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Optimal Price of Entry into a Competition*

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Abstract

A continuum of agents are choosing whether to enter a competition. Entry is controlled by a firm that charges a price for it. The mass of agents is uncertain. I analyse how the distribution of the mass of agents determines the equilibrium price and the intensity of entry. A shift of the distribution towards more mass initially induces a reduction of price, and later – a reduction in entry.

Keywords: contests, entry, university admission tests

JEL codes: C72, D82

1 Introduction

Consider a cohort of prospective students who are considering applying to university. Of those who apply, the ones with the highest ability are admitted to universities. To apply, a student needs to pay a firm to take a GRE test. What price will the firm set, and what proportion of prospective students will apply?

To answer this and similar questions, this paper models a continuum of candidates who consider whether to enter a competition for a continuum of prizes. Access to the competition is controlled by a profit-maximising firm, who charges a price for entering. Each candidate has a type, and out of those who enter the competition, candidates with the highest types

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each get one prize. Each candidate is privately informed about her type, but does not know the total mass of candidates, which is drawn from a continuous distribution – thus, a candidate is uncertain about the number of her competitors who have a higher type.

This paper is related to the literature on entry into competitive settings, such as auctions (McAfee and McMillan, 1987; Levin and Smith, 1994; Arozamena and Weinschelbaum, 2011; Moreno and Wooders, 2011; Li, 2017; Lee and Li, 2019) or contests (Fu and Lu, 2010; Kaplan and Sela, 2010; Fu et al., 2015). In particular, Morgan et al. (2017) study entry into contests with a continuum of agents and prizes; while Ginzburg (2019) analyses a costly competitive test in which, as in this paper, winning depends on the candidate's exogenous type, rather than on her endogenously chosen effort or bid. The key difference of this paper is that the cost of entry is not exogenously fixed, but is set by a firm. The firm maximises its profit from entry, and has no stake in the outcome of the competition, unlike an auction or contest designer.

The paper shows how the distribution of the mass of candidates determines optimal entry price and the proportion of candidates who enter the competition. When the mass of candidates tends to be low, the firm sets a price at which all candidates enter. A first-order shift of the distribution towards greater mass initially induces the firm to lower the price while maintaining full entry. However, after the distribution has shifted sufficiently far, a further proportional shift leads the firm to keep the price unchanged while allowing the intensity of entry to fall.

2 Model

There is a continuum of candidates, and a continuum of prizes. The value of a prize to each candidate is v. The mass of candidates $y \in (0, +\infty)$ is drawn from a smooth distribution G with density g and full support. The mass of prizes is normalised to 1. Each candidate i has a type $\theta_i \in [0, 1]$, drawn from a smooth distribution F with full support. Each candidate knows her type, but not the types of other candidates or the mass of candidates.

After learning her type, each candidate decides whether to enter the competition for prizes. Entry is controlled by a profit-maximising firm. The firm selects a price p that each candidate needs to pay for entry. The firm's marginal cost of admitting one candidate is $c \geq 0$, where $c < v$.[1]

The timing is as follows. First, the firm selects p. Then, nature draws y and the type of each candidate. Each candidate learns his type. Candidates then simultaneously decide whether to enter the competition. I will assume symmetric strategies, in the sense that candidates with the same types enter the competition with equal probabilities.

Candidates who do not enter receive a payoff of 0. Candidates who enter pay the price p. If the mass of candidates who enter is greater than 1, then out of candidates who enter, mass 1 of those who have the highest types receive one prize each. Formally, if the set of candidates who enter is S, then each candidate from the set $S \cap \left[\tilde{\theta}, 1\right]$ receives one prize, while other candidates do not receive prizes, where $\tilde{\theta}$ is a type such that the mass of $S \cap \left[\tilde{\theta}, 1\right]$ equals 1. On the other hand, if the mass of candidates who enter is smaller than 1, then each candidate who enters receives a prize.

3 Equilibrium

If $p > v$, no candidate enters, and the firm receives zero profit. Since $v > c$, the firm is better off setting some price $p \in (c, v]$, and hence every equilibrium will be of this type.

