Mathematical Economics - Marginal analysis in the consumer behavior theory

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Abstract

In the neoclassical theory, the economic value of a good is determined by the benefit that an individual consumer attributes to the last ("marginal") unit consumed. Marginal analysis was introduced to the theory of value by William Jevons, Carl Menger and Léon Walras, the founders of marginalism. Since the so-called “marginalist revolution” of the 1870s, differential (or infinitesimal) calculus has been applied to the mathematical modelling of economic theories. Our goal is to present some consumer behavior models, their advantages and limitations, using the methodology of economic science. It should be emphasized that each (re)formulation is based on different economic principles: diminishing marginal utility, diminishing marginal rate of substitution and weak axiom of revealed preference.

1. Introduction

Economics is the social science that has incorporated the most mathematics into its theories and models. The formulation and application of mathematical methods to represent economic principles gave rise to a new area of study called Mathematical Economics. The theory of value was one of the first theories to be analyzed using a mathematical framework. The so-called “marginalist revolution” in Economics, at the beginning of the 1870s, is intimately related to the use of differential (or infinitesimal) calculus. For W. Jevons, C. Menger and L. Walras, the founders of marginalism[i], the economic value of a good is determined by the benefit (satisfaction or pleasure or utility) that an individual consumer attributes to the last ("marginal") unit consumed. Jevons and Walras assumed that the marginal utility of a good could be measured by the rate of change of utility as the quantity consumed changes in infinitely smalls units. In its mathematical formulation, the marginal utility of a good is represented by a first order partial derivative of a utility function with respect to the quantity consumed. Unlike Jevons and Walras, C. Menger presented a table of consumer needs-satisfaction to describe the subjective nature of value in a more qualitative analysis. [ii] The classic diamond-water paradox is then explained by the existence of two meanings of value: value-in-use and value-in-exchange.

2. Carl Menger – An economist who kept the focus on the meaning of value

Menger kept the focus on the main point underlying the determination of value, arguing that human nature determines decisions leading to action in the economy.[iii] In the beginning of his book (Menger, 1950 [1871]) wrote: “All things are subject to the law of cause and effect”. The cause-effect relation is inherent to every decision an individual makes in the particular circumstances he faces at each moment. The definition of the economic value of a good as the benefit of the marginal unit consumed remained in Menger's mind. This is evaluated in opposition to alternative uses of other goods. In fact, economic goods are scarce and human effort must be made to provide for their availability. Not all needs associated with these goods can be satisfied by all individuals. Their value is defined by the importance that an individual attributes to the satisfaction of needs that result from the consumption of the last unit of the good that he can dispose of. Each individual establishes a scale of importance for additional units consumed of diverse economic goods.[iv] The value attributed to a good is inherently subjective, depending on the needs and preferences of each individual which are also determined by the particular context he faces at each moment and subject to rapid changes. Hence, population heterogeneity cannot be avoided and the existence of a stable function across time representing aggregated demand for the good is precluded. The optimization models created later on (see next section) are based on oversimplified assumptions such as homogeneity which is implicit in the kind of economic agent idealized in economics and for whom optimization is meant to be performed. Menger remarked that value is not an intrinsic property of goods but results from the importance attributed by individuals to concrete units of goods. He strongly stressed the distinction between value-in-use and utility. All goods, including noneconomic ones, have utility to the extent that their consumption satisfies needs, but it is only when a good is scarce for all needs in the population that it becomes economic and its units acquire value. Thus the term "value-in-use" signifies what other economists call "utility" and "total utility" is a nonsensical concept.

