Moving Forward vs. Inflicting Costs in a Random-Walk Model of War

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Abstract

How do belligerents choose and change their military strategies during war? How do these strategies shape war? To address these questions, we develop a random-walk model of war, where two belligerents fight over "forts" across periods. The random walk represents a battlefront, which moves as the war evolves, resulting in the occupation of more forts for the winning side and less forts for the losing side. Unlike existing models, ours allows the belligerents to choose an action out of moving forward, inflicting costs, and surrender in every battle. We found that equilibrium strategies are monotonic with respect to the walk—a belligerent will punish its opponent if it is sufficiently advantageous and surrender if it is too disadvantageous. Accordingly, the punishment strategy can function to shorten the war. Moreover, a severer punishment tends to make the war even shorter.

Keywords: gambler's ruin, military strategy, random walk.
JEL classifications: C73, D74, F51.

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1 Introduction

In his masterpiece *On War*, Clausewitz (1832: 90) stipulated that three broad objectives of military engagement are "the *armed forces*, the *country*, and the *enemy's will*."[1] Based on this stipulation, formal models of war can be categorized into three kinds: the combat models initiated by Lanchester (1916), where belligerents exchange attacks on *armed forces* (Bellany 1999; Langlois and Langlois 2009, 2012); the random-walk models, where war is regarded as a territorial contest over *country* (Slantchev 2003b; Smith 1998; Smith and Stam 2003, 2004); the bargaining models, where armed forces are treated as a means of coercion to influence the *enemy's will* (Fearon 2004, 2007; Filson and Werner 2002, 2004; Levetoğlu Slantchev 2007; Powell 2004a, 2004b, 2012; Slantchev 2003a, 2011; Wagner 2000). Despite the development of models of war, there have been few theoretical studies that address the *choice* among two or more of these objectives as the targets of attacks.[2] Built upon the Gambler's Ruin Problem, this article offers a random-walk model of war, where two belligerents choose the targets of attacks between the *country* and the *enemy's will* in each battle denoted by the state variable.[3]

As with other random-walk models of war shown above, the walk of our model represents a battlefront, or the distribution of "forts" between the belligerents, which moves as the war evolves. However, unlike other models, our model incorporates not only the decision to surrender (Smith 1998) but also the decision to punishment for the sake of strategic analysis. That means, our model allows the belligerents to choose an action out of moving forward, punishment, or surrender in each battle. By moving forward, a belligerent can occupy a fort from the opponent with a certain probability. With punishment, it cannot occupy a fort but can inflict a heavier cost on the opponent. By equilibrium analysis, we illuminate how the belligerents choose and change their actions throughout the war and also how their choices shape the war.

This article also aims to contribute to the theoretical literature on military strat-

[1] Among the three objectives, Clausewitz (1832: 99, 229) prioritized the armed forces. Liddell-Hart (1967: 352) notably disagreed.

[2] As exceptions, Snyder (1961) and Intriligator and Brito (1984) illuminate the choice betweeen counterforce and coutervalue attacks for nuclear deterrence. Some other models disallow the choice of targets but pertain to two of the three kinds—depicting shifts of military balance while bargaining (Langlois and Langlois 2009, 2012; Slantchev 2003b; Smith and Stam 2004).

[3] The Gambler's Ruin Problem was first posed by Blaise Pascal (Edward 1983).

egy, which have remained understudied as of today. Recent bargaining models of war have incorporated military strategies such as concealment of strength (Baliga and Sjöström 2008; Meirowitz and Sartori 2008; Slantchev 2010), indirect strategy (Lindsey 2015), and fait accompli (Tarar 2016), but they commonly presume the relative strength between belligerents to be fixed throughout war.[4] Unlike them, our model illuminates the shift of military balance by incorporating the random walk. Although our model abstracts away the bargaining aspect of war, it does allow strategic termination of war in light of developments on the battlefield.

The rest of this article proceeds as follows. Section 2 presents the random-walk models of war, to which the decision to surrender is added in Section 3 and the decision to punishment further added in Section 4. Section 5 concludes.