Intuitively, the probability that a candidate receives the prize is increasing in her type. Hence, any equilibrium is characterised by a cutoff $\hat{\theta}$ such that a candidate enters if and only if her type is above $\hat{\theta}$. The share of candidates that enter is thus $1 - F\left(\hat{\theta}\right)$. If a candidate with type $\hat{\theta}$ enters, she pays p, and receives a prize worth v if and only if $y\left[1 - F\left(\hat{\theta}\right)\right] \leq 1$. The probability of this event is $G\left[\frac{1}{1-F\left(\hat{\theta}\right)}\right]$. If she does not enter, her payoff is zero. At the equilibrium, she must be indifferent between entering and not entering. The following lemma proves this reasoning formally:

Lemma 1. *Every equilibrium is characterised by a cutoff $\hat{\theta}$ given by*

$$vG\left[\frac{1}{1 - F\left(\hat{\theta}\right)}\right] - p = 0 \tag{1}$$

[1] If $c > v$, the only kind of equilibrium involves a price p at which no candidate enters.

such that all candidates with types above $\hat{\theta}$ enter with certainty, and the mass of candidates whose types are below $\hat{\theta}$ and who enter is zero.

Proof. See Appendix. □

Let $x \equiv \frac{1}{1-F\left(\hat{\theta}\right)} \in [1, +\infty)$. Then (1) implies that at the equilibrium we have

$$p = vG(x)$$

The mass of candidates who enter equals $y\left[1 - F\left(\hat{\theta}\right)\right] = \frac{y}{x}$, and the firm receives a profit of $p - c$ from each of them. The firm's problem then is

$$\max_{p \in [0,v]} \mathrm{E}\left[\frac{y}{x}(p-c)\right] \text{ subject to } p = vG(x) \tag{2}$$

Since there is a one-to-one relationship between x and p, the firm's problem can be written as that of selecting the optimal x. A larger value of x corresponds to a lower value of $1 - F\left(\hat{\theta}\right)$, and hence to lower intensity of entry. Let x^* be the firm's equilibrium choice of x. The following result characterises it:

Proposition 1. $x^* \in \arg\max_{x \in [1,+\infty)} \frac{G(x) - \frac{c}{v}}{x}$.

Proof. Rewriting (2) yields $\max_{x \in [1,+\infty)} \frac{vG(x)-c}{x} \mathrm{E}[y]$, which has the same solution as the expression in the proposition. □

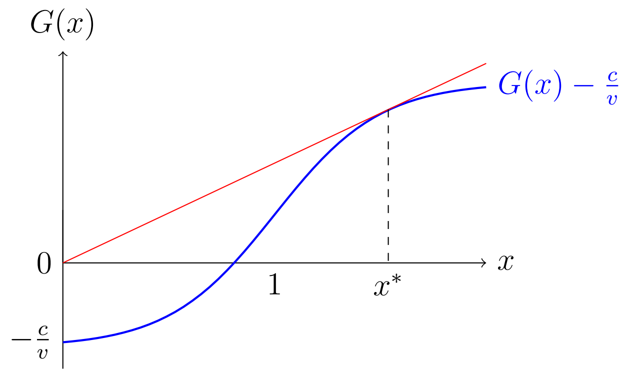
The optimal price then equals $vG(x^*)$. Differentiating the expression in Proposition 1 implies that whenever $x^* > 1$, it is given by the expression

$$x^* g(x^*) - G(x^*) = -\frac{c}{v} \tag{3}$$

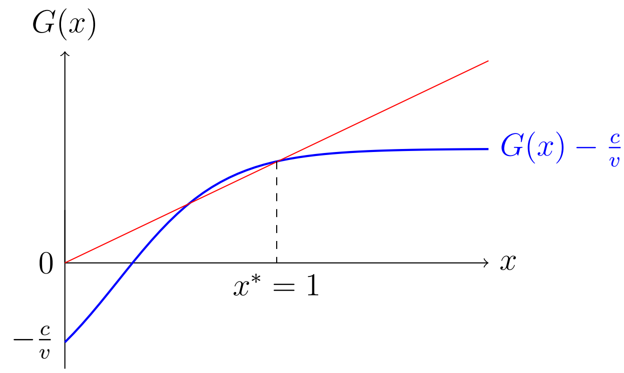
At the same time, for some shapes of G, we can have a corner solution, at which $x^* = 1$, and hence $F\left(\hat{\theta}\right) = 0$ – thus, the firm selects a price that ensures full entry. These two cases are illustrated in Figure 1.

