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Tang, Rongsheng and Wang, Gaowang and Wang, Jin

Shanghai University of Finance and Economics, Shandong Unversity, Peking University

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Rongsheng Tang[†]
Shanghai University of Finance and Economics

Gaowang Wang[‡]
Shandong University

Jin Wang[§]
Peking University

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Abstract

In this paper we prove the existence, uniqueness and saddle-point stability of the steady state (or balanced growth path) of the Romer (1990) model by utilizing the reduction of dimensionality. Furthermore, we found out a set of policy instruments to improve the monopolistic competitive equilibrium allocation up to social optimum.

Keywords: BGP; Reduction of Dimensionalty; Social Optimum.

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[†]School of Economics, Shanghai University of Finance and Eonomics. rstang@outlook.com.

[‡]Center for Economic Research, Shandong University, Jinan. E-mail: gaowang.wang@sdu.edu.cn. Corresponding author.

[§]Guanghua Shool of Management, Peking University, Beijing. E-mail: jinwang_1010@163.com.

1 Introduction

Endogenous growth theory has greatly improved economists' understanding of how technological change generates persistent economic growth. Romer's (1990) model of Endogenous Technological Change, is the most influential endogenous growth model with costly R&D activities. Romer (1990) solves a steady state (or a balanced growth path) by an ingenious conjecture and develops important economic intuitions on the steady state, leaving the proof of the uniqueness and saddle-point stability of the steady state open.

In the literature, some authors have talked in part about this issue. By simplifying the Romer model, Arnold (2000a, 2000b) examines the saddle-point stability of a conjectured steady state for the monopolistic competitive equilibrium and the social optimum respectively. This simplification misses some important information, such as how to comprehend and write down the consumer's budget constraint correctly. Asada, Semmler and Novak (1998) investigates attentively the steady state of the social optimum in the Romer model in a very complicated way. By introducing the complementarity between the intermediate goods or externalities, other authors derive more complex dynamics such as (expectational) indeterminacy (Benhabib, Perri and Xie, 1994; Asada, Semmler and Novak, 1998; Evan, Honkapohja and Romer, 1998) and Hopf bifurcation (Slobodyan, 2007).

In this paper, we solve the Romer model by changing a four-dimensional dynamic system describing the Romer economy into a three-dimensional one. This method of reduction of dimensionality is developed by Mulligan and Sala-i-Martin (1993) and used by Benhabib and Perli (1994) and Benhabib, Perri and Xie (1994). We prove that the steady state exists uniquely and is saddle-point stable in both the decentralized economy and the social planner economy. That is, we give a complete characterization of the solution of the Romer model. Besides, due to the welfare loss of the monopoly production of the producer durables, the equilibrium growth rate is lower than the optimal growth rate in the Romer model. We want to examine whether the government does play a role in reducing the welfare cost. For this purpose, we indeed introduce a set of policy instruments which improving the monopolistic competitive equilibrium allocation up to social optimum.

The remainder of the paper is organized as follows. In section 2, we solve the decentralized economy of the Romer model and prove the existence, uniqueness and stability of the steady state. In section 3, we examine the social planner economy. In section 4, we introduce a set of policy rules in the decentralized economy to support the social optimum. Finally, Section 5 concludes.

2 The decentralized economy of the Romer (1990) model

2.1 The model set-up and the equilibrium dynamic system

There are three sectors in the production side of the economy: a final-good sector, an intermediate-goods sector and a research sector. The final-good sector utilizes human capital, H_{Yt}, labor, L, and all intermediate goods, $\{x_{it}, i \in [0, A_t]\}$, to produce the final good with the generalized Cobb-Douglas production function, $Y_t = H_{Yt}^{\alpha} L^{\beta} \int_{i=0}^{A_t} x_{it}^{1-\alpha-\beta} di$. The profit maximization problem of the representative firm in the final-good sector is:

$$\max_{H_{Yt}, L, x_{it}} H_{Yt}^{\alpha} L^{\beta} \int_{i=0}^{A_t} x_{it}^{1-\alpha-\beta} di - w_{Ht} H_{Yt} - w_{Lt} L - \int_{i=0}^{A_t} p_{it} x_{it} di.$$