[4] In the context of nuclear deterrence, some strategies have been formally studied such as the risk strategy (Powell 1987, 1988), limited retaliation (Powell 1989), and counterforce first strike (Wagner 1991). Also in the context of counter-terrorism, the choice between preemption and deterrence has been investigated (Nakao 2019; Sandler and Siqueira 2006).

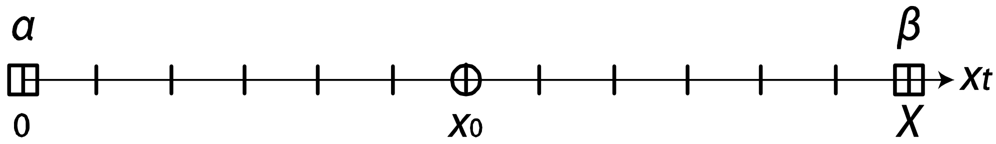

Figure 1: The random-walk model of war.

2 The Random-Walk Model of War

To explore belligerents' choices of military strategies, we develop a random-walk model of war, where the belligerents, while fighting, choose to move their forces forward or to inflict costs on the opponent. Our model is based on the Gambler's Ruin Problem.[5] We begin with the Problem and subsequently add to it the choice to surrender and further the choice to punish the opponent. In doing so, we will examine how the inclusion of these choices influences the war.

2.1 War as Gambler's Ruin

In the model, there are two belligerents (α, β), who fight each other over X "forts" in time periods $t = 0, 1, 2, \cdots$. Let x_t denote the number of forts α occupies in period t:

$$x_t \in \{0, 1, 2, \cdots, X\},$$

which can also be interpreted as α's strategic depth in t. At the war's onset $(t = 0)$, α occupies x_0 forts, whereas β occupies $X - x_0$ forts (Figure 1). As the war evolves, the battlefront x_t moves at random. In each period t, they fight a "battle," which ends in α's win, loss, or draw with respective probabilities p, q, and r such that $p + q + r = 1$. The winner of a battle captures an additional fort from the loser, so that

$$\begin{aligned} p &\equiv \Pr(x_{t+1} = x_t + 1) \\ q &\equiv \Pr(x_{t+1} = x_t - 1) \\ r &\equiv \Pr(x_{t+1} = x_t). \end{aligned}$$

[5]The Gambler's Ruin Problem is delineated as follows. At the beginning, α has x_0 tokens, while β has $X - x_0$ tokens. They play a series of bets. For each bet, α wins with probability p, and β wins with probability $1 - p$. If α wins a bet, α receives one token from β. If β wins a bet, β gets one token from α. The game ends with either player's victory if the other player loses all his tokens.

For simplicity, we assume the match to be even; i.e., $p = q$. The entire war ends when either belligerent loses all its forts; i.e., α loses the war (and β wins it) if $x_t = 0$, and α wins the war if $x_t = X$.

The model above has the following properties:

Lemma 1 *(i) The probability that α wins the war is $\frac{x_0}{X}$. (ii) The expected duration of the war is $\frac{x_0(X-x_0)}{2p}$.*

Proof. The proof is immediate from Epstein (1995) and thus is omitted. ■

3 The Decision to Surrender

We next incorporate to the model the choice to surrender. In the subsequent model, the belligerents are allowed to surrender conditional on x_t. At the war's onset, they determine their stop losses (a, b), or the numbers of forts to lose before surrendering.[6] In making the surrender decisions, they aim to maximize their own continuation payoffs (Π_α, Π_β), which consist of the lump-sum benefit $W > 0$ from winning the war, the lump-sum loss $L > 0$ of losing the war (with $W > L$), and the per-period cost $c > 0$ of fighting. Their optimization problems can be shown as:

$$\begin{aligned} \max_a \quad & W\frac{a}{a+b} - L\frac{b}{a+b} - \frac{abc}{2p} && (1) \\ & \text{s.t. } a \in \{1, 2, \cdots, x_0\} \\ \max_b \quad & W\frac{b}{a+b} - L\frac{a}{a+b} - \frac{abc}{2p}, && (2) \\ & \text{s.t. } b \in \{1, 2, \cdots, X - x_0\} \end{aligned}$$

for which the probability that α wins the war is $\frac{a}{a+b}$, and the expected duration $\frac{ab}{2p}$ (Lemma 1).