Which of these cases applies depends on the shape of G. However, when G is unimodal, we can derive necessary and sufficient conditions for either case:

Proposition 2. *Suppose G is strictly convex on $(0, k)$, and strictly concave on $(k, +\infty)$ for some $k \in (0, +\infty)$. If $k < 1$ and $G(1) \geq g(1) + \frac{c}{v}$, then $x^* = 1$. Otherwise, $x^* > 1$ and is given by (3).*



(a) $x^* > 1$



(b) $x^* = 1$

Figure 1: Firm's equilibrium strategy

Proof. See Appendix. □

Hence, when G is unimodal, the firm sets the price that induces full entry when the mass of candidates tends to be low – specifically, when (i) the modal mass of candidates is lower than the mass of prizes; and (ii) the probability that the mass of candidates is lower than the mass of prizes is sufficiently large. Otherwise, the firm sets a price at which some candidates do not enter.

4 Comparative Statics

Suppose competition between candidates increases. Formally, suppose that G is replaced by some other distribution $\tilde{G}$ that first order stochastically dominates G – that is, $\tilde{G}(y) < G(y), \forall y \in (0, +\infty)$. How does this affect the intensity of entry and the optimal price?

If the price was exogenously fixed at some level, then entry would be determined by (1). A shift from G to $\tilde{G}$ would decrease the left-hand side of (1). To restore the equality, $\hat{\theta}$ would have to increase, so the proportion of candidates that enter would fall[2].

Here, however, the price is endogenously chosen by the firm. How does a shift from G to $\tilde{G}$ affect entry and prices? Let $\tilde{x}^*$ be the optimal x under $\tilde{G}$. Consider first the case when G and $\tilde{G}$ are such that $x^* = \tilde{x}^* = 1$ – for

[2]A similar result emerges, for example, in Ginzburg (2019), in the case where entry is required for receiving the prize.

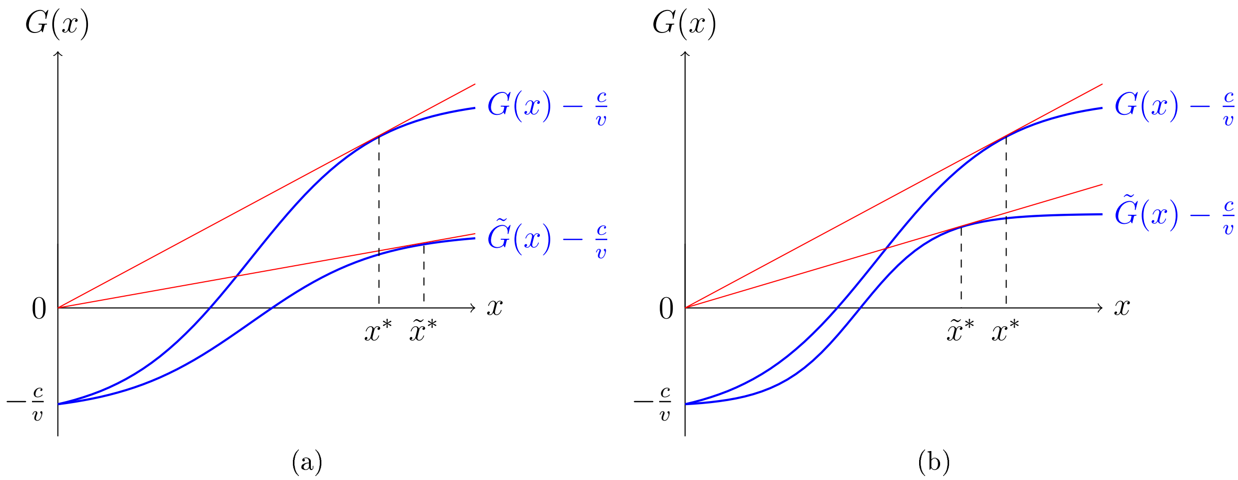


Figure 2: Effect of an increase in competition when $x^*, \tilde{x}^* > 1$.

