an equilibrium CAR, defined as the unique solution of equation

$$FOC: \; \hat{\mathcal{E}} - PD - d \cdot \Big(Ret(\mathcal{E}(k)) + (1-\hat{\mathcal{E}})k \cdot ret(\mathcal{E}(k)) \Big) = 0. \tag{1.5}$$

Proof. See Appendix A.1.

Theorem 1 implies that depending on the parameter's relation, there may be two possible cases, which cause different types of the banker's behavior.

Case 1. Let the discount d be sufficiently small, e.g., $d = 0$, so that condition (1.3) is violated. This implies an unrestricted increasing of objective function $U(L_0)$ when $L_0 \to +\infty$, which means that banker has incentives for unrestricted expansion[1] of the loan portfolio L_0.

To prevent this negative trend, the regulator restricts lending by imposing the condition

$$K_0/L_0 \geq \hat{k} \iff L_0 \leq L_0(\hat{k}) = \frac{K_0}{\hat{k}}. \tag{1.6}$$

for the exogenously given CAR $\hat{k}$. It is obvious that in this case, the modified banker problem with additional constraint (1.6) has solution $L_0^* = L_0(\hat{k})$.

Case 2. Assume that the discount d is sufficiently large to satisfy the condition (1.3) and, therefore, there exists solution $k^* \in (0,1)$ of (1.5). This situation can be interpreted as if the banker imposes self-restrain $L_0 \leq K_0/k^*$, which is active in the optimum, i.e., $L_0^* = K_0/k^* > K_0$.

Remark 2. The first order condition (1.5) may be interpreted in terms of the gains-losses as follows. Choosing the amount of the loans portfolio L_0 the banker obtains the expected total gains $(1-\hat{\mathcal{E}})K_0+$

[1] The same holds when banker is "irresponsible", i.e., does not want to pay his/her liabilities in case of default, i.e., $\hat{K}_1 = 0$.

$(\hat{\mathcal{E}} - PD)L_0$, thus the marginal gains from the further credit expansion are

$$\hat{\mathcal{E}} - PD = \frac{\tilde{r} - R}{1 + r}.$$

On the other hand, the term

$$\mathrm{Ret}(\mathcal{E}(k(L_0))) = \int_{\mathcal{E}(k(L_0))}^{1} \mathrm{ret}(z)\mathrm{d}z = \mathbb{E}(1 - \varepsilon|\ \varepsilon > \mathcal{E}(k(L_0)))$$

characterizes the share of the the loans returns *in case of the bank default.* Note that the size of the loan portfolio affects both the basis L_0 and the share of returns $\mathbb{E}(1 - \varepsilon|\ \varepsilon > \mathcal{E}(k(L_0)))$, then the total marginal losses in case of the bank default are sum of effects: the marginal losses from increasing of basis L_0 are equal to

$$d \cdot \mathrm{Ret}(\mathcal{E}(k(L_0))),$$

while the losses from change of share are equal to

$$d \cdot L_0 \left(-\mathrm{ret}(\mathcal{E}(k(L_0))) \frac{\mathrm{d}}{\mathrm{d}L_0} \mathcal{E}(k(L_0)) \right) = d \cdot (1 - \hat{\mathcal{E}}) k(L_0) \mathrm{ret}(\mathcal{E}(k(L_0))).$$

Finally, the gross marginal losses are equal to

$$d \cdot \Big(\mathrm{Ret}(\mathcal{E}(k(L_0))) + (1 - \hat{\mathcal{E}}) k(L_0) \mathrm{ret}(\mathcal{E}(k(L_0))) \Big),$$

Thus, the FOC 1.5 is equivalent to coincidence of the marginal gains and the marginal losses.

The Vasicek distribution of losses

To justify the conditions of Theorem 1, consider the Vasicek distribution of the loan losses (see [20]) with CDF

$$F(z; PD, \rho) = \Phi \left(\frac{\sqrt{1-\rho}\,\Phi^{-1}(z) - \Phi^{-1}(PD)}{\sqrt{\rho}} \right), \tag{1.7}$$

where PD is a probability of a borrower's default, ρ is a correlation coefficient of a borrower's assets. The corresponding PDF is as follows

$$f(z; PD, \rho) = \sqrt{\frac{1-\rho}{\rho}} \exp \left(\frac{1}{2} \left[(\Phi^{-1}(z))^2 - \left(\frac{\sqrt{1-\rho}\,\Phi^{-1}(z) - \Phi^{-1}(PD)}{\sqrt{\rho}} \right)^2 \right] \right). \tag{1.8}$$

Lemma 1. *Let $0 < \rho < 1$, $0 < PD < 1$ and $f(z; PD, \rho)$ be the density function of the Vasicek distribution of losses. Then the function ret$(z; PD, \rho) \equiv (1 - z) f(z; PD, \rho)$ decreases with respect to z*

in interval $PD < z < 1$.

Proof. See Appendix A.2[2].

1.1 Comparative statics of equilibrium

Now we will study how the equilibrium reacts to the changes of the parameters d, R and r.

Proposition 1. *The signs of partial derivatives of the equilibrium values of* k^**,* L_0^**,* D_0^* *with respect to* d*,* R*, and* r *are as follows:*

$$\frac{\partial k^*}{\partial d} > 0, \ \frac{\partial k^*}{\partial R} > 0, \ \frac{\partial k^*}{\partial r} < 0$$
$$\frac{\partial L_0^*}{\partial d} < 0, \ \frac{\partial L_0^*}{\partial R} < 0, \ \frac{\partial L_0^*}{\partial r} > 0$$
$$\frac{\partial D_0^*}{\partial d} < 0, \ \frac{\partial D_0^*}{\partial R} < 0, \ \frac{\partial D_0^*}{\partial r} > 0$$

Proof. See Appendix A.3.

These results comply with intuitive expectations. For example, increasing of discount d suppress the banker's activity, forcing to reduce the loan portfolio L_0^* and the attraction of deposits. As expected, an increasing in the deposit interest rate R reduces the demand of deposit, while increasing in the loan interest rate r increases supply of loans, etc.