The marginal productivity conditions for human capital and raw labor force are[1]:

$$\alpha H_{Yt}^{\alpha-1} L^{\beta} x_t^{1-\alpha-\beta} A_t = w_{Ht}, H_{Yt}^{\alpha} \beta L^{\beta-1} x_t^{1-\alpha-\beta} A_t = w_{Lt}, \tag{1}$$

and the (inverse) demand function for intermediate good i is

$$p_{it} = H_{Yt}^{\alpha} L^{\beta} (1-\alpha-\beta) x_{it}^{-\alpha-\beta}, i \in [0, A_t]. \tag{2}$$

Each intermediate good is produced by a monopolistic firm. The decision process of any monopolistic firm can be separated into two steps. Step 1: it pays the price P_{At} to buy the patent for producing intermediate good i in the competitive patents market, which is the sunk cost for the monopolistic firm. Since the patents market is competitive, the price of new design i is the discounted present value of the profits flow extracted by firm i, i.e., $P_{At} = \int_{\tau=t}^{\infty} \pi_{\tau} \exp\left(-\int_{s=t}^{\tau} r_s ds\right) d\tau$.[2] Differentiating it on both sides with respect to t yields the differential equation of P_{At},

$$\dot{P}_{At} = r_t P_{At} - \pi_t. \tag{3}$$

Step 2: in the monopoly pricing problem, monopolistic firm i rents capital (as variable costs) and produces intermediate good i to meet the demand of the final-good sector for its products, i.e., (2). It is assumed that the unit cost for any intermediate good is the same $\eta\,(> 0)$ units of capital. Solving the monopoly pricing problem of any intermediate good i, namely, $\pi_{it} = \max_{p_{it}, x_{it}} p_{it} x_{it} - r_t \eta x_{it}$, we have the symmetric monopolistic pricing formula:

$$p_{it} = \frac{r_t \eta}{(1-\alpha-\beta)} \equiv p_t, \tag{4}$$

where $r_t \eta$ is the marginal cost for producing additional unit of any intermediate good, and $1/(1-\alpha-\beta)\,(> 1)$ is the mark-up over the marginal cost. Thus all monopolistic firms set the

[1] Notice that the expressions for the marginal productivity conditions have utilized the symmetric property of the model that will be derived in the subsequent analysis.

[2] We omit the superscript of P_{At}^{i} because of the derived symmetric property of the model.

same monopoly price p_t, produce the same amount x_t (due to (2)), and earn the same monopoly profit $\pi_t = (\alpha+\beta)\, p_t x_t$. Furthermore, the total capital stock is related to the durable goods that are actually used in production by the rule $K_t = \int_{i=0}^{A_t} \eta x_{it} di = \eta x_t A_t$.

The research sector uses the knowledge stock A_t and human capital H_{At} to produce new knowledge, namely,

$$\dot{A}_t = \delta A_t H_{At} = \delta A_t \left(H - H_{Yt}\right), \tag{5}$$

where the second equality follows from the fact that the sum of the human capital used in the research sector H_{At} and in the final-good sector H_{Yt} must be equal to the total stock of human capital in the economy H. Free mobility and no arbitrage require that the rental rate of human capital must be equal in the research sector and in the final-good sector, namely,

$$P_{At}\delta A_t = w_{Ht} = \alpha H_{Yt}^{\alpha-1} L^{\beta} x_t^{1-\alpha-\beta} A_t. \tag{6}$$

The representative consumer's utility maximization problem is summerized as follows:

$$\max_{C_t, K_t} \int_{t=0}^{\infty} e^{-\rho t} \frac{C_t^{1-\sigma}-1}{1-\sigma} dt,$$

subject to the flow budget constraint (FBC)[3]:

$$\dot{K}_t = w_{Ht} H_{Yt} + w_{Lt} L + r_t K_t + \int_{i=0}^{A_t} \pi_{it} di - C_t, \tag{7}$$

where $\rho \in (0,1)$ is the time discount rate and $1/\sigma \in (0, +\infty)$ is the elasticity of intertemporal substitution (EIS). The Euler equation is: $\dot{C}_t / C_t = \frac{1}{\sigma}\left(r_t - \rho\right)$. Substituting (2), (4), and $K_t = \eta x_t A_t$ into the Euler equation leads to the dynamic equation of consumption

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\sigma}\left[(1-\alpha-\beta)^2 \eta^{\alpha+\beta-1} H_{Yt}^{\alpha} L^{\beta} K_t^{-(\alpha+\beta)} A_t^{(\alpha+\beta)} - \rho\right]. \tag{8}$$

Putting (1), (2) and (4) into (7) gives us the dynamic equation of physical capital

$$\frac{\dot{K}_t}{K_t} = \eta^{\alpha+\beta-1} H_{Yt}^{\alpha} L^{\beta} K_t^{-(\alpha+\beta)} A_t^{(\alpha+\beta)} - \frac{C_t}{K_t}, \tag{9}$$

which is essentially the resource constraint.