example, when G and $\tilde{G}$ satisfy the conditions in Proposition 2. Then we have the following result:

Proposition 3. *Suppose G and $\tilde{G}$ are such that $x^* = \tilde{x}^* = 1$. If $\tilde{G}$ first order stochastically dominates G, then the optimal price is lower under $\tilde{G}$ than under G.*

Proof. The optimal price under G and under $\tilde{G}$ equals $vG(1)$ and $v\tilde{G}(1)$, respectively. Furthermore, $\tilde{G}(1) < G(1)$, implying the result. □

Hence, when the mass of candidates tends to be low, a shift of G towards greater mass decreases the optimal price, while maintaining full entry.

Now consider the case when G and $\tilde{G}$ are such that $x^* > 1$ and $\tilde{x}^* > 1$. Then a shift from G and $\tilde{G}$ can both increase and decrease entry, as Figure 2 shows.

As a specific example of a shift in distribution, suppose $\tilde{G}(x) = G(ax)$ for some $a \in (0, 1)$. A shift from G to $\tilde{G}$ then represents a proportional increase in the mass of candidates for every draw made by nature[3]. We can

[3]Specifically, consider again the example of university admission test in the introduction, and suppose that $y = \lambda Y$, where Y is the total mass of people of a particular age drawn from the distribution $G\left(\frac{Y}{\lambda}\right)$, and λ is the proportion of people for whom the benefit of higher education is higher than the cost. Then an increase in that proportion from λ to

show that such a shift does not change the optimal price, while reducing the intensity of entry:

Proposition 4. *Suppose that* $\tilde{G}(x) = G(ax)$ *for* $a \in (0,1)$*, and that* G *and* $\tilde{G}$ *each induce a unique equilibrium with* $x^* > 1$ *and* $\tilde{x}^* > 1$*. Then the optimal price is the same under* $\tilde{G}$ *as under* G*. Furthermore,* $\tilde{x}^* > x^*$*.*

Proof. See Appendix. □

Informally, Propositions 3 and 4 together imply that if the mass of candidates tends to be low, a shift in the distribution towards more mass initially induces the firm to lower the price while maintaining full entry. Eventually, however, a further proportional shift leads the firm to maintain the same price while allowing the proportion of candidates who enter to fall.

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$\frac{\lambda}{a}$ – for example, as a result of an increase in returns to education, or a decrease in tuition fees – corresponds to a shift from G to $\tilde{G}$.

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Appendix

Proof of Lemma 1. Let $h(\theta)$ be the equilibrium probability that a candidate with type θ enters the competition. If a candidate with type θ enters, her expected payoff is

$$v \Pr\left[y \int_{\theta}^{1} h(t)\, dF(t) \leq 1\right] - p = vG\left[\frac{1}{\int_{\theta}^{1} h(t)\, dF(t)}\right] - p \equiv w(\theta)$$

which is weakly increasing in θ. Note that $w(\theta) \geq 0$ for all types for which $h(\theta) > 0$, and $w(\theta) \leq 0$ for all types for which $h(\theta) < 1$. Denote $\hat{\theta} \equiv \sup\{t \mid h(t) < 1\}$, that is, the highest type that does not enter with certainty.

Suppose first that $\hat{\theta} = 0$. Then $w\left(\hat{\theta}\right) = w(0) = vG(1) - p$. If $w(0) > 0$, then $w(\theta) > 0, \forall\theta$, so the firm can raise its profit by increasing p. Hence, we must have $w\left(\hat{\theta}\right) = vG(1) - p = 0$, so (1) holds.