3. Mathematical Models and Methodologies

Like in Physics, neoclassical economic theories focus in the equilibrium concept (Mirowski, 1991). The main goal of consumer behavior models is to explain the relationship between the prices and quantities of goods demanded by markets.[v] Mainstream economics preferred the partial equilibrium analysis developed by Marshall[vi] (Marshall, 1920) over the general equilibrium analysis of Walras. The highlight on mathematical representation is the Marshallian cross diagram illustrated in the introductory economics textbooks. It is used to show the equilibrium price of a good that results from the intersection of the demand and supply curves. We will present three models[vii] of consumer behaviour using distinct methodologies. The first two are neoclassical consumer models based on cardinal and ordinal utility theories, respectively. The demand curves can be derived from utility maximization in both models. The third model is formulated from restrictions on observable data (choices) and is called Samuelson's revealed preferences model.

3.1 The neoclassical consumer model based on the cardinal utility theory

Neoclassical microeconomics adopted the following definition (Robbins, 1984): "Economics is the science which studies human behavior as a relationship between ends and scarce means which have alternative uses". The neoclassic consumer model is a theoretical model in which an individual consumer is an economic agent whose behavior is influenced by three assumptions. First, there is an allocation of scarce means, that is, the consumer spends his (or her) income I buying a vector of quantities of n goods, $q = (q_1, q_2, \dots, q_n)$, at given unit prices vector $p = (p_1, p_2, \dots, p_n)$ in a market. The consumer's behavior then depends on a subjective utility that he (or she) attributes to goods, which is represented by a unique function, $u(q_1, q_2, \dots, q_n)$, in the cardinal utility theory. Finally, the consumer will maximize utility by the rationality principle. In order to use marginal analysis, the utility function is assumed to be twice continuously differentiable that satisfies the following three axioms: First, goods are continuously divisible which implies continuity[viii] of the utility function; Second, the marginal utility of each good is positive, which means the consumer prefers more rather than fewer goods, so that utility increases as the consumption of one good increases, holding the consumption of the other goods constant; Third, the diminishing marginal utility principle states that if the consumption of one good increases, then its marginal utility decreases, holding the consumption of the other goods constant. Thus, it is supposed that

$$\frac{\partial u}{\partial q_i}(q) > 0 \qquad \wedge \qquad \frac{\partial^2 u}{\partial {q_i}^2}(q) < 0, \text{ for all } q = (q_1, q_2, \dots, q_n),\ i = 1, \dots, n.$$

The consumer problem is to choose a vector of goods that maximizes the utility function $u(q)$ subject to the budget constraint $p_1 q_1 + p_2 q_2 + \cdots + p_n q_n = I$. The mathematical model is then represented by a constrained optimization problem on the set $\mathbb{R}_n^+ = \{(q_1, q_2, \dots, q_n): q_i > 0\}$, which can be solved by applying the Lagrange multipliers method. The first order necessary conditions are given by

$$\frac{\frac{\partial u}{\partial q_1}(q)}{p_1} = \frac{\frac{\partial u}{\partial q_2}(q)}{p_2} = \cdots = \frac{\frac{\partial u}{\partial q_n}(q)}{p_n} = \lambda \quad \wedge \ p_1 q_1 + p_2 q_2 + \cdots + p_n q_n = I.$$

It is well known that if the utility function is strictly quasiconcave, then the bordered Hessian matrix is negative definite, hence the problem has a unique solution $P_e = ({q_1}^e, {q_2}^e, \dots, {q_n}^e)$. It is said that P_e is the optimal bundle in the market for the consumer. For arbitrary prices and income, n equilibrium demand functions on the set $\mathbb{R}_{n+1}^+ = \{(p_1, p_2, \dots, p_n, I): p_i > 0 \wedge I > 0\}$ are deduced, defined by ${q_i}^e = f_i(p_1, p_2, \dots, p_n, I)$, by solving the first order necessary conditions explicitly in order to determine q_i.

