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# A Generalized Growth Model and the Direction of Technological Progress\*

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**Abstract:** Based on a general growth model, this paper finds that the steady-state direction of technological progress is determined by the scale return of the production function and the relative factor supply elasticities. A specific version of that model extends Acemoglu (2002) to provide the underlying determinants of the supply elasticities and demonstrates that the relative price (Hicks, 1932) and relative market size (Acemoglu, 2002) have only a short-term impact on the direction of technological progress. A consequence of the analysis is that the steady-state technological progress is purely labor-augmenting (i.e. delivers Uzawa's steady-state theorem) if and only if the scale return of the production function is constant and the supply elasticity of capital is infinite. Analogously, an infinite labor supply elasticity is required if labor-augmenting technological progress is to be excluded prior to the Industrial Revolution. Accordingly, changing factor supply elasticities may have induced the Industrial Revolution.

**Key Words:** Economic Growth, Direction of Technological Progress, Returns to Scale, Factor Supply Elasticities, Uzawa's Steady-State Theorem, Industrial Revolution

**JEL:** O33; O41; E13; E25

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**Comments are welcome!**

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# A Generalized Growth Model and the Direction of Technological Progress

## I. Introduction

There is a large and influential body of literature concerning the determinants of the *rate* of technological progress (see, e.g., Romer, 1990; Aghion and Howitt, 1992). However, the *direction* of technological progress is not as well understood. In particular, according to Kaldor (1961), post Industrial Revolution economic growth in developed countries has been characterized by increasing per-capita output and physical capital, whereas the capital/output ratio and factor income shares have remained basically constant.<sup>3</sup> These facts are generally interpreted as indicating that technological progress has been purely labor-augmenting. In contrast, Ashraf and Galor (2011) show that during the preindustrial era, technological progress generated population growth and higher density, but not higher per-capita income, which may imply that during that period technological progress lacked labor augmentation. What are the underlying factors implying that labor-augmentation played no role before the industrial revolution but became dominant afterwards? More generally, what determines the direction of technological progress?

Unlike the existing literature (Acemoglu, 2002; Irmen and Tabakovic, 2017), which addresses these questions through the lens of specific growth models, here we first adopt a general approach, assuming a general production function and general factor accumulation processes. Using this setup, we find that, the crucial determinants of the steady-state direction of technological progress are the returns to scale of the production function and the relative factor supply elasticities. In particular, if the production function has constant returns to scale, the steady-state direction of technological progress depends on the relative factor supply elasticities, and is biased towards the factor with the lower elasticity.

A specific version of that model extends Acemoglu's (2002) micro-mechanism of technological progress to further identify the determinants of the supply elasticities. This version also shows that changing relative factor prices (as suggested by Hicks 1932) and the relative market size (as argued by Acemoglu 2002) do affect the direction of technological progress in the short run, but in steady-state they turn out to have no impact on that direction. The intuition behind this result is the following. In the short run, a higher factor price may encourage not only invention to economize that factor's use, but also its accumulation. If the supply elasticity of the factor is very large, the

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<sup>3</sup>These stylized characteristics are further supported by Jones (2015) using the latest available data.

invention incentive may be reversed. Furthermore, to offset that factor's abundance, balanced growth requires an increased investment in technologies that augment the efficiency of the factor with the smaller supply elasticity, which leads to the aforementioned extreme configurations of technological progress.

To fix ideas, consider the case of oil. During a long time period, oil was abundant, and hardly any effort was put into economizing its use, as evidenced by the MPG (Miles per Gallon) figures of cars produced in the U.S. before the 1973 oil crisis.<sup>4</sup> That crisis has caused a sharp increase in oil prices, inducing investment in energy-saving technologies (e.g., increasing MPG). However, the same price increase also induced search for new oil sources, such as shale oil. These new sources have again increased the supply of oil, eventually contributing to sharp price decreases. Consequently, incentives to further invest in energy-saving technologies have decreased.

With this intuition in mind, the paper suggests the following answers to the aforementioned questions. In the pre-industrial era technological progress did not increase labor productivity because labor supply was very elastic (as described by Malthus 1798). Approximately concurrent with the industrial revolution, the demographic transition reduced the supply elasticity of labor. Moreover, the industrial revolution has replaced land by reproducible physical capital. As the supply elasticity of capital increased, there were no incentives to economize on its use and improve its productivity. Consequently, technological progress was biased towards improving human capital, thereby increasing labor productivity. Hence, according to this interpretation, it is the reduction of the labor supply elasticity that has led to the changing direction of technological progress.

Altogether, the contributions of this paper to the literature are as follows: **first**, within the generalized growth model, it identifies the general factors that determine the direction of technological progress irrespective of the specific parameters of the factor accumulation processes; **second**, it suggests a specific version of that model with micro-founded technological progress in order to analyze the impact of the production function parameters and those of the factor accumulation processes on the direction of technological progress; **third**, it delineates the necessary and sufficient conditions that deliver Uzawa's (1961) steady-state theorem in the neoclassical growth model; **fourth**, it highlights the simultaneous impact of the production technology and factor accumulation processes on the direction of technological progress.

The plan of the paper is as follows. Section II discusses the related literature;

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<sup>4</sup> According to the PEW Environment Group, the model-year 1975 cars drove about 14 miles per gallon. This figure has doubled by 1985, and stayed roughly stagnant for the next two decades, rising to about 33 by 2005 (see <http://www.pewtrusts.org/~media/assets/2011/04/history-of-fuel-economy-clean-energy-factsheet.pdf>), constructed

Section III analyzes the determinants of technological change in a general growth model ; Section IV turns to the same question within a specific model; Section V focuses on some applications: first, Uzawa's steady-state theorem and second, the Industrial Revolution; Section VI contains concluding remarks.