[3] Note that H_{Yt} (rather than H) enters the budget constraint of the representative consumer. The part H_{Yt} of the total human capital stock H is determined endogenously both by the utility-maximizing behaviors of consumers and the profit-maximizing behaviors of the firms in the final-good sector. The other part H_{At} of H is pinned down by the market-clearing condition of human capital rather than the optimum in the research sector. If we replace H_{Yt} by H in the FBC, then there will be inconsistancy between the FBC and the resource constraint in competitive equilibrium.

Substituting $K_t = \eta x_t A_t$ into (6) and taking logrithmic derivative on both sides with respect to t give us

$$\frac{\dot{P}_{At}}{P_{At}} = (\alpha + \beta - 1) \frac{\dot{A}_t}{A_t} + (\alpha - 1) \frac{\dot{H}_{Yt}}{H_{Yt}} + (1 - \alpha - \beta) \frac{\dot{K}_t}{K_t}.$$

Plugging (2), (4), and (6) into (3) turns out to

$$\frac{\dot{P}_{At}}{P_{At}} = (1 - \alpha - \beta)^2 \eta^{\alpha+\beta-1} H_{Yt}^{\alpha} L^{\beta} K_t^{-(\alpha+\beta)} A_t^{(\alpha+\beta)} - \frac{\delta}{\Lambda} H_{Yt},$$

where $\Lambda = \alpha / (\alpha + \beta)(1 - \alpha - \beta)$. Combining the above two equations and using (5) and (9) yield us the dynamic equation of H_{Yt}:

$$\frac{\dot{H}_{Yt}}{H_{Yt}} = \left\{ \begin{array}{c} \frac{(1-\alpha-\beta)(\alpha+\beta)}{1-\alpha} \eta^{\alpha+\beta-1} H_{Yt}^{\alpha} L^{\beta} K_t^{-(\alpha+\beta)} A_t^{(\alpha+\beta)} + \\ \frac{\delta\Lambda(1-\alpha-\beta)+\delta}{\Lambda(1-\alpha)} H_{Yt} - \frac{(1-\alpha-\beta)}{1-\alpha} \frac{C_t}{K_t} - \frac{(1-\alpha-\beta)\delta H}{1-\alpha} \end{array} \right\}. \tag{10}$$

The dynamic system composed of the four differential equations (5), (8), (9) and (10) describes the equilibrium dynamics of the model economy, with two state variables (K, A), two control variables (P_A, H_Y), and two initial conditions K_0, A_0.

2.2 Saddle-point stability of the balanced growth path

To study the transitional dynamics implied by the model, we reduce the dimensionality of the problem from four to three by a change of variable very similar to those used in Mulligan and Sala-i-Martin (1993), Benhabib and Perli (1994), and Benhabib, Perli and Xie (1994). Thus we define $y_t \equiv \eta^{(1-\alpha-\beta)/(\alpha+\beta)} K_t / A_t$ and $q_t \equiv C_t / K_t$. Since $\dot{y}_t / y_t = \dot{K}_t / K_t - \dot{A}_t / A_t$ and $\dot{q}_t / q_t = \dot{C}_t / C_t - \dot{K}_t / K_t$, we have:

$$\frac{\dot{y}_t}{y_t} = y_t^{-(\alpha+\beta)} H_{Yt}^{\alpha} L^{\beta} - q_t - \delta (H - H_{Yt}), \tag{11}$$

$$\frac{\dot{H}_{Yt}}{H_{Yt}} = \frac{(1 - \alpha - \beta)(\alpha + \beta)}{1 - \alpha} y_t^{-(\alpha+\beta)} H_{Yt}^{\alpha} L^{\beta} + \frac{\delta\Lambda (1 - \alpha - \beta) + \delta}{\Lambda (1 - \alpha)} H_{Yt} - \frac{1 - \alpha - \beta}{1 - \alpha} q_t - \frac{1 - \alpha - \beta}{1 - \alpha} \delta H, \tag{12}$$

$$\frac{\dot{q}_t}{q_t} = \left(\frac{(1 - \alpha - \beta)^2}{\sigma} - 1 \right) y_t^{-(\alpha+\beta)} H_{Yt}^{\alpha} L^{\beta} - \frac{\rho}{\sigma} + q_t. \tag{13}$$