How CAR k^* depends on the correlation ρ

Now we focus on the case of Vasicek distribution of the loan losses (1.7), which is characterized by two parameters — ρ and PD — the borrower's asset correlation and the probability of borrower's asset default, respectively. From intuitive point of view, the larger is correlation ρ, the more restrictive banking policy is required. In other words, $k^*(\rho)$ should be increasing function, but it is not clear, whether the presented model catches this effect? The analytical way, like in Proposition 1, failed due to very tedious calculations, thus, the Figures 1a and 1b show the series of computer simulations. Figure 1a shows the curves, corresponding to the fixed discount $d = 0.5$ and three values of the default probability $PD = 0.06, \ 0.07, \ 0.08$. Similarly, Figure 1b shows the curves, corresponding to the fixed probability $PD = 0.075$ and three values of discount $d = 0.3, \ 0.5, \ 0.75$. As we see, increasing in both d and PD shifts the curves upwards. An interpretation of this effect is quite natural. Increasing in both cases implies the risk of default and/or the associated losses, which forces the "responsible" banker to be more safe and conservative. Note that Figure 1b agrees with Proposition 1 statement on $\frac{\partial k^*}{\partial d} > 0$.

[2] This statement was proved mostly by Dirk Tasche. Authors are grateful to him for the kind assistance.

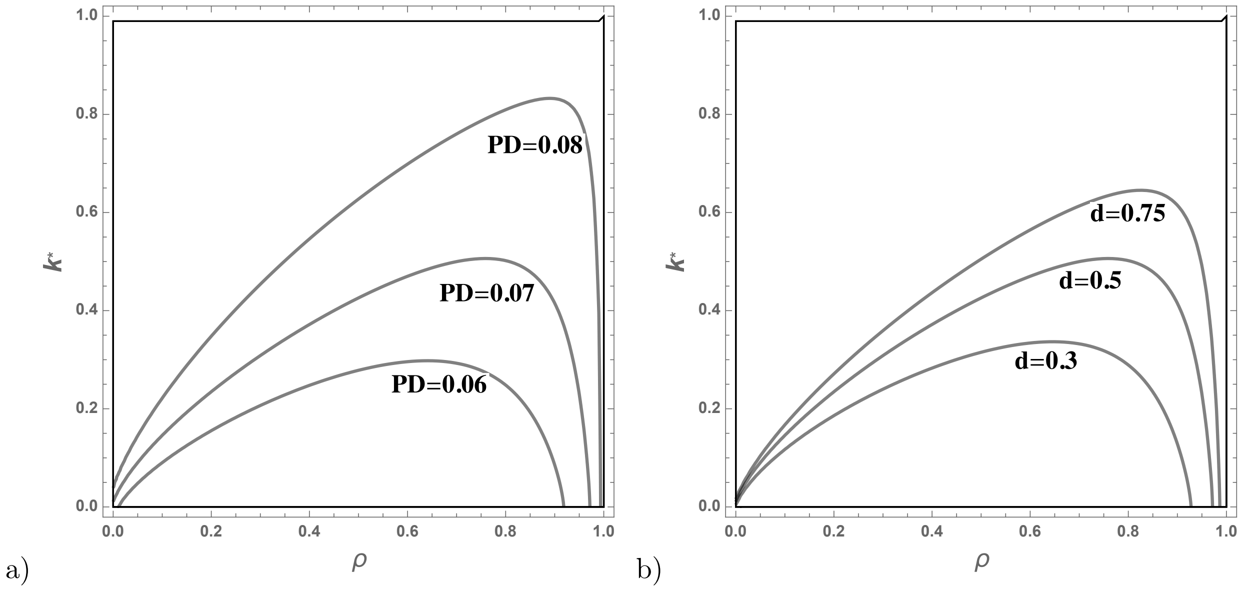


Figure 1: CAR k^* as a function of ρ, $R = 0.05$, $r = 0.15$, a) $d = 0.5$, b) $PD = 0.07$

The lack of solution in neighborhood of $\rho = 0$ and $\rho = 1$ is result of violation of the solvability condition (1.3). The direct calculations show that for all ρ sufficiently close to 0 or 1 the fraction

$$\frac{\hat{\mathcal{E}} - PD}{\mathrm{Ret}(\hat{\mathcal{E}})}$$

exceeds $d = 0.5$ and even it may be larger than 1. The values $k^* = 0$, i.e., the bottom points of the "arcs", correspond to the threshold values of ρ, that satisfy the identity

$$\frac{\hat{\mathcal{E}} - PD}{\mathrm{Ret}(\hat{\mathcal{E}}; PD, \rho)} = d$$

and delimit the areas of ρ with the equilibrium with finite size of L_0^*, which corresponds to the strictly positive CAR $k^* > 0$, from the "bubble" ρ with the unrestricted credit expansion $L_0 \to \infty$, which may be associated with $k^* = 0$. Therefore, we can extend the function $k^*(\rho)$ on the "non-existence" areas putting $k^*(\rho) = 0$.

1.2 Probability of the bank's default

The considered above optimum asset liability management is based on the risk-neutral behavior, targeted to maximize the expected terminal capital $\mathbb{E}(K_1)$, which is nominally greater than initial capital K_0, due to Theorem 1. However, the risk of default persists even if the management decisions are optimal. The probability of event $K_1 \leq 0$ may be calculated as follows

$$p = \mathbb{P}(\varepsilon \geq \mathcal{E}(k^*)) = \int_{\mathcal{E}(k^*)}^{1} f(z)\mathrm{d}z = 1 - F(\mathcal{E}(k^*)), \tag{1.9}$$