3.2 The neoclassical consumer model based on the ordinal utility theory

Following Pareto's idea of ordinal utility (Pareto, 1909), Hicks asserted: "The quantitative concept of utility is not necessary in order to explain market phenomena".[ix] Rejecting the marginal utility notion and consequently the diminishing marginal utility principle, the concept of the marginal rate of substitution (MRS) between goods was introduced by Hicks and Allen (Hicks, 1934) to develop indifference curves analysis. Given any two goods X e Y, the MRS of Y for X measures an amount of good Y that the consumer is willing to give up in order to gain an incremental increase of consumption of X. The neoclassical consumer model is a theoretical model in which a rational consumer seeks to maximize her (or his) utility subject to the budget constraint $p_1 q_1 + p_2 q_2 + \cdots + p_n q_n = I$. In the framework of the ordinal utility theory,[x] an individual consumer has a scale of preferences, which could be represented by a utility function, $u(q_1, q_2, \dots, q_n)$, (Debreu, 1959). Consumer behavior is limited by three assumptions: First, if consumer's preferences are defined on the set $\mathbb{R}_n^+ = \{(q_1, q_2, \dots, q_n): q_i > 0\}$, then given any two bundles, she (or he) will prefer one of those or will be indifferent (an indifference hypersurface is defined as the set of all bundles of goods which have the same preference rank or utility level); Second, the consumer will prefer more to fewer goods, meaning that she (or he) will choose the vector of goods that belongs to the indifference hypersurface with the highest rank among those she (or he) can afford; Third, the diminishing MRS principle states that the rate will decrease as Y good is substituted for X along an indifference hypersurface. Given n goods, X_i, $\tau_i(q)$ denotes the MRS of X_n for X_i, $i \neq n$. If we suppose that the marginal rate of substitution, $\tau(q) = (\tau_1(q), \tau_2(q), \dots, \tau_{n-1}(q))$, is a continuously differentiable mapping satisfying the properties of positivity and convexity, then there is an indifference map that consists of a one-parameter family of indifference hypersurfaces. From the economic point of view, the best bundle P_e satisfies $\tau_i(q) = \frac{p_i}{p_n}$, and from the geometric point of view, the optimal bundle P_e, solution of the constrained maximization problem, belongs to the indifference hypersurface with highest parameter (utility level). We note that, assuming that the expression of a utility function is unknown, there is an alternative approach in which the consumer's preferences can be characterized by the marginal rate of substitution between goods, $\tau(q)$, using ordinary differential equations (we can observe the particular case of $n = 2$ goods in (Marques, 2014)).

3.3 Samuelson's revealed preferences model

Samuelson[xi] provided a step forward in getting rid of the unnecessary and explicit reference to the utility concept. He proposed a new methodology based on observable market data. In his approach, called "revealed preferences", it is assumed that an individual's choices (rather than preferences) are empirically determinable from the prices of goods and the income available for consumption. Samuelson's revealed preferences model is designed to deduce the conditions to be imposed on demand by formulating three axioms: First, the existence of n continuously differentiable demand functions $q_i = f_i(p_1, p_2, \dots, p_n, I)$, subject to the budget constraint $p_1 q_1 + p_2 q_2 + \cdots + p_n q_n = I$, is assumed; Second, it is assumed that demand functions are homogeneous of degree zero, meaning that these functions are independents of monetary units[xii]; The third axiom is known as the "weak axiom of revealed preference" (WARP), which states that, for any pair of bundles q^1 and q^2, if q^1 is preferred to q^2, $q^2 < q^1$, then $q^1 \nless q^2$. From these axiom, expressing the consistency of consumer behavior, Samuelson deduced that the Slutsky substitution matrix must be negative semidefinite (see (Mas-Colell, 1995)). This is deduced as follows: given prices vector $p = (p_1, p_2, \dots, p_n)$, it is supposed that the bundles $q^1 = (q_{11}, q_{21}, \dots, q_{n1})$ and $q^2 = (q_{12}, q_{22}, \dots, q_{n2})$ have the same total cost, that is, $\sum_{i=1}^{n}(q_{i1} - q_{i2})p_i = 0$. If $q^2 < q^1$ (so that, at price p, q^1 was chosen instead q^2) then WARP implies that when prices change (from p to p'), consumer preferences are unchanged so that $\sum_{i=1}^{n}(q_{i1} - q_{i2})\, p'_i > 0$. Let $q_{i2} = q_{i1} + \Delta\, q_i$ and $p'_i = p_i + \Delta\, p_i$, after some algebraic calculus, we have $\sum_{i=1}^{n} p_i \Delta q_i = 0$ and $\sum_{i=1}^{n} \Delta p_i \Delta q_i < 0$. Taking these expressions to the limit and using $dI = \sum_{j=1}^{n} q_{j1} \mathrm{d}p_j$ we obtain

