The geometry of consistent majoritarian judgement aggregation

Pivato, Marcus

Department of Mathematics, Trent University

16 July 2008

The geometry of consistent majoritarian judgement aggregation

Marcus Pivato
Department of Mathematics, Trent University, Canada

July 16, 2008

Abstract

Given a set of propositions with unknown truth values, a 'judgement aggregation rule' is a way to aggregate the personal truth-valuations of a set of jurors into some 'collective' truth valuation. We introduce the class of 'quasimajoritarian' judgement aggregation rules, which includes majority vote, but also includes some rules which use different weighted voting schemes to decide the truth of different propositions. We show that if the profile of jurors' beliefs satisfies a condition called 'value restriction', then the output of any quasimajoritarian rule is logically consistent; this directly generalizes the recent work of Dietrich and List (2007). We then provide two sufficient conditions for value-restriction, defined geometrically in terms of a lattice ordering or an ultrametric structure on the set of jurors and propositions. Finally, we introduce another sufficient condition for consistent majoritarian judgement aggregation, called 'convexity'. We show that convexity is not logically related to value-restriction.

Let $\mathcal{P}$ be a finite set of propositions and let $\mathcal{J}$ be a finite jury. For all $j \in \mathcal{J}$, let $\mathcal{P}_j \subset \mathcal{P}$ be j's ***judgement set***: the set of propositions which j believes are true. Assume $\mathcal{P}_j$ is logically consistent, for each $j \in \mathcal{J}$. The list $\mathfrak{P} := (\mathcal{P}_j)_{j\in\mathcal{J}}$ is called a ***judgement profile***. A ***judgement aggregation rule*** is a function R which converts any judgement profile $\mathfrak{P}$ into an aggregate judgement set $R(\mathfrak{P}) \subset \mathcal{P}$; heuristically, $R(\mathfrak{P})$ is the set of propositions which are judged to be 'true' by the jury $\mathcal{J}$ as a whole.

For example, the ***simple majoritarian*** rule R_{maj} works as follows: For all $p \in \mathcal{P}$, let $\mathcal{J}_p := \{j \in \mathcal{J} \; ; \; p \in \mathcal{P}_j\}$. Then define $R_{\text{maj}}(\mathfrak{P}) := \{p \in \mathcal{P} \; ; \; |\mathcal{J}_p| > |\mathcal{J}|/2\}$. The problem is that $R_{\text{maj}}(\mathfrak{P})$ may be inconsistent; this phenomenon was called the *Doctrinal Paradox* by Kornhauser and Sager (1986, 1993) in the context of jurisprudence. List and Pettit (2002) called this phenomenon the *Discursive Dilemma*, and showed that it is inevitable using any 'reasonable' judgement aggregation rule (not just R_{maj}). Since then, the Dilemma has been the subject of intense investigation; see List and Puppe (2007) for a survey.

Dietrich and List (2007; Proposition 16) have shown that if the profile $\mathfrak{P}$ satisfies a structural condition called *value restriction*, then $R_{\text{maj}}(\mathfrak{P})$ will be consistent. Value restriction is a somewhat abstract property without any clear social or epistemological interpretation, but Dietrich and List also provide several geometrically appealing sufficient conditions for value restriction, which involve some linear ordering of the elements of $\mathcal{J}$ and/or $\mathcal{P}$; these include the sufficient condition of 'unidimensional alignment' earlier proposed by List (2003, 2006). These conditions can be plausibly interpreted as arranging jurors and/or propositions along some 'ideological continuum' (e.g. from 'liberal' to 'conservative'; from 'religious fundamentalist' to 'scientific rationalist', etc.). However, there are many judgement aggregation problems where such a one-dimensional ordering of jurors and/or propositions may not be possible. We will generalize the conditions of Dietrich and List to a much broader class of geometric realizations.[1]

In §1 we show that value-restriction guarantees logical consistency using any *quasi-majoritarian* judgement aggregation rule; this is a somewhat broader class than the rule R_{maj} considered in Dietrich and List (2007). In §2, we introduce *diamond* profiles, which involve arranging $\mathcal{J}$ into a *lattice* (a partially ordered set with 'meet' and 'join' operators); this generalizes the *unidimensional order* condition of Dietrich and List (2007). We show that any diamond profile is value-restricted. In §3 we show that any *ultrametric profile* is value-restricted; ultrametric profiles are defined by geometrizing $\mathcal{J}$ and $\mathcal{P}$ in terms of an ultrametric space. In §4, we introduce define the class of *convex profiles*, by embedding $\mathcal{J}$ in a vector space $\mathcal{V}$ and identifying $\mathcal{P}$ with convex subsets of $\mathcal{V}$; this again generalizes the 'unidimensional order' condition of Dietrich and List (2007). We show that R_{maj} is consistent on any convex profile; however, convex profiles are *not* necessarily value-restricted, so this result falls outside the scope of the theory developed by Dietrich and List (2007).