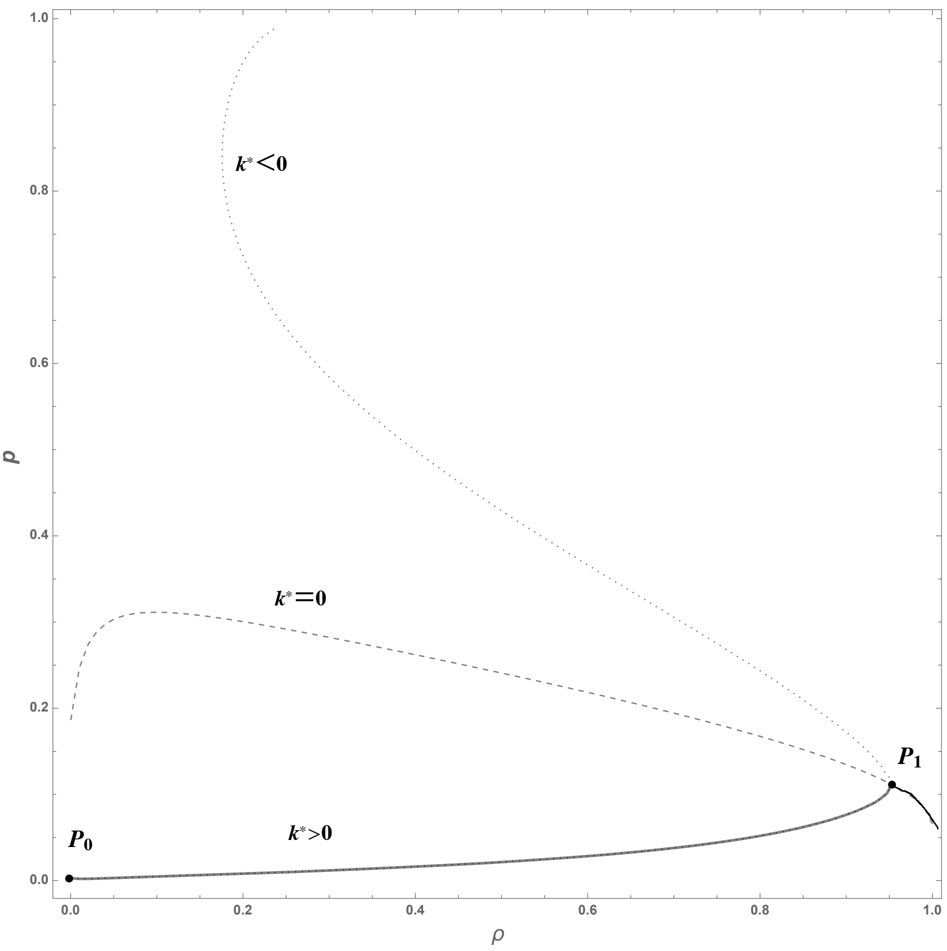


Figure 2: Probability of the banker's default as function of ρ, $d = 0.25$, $PD = 0.075$

where k^* is solution of equation (1.5). Function $1 - F(\mathcal{E}(k))$ strictly decreases with respect to k, therefore, Proposition 1 implies that

$$\frac{\partial p}{\partial d} < 0, \ \frac{\partial p}{\partial R} < 0, \ \frac{\partial p}{\partial r} > 0,$$

which is quite intuitive. Note that the equilibrium values of k^* must be positive, therefore, the feasible values of the bank's probability of default satisfy inequality $p = 1 - F(\mathcal{E}(k^*)) < 1 - F(\hat{\mathcal{E}})$.

Focusing on the Vasicek distribution of losses, we can consider the comparative statics of probability p with respect to specific parameters — the correlation ρ and the probability of borrower's default PD. Unfortunately, the analytic study of this question is problematic. The Figure 2 shows the result of the computer simulations with $d = 0.25$, $r = 0.15$, $R = 0.05$, $PD = 0.075$.

Given the FOC

$$G(k, \rho) \equiv \hat{\mathcal{E}} - PD - d \cdot \Big(\mathrm{Ret}(\mathcal{E}(k); \rho) + (1 - \hat{\mathcal{E}})k \cdot \mathrm{ret}(\mathcal{E}(k); \rho)\Big) = 0,$$

we substitute $k = \frac{F^{-1}(1-p;\rho)-\hat{\mathcal{E}}}{1-\hat{\mathcal{E}}}$ obtaining the equation

$$H(\rho, p) = G\left(\frac{F^{-1}(1-p;\rho)-\hat{\mathcal{E}}}{1-\hat{\mathcal{E}}}, \rho\right) = 0,$$

which determines the implicit function $p(\rho)$. The set of all solutions (ρ, p) of this equation contains the "fictive" roots, violating the feasibility condition

$$k > 0 \iff p(\rho) < 1 - F(\hat{\mathcal{E}}, \rho).$$

To screen the fictive solutions we draw the delimiting border

$$k^* = 0 \iff p(\rho) = 1 - F(\hat{\mathcal{E}}, \rho),$$

which is depicted by dashed curve on Figure 2. The solid curve P_0P_1 is a set of all feasible solution (ρ, p), satisfying both $H(\rho, p) = 0$ and $p(\rho) < 1 - F(\hat{\mathcal{E}}, \rho) \iff k^* > 0$. The points (ρ, p) of the pointed curve above the border $p = 1 - F(\hat{\mathcal{E}}, \rho)$, satisfying $H(\rho, p) = 0$ and $p(\rho) > 1 - F(\hat{\mathcal{E}}, \rho) \iff k^* < 0$, are non-feasible.

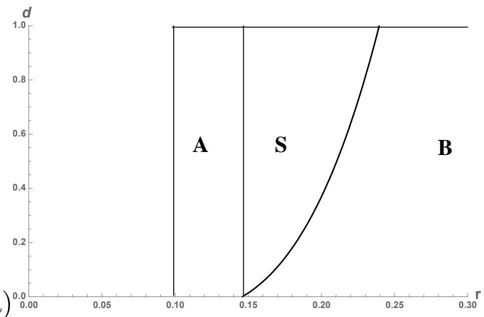
As for definition of the default probability for ρ rightward to P_1, let's to recall that in these cases the banker can not impose the self-restriction at some finite amount of the loan portfolio, which implies $L_0 \to \infty \iff k(L_0) \to 0$. Thus we may define the function $k(\rho)$ as folows

$$k^*(\rho) = 0 \Rightarrow p(\rho) = 1 - F^{-1}\left(\hat{\mathcal{E}}; \rho\right),$$

i.e., the continuation of the probability of the bank default belongs to the delimiting curve $p = 1 - F(\hat{\mathcal{E}}, \rho)$.