Suppose instead that $\hat{\theta} > 0$. Take some $\theta' < \hat{\theta}$. If $h(\theta') > 0$, then $w(\theta') \geq 0$. But $w\left(\hat{\theta}\right) \leq 0$, and $w(\cdot)$ is strictly increasing over any interval over which $h(\cdot) > 0$, so we must have $h(t) = 0$ for almost all $t \in \left(\theta', \hat{\theta}\right)$. Thus, $h(\theta) = 1$ for all $\theta > \hat{\theta}$, and $\int_0^{\hat{\theta}} h(\theta)\, dF(\theta) = 0$. Hence, $\hat{\theta}$ is given by $vG\left[\frac{1}{\int_{\hat{\theta}}^{1} dF(t)}\right] = p$, which is equivalent to (1). □

Proof of Proposition 2. If $k \geq 1$, then, at $x = 1$, G is convex, so $\frac{G(x)}{x}$ is increasing. Since the firm's profit equals $\frac{vG(x)-c}{x}\text{E}[y] = \left(\frac{vG(x)}{x} - \frac{c}{x}\right)\text{E}[y]$, it is increasing in x at $x = 1$. Thus, $x^* > 1$.

If $G(1) < g(1) + \frac{c}{v}$, then

$$\left.\frac{d}{dx}\right|_{x=1}\left(\frac{vG(x)-c}{x}\text{E}[y]\right) = [vg(1) - vG(1) + c]\,\text{E}[y] > 0$$

Hence, profit is again increasing in x at $x = 1$. Thus, $x^* > 1$.

Now consider the case when $G(1) \geq g(1) + \frac{c}{v}$ and $k < 1$. The former implies that the left-hand side of (3) is weakly smaller than $-\frac{c}{v}$ at $x^* = 1$. The latter implies that $g'(x) < 0$ for all $x > 1$, and thus for all $x^* \in (1, +\infty)$ the derivative of the left-hand side of (3) with respect to x^* equals

$$g(x^*) + x^* g'(x^*) - g(x^*) = x^* g'(x^*) < 0$$

Hence, the left-hand side of (3) is decreasing in x^*. Thus, (3) is not satisfied for any $x^* > 1$, so $x^* = 1$. □

Proof of Proposition 4. If $\tilde{G}(x) = G(ax)$, then the associated density equals $\tilde{g}(x) = ag(ax)$. Then $\tilde{x}^*$ is given by (3) as

$$G(a\tilde{x}^*) = \frac{c}{v} + a\tilde{x}^* g(a\tilde{x}^*)$$

To show that that $\tilde{x}^*$ is decreasing in a, differentiate both sides with respect to a to obtain

$$g(a\tilde{x}^*)\left(\tilde{x}^* + a\frac{\partial\tilde{x}^*}{\partial a}\right) = g(a\tilde{x}^*)\left(\tilde{x}^* + a\frac{\partial\tilde{x}^*}{\partial a}\right) + a\tilde{x}^* g'(a\tilde{x}^*)\left(\tilde{x}^* + a\frac{\partial\tilde{x}^*}{\partial a}\right)$$

Hence, $a\tilde{x}^* g'(a\tilde{x}^*)\left(\tilde{x}^* + a\frac{\partial\tilde{x}^*}{\partial a}\right) = 0$, which implies that $\frac{\partial\tilde{x}^*}{\partial a} = -\frac{\tilde{x}^*}{a} < 0$.

At the same time, the optimal price equals

$$vG(a\tilde{x}^*) = c + va\tilde{x}^* g(a\tilde{x}^*)$$

Hence,

$$\frac{\partial\left[vG(a\tilde{x}^*)\right]}{\partial a} = vg(a\tilde{x}^*)\left(\tilde{x}^* + a\frac{\partial\tilde{x}^*}{\partial a}\right) + va\tilde{x}^* g'(a\tilde{x}^*)\left(\tilde{x}^* + a\frac{\partial\tilde{x}^*}{\partial a}\right) = 0$$

where the last equality follows from the fact that $\tilde{x}^* + a\frac{\partial\tilde{x}^*}{\partial a} = 0$. □