Problems (1, 2) are solvable in a closed form if a certain condition is met:

Proposition 1 *Suppose there exists a natural number Ψ such that $\Psi \equiv \left(\frac{2p(W+L)}{c}\right)^{\frac{1}{2}}$. If both x_0 and $X - x_0$ are sufficiently large, any pair (a^*, b^*) forms a Markov perfect equilibrium such that*

$$a^* + b^* = \Psi. \tag{3}$$

[6] For instance, α continues to fight as far as it loses no more than a forts and surrenders if it loses $a + 1$ forts.

Proof. The proof appears in Appendix. ■

Although Proposition 1 suggests multiple equilibria, each of them actually forms an equilibrium of every subgame in a single Markov perfect equilibrium—the players' incentives remain compatible with the equilibrium behavior even when the state variable x_t moves. For instance, even after β wins battles and occupies more forts, they continue to fight as far as the number of forts α lost is less than a^*. The proposition also shows that the equilibrium strategies are monotonic with respect to the state variable x_t—a belligerent's decision depends on a threshold of the number of its forts, above which it fights and below which it surrenders (Smith 1998).

In light of the surrender decision, states are better represented by the number of forts that each belligerent can lose before surrendering rather than by the absolute size of x_t.

Definition 1 *State (a, b) denotes the one where α has a forts to lose before surrendering without punishment, and β has b corresponding forts.*

For instance, as β wins a battle in state (a, b), they move to $(a-1, b+1)$. With additional moves, α will surrender once they reach $(0, a+b)$.

4 The Decision to Punishment

We further introduce to the model the choice of punishment. In the next model, the players have the choice to punish each other conditional on state (a, b). A punishment can inflict a cost c^P larger than c on the opponent, but it cannot win a battle. Accordingly, the probability that the punished party wins a battle doubles if it fights.[7] If both the parties adopt punishments, neither party can win a battle, as in the model of war of attrition (Maynard Smith 1974).[8]

A belligerent is willing to adopt punishment to which the targeted opponent surrenders if punishment is so severe that the target's continuation payoff from fighting falls below the loss of surrendering $-L$:

[7] For instance, if α moves forward while β adopts punishment in (a, b), the probability distribution of battle outcomes is: $2p = \Pr(a+1, b-1)$; $0 = \Pr(a-1, b+1)$; $r = \Pr(a, b)$.

[8] As we rule out mixed strategies, no equilibrium emerges where both the belligerents adopt punishments.

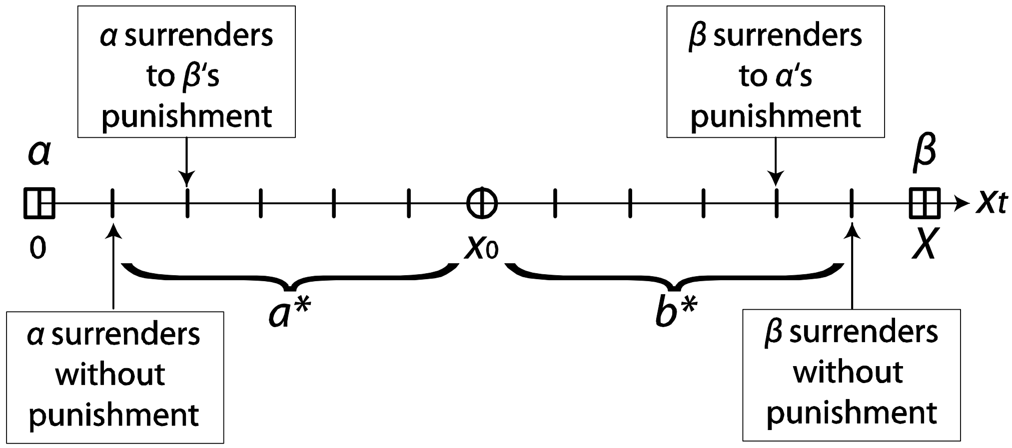

Figure 2: The thresholds of surrender and punishment.