1 Value Restriction and Quasimajoritarianism

Let $\mathcal{P}$ be a set of propositions. A *logic* on $\mathcal{P}$ is a collection $\mathfrak{Y}$ of nonsingleton finite subsets of $\mathcal{P}$, called *minimal inconsistent sets*. A subset of $\mathcal{P}$ is *inconsistent* if it contains some element of $\mathfrak{Y}$, and *consistent* if it doesn't.[2] The set $\mathcal{P}$ is *symmetric* if, for every $p \in \mathcal{P}$, its negation $\neg p$ is also in $\mathcal{P}$ (note that we identify $\neg\neg p$ with p). In this case, a judgement set $\mathcal{P}_j \subset \mathcal{P}$ is called *logically complete* if, for every $p \in \mathcal{P}$, either $p \in \mathcal{P}_j$ or $\neg p \in \mathcal{P}_j$. We do not assume that either the individual or collective judgement sets are complete. (Thus, individual jurors and the whole jury can 'abstain from judgement' on some propositions.)

A profile $\mathfrak{P} := (\mathcal{P}_j)_{j\in\mathcal{J}}$ is *value-restricted* if, for any $\mathcal{Y} \in \mathfrak{Y}$, there exist $y_1, y_2 \in \mathcal{Y}$ such that $\mathcal{J}_{y_1} \cap \mathcal{J}_{y_2} = \emptyset$. (That is: for all $j \in \mathcal{J}$, either $y_1 \notin \mathcal{P}_j$ or $y_2 \notin \mathcal{P}_j$).

A *voting rule* is a collection $\mathfrak{R}$ of subsets of $\mathcal{J}$ such that, if $\mathcal{R} \in \mathfrak{R}$ and $\mathcal{R} \subseteq \mathcal{R}'$, then $\mathcal{R}' \in \mathfrak{R}$ also. An element $\mathcal{R} \in \mathfrak{R}$ is a *ruling coalition*; for example, if $\mathcal{J}_p$ is the set of jurors supporting a proposal p, and $\mathcal{J}_p \in \mathfrak{R}$, then p is approved by the jury.

$\mathfrak{R}$ is called *supermajoritarian* if for all $\mathcal{R} \in \mathfrak{R}$, we have $|\mathcal{R}| > |\mathcal{J}|/2$. For example, the *simple majoritarian* rule $\mathfrak{R}_{\mathrm{maj}} := \{\mathcal{R} \subseteq \mathcal{J}\ ;\ |\mathcal{R}| > |\mathcal{J}|/2\}$ is supermajoritarian. If $j \in \mathcal{J}$, we say that j has a *veto* in $\mathfrak{R}$ if $j \in \mathcal{R}$ for all $\mathcal{R} \in \mathfrak{R}$.

[1]A different 'geometric' approach to judgement aggregation has recently been introduced by Eckert and Klamler (2008). Our model is unrelated to their work.

[2]Presumably the elements of $\mathcal{P}$ are embedded in some logico-deductive framework —e.g. predicate calculus —and $\mathfrak{Y}$ is defined using this framework. However, the actual manner in which $\mathfrak{Y}$ is defined is unimportant to us.

A *propositionwise judgement aggregation rule* (or *rule* for short) for a set of propositions $\mathcal{P}$ is a collection $R = \{\mathfrak{R}_p\}_{p \in \mathcal{P}}$, where $\mathfrak{R}_p$ is a voting rule for each $p \in \mathcal{P}$. If $\mathfrak{P}$ is a judgement profile on $\mathcal{P}$, then we define $R[\mathfrak{P}] := \{p \in \mathcal{P} \ ; \ \mathcal{J}_p \in \mathfrak{R}_p\}$ —the set of all propositions p which are approved by a ruling coalition (according to the voting rule $\mathfrak{R}_p$ specific to p). We say R is *supermajoritarian* if each $\mathfrak{R}_p$ is supermajoritarian. We say R is *quasimajoritarian* if, for any $p_1, p_2 \in \mathcal{P}$ and any $\mathcal{R}_1 \in \mathfrak{R}_{p_1}$ and $\mathcal{R}_2 \in \mathfrak{R}_{p_2}$, we have $\mathcal{R}_1 \cap \mathcal{R}_2 \neq \emptyset$.

Example 1.1: (a) The *simple majoritarian* aggregation rule R_{maj} simply sets $\mathfrak{R}_p := \mathfrak{R}_{\text{maj}}$ for all $p \in \mathcal{P}$; this rule is supermajoritarian.

(b) Any supermajoritarian rule is quasimajoritarian, because if $|\mathcal{R}_1| > |\mathcal{J}|/2$ and $|\mathcal{R}_2| > |\mathcal{J}|/2$, then $\mathcal{R}_1 \cap \mathcal{R}_2 \neq \emptyset$ by the pigeonhole principle.