2 Parametric zoning by the solution types

The main aim of the present section is to visualize the various types of equilibria in terms of the model primitives. First, assume that the deposit interest rate R, and CDF $F(z)$ for the loan losses ε are given and its PDF $f(z)$ satisfies the condition $\mathrm{ret}(z)$ decreases for all $PD < z < 1$. Consider the set $\mathcal{S}$ of feasible points $r > R$, $0 \leq d \leq 1$ of the parameter plane (r, d). With any point of this set we associate specific type of equilibrium, which corresponds to the whole set of parameters, including the given ones. Figure 3 shows two examples of such zoning of $\mathcal{S}$ for the Vasicek distribution of losses, which is characterized by two additional parameters, ρ and PD.



a)

Figure 3: Vasicek distribution of loan losses: $R = 0.1$, $PD = 0.04$, $\rho = 0.1$

There are three, or, conditionally, four areas in parameters space, which may be described as follows.

I. Bubble area B corresponds to the unrestricted credit expansion. It consists of points $(r, d) \in \mathcal{S}$, that violate condition (1.3).

II. Self-Constrained area S corresponds to case when the bank attracts deposits and places funds to the loan portfolio of the limited size. It consists of points $(r, d) \in \mathcal{S}$ that satisfy conditions (1.3) and $\tilde{r} > R$

III. Autarchy area A consists of points $(r, d) \in \mathcal{S}$ that satisfy the inequality $\tilde{r} < R$, $0 \leq d \leq 1$, which means that condition (1.3) trivially holds and the banker's optimum solution is degenerate: the bank does not attracts deposits, i.e., $D_0 = 0$, while the the loan portfolio $L_0 = K_0 > 0$.

Remark 3. For any given positive value of discount $d_+ > 0$, no matter how small is it, we obtain the nonempty intersection of the line $d = d_+$ with all three areas. If $d = 0$, that corresponds to the linear model of Section 1, the Self-Constrained area **S** vanishes and we obtain only two generic cases —Bubble area **B** and Autarchy area **A**, which agrees with result obtained in Section 1.

The main result of this subsection is that the shapes of this zoning does not depend, on choice of distribution function

Proposition 2. *The structure of areas* **B**, **S**, **A** *is persisting.*

Proof. See Appendix A.4.

2.1 The Basel III requirements

The Basel III requires that the probability of the bank's default

$$p = 1 - F(\mathcal{E}(k^*))$$

must not exceed 0.001, which implies the inequality

$$k^* \geq \bar{k} = \frac{\mathrm{VaR}_{99.9} - \hat{\mathcal{E}}}{1 - \hat{\mathcal{E}}}.$$

where $\mathrm{VaR}_{99.9} = F^{-1}(0.999)$.

The Basel III analysis uses the Vasicek loan losses distribution (1.7), therefore,

$$\mathrm{VaR}_{99.9} = F^{-1}(0.999; PD, \rho) = \Phi\left(\sqrt{\frac{\rho}{1-\rho}}\Phi^{-1}(0.999) + \sqrt{\frac{1}{1-\rho}}\Phi^{-1}(PD)\right), \tag{2.1}$$

which allows to calculate the corresponding required value of CAR. Now we are going to identify sets of the bank parameters d, r, R, PD, ρ which guarantee that the banker complies voluntarily with Basel III requirements, or, on the contrary, the external regulation is needed. Substituting $\mathrm{VaR}_{99.9} = \mathcal{E}(\bar{k})$ into equation (1.5), we can determine the minimum value of discount d_B, as a function of r, guaranteeing the precise discharge of Basel III requirements, as follows

$$d_B(r) = \frac{\hat{\mathcal{E}} - PD}{\mathrm{Ret}(\mathrm{VaR}_{99.9}) + (\mathrm{VaR}_{99.9} - \hat{\mathcal{E}})\mathrm{ret}(\mathrm{VaR}_{99.9})}. \tag{2.2}$$

Let parameters R, PD, ρ be given, consider the curve $d = d_B(r)$ in the parameter plane (r, d). Obviously it starts from point $d = 0$, $r = \frac{PD+R}{1-PD}$, moreover, function $d_B(r)$ strictly increases, because function $\hat{\mathcal{E}} = \frac{r-R}{1+r}$ is increasing with respect to r. To illustrate this division, consider the following example with $R = 0.1$, $PD = 0.04$, $\rho = 0.2$, presented on Figure 4. The dashed "Basel curve" $d = d_B(r)$ divides area $\mathbf{S}$ into two sub-areas: $\mathbf{S}_A$, where Basel III requirements are violated, and $\mathbf{S}_B$, where they are complied.

The "Basel friendly" combination of parameters admits an arbitrary value of discount d, while the loan interest rates should not be too large. The existence of area $\mathbf{S}_B$ may explain the paradoxical dispersion of real values of CAR.

2.2 Generalized Basel and the equiprobability curves

The curve dividing area $\mathbf{S}$ into two subareas in Figure 5 was determined by specific Basel III requirement. Let's generalize this approach considering an arbitrary value of the bank's default probability p as a parameter and determining the equibrobability curve $\mathcal{I}_p$ associated with the value of p, as a set of pairs (r, d), which generate the equilibrium with the probability of default equal to p, provided that the rest of the model parameters, including CDF $F(z)$, are given. We also keep the assumption on decreasing of the function $\mathrm{ret}(z)$ on the interval $PD < z < 1$