This is a reduced three-dimensional dynamic system in y_t, H_{Yt} and q_t; its dynamics is equivalent to that of the original four-dimensional system in the sense that its steady states correspond to the BGPs of the original four-dimensional model. Then the unique steady state (BGP) is solved as

$$y^* = \left[\frac{H_Y^{*\alpha} L^\beta (1-\alpha-\beta)^2 (1+\sigma\Lambda)}{\rho+\delta H\sigma}\right]^{\frac{1}{\alpha+\beta}}, H_Y^* = \frac{\Lambda(\rho+\delta H\sigma)}{\delta(1+\sigma\Lambda)}, q^* = \frac{(\rho\Lambda-\delta H)(1-\alpha-\beta)^2+(\rho+\delta H\sigma)}{(1-\alpha-\beta)^2(1+\sigma\Lambda)}. \tag{14}$$

Substituting (14) into (5), we solve the equilibrium growth rate on the BGP as

$$g^* = \frac{\delta H - \rho\Lambda}{1+\sigma\Lambda}, \tag{15}$$

which is exactly the conjectured equilibrium BGP in Romer (1990).

Before studying the stability of the equilibrium BGP, we talk about the parameter values in the model. Due to $H_Y^* \in (0, H)$, we know from (14) that

$$\delta H - \rho\Lambda > 0. \tag{16}$$

For the convergence of the objective function on the BGP, we need to impose the restriction $\rho + (1-\sigma) g^* > 0$, impliying that

$$\sigma > \frac{\delta H - \rho(1+\Lambda)}{\delta H}. \tag{17}$$

Hence if the equilibrium BGP in Romer model makes sense, then the two restrictions on parameter values, (16) into (17), will be implicitly assumed.

Then we examine the stability of the BGP. For this purpose, we define $z_t \equiv y_t^{-(\alpha+\beta)} H_{Yt}^\alpha L^\beta$, $w_1 \equiv (1-\alpha-\beta)(\alpha+\beta)/(1-\alpha)$ and $w_2 \equiv (1-\alpha-\beta)^2/\sigma - 1$. Linearizing the three-dimensional dynamic system composed of (11)-(13) around the steady state (14), we obtain the Jacobian matrix evaluated at the steady state, namely,

$$J = \begin{bmatrix} -(\alpha+\beta) z^* & \left(\frac{\alpha z^*}{H_Y^*} + \delta\right) y^* & -y^* \\ -(\alpha+\beta) w_1 z^* \frac{H_Y^*}{y^*} & \alpha w_1 z^* + \frac{\delta\Lambda(1-\alpha-\beta)+\delta}{\Lambda(1-\alpha)} H_Y^* & -\frac{1-\alpha-\beta}{1-\alpha} H_Y^* \\ -(\alpha+\beta) w_2 z^* \frac{q^*}{y^*} & \alpha w_2 z^* \frac{q^*}{H_Y^*} & q^* \end{bmatrix}.$$

It is easy to konw that

$$\det(J) = -\frac{\delta(\alpha+\beta)(1-\alpha-\beta)^2(1+1/\sigma\Lambda)}{1-\alpha} z^* q^* H_Y^* = \Pi_{i=1}^3 \lambda_i < 0. \tag{18}$$

The negative determinant of the Jacobian matrix establishes that two possibilities will occur: (i) there is one negative eigenvlue and two other eigenvalues with negative real parts; (ii) there is one negative real eigenvalue and two other eigenvalues with positive real parts. Now we examine the sign of the trace of the Jacobian matrix,

$$trace(J) = \frac{(\rho+\delta H\sigma)\left\{\begin{array}{c}\left[2-2\alpha-\beta+\alpha(\alpha+\beta)+\Lambda(1-\alpha-\beta)^2\right] \\ -(\delta H-\rho\Lambda)(1-\alpha)(1-\alpha-\beta)\end{array}\right\}}{(1+\sigma\Lambda)(1-\alpha)(1-\alpha-\beta)} = \sum_{i=1}^3 \lambda_i. \tag{19}$$