(c) Suppose j has a veto in $\mathfrak{R}_p$ for every $p \in \mathcal{P}$; then R is quasimajoritarian. (In this case we say j has a *global veto*).

(d) Let $N \geq 2$, and let $\mathcal{J}_1, \ldots, \mathcal{J}_N \subseteq \mathcal{J}$ be subsets such that $\mathcal{J}_n \cap \mathcal{J}_m \neq \emptyset$ for all $n, m \in [1...N]$. Suppose that, for each $p \in \mathcal{P}$, there is some $n = n(p) \in [1...N]$ such that the rule $\mathfrak{R}_p$ requires unanimous approval of all members of $\mathcal{J}_n$. Then R is quasimajoritarian. Note that R is not necessarily supermajoritarian (because $\mathcal{J}_1, \ldots, \mathcal{J}_N$ need not be majorities). Also, there might be no juror with a global veto (if $\mathcal{J}_1 \cap \cdots \cap \mathcal{J}_N = \emptyset$).

(e) Let $\mathcal{J} := \{1, 2, 3, 4, 5\}$, let $\mathcal{P} := \{p_1, p_2\}$, let $\mathfrak{R}_{p_1} := \{\mathcal{R} \subseteq \mathcal{J} \ ; \ |\mathcal{R}| \geq 4\}$ and let $\mathfrak{R}_{p_2} := \{\mathcal{R} \subseteq \{1, 2, 3\} \ ; \ |\mathcal{R}| \geq 2\}$. Then R is quasimajoritarian. However, rule $\mathfrak{R}_{p_2}$ is not supermajoritarian, and no one has a veto anywhere. $\diamondsuit$

Dietrich and List (2007, Theorem 1) show that R_{maj} is the only rule which is anonymous, 'neutral' about the acceptance/rejection of each proposal in $\mathcal{P}$, and which produces consistent outcomes on at least some restricted classes of judgement profiles. However, in some cases, we might reject anonymity (e.g. if certain jurors have special 'expertise' about certain propositions). In other cases, we might reject neutrality (e.g. in criminal law, the defendant is 'presumed innocent' until proven guilty; in medicine, the 'precautionary principle' says that a drug or treatment should be regarded as unsafe until it is proven safe). Thus, it is sometimes appropriate to consider aggregation rules other than R_{maj}. Therefore, our first result extends the proof of Proposition 16 in Dietrich and List (2007) to the class of quasimajoritarian rules.

Proposition 1.2 *If $\mathfrak{P}$ is value-restricted, and R is a quasimajoritarian aggregation rule, then the judgement set $R(\mathfrak{P})$ is logically consistent.*

Proof: (by contradiction) Suppose $R(\mathfrak{P})$ was logically inconsistent, and find $\mathcal{Y} \subset R(\mathfrak{P})$ with $\mathcal{Y} \in \mathfrak{Y}$. Then there exist $y_1, y_2 \in \mathcal{Y}$ such that $\mathcal{J}_{y_1} \cap \mathcal{J}_{y_2} = \emptyset$ (because $\mathfrak{P}$ is value-restricted). But $\mathcal{J}_{y_1} \in \mathfrak{R}_{y_1}$ and $\mathcal{J}_{y_2} \in \mathfrak{R}_{y_2}$; hence $\mathcal{J}_{y_1} \cap \mathcal{J}_{y_2} \neq \emptyset$ because R is quasimajoritarian. Contradiction. $\square$

2 Diamond profiles

A *lattice* is a partially ordered set $(\mathcal{L}, \preceq)$ such that, for any $\ell_1, \ell_2 \in \mathcal{L}$:

- There is a unique element $\ell_1 \vee \ell_2$ in $\mathcal{L}$ (the *join* of ℓ_1 and ℓ_2) such that, for all $\ell \in \mathcal{L}$, $[\ell_1 \preceq \ell \text{ and } \ell_2 \preceq \ell] \iff [(\ell_1 \vee \ell_2) \preceq \ell]$.
- There is a unique element $\ell_1 \wedge \ell_2$ in $\mathcal{L}$ (the *meet* of ℓ_1 and ℓ_2) such that, for all $\ell \in \mathcal{L}$, $[\ell_1 \succeq \ell \text{ and } \ell_2 \succeq \ell] \iff [(\ell_1 \wedge \ell_2) \succeq \ell]$.

If $\ell_1 \preceq \ell_2$ then the *diamond* between ℓ_1 and ℓ_2 is the set $[\ell_1, \ell_2] := \{\ell \in \mathcal{L} \ ; \ \ell_1 \preceq \ell \preceq \ell_2\}$. The operations $\wedge$ and $\vee$ are commutative and associative. Thus, if $\mathcal{Y} = \{y_1, y_2, \ldots, y_N\} \subseteq \mathcal{L}$ is any finite subset, we can define $\bigwedge \mathcal{Y} := y_1 \wedge y_2 \wedge \cdots \wedge y_N$ and $\bigvee \mathcal{Y} := y_1 \vee y_2 \vee \cdots \vee y_N$. If $\mathcal{L}$ is finite, then it has a global maximum $\bigvee \mathcal{L}$ and global minimum $\bigwedge \mathcal{L}$.