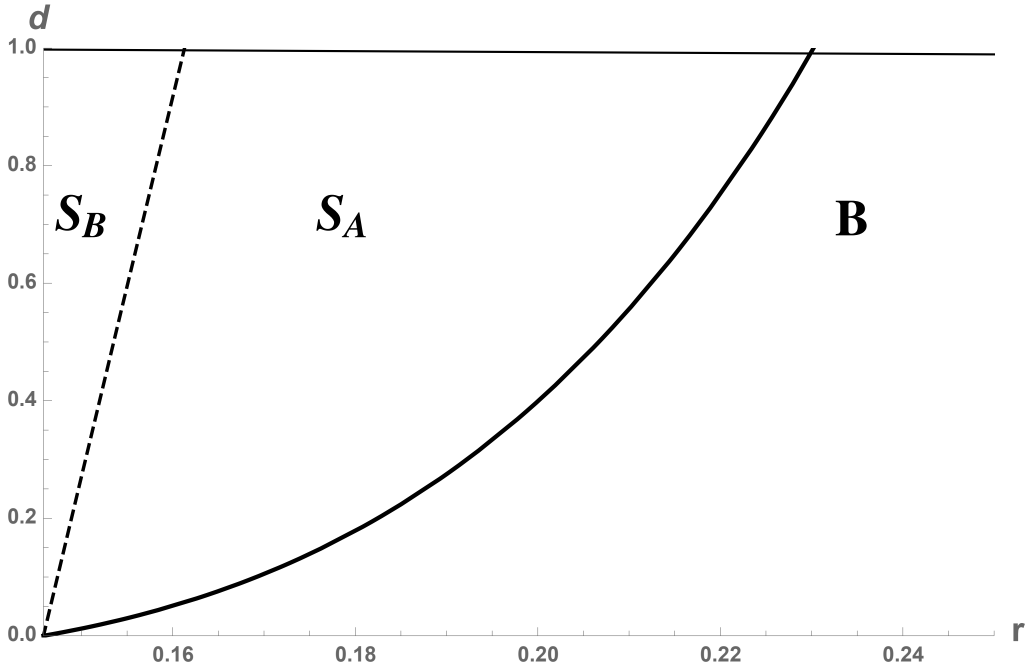


Figure 4: Dichotomy "self-restriction – external restriction"

Theorem 2. *The assemblage of curves $\mathcal{I}_p$ is characterized by the following properties:*

1. All curves $\mathcal{I}_p$ associated with different probabilities p start from the same point $r = \frac{R+PD}{1-PD}$, $d = 0$.

2. If probability of the bank's default p converges to zero, then the curves $\mathcal{I}_p$ converge to the vertical line $r = \frac{R+PD}{1-PD}$, $0 \leq d \leq 1$.

3. The assemblage of curves $\mathcal{I}_p$ for all $p < 1 - F(PD)$ fills the whole Self-Constrained area **S***, moreover, for $p < p'$ the curve $\mathcal{I}_p$ resides leftward and above the curve $\mathcal{I}_{p'}$.*

4. For all sufficiently small p the equiprobability curve $\mathcal{I}_p$ does not intersect the border of the areas **B** *and* **S** *for $0 < d \leq 1$.*

Proof. See Appendix A.5.

Figure 5 illustrates the three possible cases of the equiprobability curves described in Theorem 2 for the Vasicek distribution of losses with $PD = 0.04$, $\rho = 0.2$, and $R = 0.1$. Solid curve is the border between Self-Constrained and Bubble areas, while three dashed lines are the equiprobability curves $\mathcal{I}_p$ associated with three values of the bank probability of default: for the very large probability $p = 0.25$ the curve $\mathcal{I}_p$ leave the Self-Constrained area immediately, for the intermediate value $p = 0.05$ the curve $\mathcal{I}_p$ intersect the border, while the small probability of the bank default $p = 0.02$ generates the curve $\mathcal{I}_p$ intersecting the line $d = 1$.

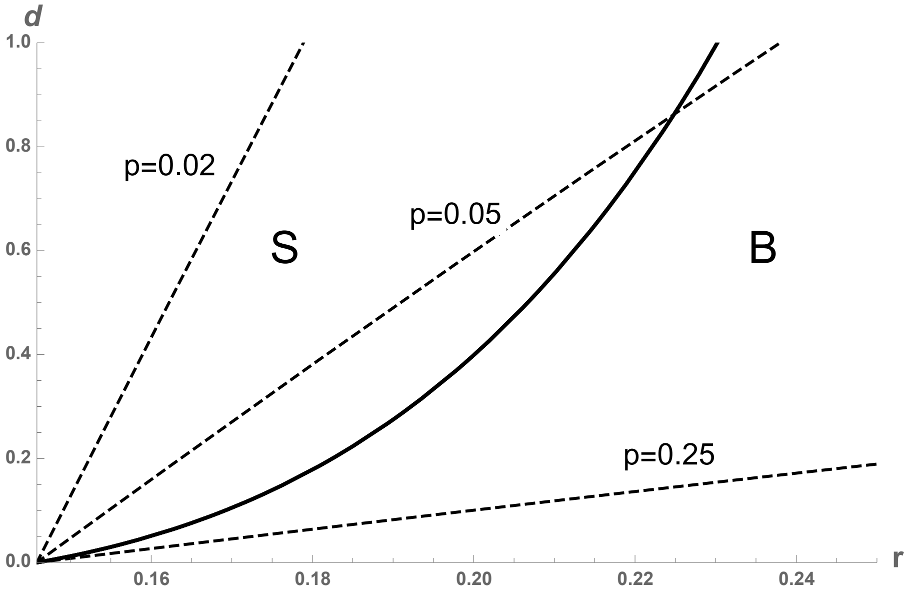


Figure 5: Equiprobability curves $\mathcal{I}_p$ for $p = 0.02,\ 0.05,\ 0.25$.

3 Conclusion

The banking is one of the most over-regulated and over-supervised industries, and the pressure on banks continues to grow. A natural question arises: can banks do without a regulator - at least in some aspects of their activities that are now under strict regulation and supervision?

For example, can banks limit their credit expansion on their own, without intervention of a regulator? To answer this question, we built a simple microeconomic model with one stochastic factor – the share of non-performing loans. It turned out that if, in the event of a bank default, a loan portfolio can be sold without a discount, then the banker has no incentives to limit the credit expansion, even despite the prospect of incurring of huge losses. This means that in this case, banking cannot do without a regulator, only the state can restrict the credit expansion.

The situation changes drastically, when we assume that in the event of a bank failure, its loan portfolio is sold at some non-zero discount. In this case, when certain limitations on the model parameters are satisfied, an endogenous restriction of credit expansion arises. Unlike external restrictions that banks have learned to successfully circumvent, these restrictions are internal, and deceiving oneself is usually not beneficial. However, from the point of view of the regulator, which evaluates the result in terms of CAR, the level of bank self-restraint may seem unacceptable, for example, if the ratio has a too low value. In this paper we derive the conditions of the existence and uniqueness of equilibrium (FOC and SOC), which have the clear economic interpretation and appropriate for both analytical and numerical study.