Obviously, the denominator of the $trace\,(J)$ is positive. Then the sign of $trace\,(J)$ is identical to its numerator. And the positivity of the numerator is equivalent to the inequality

$$\sigma > \frac{(1-\alpha)(1-\alpha-\beta)}{A} - \frac{\rho}{\delta H}\left[1 + \frac{\Lambda(1-\alpha)(1-\alpha-\beta)}{A}\right] \equiv \Xi, \tag{20}$$

where

$$A \equiv 2 - 2\alpha - \beta + \alpha(\alpha+\beta) + \Lambda(1-\alpha-\beta)^2 > 0.$$

Due to (16), we have

$$\frac{\delta H - \rho(1+\Lambda)}{\delta H} - \Xi = \frac{\delta H - \rho\Lambda}{\delta H}\frac{1+\Lambda(1-\alpha-\beta)^2}{A} > 0. \tag{21}$$

Combining equations (17) and (21), we know that (20) holds, which tells that the trace of the Jacobian matrix is pisitive. Hence case (ii) holds. The number of the stable eigenvalue is equal to the number the state variable, which establishes that the BGP is saddle-point stable. Given the initial values of the state variables, the economy converges to the unique steady state along the uniqe stable manifold.

3 Social planner economy and social optimum

3.1 Optimal growth path

In this section we present the optimal optimum of the Romer (1990) by reviewing the social planner economy. The social planner maximizes the representative agent's objective function

$$\max_{C_t,K_t}\int_{t=0}^{\infty} e^{-\rho t}\frac{C_t^{1-\sigma}-1}{1-\sigma}dt,$$

subject to the social resource constraint

$$\dot{K}_t = \eta^{\alpha+\beta-1}H_{Yt}^{\alpha}L^{\beta}K_t^{1-(\alpha+\beta)}A_t^{(\alpha+\beta)} - C_t,$$

and the knowledge accumulation equation

$$\dot{A}_t = \delta A_t(H - H_{Yt}),$$

with the given initial values of capital and knowledge (K_0, A_0). Applying Pontryagin's maximum principle and arranging these necessary conditions, we derive the following four-dimensional dynamic system with respect to C_t, K_t, A_t and H_{Yt} as follows:

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\sigma}\left[(1-\alpha-\beta)\eta^{\alpha+\beta-1}H_{Yt}^{\alpha}L^{\beta}K_t^{-(\alpha+\beta)}A_t^{(\alpha+\beta)} - \rho\right], \tag{22}$$

$$\frac{\dot{K_t}}{K_t} = \eta^{\alpha+\beta-1} H_{Yt}^{\alpha} L^{\beta} K_t^{-(\alpha+\beta)} A_t^{(\alpha+\beta)} - \frac{C_t}{K_t}, \tag{23}$$

$$\frac{\dot{A_t}}{A_t} = \delta\left(H - H_{Yt}\right), \tag{24}$$

$$\frac{\dot{H_{Yt}}}{H_{Yt}} = \frac{(1-\alpha-\beta)+\beta/\alpha}{(1-\alpha)}\delta H_{Yt} - \frac{(1-\alpha-\beta)}{1-\alpha}\frac{C_t}{K_t} + \frac{(\alpha+\beta)\,\delta H}{1-\alpha}. \tag{25}$$

Using the reduction of dimension similar to the above section and setting $y_t \equiv \eta^{(1-\alpha-\beta)/(\alpha+\beta)} K_t/A_t$ and $q_t \equiv C_t/K_t$, we obtain the following equivalent three-dimensional dynamic system:

$$\frac{\dot{y_t}}{y_t} = y_t^{-(\alpha+\beta)} H_{Yt}^{\alpha} L^{\beta} - q_t - \delta\left(H - H_{Yt}\right), \tag{26}$$

$$\frac{\dot{H_{Yt}}}{H_{Yt}} = \frac{(1-\alpha-\beta)+\beta/\alpha}{(1-\alpha)}\delta H_{Yt} - \frac{(1-\alpha-\beta)}{1-\alpha} q_t + \frac{(\alpha+\beta)\,\delta H}{1-\alpha}, \tag{27}$$

$$\frac{\dot{q_t}}{q_t} = \left(\frac{(1-\alpha-\beta)}{\sigma} - 1\right) y_t^{-(\alpha+\beta)} H_{Yt}^{\alpha} L^{\beta} - \frac{\rho}{\sigma} + q_t. \tag{28}$$