Example 2.1: (a) Let $\mathcal{L}$ be any finite, totally ordered set (e.g. a finite subset of $\mathbb{R}$). If $\ell_1, \ell_2 \in \mathcal{L}$ then $\ell_1 \vee \ell_2 = \max\{\ell_1, \ell_2\}$ and $\ell_1 \wedge \ell_2 = \min\{\ell_1, \ell_2\}$.

(b) Let $\mathcal{S}$ be either (i) a set, or some 'mathematical structure' such as (ii) a topological space, or (iii) a measure space, or an algebraic structure such as a (iv) group, (v) ring, (vi) module, (vii) vector space, (viii) convex set, etc. Let $\mathcal{L}$ be either (i) the set of all subsets of $\mathcal{S}$, or the set of suitable 'substructures' of $\mathcal{S}$, such as (ii) all open sets, *or* all closed sets, or (iii) all measurable sets or (iv) all subgroups, (v) subrings, (vi) submodules, (vii) linear subspaces, (viii) convex subsets of $\mathcal{S}$, etc. For all $\ell_1, \ell_2 \in \mathcal{L}$, let $(\ell_1 \preceq \ell_2) \iff (\ell_1 \subseteq \ell_2)$, and $\ell_1 \wedge \ell_2 := \ell_1 \cap \ell_2$. In cases (i,ii,iii), let $\ell_1 \vee \ell_2 := \ell_1 \cup \ell_2$. Otherwise, let $\ell_1 \vee \ell_2$ be the smallest substructure of $\mathcal{L}$ containing $\ell_1 \cup \ell_2$ (e.g. (iv) the 'subgroup generated by', or (vii) the 'subspace spanned by', or the (viii) 'convex hull of' $\ell_1 \cup \ell_2$, etc.).

(c) Let $\mathcal{S} = \mathbb{N}$, with $(\ell_1 \preceq \ell_2) \iff (\ell_1 \text{ divides } \ell_2)$. Then $\ell_1 \vee \ell_2 = \mathrm{lcm}(\ell_1, \ell_2)$, and $\ell_1 \wedge \ell_2 = \gcd(\ell_1, \ell_2)$.

(d) Let $\mathcal{L} \subseteq \mathbb{R}^N$ for some $N \in \mathbb{N}$. For any $\mathbf{q}, \mathbf{r} \in \mathcal{L}$, let $(\mathbf{q} \preceq \mathbf{r}) \iff (q_n \leq r_n$ for all $n \in [1...N])$. Let $\mathbf{q} \vee \mathbf{r} := \mathbf{s}$, where $s_n := \max\{q_n, r_n\}$ for all $n \in [1...N]$. Let $\mathbf{q} \wedge \mathbf{r} := \mathbf{t}$, where $t_n := \min\{q_n, r_n\}$ for all $n \in [1...N]$.

For example, suppose there are N independent 'ideological dimensions' (e.g. socialist vs. laissez-faire; social liberal vs. social conservative; pacifist vs. militarist; cosmopolitan vs. nationalist; rehabilitationist vs. punitivist, etc.) corresponding to various aspects of what is usually called the 'left vs. right' ideological continuum. Then each point in $\mathcal{L} \subset \mathbb{R}^N$ could represents a person who is assigned a position on each axis. Thus, $\mathbf{q} \preceq \mathbf{r}$ if $\mathbf{r}$ is 'more right-wing' in every ideological dimension than $\mathbf{q}$ is. $\diamondsuit$

The profile $\mathfrak{P}$ is *diamond* if $\mathcal{J}$ is a lattice, and for all $p \in \mathcal{P}$, the set $\mathcal{J}_p$ is a diamond in $\mathcal{J}$. If $\mathcal{J} \subset \mathbb{R}$ [Example 2.1(a)], then this is equivalent to the condition of *unidimensional order* from Dietrich and List (2007).

Example 2.2: Suppose the jurors in $\mathcal{J}$ can be partially ordered along some 'ideological continuum' [e.g. as in Example 2.1(d)], such that $(\mathcal{J}, \preceq)$ forms a lattice. Each $p \in \mathcal{P}$ can also be located somewhere in this ideological continuum; thus p is only acceptable to jurors in some 'ideological range', which we assume is a diamond. For example, if $\underline{j} := \bigwedge \mathcal{J}$ and $\overline{j} := \bigvee \mathcal{J}$, and p was an 'extreme left-wing' (respectively, right-wing) proposition, then we would have $\mathcal{J}_p = [\underline{j}, j]$ (resp. $\mathcal{J}_p = [j, \overline{j}]$) for some $j \in \mathcal{J}$. If p was a 'centrist' proposition, then $\mathcal{J}_p = [j_1, j_2]$ for some $j_1 \preceq j_2 \in \mathcal{J}$. $\diamondsuit$