There is a problem of identifying the outcome in terms of the basic parameters of the model. This task received a comprehensive solution. A procedure has been formulated and justified, which makes it possible to unambiguously determine the type of outcome according to the model exogenous parameters and the known loss distribution function. It was shown that with sufficiently weak and natural restrictions on the loss distribution function, that the parameter space is divided into 3 non-empty regions in which one of the three possible outcomes is realized: **B** ("Bubble") - there are no bounded solutions (i.e., we get an analogue of the linear model with zero discount); **S** ("Self-Constraned") with limited solutions; and, finally, **A** - autarchy solutions - deposits are not attracted, loans are placed within their own funds as a result, credit expansion does not occur due to unfavorable conditions. In addition, a more subtle identification of compliance with the requirements established by Basel III in the area **S** was carried out.

The influence of exogenous factors on solutions, both analytically and, in particularly complex cases, using computer simulations, has been studied. In all cases, the results of the study of comparative statics are consistent with intuitive expectations.

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A Appendix

Notations and abbreviations

K_t	capital
D_t	deposits
M_t	cash
L_t	loans
r	loan interest rate
R	deposit interest rate
ε	share of nonperforming loans (the portfolio percentage loss)
$PD = \mathbb{E}(\varepsilon)$	probability of default of a borrower
$\tilde{r} = r - (1+r)PD$	loan risk-adjusted interest rate
d	discount of loan nominal value in case of selling of the loan
$k(L_0) = \frac{K_0}{L_0}$	CAR (capital adequacy ratio) – capital/risk weighted assets
$\hat{\mathcal{E}} = \frac{r-R}{1+r}$	the limit threshold for the loan losses
$\mathcal{E}(k) = \hat{\mathcal{E}} + (1-\hat{\mathcal{E}})k$	the threshold for the loan losses
$\mathrm{ret}(z)$	the weighted share of the returned loans
$\mathrm{Ret}(\mathcal{E})$	the expected returns of loans in case of the bank default
$F(x)$	CDF (cumulative density function)
$f(x) = F'(x)$	PDF (probability density function)
$\Phi(z)$	standard normal distribution
FOC	First-Order Condition
SOC	Second-Order Condition
ρ	asset correlation
$U(L_0)$	objective function
r_f	risk-free rate

A.1 Proof of Theorem 1

Proof. Differentiating the function (1.2), we obtain the derivatives

$$U'(L_0) = \hat{\mathcal{E}} - PD - d \cdot \Big(\text{Ret}(\mathcal{E}(k(L_0))) + (1-\hat{\mathcal{E}})k(L_0)\text{ret}(\mathcal{E}(k(L_0))\Big) \tag{A.1}$$

$$U''(L_0) = -d \cdot \left[-\text{ret}(\mathcal{E}(k(L_0))\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}L_0} + (1-\hat{\mathcal{E}})\text{ret}(\mathcal{E}(k(L_0))\frac{\mathrm{d}k}{\mathrm{d}L_0} + (1-\hat{\mathcal{E}})k(L_0)\text{ret}'(\mathcal{E}(k(L_0))\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}L_0}\right] =$$

$$= \frac{d}{L_0} \cdot (1-\hat{\mathcal{E}})^2 k^2(L_0)\text{ret}'(\mathcal{E}(k(L_0)) < 0. \tag{A.2}$$

Furthermore,

$$U'(K_0) = \hat{\mathcal{E}} - PD - d\,(1 - F(1) - PD + \text{Ret}(0)) = \hat{E} - PD > 0,$$

$$\lim_{L_0\to\infty} U'(L_0) = \hat{\mathcal{E}} - PD - d \cdot \text{Ret}(\hat{\mathcal{E}}).$$

Given the SOC $U'' < 0$, we obtain that a necessary and sufficient condition for the existence and uniqueness of the FOC $U'(L_0) = 0$ is the inequality

$$\lim_{L_0\to\infty} U'(L_0) < 0 \iff d > \frac{\hat{\mathcal{E}} - PD}{\text{Ret}(\hat{\mathcal{E}})}.$$

Due to (A.1) we may represent the FOC $U'(L_0) = 0$ as the equation

$$\hat{\mathcal{E}} - PD - d \cdot \Big(\text{Ret}(\mathcal{E}(k)) + \Big(1 - \hat{\mathcal{E}}\Big)\, k \cdot \text{ret}(\mathcal{E}(k))\Big) = 0$$

of variable $k = K_0/L_0$, which sets the correspondence between solution of this equation k^* and the equilibrium size of the loan portfolio L_0^*.

Q.E.D.

A.2 Proof of Lemma 2

Assume first that $\rho < 1/2$ and $PD \le 1/2$, then in this case the PDF (1.8) is unimodal with mode at

$$z_{\text{mode}} = \Phi\left(\frac{\sqrt{1-\rho}}{1-2\rho}\Phi^{-1}(PD)\right),$$

(see, e.g., [20]). Moreover, $\rho < 1/2$ and $PD \le 1/2$ imply

$$\frac{\sqrt{1-\rho}}{1-2\rho} > 1 \Rightarrow \frac{\sqrt{1-\rho}}{1-2\rho}\Phi^{-1}(PD) \le \Phi^{-1}(PD) \Rightarrow z_{\text{mode}} \le \Phi(\Phi^{-1}(PD)) = PD,$$

because $\Phi^{-1}(PD) \le 0$, therefore, $f(z)$ decreases with respect to z, as well as $(1-z)f(z)$ does.