Proposition 2 *In state $(1, \Psi - 1)$, β adopts punishment while α surrenders if the cost of being punished is so large that*

$$c^P > \frac{2p\,(W+L)}{\Psi - 1} - (\Psi - 2)\,c. \tag{4}$$

Symmetrically, in state $(\Psi - 1, 1)$, α adopts punishment while β surrenders if Condition (4) holds.

Proof. The proof appears in Appendix. ■

While Proposition 1 implies that the war can end earlier if surrender is an option—it ends in $(0, \Psi)$ or $(\Psi, 0)$, Proposition 2 further suggests that the war can be even shorter in light of severe punishment—it ends in $(1, \Psi - 1)$ or $(\Psi - 1, 1)$. The disadvantageous party would surrender one-fort earlier if punishment is an option (Figure 2).[9]

The proposition also delineates how the war evolves in light of punishment. In early stages of the war, when both the belligerents retain enough forts in reserve, they fight conventionally to move their forces forward. However, toward the last stage when either side establishes its advantageous position by pushing the battlefront forward, it

[9] For Figure 2, the following parameter values are adopted: $x_0 = 6$, $X = 12$, $W = 200$, $L = 0$, $c = 1$, and $p = \frac{1}{4}$. Then, the symmetric equilibrium without punishment is: $(a^*, b^*) = (5, 5)$ with $\Psi = 10$. A belligerent would surrender with at least one fort more (i.e., earlier) in light of punishment with $c^P > \frac{28}{9}$.

will introduce the punishment strategy to coerce its opponent into capitulation. This pattern could be found at least in some past wars, where the prevailing side with advanced military technology and affluent resources exerted punishment strategies toward the end (e.g., Pacific War, Vietnam War).[10]

More generally, in any state (a^*, b^*) with $a^* + b^* = \Psi$, the war can end with the prevailing side's punishment if it is severe enough:

Corollary 1 *In state* $(k, \Psi - k)$ *with* $k \in \{1, 2, \cdots, a^* - 1\}$, β *adopts a punishment while* α *surrenders if the cost of being punished is so large that*

$$c^P > \frac{2p\,(W + L)}{(\Psi - k)} - (\Psi - (k + 1))\,c. \tag{5}$$

Symmetrically, in state $(\Psi - k', k')$ *with* $k' \in \{1, 2, \cdots, b^* - 1\}$, α *adopts punishment while* β *surrenders if Condition (5) holds.*

Proof. The proof can be derived in a way similar to that of Proposition 2 and thus is omitted. ■

Corollary 1 holds the monotonicity of the equilibrium strategies in terms of punishment; as with surrender (Proposition 1), there exists a certain threshold of forts occupied, above which a belligerent punishes, and below which it moves forward. To summarize, the equilibrium strategies are monotonic in terms of all punishment, conventional fight, and surrender—a belligerent adopts punishment when it occupies enough forts to prevail, fight conventionally to move forward when it lies midway between victory and defeat, and surrenders when it retains few forts to spare. In short, the belligerents hinge their actions on their strategic positions, or on how advantageous they waged the war.

Moreover, the timing of punishment can also depend on its severity:

Proposition 3 *If the cost* c^P *of being punished is greater, a punishment tends to be adopted earlier, and the war tends to end sooner.*

[10] Both the Pacific and Vietnamese Wars involved punishments toward their ends. The Pacific War ended with the drops of two atomic bombs on Hiroshima and Nagasaki before the execution of the Operation Downfall, or the U.S. campaign to invade the mainland of Japan. The Vietnamese War was settled shortly after Linebacker II, which aimed to produce psychological impacts on the Northern leaders by inflicting the utmost civilian distress (Clodfelter 1989: 182-184).

Proof. Let $\overline{c}(k)$ denote the threshold of Condition (5):

$$\overline{c}(k) \equiv \frac{2p(W+L)}{(\Psi - k)} - (\Psi - (k+1))\,c,$$

which increases with k, or

$$\frac{d\overline{c}(k)}{dk} = \frac{2p(W+L)}{(\Psi - k)^2} + c > 0.$$

This implies that for α to surrender in state $(k, \Psi - k)$ with a larger k, a larger c^P is needed. ■

Proposition 3 implies that the timing of punishment depends on its severity, which in turn influences the duration of war. If one retains a more powerful punitive measure, it might not need to occupy many forts to initiate punishment. In other words, the proposition suggests a tradeoff between the occupation of forts and the severity of punishment. A harder punisher could end the war sooner.