3.2 Stability of the BGP

The steady state (or BGP) of the social planner economy is solved as

$$H_Y^o = \frac{\rho - \delta H(1-\sigma)}{\delta(\sigma+\beta/\alpha)}, y^o = \left[\frac{H_Y^o L^{\beta}(1-\alpha-\beta)(\beta+\alpha\sigma)}{\beta\rho + \delta H\sigma(\alpha+\beta)}\right]^{\frac{1}{\alpha+\beta}}, q^o = \frac{\left\{\begin{array}{c}\rho\left[\alpha(1-\alpha-\beta)+\beta\right] - \\ \delta H(\alpha+\beta)(1-\alpha-\beta-\sigma)\end{array}\right\}}{(1-\alpha-\beta)(\beta+\alpha\sigma)}, \tag{29}$$

with the optimal growth rate

$$g^o = \frac{\delta H - \rho\Theta}{\sigma\Theta + (1-\Theta)}, \tag{30}$$

where $\Theta \equiv \alpha/(\alpha+\beta)$. Due to $H_Y^o \in (0, H)$ and $\rho + (1-\sigma)g^o > 0$, we need to impose the following two assumptions:

$$\delta H > \rho\Theta, \sigma > 1 - \frac{\rho}{\delta H}. \tag{31}$$

Define $z_t \equiv y_t^{-(\alpha+\beta)} H_{Yt}^{\alpha} L^{\beta}$, $\phi_1 \equiv (1-\alpha-\beta+\beta/\alpha)/(1-\alpha)$, $\phi_2 = (1-\alpha-\beta)/(1-\alpha)$, and $\phi_3 \equiv (1-\alpha-\beta)/\sigma - 1$. To examine the stability property of the steady state, we linearize the dynamic system (26)-(28) around the steady state (29) and derive the Jacobian matrix

$$J^o = \begin{bmatrix} -(\alpha+\beta) z^o & \left(\frac{\alpha z^o}{H_Y^o} + \delta\right) y^o & -y^o \\ 0 & \delta\varpi_1 H_Y^o & -\varpi_2 H_Y^o \\ -(\alpha+\beta)\varpi_3 z^o \frac{q^o}{y^o} & \alpha\varpi_3 z^o \frac{q^o}{H_Y^o} & q^o \end{bmatrix}.$$

Under Assumption (31), we find that the determinant of the Jacobian matrix J^o is negative,

$$\det(J^o) = -\frac{\delta(\alpha+\beta)\left[1-\alpha-\beta+\beta(1-\alpha-\beta)/(\alpha\sigma)\right]}{1-\alpha} z^o q^o H_Y^o < 0,$$

and the trace of the Jacobian matrix J^o is positive,

$$trace(J^o) = \frac{\sigma-1}{\sigma}\frac{\beta\rho+\delta H\sigma(\alpha+\beta)}{\beta+\alpha\sigma} + \frac{\rho}{\sigma} + \frac{(1-\alpha-\beta)\alpha+\beta}{1-\alpha}\frac{\rho-\delta H(1-\sigma)}{\beta+\alpha\sigma} > 0,$$

which establish that there is a stable eigenvalue corresponding to the unique state variable y_t of the dynamic system (26)-(28). Therefore the steady state (or BGP) of the social planner economy of the Romer model is also a local saddle.

4 Policy analysis

It is easy to know from (15) and (30) that the optimal growth rate is larger than the equilibrium growth rate, i.e., $g^o > g^*$, which displays that monopoly brings about welfare cost in the decentralized economy. Whether there exist appropriate policy instruments improving equilibrium growth, we will give a definite answer to this question.

Similar to Arnold (2000), we assume that the government has two policy instruments at its disposal: subsiding each intermediate-good producer $s\,(>0)$ dollars per dollar of revenues and paying a fraction $1-\theta_t$ of the R&D outlays in the research sector. Then the profit-maximization problem of any intermediate good i is changed into

$$\pi_{it}^s = \max_{x_{it}} (1+s) H_{Yt}^{\alpha} L^{\beta} (1-\alpha-\beta) x_{it}^{1-\alpha-\beta} - r_t \eta x_{it}.$$