Now assume that $1 > \rho \geq 1/2$ and $PD \leq 1/2$. Substituting $x = \Phi^{-1}(z)$ we obtain the following problem: to prove that the function

$$h(x) = \sqrt{\frac{1-\rho}{\rho}} \exp\left(\frac{1}{2}\left[x^2 - \left(\frac{\sqrt{1-\rho}x - c}{\sqrt{\rho}}\right)^2\right]\right)(1 - \Phi(x)) = \varphi\left(\frac{\sqrt{1-\rho}x - c}{\sqrt{\rho}}\right)\frac{\Phi(-x)}{\varphi(x)}$$

is decreasing with respect to x, where $c = \Phi^{-1}(PD) < 0$, $\varphi(z) = \Phi'(z) > 0$ is the density function of the standard normal distribution satisfying the identity $\varphi'(x) = -x\varphi(x)$. Differentiating $h(x)$ we obtain

$$h'(x) = \frac{\varphi\left(\frac{\sqrt{1-\rho}x-c}{\sqrt{\rho}}\right)}{\varphi(x)}\left[\left(\frac{c\sqrt{1-\rho}}{\rho} + \frac{2\rho - 1}{\rho}x\right)\Phi(-x) - \varphi(x)\right].$$

Assume first that $x \leq 0$. Given $c = \Phi^{-1}(PD) \leq 0$, we obtain

$$\left(\frac{c\sqrt{1-\rho}}{\rho} + \frac{2\rho - 1}{\rho}x\right)\Phi(-x) - \varphi(x) < 0 \Rightarrow h'(x) < 0.$$

Now let $x > 0$, then

$$\left(\frac{c\sqrt{1-\rho}}{\rho} + \frac{2\rho - 1}{\rho}x\right)\Phi(-x) - \varphi(x) < x\Phi(-x) - \varphi(x) < 0,$$

due to

$$\frac{2\rho - 1}{\rho} = 1 - \frac{1-\rho}{\rho} < 1$$

and $\varphi(x) > x\Phi(-x)$ for all $x \geq 0$. Indeed, $\varphi(0) > 0 = 0 \cdot \Phi(-0)$ and for all $x > 0$ the inequality

$$\varphi'(x) = -x\varphi(x) = -x\varphi(-x) > \Phi(-x) - x\varphi(-x) = (x\Phi(-x))'$$

holds, which completes the proof of this case.

Finally, assume that $PD > 1/2$, then $c = \Phi^{-1}(PD) > 0$, and $z > PD$ implies $x = \Phi^{-1}(z) > c > 0$ and, consequently,

$$\frac{c\sqrt{1-\rho}}{\rho} + \frac{2\rho - 1}{\rho}x < \frac{2\rho - 1 + \sqrt{1-\rho}}{\rho}x = x - \frac{\sqrt{1-\rho}(1 - \sqrt{1-\rho})}{\rho}x < x$$

for all $0 < \rho < 1$. The rest of proof is similar. Q.E.D.

A.3 Proof of Proposition 1

To simplify calculations, consider the following substitution of variables $E = \hat{\mathcal{E}} + (1-\hat{\mathcal{E}})k$. Then the FOC (1.5) is equivalent to the equation

$$G(E) \equiv \hat{\mathcal{E}} - PD - d \cdot \left(\text{Ret}(E) + \left(E - \hat{\mathcal{E}}\right)\text{ret}(E)\right) = 0.$$

Let E^* be the solution of this equation, considered as an implicit function of all parameters. The corresponding derivative with respect to an arbitrary parameter a is as follows

$$\frac{\partial E^*}{\partial a} = -\frac{\partial G}{\partial a} \Big/ \frac{\partial G}{\partial E},$$

where

$$\frac{\partial G}{\partial E} = d \cdot \left(E - \hat{\mathcal{E}}\right)\left(f(E) - (1-E)f'(E)\right) = -d \cdot \left(E - \hat{\mathcal{E}}\right)\text{ret}'(E) > 0,$$

because $E > \hat{\mathcal{E}}$ and ret(E) is a decreasing function. Moreover,

$$\frac{\partial G}{\partial d} = -\left(\text{Ret}(E) + \left(E - \hat{\mathcal{E}}\right)\text{ret}(E)\right) < 0,$$

which implies $\frac{\partial E^*}{\partial d} > 0$.

Now let $a = R$, given

$$\hat{\mathcal{E}} = \frac{r-R}{1+r}$$

we obtain

$$\frac{\partial G}{\partial R} = -\frac{1}{1+r} - d \cdot \frac{\text{ret}(E)}{1+r} < 0,$$

which implies $\frac{\partial E^*}{\partial R} > 0$. Furthermore, the inequality

$$\frac{\partial G}{\partial r} = \frac{1+R}{(1+r)^2} + d \cdot \frac{(1+R)\text{ret}(E)}{(1+r)^2} > 0$$

implies $\frac{\partial E^*}{\partial r} < 0$.

Given $E^* = (1-\hat{\mathcal{E}})k^* + \hat{\mathcal{E}}$ and $\hat{\mathcal{E}} = \frac{r-R}{1+r}$, we obtain that

$$k^* = \frac{(1+r)E^* - (r-R)}{1+R}, \; L_0^* = \frac{(1+R)K_0}{(1+r)E^* - (r-R)}, \; D_0^* = L_0^* - K_0,$$

therefore,

$$\frac{\partial k^*}{\partial d} > 0, \frac{\partial L_0^*}{\partial d} < 0, \frac{\partial D_0^*}{\partial d} < 0$$

Moreover,

$$\frac{\partial k^*}{\partial R} = -\frac{1+r}{(1+R)^2}E^* + \frac{1+r}{1+R}\frac{\partial E^*}{\partial R} + \frac{1+r}{(1+R)^2} = \frac{1+r}{(1+R)^2}(1-E^*) + \frac{1+r}{1+R}\frac{\partial E^*}{\partial R} > 0,$$

$$\frac{\partial L_0^*}{\partial R} = \frac{\partial D_0^*}{\partial R} = \frac{K_0\left((1+r)E^* - (r-R)\right) - (1+R)K_0\left((1+r)\frac{\partial E^*}{\partial R}+1\right)}{\left((1+r)E^* - (r-R)\right)^2} =$$
$$= -\frac{(1+r)K_0\left((1-E^*) + (1+R)\frac{\partial E^*}{\partial R}\right)}{\left((1+r)E^* - (r-R)\right)^2} < 0,$$

because $E^* < 1$, $\frac{\partial E^*}{\partial R} > 0$.

Finally,

$$\frac{\partial k^*}{\partial r} = \frac{1}{1+R}\left[(1+r)\frac{\partial E^*}{\partial r} - (1-E^*)\right] < 0$$

$$\frac{\partial L_0^*}{\partial r} = \frac{\partial D_0^*}{\partial r} = -\frac{(1+R)K_0}{\left((1+r)E^* - (r-R)\right)^2}\left[E^* - 1 + (1+r)\frac{\partial E^*}{\partial r}\right] > 0,$$

because of $E^* < 1$ and $\frac{\partial E^*}{\partial r} < 0$. Q.E.D.