5 Conclusion

Despite the growing theoretical literature on the process of war especially since Smith (1998) and Wagner (2000), there have been limited theoretical studies on how military strategies are adopted during war (Intriligator and Brito 1984; Lindsey 2015; Powell 1987, 1988, 1989; Wagner 1991; Tarar 2016). We have conducted the first theoretical attempt to incorporate military strategies into a random-walk model of war, where two belligerents choose to move their forces forward, punish each other, or surrender in each battle. Unlike bargaining models of war, which commonly regard armed forces as a means of coercion (Fearon 2004, 2007; Filson and Werner 2002, 2004; Leventoğlu Slantchev 2007; Powell 2004a, 2004b, 2012; Slantchev 2003a, 2011; Wagner 2000), our model delineates both physical and psychological elements of armed forces—one to seize enemy "forts" in a territorial contest, and the other to compel the opponent to capitulation through inflicting pain.

Our theory predicts some patterns in the choices of military strategies. At the early stage of war, when the distribution of forts is more or less equal between the two belligerents, they both aim at occupying more forts to produce military imbalance in their favor, while refraining from the punishment strategy. Because punishment entails the loss of opportunities to seize more forts, punishment could put the punisher himself in a disadvantageous position unless it could bring about the opponent's surrender. In contrast, toward the end of war, when either belligerent overpowers the other in terms of the distribution of forts, the prevailing side might resort to punishment, to which the prevailed side is expected to give in. With enough severity, punishment could reduce the cost of prosecuting a war by shortening it. These patterns confirm the monotonicity of equilibrium strategies with respect to the state variable (Smith 1998)—punishment is employed when a belligerent is sufficiently advantageous and surrender is chosen when he is too disadvantageous.

The Pacific War comports with our theoretical predictions. During the War, the U.S. initiated the "leapfrogging" strategy in 1943 to capture successive islands with high strategic values from the South Sea to the Japanese mainland. In the summer of 1944 when Japan lost the Mariana Islands (the Absolute Zone of National Defense), the U.S. military victory of the War became a matter of time (Alperovitz 1995: ch. 2). Shortly after the Battle of Iwo Jima, which was critical to its strategic air campaigns, the U.S. introduced a series of punishment strategies in 1945, including naval blockades of major ports, incendiary bombings on cities, and drops of two atomic bombs, to which Japan finally surrendered (Asada 1998).

Our theory suggests that the coerciveness of punishment depends on the relative military balance (i.e., the distribution of forts). Therefore, punishment can have a greater coercive effect if the punisher occupies more forts, but even without occupying many forts, punishment can end the war if it can inflict a large cost by itself. In this regard, a war could be shorter if punishments are severer; in other words, the duration of war would be negatively associated with the severity of punishment. Ultimately, in light of punishment with extreme severity, a war could not occur in the first place, as the nuclear peace theory claims (Waltz 1981).

To conclude, we have explored the choice of military strategies during war and found that the choice depends on the relative military balance—as measured by the distribution of forts in our model—which is determined by the past battle outcomes. By contrast, the strategies themselves can also affect how the war will further evolve (Arreguin-Toft 2011; Bennett and Stam 1996; Horowitz and Reiter 2001; Mearsheimer 1983; Pape 1996; Reiter 1999; Reiter and Stam 1998, 2002; Stam 1996; Toft and Zhukov 2012). Therefore, the analysis of military strategy demands the consideration of the mutual influence between military strategy and war. For instance, the seeming association between the punishment strategy and the punisher's victory, as found in our model, does not necessarily guarantee that punishment can bring about a victory (Reiter and Stam 1998, 2002; Stam 1996). Opposingly, it might be the military imbalance that induces the prevailing side to adopt punishment (Downes 2008: ch. 2). This simultaneity problem makes the study of military strategy a difficult but promising agenda for further research.

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APPENDIX

Proof of Proposition 1. Problems (1, 2) give the identical first-order condition, or Equation (3), implying multiple equilibria. To ensure the existence of them, we examine α's incentive in each state. Because the game is symmetric with respect to the players, α's incentive compatibility suffices β's.