Proposition 2.3 *If $\mathfrak{P}$ is diamond, then $\mathfrak{P}$ is value-restricted; hence the outcome of any quasimajoritarian judgement rule is consistent.*

Proof: [based on Prop. 17 of Dietrich and List (2007)]

Let $\mathcal{Y} \in \mathfrak{Y}$. Suppose (for a contradiction) that $\mathcal{Y}$ violated value-restriction. That is:

$$\forall\, y_1, y_2 \in \mathcal{Y}, \quad \mathcal{J}_{y_1} \cap \mathcal{J}_{y_2} \quad \neq \quad \emptyset. \tag{1}$$

For any $y \in \mathcal{Y}$, there are $\underline{j}_y \preceq \overline{j}_y$ in $\mathcal{J}$ such that $\mathcal{J}_y = [\underline{j}_y, \overline{j}_y]$ (because $\mathfrak{P}$ is diamond). Let $\underline{j} := \bigvee\{\underline{j}_y\,;\ y \in \mathcal{Y}\}$ and $\overline{j} := \bigwedge\{\overline{j}_y\,;\ y \in \mathcal{Y}\}$ (well-defined because $\mathcal{Y}$ is finite). For all $y_1, y_2 \in \mathcal{Y}$, eqn.(1) yields some $j \in \mathcal{J}_{y_1} \cap \mathcal{J}_{y_2}$; thus, $\underline{j}_{y_1} \preceq j \preceq \overline{j}_{y_2}$, so $\underline{j}_{y_1} \preceq \overline{j}_{y_2}$. Thus, for all $y \in \mathcal{Y}$, we have $\underline{j}_y \preceq \overline{j}$. Thus, $\underline{j} \preceq \overline{j}$. Thus, $[\underline{j}, \overline{j}] \neq \emptyset$. Let $j \in [\underline{j}, \overline{j}]$. Then for any $y \in \mathcal{Y}$, we have $\underline{j}_y \preceq \underline{j} \preceq j \preceq \overline{j} \preceq \overline{j}_y$; hence $j \in [\underline{j}_y, \overline{j}_y] = \mathcal{J}_y$; hence $y \in \mathcal{P}_j$. Thus, $\mathcal{Y} \subseteq \mathcal{P}_j$. But $\mathcal{Y} \in \mathfrak{Y}$, while $\mathcal{P}_j$ is consistent. Contradiction. $\Box$

3 Ultrametric profiles

Let $\mathcal{X}$ be a set, and let d be an *ultrametric* on $\mathcal{X}$. That is, $d : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ is a function such that:

- For all $x \in \mathcal{X}$, $d(x, x) = 0$ and $d(x, y) > 0$ for all $y \neq x$.
- For all $x, y \in \mathcal{X}$, $d(x, y) = d(y, x)$.
- For all $x, y, z \in \mathcal{X}$, $d(x, z) \leq \max\{d(x, y),\ d(y, z)\}$.

(It is the third property —a 'strong' form of the triangle inequality —which puts the 'ultra' in 'ultrametric'). If $x \in \mathcal{X}$, and $r > 0$, let $\mathcal{B}(x, r) := \{y \in \mathcal{X}\ ;\ d(x, y) \leq r\}$ be the closed ball of radius r around x.

Example 3.1: (a) Let $\mathcal{A}$ be a finite set, and let $\mathcal{A}^{\mathbb{N}}$ be the set of all infinite sequences $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ where $a_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. For any $\mathbf{a}, \mathbf{b} \in \mathcal{A}^{\mathbb{N}}$, let $\Delta(\mathbf{a}, \mathbf{b}) := \min\{n \in \mathbb{N}\ ;\ a_n \neq b_n\}$. Define ultrametric $d : \mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} \longrightarrow [0, \infty)$ by $d(\mathbf{a}, \mathbf{b}) := 1/\Delta(\mathbf{a}, \mathbf{b})$. In this case, $\mathcal{B}(\mathbf{a}, r) := \left\{\mathbf{b} \in \mathcal{A}^{\mathbb{N}}\ ;\ b_n = a_n, \forall n \leq 1/r\right\}$.

Intuitively, if $\mathcal{A}$ is an 'alphabet', then a sequence $\mathbf{a} \in \mathcal{A}^{\mathbb{N}}$ is a 'text'. The ultrametric d says that two texts are 'close' if they agree on a long initial segment.