The monopoly pricing formular is solved as

$$p_t^s = \frac{1}{(1-\alpha-\beta)(1+s)} r_t \eta, \tag{32}$$

where $1/(1-\alpha-\beta)(1+s)$ is the new mark-up. Thus all monopolistic firms earn the same profit $\pi_t = (1+s)(\alpha+\beta) p_t x_t$. Since the research sector only affords the share $\theta_t \in (0,1)$ of the wage cost of the human capital employed in the research sector, the no-arbitrage condition of the allocation of human capital is changed as

$$P_{At}\delta A_t = \theta_t w_{Ht} = \theta_t \alpha \eta^{\alpha+\beta-1} H_{Yt}^{\alpha-1} L^{\beta} K_t^{1-\alpha-\beta} A_t^{\alpha+\beta}. \tag{33}$$

The flow budget constraint of the representative consumer is changed as follows

$$\dot{K}_t = w_{Ht} H_{Yt} + w_{Lt} L + r_t K_t + \int_{i=0}^{A_t} \pi_{it} di - C_t - T_t, \tag{34}$$

where T_t is the lump-sum tax. Thus the Euler equation is also

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\sigma}\left(r_t - \rho\right). \tag{35}$$

The balanced budget constraint of the government is

$$s\int_{i=0}^{A_t} p_{it}x_{it}di = T_t. \tag{36}$$

Combining (2), (32), and (35) gives us the dynamic equation of consumption

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\sigma}\left[(1+s)\left(1-\alpha-\beta\right)^2\eta^{\alpha+\beta-1}H_{Yt}^{\alpha}L^{\beta}K_t^{-(\alpha+\beta)}A_t^{(\alpha+\beta)} - \rho\right]. \tag{37}$$

Plugging (1), (32), and the balanced budget of the government into (34) yields

$$\frac{\dot{K}_t}{K_t} = \eta^{\alpha+\beta-1}H_{Yt}^{\alpha}L^{\beta}K_t^{-(\alpha+\beta)}A_t^{(\alpha+\beta)} - \frac{C_t}{K_t}. \tag{38}$$

Substituting (33) and (32) into the differential equation $\dot{P}_{At} = r_tP_{At} - \pi_t$ leads to

$$\frac{\dot{P}_{At}}{P_{At}} = (1+s)\left(1-\alpha-\beta\right)^2\eta^{\alpha+\beta-1}H_{Yt}^{\alpha}L^{\beta}K_t^{-(\alpha+\beta)}A_t^{(\alpha+\beta)} - \frac{(1+s)\,\delta}{\Lambda}\frac{H_{Yt}}{\theta_t}. \tag{39}$$

Taking logrithmic derivatives with respect to t on both sides of (33) and combining it with (5), (38) and (39), we know that

$$\frac{\dot{H}_{Yt}}{H_{Yt}} = \left\{\begin{array}{c} \frac{1}{1-\alpha}\frac{\dot{\theta}_t}{\theta_t} + \frac{(1-\alpha-\beta)}{1-\alpha}\left[1-(1-\alpha-\beta)(1+s)\right]\eta^{\alpha+\beta-1}H_{Yt}^{\alpha}L^{\beta}K_t^{-(\alpha+\beta)}A_t^{(\alpha+\beta)} \\ -\frac{(1-\alpha-\beta)}{1-\alpha}\frac{C_t}{K_t} - \frac{(1-\alpha-\beta)\delta H}{1-\alpha} + \left[(1-\alpha-\beta) + \frac{1+s}{\Lambda\theta_t}\right]\frac{\delta H_{Yt}}{1-\alpha} \end{array}\right\}. \tag{40}$$

Using the same definition as Section 2, we obtain the following dynamic system about (y, H_y, q) as follows:

$$\frac{\dot{y}_t}{y_t} = y_t^{-(\alpha+\beta)}H_{Yt}^{\alpha}L^{\beta} - q_t - \delta\left(H - H_{Yt}\right), \tag{41}$$

$$\frac{\dot{H}_{Yt}}{H_{Yt}} = \left\{\begin{array}{c} \frac{1}{1-\alpha}\frac{\dot{\theta}_t}{\theta_t} + \frac{(1-\alpha-\beta)}{1-\alpha}\left[1-(1-\alpha-\beta)(1+s)\right]y_t^{-(\alpha+\beta)}H_{Yt}^{\alpha}L^{\beta} \\ -\frac{1-\alpha-\beta}{1-\alpha}q_t - \frac{1-\alpha-\beta}{1-\alpha}\delta H + \left[(1-\alpha-\beta) + \frac{1+s}{\Lambda\theta_t}\right]\frac{\delta H_{Yt}}{1-\alpha} \end{array}\right\}, \tag{42}$$

$$\frac{\dot{q}_t}{q_t} = \left[\frac{\left(1-\alpha-\beta\right)^2}{\sigma}(1+s) - 1\right]y_t^{-(\alpha+\beta)}H_{Yt}^{\alpha}L^{\beta} + q_t - \frac{\rho}{\sigma}. \tag{43}$$