In state $(1, \Psi-1)$, α's continuation payoff from fighting is:

$$\begin{aligned}\Pi_\alpha(1,\Psi-1) &= \frac{W-(\Psi-1)L}{\Psi}-(\Psi-1)c \\ &= c-L>-L,\end{aligned}$$

guaranteeing that α is willing to fight in state $(1, \Psi-1)$, which corresponds to α's least advantageous state on the equilibrium path. In addition, α's continuation payoff from fighting increases as a rises and b falls while $a+b=\Psi$ kept as a constant:[11]

$$\begin{aligned}\frac{d\Pi_\alpha(a,\Psi-a)}{da} &= \frac{W+L}{\Psi}-\frac{\Psi-2a}{2p}c \\ &= \frac{ac}{p}>0,\end{aligned}$$

which guarantees that α is willing to fight as it wins more forts. In contrast, α is unwilling to fight when it loses one more additional fort than what the equilibrium specifies, or when it is in state $(1, \Psi)$, because

$$\begin{aligned}\Pi_\alpha(1,\Psi) &= \frac{W-L\Psi}{\Psi+1}-\frac{\Psi}{2p}c \\ &= \frac{W+L}{\left(\frac{2p(W+L)}{c}\right)^{\frac{1}{2}}+1}-L-\frac{W+L}{\left(\frac{2p(W+L)}{c}\right)^{\frac{1}{2}}}<-L.\end{aligned}$$

Moreover, α remains unwilling to fight as it loses even more forts, or

$$\frac{d\Pi_\alpha(1,\Psi)}{d\Psi}=-\frac{W+L}{(\Psi+1)^2}-\frac{c}{2p}<0.$$

To conclude, given β's strategy b, α is willing to fight until it loses $\Psi-b$ forts and surrenders if it loses more than $\Psi-b$. ■

[11] This condition differs form the first-order condition in that for the latter, b (instead of Ψ) is kept as a constant.

Proof of Proposition 2. We examine α's incentive to surrender in state $(1, \Psi - 1)$ when β adopts punishment.

Being punished when it has only one fort, α's continuation payoff in $(1, \Psi - 1)$ is:

$$\Pi_{\alpha}^{P-1}(1, \Psi - 1) = \Pi_{\alpha}^{P-1}(2, \Psi - 2) - \frac{c^P}{2p}, \tag{6}$$

for which α can go to $(2, \Psi - 2)$ for sure at the expected cost $\frac{c^P}{2p}$ of being punished, because β cannot win a battle with punishment.

As α wins a fort in $(1, \Psi - 1)$, α's continuation payoff in $(2, \Psi - 2)$ is:

$$\begin{aligned} \Pi_{\alpha}^{P-1}(2, \Psi - 2) &= \frac{1}{\Psi - 1} W + \left(\frac{\Psi - 2}{\Psi - 1}\right)\left(\Pi_{\alpha}(2, \Psi - 2) - \frac{c^P}{2p}\right) - \frac{(\Psi - 2)\, c}{2p} \\ &= W - \frac{(\Psi - 2)\, c^P}{2p} - \frac{(\Psi - 1)(\Psi - 2)}{2p} c, \end{aligned} \tag{7}$$

for which a single loss results in $(1, \Psi - 1)$, and α has $\Psi - 2$ more forts to occupy to win the war (Lemma 1).

By plugging Equation (7) into (6), the condition for α's surrender can be derived; α yields to β's punishment in $(1, \Psi - 1)$ if $\Pi_{\alpha}^{P-1}(1, \Psi - 1) < -L$, or if

$$\begin{aligned} \Pi_{\alpha}^{P-1}(1, \Psi - 1) &= \left(W - \frac{(\Psi - 2)\, c^P}{2p} - \frac{(\Psi - 1)(\Psi - 2)}{2p} c\right) - \frac{c^P}{2p} \\ &= W - \frac{(\Psi - 1)\, c^P}{2p} - \frac{(\Psi - 1)(\Psi - 2)}{2p} c < -L, \end{aligned}$$

which is equivalent to Inequality (4). ■