$$\sum_{i=1}^{n} p_i \mathrm{d}q_{i1} = 0 \ \wedge \ \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial q_{i1}}{\partial I} q_{j1} + \frac{\partial q_{i1}}{\partial p_j} \right) dp_j\, dp_i < 0.$$

A decade later, Samuelson (Samuelson, 1948), recognized that the revealed preferences logic is complementary to the preferences theory based on ordinal utility. Indeed, Houthakker (Houthakker, 1950) has shown that if consumer preferences are transitive[xiii], then the revealed preferences approach should be able to empirically reconstruct the indifference map on which the ordinal utility theory relied.

Samuelson's model is an economic choice model that draws conclusions exclusively based on observed behavior, making no psychological or philosophical considerations which may be misleading if based on more or less arbitrary assumptions. Samuelson claimed that what matters are the exchanges that a consumer really makes, not the exchanges he claims he would make. The WARP formulation eloquently exhibits the flaw pointed out by the Austrian School, that is, it assumes time stability in individuals' choices, which may be hard to justify except in special circumstances. We further suggest reading Wong's book (Wong, 2006) for a critical analysis of Samuelson's model using Popper's method of rational reconstruction.

4. Karl Menger – A mathematician with a heavy heritage

Karl Menger, Carl Menger's son, was a mathematician with some works in mathematical economics, but his father, the founder of the Austrian School of economics, gave priority to other methods of economic research rather than mathematics. He was led to try to connect these two antagonistic perspectives (Menger, 2003). In his opinion these were, above all, two different forms of expressing ideas on subjects they could agree. While mathematical economics used mostly mathematics, descriptive language was the privileged means for the Austrian School. On the issue of goods valuation, he considered that, unlike what mathematical economists might think, formal mathematical presentation did not add anything in generality and precision to Austrian reasoning. On the contrary, he argued that mathematical analysis often imposed unnecessary assumptions, for instance, continuity and differentiability properties, which are not based on observed facts in the economy. In the case of the marginal utility of a good, this concept is interpreted as the limit of the rate of change of utility when the quantity increment of that good tends toward zero. Karl Menger assayed the mathematical formalization of the Austrian reasoning by defining a non-decreasing and convex utility function, f, to express the idea of a decreasing rhythm of utility change. For simplicity, f is assumed to be a function of quantity consumed of only one good such that

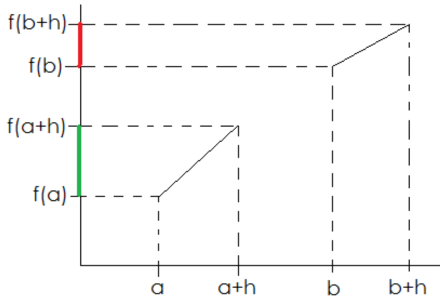
$$\frac{f(c)-f(b)}{c-b} \leq \frac{f(b)-f(a)}{b-a}, \text{for } a < b < c.$$

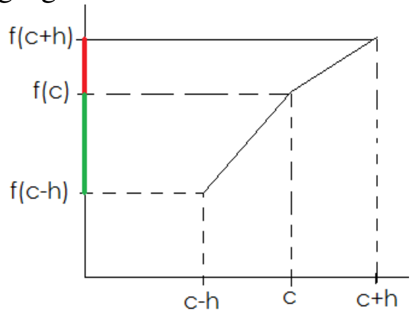
He also considered generalizations given by

(i) $For\ a < b, h > 0, \quad |f(b+h) - f(b)| \leq |f(a+h) - f(a)|;$

(ii) $For\ h > 0, \quad f(c+h) - f(c) \leq f(c) - f(c-h);$

with corresponding graphic representations in the following figures.