To obtain the same allocation as the one of optimal growth path, we compare the dynamic system with (41)-(43) with the one with (11)-(13). Obviously, if $(1+s)\left(1-\alpha-\beta\right)^2/\sigma - 1 = (1-\alpha-\beta)/\sigma - 1$, i.e., $s = (\alpha+\beta)/(1-\alpha-\beta)$, then (43) is the same as (13). Substituting

$s = (\alpha + \beta) / (1 - \alpha - \beta)$ into (42) and comparing it with (10), we know that they are the same thing if θ_t follows the following differential equation

$$\frac{\dot{\theta}_t}{\theta_t} = \left(\frac{\beta}{\alpha} - \frac{1+s}{\Lambda \theta_t} \right) \delta H_{Yt} + \delta H. \tag{44}$$

Then the steady state of the dynamic system composed of (41)-(44) can be solved as follows:

$$H_Y^{**} = \frac{\rho - \delta H (1-\sigma)}{\delta (\sigma + \beta/\alpha)}, y^{**} = \left(\frac{H_Y^{**} L^\beta (1-\alpha-\beta)(\beta + \alpha\sigma)}{\beta\rho + \delta H \sigma (\alpha+\beta)} \right)^{\frac{1}{\alpha+\beta}},$$
$$\theta^{**} = \frac{\rho - \delta H (1-\sigma)}{\delta H \sigma + \frac{\beta}{\alpha+\beta}\rho}, q^{**} = \frac{\rho [\alpha (1-\alpha-\beta) + \beta] - \delta H (\alpha+\beta)(1-\alpha-\beta-\sigma)}{(1-\alpha-\beta)(\beta+\alpha\sigma)}.$$

Then the associated equilibrium growth rate is derived as the optimal growth rate, namely,

$$g^{**} = g^o = \frac{\delta H - \rho\Theta}{\sigma\Theta + (1-\Theta)}, \tag{45}$$

where $\Theta \equiv \alpha / (\alpha + \beta)$. Note that under the implied parameter values in the social planner economy, we have $\theta^{**} \in (0,1)$. Therefore, we have found out a monopolistic competition with a set of policy rules (s, θ_t, T_t) satisfying (36) and (44), which supports the social optimum allocation.

5 Conclusion

In this note, by utilizing the reduction of dimensionality, we prove the existence, uniqueness and saddle-point stability of the steady state (or BGP) of the Romer (1990) model in both the decentralized economy and social planner economy. That is, given the initial values of the state variable, there is a unique stable manifold converging to the unique steady state. Based on this result, we find out a set of policy instruments to reduce the welfare cost of the monopoly production for the producer durables up to social optimum.

References

[1] Arnold, G., 2000(a). Stability of the Market Equilibrium in Romer's Model of Endogenous Technological Change: a Complete Characterization. *Journal of Macroeconomics* 22 (1), 69-84.

[2] Arnold, G., 2000(b). Endogenous Technological Change: a Note on Stability. *Economic Theory* 16, 219-226.

[3] Asada, T., W. Semmler and J. Novak, 1998. Endogenous Growth and the Balanced Growth Equilibrium. *Research in Economics* 52, 189-212.

[4] Benhabib, J., and Perli, R., 1994. Uniqueness and Indeterminacy: on the Dynamics of Endogenous Growth. *Journal of Economic Theory* 63, 113-142.

[5] Benhabib, J., R. Perli and D. Xie, 1994. Monopolistic Competition, Indeterminacy and Growth. *Ricerche Economiche* 48, 279-298.

[6] Evans, G.W., S. Honkapohja and P. Romer, 1998. Growth Cycles. *American Economic Review* 88, 495-515.

[7] Mulligan, C.B., and Sala-i-Martin, X., 1993. Transitional Dynamics in Two-sector Models of Endogenous Growth. *Quarterly Journal of Economics* 108, 739-773.

[8] Romer, P., 1990. Endogenous Technological Change. *Journal of Political Economy* 98(5), 71-102.

[9] Slobogyan, S., 2007. Indeterminacy and Stability in a Modified Romer Model. *Journal of Macroeconomics* 29, 169-177.