(b) Again let $\mathcal{A}$ be a finite set, let $\mathbb{M}$ be any 'indexing' set, and let $\mathcal{A}^{\mathbb{M}}$ be the set of all functions $\mathbf{a} : \mathbb{M} \longrightarrow \mathcal{A}$; we will indicate such a function as $\mathbf{a} = [a_m]_{m \in \mathbb{M}}$, where

$a_m := \mathbf{a}(m)$. Let $f : \mathbb{M} \longrightarrow (0, \infty)$ be some function, and for any $\mathbf{a}, \mathbf{b} \in \mathcal{A}^{\mathbb{M}}$, let $\Delta(\mathbf{a}, \mathbf{b}) := \min\{f(m) \; ; \; m \in \mathbb{M} \text{ and } a_m \neq b_m\}$. Define ultrametric $d : \mathcal{A}^{\mathbb{M}} \times \mathcal{A}^{\mathbb{M}} \longrightarrow [0, \infty)$ by $d(\mathbf{a}, \mathbf{b}) := 1/\Delta(\mathbf{a}, \mathbf{b})$. In this case, $\mathcal{B}(\mathbf{a}, r) := \left\{\mathbf{b} \in \mathcal{A}^{\mathbb{M}} \; ; \; b_m = a_m, \forall m \in \mathbb{M} \text{ with } f(m) \leq 1/r\right\}$.

If $\mathbb{M} = \mathbb{N}$, and $f(m) := m$, this coincides with example (a). Suppose $\mathbb{M}$ is a set of 'elementary statements' about the world, and let $\mathcal{A} = \{T, F\}$; thus, any element of $\mathcal{A}^{\mathbb{M}}$ assigns a truth value to each element of $\mathbb{M}$, which we can suppose describes a 'world-view'. Suppose that $f(m)$ is inversely proportional to the 'importance' or 'priority' of statement m. Thus, $1/\Delta(\mathbf{a}, \mathbf{b})$ measures the importance of the most important statement on which the world-views $\mathbf{a}$ and $\mathbf{b}$ disagree. Thus, $d(\mathbf{a}, \mathbf{b})$ is small if $\mathbf{a}$ and $\mathbf{b}$ agree on all 'important' statements. ♢

A judgement profile $\mathfrak{P}$ is called ***ultrametric*** if there is some ultrametric space $(\mathcal{X}, d)$ such that $\mathcal{P} \subseteq \mathcal{X}$ and $\mathcal{J} \subseteq \mathcal{X}$, and for all $j \in \mathcal{J}$, $\mathcal{P}_j = \mathcal{P} \cap \mathbb{B}(j, r_j)$ for some $r_j > 0$. Intuitively, juror j endorses all propositions in $\mathcal{P}$ which are close enough to her own 'world-view' (represented by the position of j in $\mathcal{X}$).

Proposition 3.2 *If $\mathfrak{P}$ is ultrametric, then $\mathfrak{P}$ is value-restricted; hence the outcome of any quasimajoritarian judgement rule is consistent.*

Proof: Let $\mathcal{Y} \in \mathfrak{Y}$, and let $R := \operatorname{diam}(\mathcal{Y}) := \sup\{d(y_1, y_2) \; ; \; y_1, y_2 \in \mathcal{Y}\}$. Now, $\mathcal{Y}$ is finite, so there exist $y_1, y_2 \in \mathcal{Y}$ with $d(y_1, y_2) = R$. We claim that, for any $j \in \mathcal{J}$, either $y_1 \notin \mathcal{P}_j$ or $y_2 \notin \mathcal{P}_j$.

By contradiction, suppose $\{y_1, y_2\} \subseteq \mathcal{P}_j$. Now, $\mathcal{P}_j = \mathcal{P} \cap \mathbb{B}(j, r_j)$ for some $r_j > 0$. Thus,

$$R \quad = \quad d(y_1, y_2) \quad \underset{(\mathrm{u})}{\leq} \quad \max\{d(y_1, j),\, d(j, y_2)\} \quad \underset{(\flat)}{\leq} \quad r_j. \tag{2}$$

Here, (u) is because d is an ultrametric; ($\flat$) is because $\{y_1, y_2\} \subseteq \mathcal{P}_j \subseteq \mathbb{B}(j, r_j)$. Thus,

$$\forall\, y \in \mathcal{Y}, \qquad d(j, y) \quad \underset{(\mathrm{u})}{\leq} \quad \max\{d(j, y_1),\, d(y_1, y)\} \quad \underset{(*)}{\leq} \quad \max\{r_j, R\} \quad \underset{(\dagger)}{=} \quad r_j. \tag{3}$$

Here, (u) is because d is an ultrametric, and ($*$) is because $d(j, y_1) \leq r_j$ (because $y_1 \in \mathcal{P}_j \subseteq \mathbb{B}(j, r_j)$) and $d(y_1, y) \leq \operatorname{diam}(\mathcal{Y}) = R$. Meanwhile (†) is by eqn.(2).