It is highly doubtful that Karl Menger succeeded in making the two perspectives compatible. In fact, this formalization does not account for specific dimensions of Austrian analysis such as subjectivity in valuation or the importance of time in decision-making, namely the implications of the absence of a time-stable utility function.

5. Conclusion

In a famous quote, Hayek said: "The curious task of economics is to demonstrate to men how little they really know about what they imagine they can design" (Hayek, 1988). At first sight, it would seem that it is just the definition (Robbins, 1984) of scarce means to unlimited ends, but this is not so. Taking a more humanistic approach to economics, Hayek's logic goes far beyond Robbin's since an affectation of scarce means to multiple ends does not imply only a "mechanic" model of constrained optimization. In this approach, it would be necessary to impose more effective and realistic assumptions. For instance, exploring the motivations of each individual in his complexity, taking into account his specificity and subjectivity. The subjectivity concerns not only individual's idiosyncratic preferences but also the unique environmental circumstances he faces at every moment. In the richness of everyone's freely lived life, there necessarily exists a highly heterogeneous population, most of the time not represented in a representative agent model reflecting everybody's choices. Another important point is the recognition that economic individuals are limited in their resources as well as economic researchers and political decision makers. In a highly complex framework with constant novel information, economic analysis is more efficient using a network of individual decision makers where each one manages little information, rather than using central planning where effective decisions are usually not available even to the most potent supercomputer. However, the mathematical models presented here have made important contributions to understanding consumer behavior theory. Nowadays, an interdisciplinary approach involving concepts from all social sciences concerned with human nature is taken to study this complex subject.

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[i] New ideas about the economic value of goods were expressed in independent works by Jevons in England (Jevons, 1871), Menger in Austria (Menger, 1950 [1871]) and Walras in Switzerland (Walras, 1874).

[ii] The marginalist analysis refuted the classic "labor theory" of value.

[iii] The method leading to the study of human action is called Praxeology, used as a research method by the Austrian School of Economics, founded by Menger. This school defends that the use of differential calculus is deemed excessively simplistic for analyzing the complexity of economic decisions.

[iv] For instance, the first unit of the most essential good, e.g., food, has the highest importance, its second unit has less importance but may have as much importance as the first unit of the second most relevant good, and so on.

[v] On the supply side, producers sell goods in markets by minimizing their costs.

[vi] Marshall introduced a fundamental assumption in economic analysis, known as *ceteris paribus*, to study a relationship between two variables while holding others constant in a short period of time.

[vii] These models are normative models that only describe what rational consumers should do (Thaler, 1980).

[viii] Marshall adopted the expression *Natura non facit saltum* in his book (Marshall, 1920).

[ix] “The equilibrium conditions [first order conditions] and the stability conditions [second order conditions] for an individual consumer have been written out assuming the existence of a particular utility function u. This is, indeed, the most convenient way of writing them; but it is important to observe that they do not depend upon the existence of any unique utility function” (Hicks, 1939).

[x] In the framework of ordinal utility, a utility function u is not unique because $U = g \circ u$ is also a utility function whenever g to be is a strictly increasing function.

[xi] “I propose, therefore, that we start anew in direct attack upon the problem, dropping off the last vestiges of the utility analysis. This does not preclude the introduction of utility by any who may care to do so, nor will it contradict the results attained by use of related constructs. It is merely that the analysis can be carried on more directly, and from a different set of postulates.” (Samuelson, 1938).

[xii] It allows the expressing of demand functions in terms of relative prices of each good with respect to a numeraire good having a price equal to one.

[xiii] It means that the “strong axiom of revealed preference” holds.