A.4 Proof of Proposition 2

The statement about area **A** is obvious. The rest is to show the robustness of shapes of areas **B** and **S**. Note that the function

$$d_S(r) = \frac{\hat{\mathcal{E}}(r) - PD}{\mathrm{Ret}(\hat{\mathcal{E}}(r))},$$

where $\hat{\mathcal{E}}(r) = \frac{r-R}{1+r}$, satisfies the following conditions:

1. $d_S\left(\frac{PD+R}{1-PD}\right) = 0$,

2. $d_S\left(r\right)$ strictly increases for all $r > \frac{PD+R}{1-PD}$.

The first statement is obvious by due to definition of $d_S\left(r\right)$. Then, representing the function $d_N(r)$ as follows

$$d_S(r) = \frac{\tilde{r} - R}{1+r} \cdot \frac{1}{Ret\left(\frac{r-R}{1+r}\right)},$$

and given the functions $\frac{\tilde{r}-R}{1+r}$, $\frac{r-R}{1+r}$ are positive and strictly increasing with respect to r we obtain that the function $d_0(r)$ is also strictly increasing. Finally, the function $d_S\left(r\right)$ is unrestrictedly increasing with $r \to \infty$, because

$$\frac{\tilde{r} - R}{1+r} \to 1 - PD, \ \frac{r-R}{1+r} \to 1.$$

Therefore, its graph intersects the line $d = 1$ in finite point $r_S > \frac{PD+R}{1-PD}$, which determine the base of the curvilinear triangle **S**. Q.E.D.

A.5 Proof of Theorem 2

Consider the probability of the bank's default p as a parameter with possible values from the interval $[0,1]$. Formula (1.9) implies that for any given p, the equation $p = 1 - F(E)$ determines the value $E_p = F^{-1}(1-p)$. This means that the equiprobability curve $\mathcal{I}_p$ is determined by equation

$$\hat{\mathcal{E}}(r) - PD - d \cdot \Big(\mathrm{Ret}(E_p) + \Big(E_p - \hat{\mathcal{E}}(r)\Big)\,\mathrm{ret}(E_p)\Big) = 0,$$

or, equivalently,

$$d = d_p(r) \equiv \frac{\hat{\mathcal{E}}(r) - PD}{\mathrm{Ret}(E_p) + \Big(E_p - \hat{\mathcal{E}}(r)\Big)\,\mathrm{ret}(E_p)}, \tag{A.3}$$

where $\hat{\mathcal{E}}(r) = \frac{r-R}{1+r}$. It is obvious that $d_p\left(\frac{PD+R}{1-PD}\right) = 0$ for all p, which completes the first statement of the theorem. Moreover, $p \to 0$ implies $E_p \to 1$, therefore, $\lim_{p\to 0} d_p(r) = +\infty$ for any $r > \frac{R+PD}{1-PD} \iff \hat{\mathcal{E}}(r) - PD > 0$, which completes the second statement.

Recall that the border of areas **S** and **B** is determined by the function

$$d_S\,(r) = \frac{\hat{\mathcal{E}}(r) - PD}{\mathrm{Ret}(\hat{\mathcal{E}}(r))}.$$

Calculating and comparing the derivatives of $d_S(r)$ and $d_p(r)$ at the starting point $r_0 = \frac{R+PD}{1-PD}$

$$\frac{\mathrm{d}}{\mathrm{d}r} d_S(r_0) = \frac{(1-PD)^2}{(1+R)\mathrm{Ret}(PD)},$$
$$\frac{\mathrm{d}}{\mathrm{d}r} d_p(r_0) = \frac{(1-PD)^2}{(1+R)\left(\mathrm{Ret}(E_p) + (E_p - PD)\mathrm{ret}(E_p)\right)},$$

we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}r} d_p(r_0) > \frac{\mathrm{d}}{\mathrm{d}r} d_S(r_0) \iff \mathrm{Ret}(PD) > \mathrm{Ret}(E_p) + (E_p - PD)\mathrm{ret}(E_p).$$

Consider the function

$$G(x) = \mathrm{Ret}(x) + (x - PD)\mathrm{ret}(x),$$

which obviously satisfies $G(PD) = \mathrm{Ret}(PD)$. Moreover,

$$G'(x) = (x - PD)\mathrm{ret}'(x) < 0$$

for all $x > PD$. This implies that

$$x = E_p = F^{-1}(1-p) > PD \iff p < 1 - F(PD)$$

is necessary and sufficient condition for the curve $\mathcal{I}_p$ to belong the area **S**, at least in some neighborhood of r_0.

Let's determine the point of intersection of the equiprobability curve $\mathcal{I}_p$ with the border of areas **S** and **B** from the following equation

$$d_S(r) = d_P(r) \iff \mathrm{Ret}(\hat{\mathcal{E}}(r)) = \mathrm{Ret}(E_p) + \Big(E_p - \hat{\mathcal{E}}(r)\Big)\,\mathrm{ret}(E_p).$$

The unique solution $r(p)$ of this equation is determined by identity

$$E_p = \hat{\mathcal{E}}(r(p)) \iff r(p) = \frac{E_p + R}{1 - E_p} = \frac{F^{-1}(1-p) + R}{1 - F^{-1}(1-p)} > r_0 = \frac{PD + R}{1 - PD}$$

because $F^{-1}(1-p) > PD$. Note that this point of intersection is actual only in case

$$d_S(r(p)) = d_p(r(p)) \leq 1,$$

otherwise, the equiprobability curve will intersect the line $d = 1$ instead of $d_S(r)$. This happens if and only if

$$d_S(r(p)) > 1 \iff E_p - PD - \mathrm{Ret}(E_p) > 0.$$

Note that the function

$$H(x) = x - PD - \mathrm{Ret}(x)$$

for $x \geq PD$ satisfies the following conditions: $H(PD) < 0$, $H(1) = 1 - PD > 0$, and $H'(x) = 1 + (1-x)f(x) > 0$. This implies that there is $x^* \in (PD, 1)$ such that for all $x > x^*$ the function $H(x) > 0$, which is equivalent to

$$p < 1 - F(x^*) \Rightarrow E_p - PD - \mathrm{Ret}(E_p) > 0.$$

Q.E.D.