Equation (3) implies that $\mathcal{Y} \subseteq \mathbb{B}(j, r_j)$; hence $\mathcal{Y} \subseteq \mathcal{P}_j$. But $\mathcal{Y} \in \mathfrak{Y}$, while $\mathcal{P}_j$ is consistent. Contradiction. □

4 Convex profiles

We will now introduce another 'geometric' sufficient condition for consistent majoritarian judgement aggregation. This condition can be seen as another generalization of 'unidimensional order', but we will demonstrate (by a counterexample) that it is *not* a special case of value-restriction.

Let $\mathcal{V}$ be a real vector space, and let $\mathcal{J} \subset \mathcal{V}$ be a finite symmetric subset of $\mathcal{V}$ —that is, for all $j \in \mathcal{J}$, we have $-j \in \mathcal{J}$ also. We also assume $0 \in \mathcal{J}$. The judgement profile $\mathfrak{P}$ is ***convex*** if, for all $p \in \mathcal{P}$, there is some convex subset $\mathcal{C}_p \subset \mathcal{V}$ such that $\mathcal{J}_p = \mathcal{C}_p \cap \mathcal{J}$.

Example 4.1: (a) (*Unidimensional order*) Let $\mathcal{V} = \mathbb{R}$; then a convex subset is an interval. Thus, if $|\mathcal{J}|$ is odd, then convexity is equivalent to the 'unidimensional order' condition in Dietrich and List (2007).

(b) (*World-views*) Let $\|\bullet\|$ be a norm on $\mathcal{V}$ and suppose $\mathcal{P} \subset \mathcal{V}$; thus, for any $p \in \mathcal{P}$ and any $r > 0$, the closed ball $\mathcal{B}(p, r) = \{v \in \mathcal{V} \; ; \; \|v - p\| \leq r\}$ is a convex set. Suppose that for all $p \in \mathcal{P}$, $\mathcal{J}_p = \mathcal{B}(p, r_p) \cap \mathcal{J}$ for some $r_p > 0$; then $\mathfrak{P}$ is convex. (Intuitively, $\mathcal{V}$ is a space of 'world-views'; the world-view of juror j is represented by the location of j in $\mathcal{V}$. The proposition p obtains the endorsement of all jurors j whose own world-view is close enough to p).

(c) (*Voronoi model*) Suppose $\mathcal{P} = \mathcal{P}^1 \sqcup \mathcal{P}^2 \sqcup \cdots \sqcup \mathcal{P}^K$, where for each $k \in [1...K]$, $\mathcal{P}^k$ is a set of mutually exclusive propositions. (For example, $\mathcal{P}^k = \{P, \neg P\}$ for some proposition P.) Thus, for all $j \in \mathcal{J}$, and $k \in [1...K]$, the intersection $\mathcal{P}_j \cap \mathcal{P}^k$ can contain at most one element. Let $\|\bullet\|$ be a norm on $\mathcal{V}$, and let $\mathcal{P} \subset \mathcal{V}$. For all $j \in \mathcal{J}$ and $k \in \mathcal{K}$, suppose $\mathcal{P}_j \cap \mathcal{P}^k = \{p_j^k\}$ where p_j^k is the element in $\mathcal{P}^k$ which is closest to j with respect to norm $\|\bullet\|$. For any $k \in [1...K]$, the set $\mathcal{P}^k$ partitions $\mathcal{V}$ into *Voronoi cells* $\{\mathcal{C}_p\}_{p \in \mathcal{P}^k}$, where, for all $p \in \mathcal{P}^k$, we define $\mathcal{C}_p := \left\{v \in \mathcal{V} \; ; \; \|v - p\| \leq \|v - p'\|, \; \forall p' \in \mathcal{P}^k\right\}$. Each $\mathcal{C}_p$ is convex. (For example, if $\mathcal{P}^k = \{P, \neg P\}$, then $\mathcal{C}_P$ and $\mathcal{C}_{\neg P}$ are half-spaces divided by the hyperplane which perpendicularly bisects the line from P to $\neg P$ in $\mathcal{V}$). Assume that no element of $\mathcal{J}$ lies on the boundary between two Voronoi cells (generically, this is true). Then $\mathcal{J}_p = \mathcal{J} \cap \mathcal{C}_p$ for all $p \in \mathcal{P}$, so the profile $\mathfrak{P}$ is convex. ◇

Proposition 4.2 *If $\mathfrak{P}$ is convex, and R is any supermajoritarian judgement rule, then $R(\mathfrak{P})$ is consistent.*

Proof: We claim that, for all $p \in R(\mathfrak{P})$, $\mathcal{J}_p$ contains 0. To see this, note that $|\mathcal{J}_p| > |\mathcal{J}|/2$, so there exists $j \in \mathcal{J}_p$ such that $-j \in \mathcal{J}_p$ also. But then $0 = (j - j)/2 \in \mathcal{J}_p$ by convexity.

It follows that $R(\mathfrak{P}) \subseteq \mathcal{P}_0$; hence $R(\mathfrak{P})$ is consistent, because $\mathcal{P}_0$ is consistent. □

Proposition 4.3 *A convex profile need not be value-restricted.*

Proof: Figure 1 portrays a counterexample. Here, $\mathcal{V} = \mathbb{R}^2$ and $\mathcal{J} = \{0, \pm i, \pm j\}$, where $i = (1, 0)$ and $j = (0, 1)$. Let $\mathcal{P} := \{p, \neg p, q, \neg q, (p \Rightarrow q), \neg(p \Rightarrow q)\}$. Let $\mathcal{J}_p = \{i, -j\}$, $\mathcal{J}_q = \{-i, 0, -j\}$, and $\mathcal{J}_{(p \Rightarrow q)} = \{-i, 0, +j, -j\}$. In each case, let $\mathcal{J}_{\neg x} = \mathcal{J}_x^{\complement}$. The figure shows how $\mathcal{J}_x$ and $\mathcal{J}_{\neg x}$ can be separated by a line; thus, each is the intersection of $\mathcal{J}$ with a half-plane (i.e. a convex set). Thus, the profile is convex. Note that every juror has complete and logically consistent beliefs; we have

$$\begin{array}{rclcrclcl} \mathcal{P}_i &=& \{p, \neg(p \Rightarrow q), \neg q\}; & & \mathcal{P}_{-i} &=& \mathcal{P}_0 &=& \{\neg p, (p \Rightarrow q), q\}; \\ \mathcal{P}_j &=& \{\neg p, (p \Rightarrow q), \neg q\}; & & \text{and } \mathcal{P}_{-j} && &=& \{p, (p \Rightarrow q), q\}. \end{array}$$

(Indeed, any 'inconsistent' juror would have to be located inside the shaded region in the figure). Now, let $\mathcal{Y} = \{p, (p \Rightarrow q), \neg q\}$. This set is logically inconsistent, but it violates 'value restriction', because

$$\mathcal{J}_p \cap \mathcal{J}_{\neg q} = \{i\}; \qquad \mathcal{J}_p \cap \mathcal{J}_{(p \Rightarrow q)} = \{-j\}; \qquad \text{and} \qquad \mathcal{J}_{(p \Rightarrow q)} \cap \mathcal{J}_{\neg q} = \{+j\}.$$

Each intersection is nonempty; hence the profile is not value-restricted. □

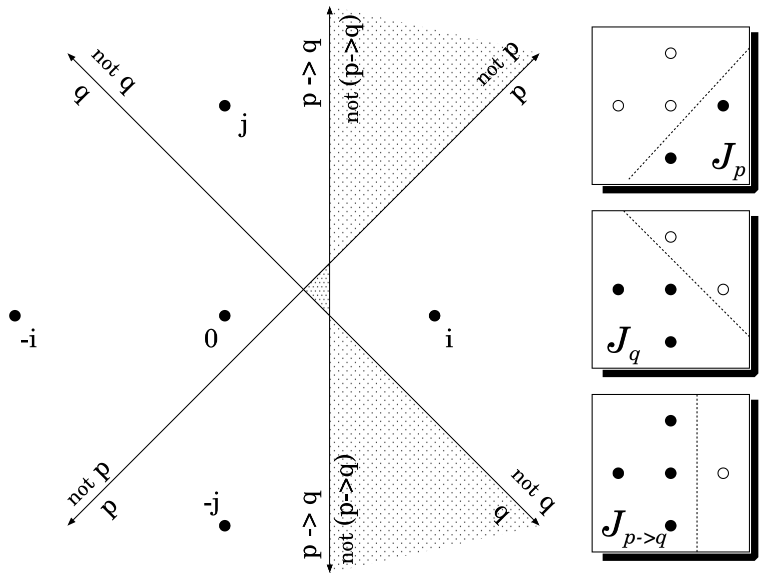

Figure 1: A convex profile which is not value-restricted; see the proof of Proposition 4.3

References

Dietrich, F., List, C., November 2007. Majority voting on restricted domains. Presented at SCW08; see `http://personal.lse.ac.uk/LIST/PDF-files/MajorityPaper22November.pdf`.

Eckert, D., Klamler, C., 2008. A geometric approach to judgement aggregation. Presented at SCW08; see `http://www.accessecon.com/pubs/SCW2008/SCW2008-08-00214S.pdf` .

Kornhauser, L., Sager, L., 1986. Unpacking the court. Yale Law Journal.

Kornhauser, L., Sager, L., 1993. The one and the many: adjudication in collegial courts. California Law Review 91, 1–51.

List, C., 2003. A possibility theorem on aggregation over multiple interconnected propositions. Math. Social Sci. 45 (1), 1–13.

List, C., 2006. Corrigendum to: "A possibility theorem on aggregation over multiple interconnected propositions". Math. Social Sci. 52 (1), 109–110.

List, C., Pettit, P., 2002. Aggregating sets of judgements: an impossibility result. Economics and Philosophy 18, 89–110.

List, C., Puppe, C., September 2007. Judgment aggregation: a survey. (preprint).