## II. Related Literature

The direction of technological progress has been a long-standing object of economics research. Hicks (1932) pointed out that changing relative factor prices may affect that direction. Brozen (1953) also pointed out that the direction of technological progress was endogenously determined by economic forces. However, lacking a dynamic growth framework (to be developed by Solow a few years later), these early contributions could not distinguish between short-term and long-term effects.

The neoclassical growth models (Solow, 1956; Swan, 1956) provided this perspective and pointed out that technological progress was the key factor of economic growth in the long run. However, the direction of technological progress turned out to be a cumbersome issue in the neoclassical growth models. As pointed out by Uzawa's (1961) steady-state theorem, if a neoclassical growth model exhibits steady-state growth, then technological progress must be labor augmenting.<sup>5</sup> Although the steady-state equilibrium with purely labor-augmenting technological progress is consistent with Kaldor's (1961) stylized facts, there are no compelling intuitive reasons as to why technological progress should per-force take this specific form.

The introduction of an innovation possibility function (Weizsacker, 1962; Kennedy, 1964) coupled with cost reduction maximization, has seemingly enabled the induced innovation literature of the 1960s (Samuelson, 1965; Drandakis and Phelps, 1966) to resolve this issue. However, Nordhaus (1973) questioned the validity of this resolution for its lack of micro-mechanisms generating technological progress. The assumption that enterprises maximize the current rate of cost reduction rather than profits was also criticized (Acemoglu, 2001). In addition, the approach does not clarify the key factors responsible for Uzawa's theorem.

After nearly 30 years of silence, Acemoglu (1998, 2002, 2003) tried to reactivate the research on the determinants of technological progress based on endogenous growth models (Romer, 1990; Aghion and Howitt, 1992). In these settings, the one-dimensional technology was expanded to two dimensions, enabling researchers and developers to choose between two types of technological innovations. Within this framework,

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<sup>5</sup> Or the production function must be of the Cobb-Douglas type. In the sequel we view this conclusion as too restrictive and ignore that caveat.

Acemoglu (2002) proposes a market size effect as another key factor affecting the direction of technological progress besides the price effect of Hicks (1932). However, Acemoglu (2002) focuses on the determinants of the steady-state *relative* level of technology which remains constant only with Hicks-neutral technological progress. When Acemoglu (2003) incorporated the framework into the neoclassical growth model, it again yielded a steady-state growth path with purely labor-augmenting technological progress, failing to explain why enterprises pursuing maximum profits choose *only* that form of technological progress.

Some additional authors (Funk, 2002; Irmen and Tabakovic, 2017) have constructed growth models which endogenize the direction of technological progress. These models are based on perfect rather than monopolistic competition. However, because the key parameters determining the direction of technological progress are the same as those of the neoclassical growth model, in the steady-state technological progress must still be purely labor-augmenting. Other papers tried to prove that profit-maximizing firms *choose* to implement purely labor-augmenting technological progress or that the production function converges to the Cobb-Douglas specification (where there is no distinction between various modes of technological progress) based on technology choices among firms (Jones, 2005; Leon-Ledesma and Satchi, 2018). However, these contributions do not discuss what determines the direction of technological progress.

All of the above models share the constraints under which profit-maximizing research and development enterprises choose the direction of technological innovation. Since these constraints are identical to those of the neoclassical growth model, the direction of steady-state technological progress will have to be purely labor-augmenting. Therefore, the direction of technological progress is not determined by whether research and development enterprises can choose technological innovation in different directions, but by the resource endowment conditions that restrict their choices.

Contrary to the above literature which attempts to provide a theoretical basis for Uzawa's theorem, many authors, especially in recent years, have proposed growth models avoiding the implication that steady-state technological progress is purely labor-augmenting. Already in the 1960s, in an argument akin to the one presented below, Samuelson (1965) pointed out that if factor supplies remain in balance, then technological progress will be Hicks-neutral in steady state, otherwise it would be tilted towards labor- or capital-augmentation. Sato (1996) showed that a non-linear capital accumulation process which allows for diminishing returns is necessary if the steady-

state is to include also capital-enhancing technological progress. Sato et al. (1999, 2000) also proposed specific models of that nature where steady-states include capital-enhancing technological progress. Irmen (2013) proved that technological progress could include capital-augmentation if adjustment costs become a part of the capital accumulation process. However, these papers do not consider the impact of the production technology on the direction of technological progress. Grossman et al. (2017) introduce an educational input besides capital and labor, and present a growth model that in steady-state admits both capital-enhancing and capital-embodied technological change. Casey and Horii (2019) also build a model with steady-state capital-augmenting technological progress under a linear capital accumulation process by introducing new factors (such as land) into the production function and decreasing returns to scale for capital and labor. It turns out that these papers all circumvent the constraints underlying Uzawa's theorem without explicitly showing what these constraints are to begin with.

In sum, to the best of our knowledge, the existing literature does not consider the simultaneous impact of the production technology and factor accumulation processes on technological progress, and does not provide an overview concerning the determinants of that direction that would be applicable across different models. This is what we do in the remainder.

### **III. The Determinants of Technological Progress in a General Growth Model**

#### **1. Preliminaries**

Consider an economy that has two factors of material production, denoted by K and L respectively. What K and L stand for can be determined according to the context of the research questions. For example, K can be capital or land, L can be labor or human capital. For simplicity, we refer to them as “capital” and “labor”. There are two kinds of technologies that can enhance the corresponding material elements, namely a labor-enhancing technology, A, and a capital-enhancing technology, B. The economy has a final output, expressed as Y with a production function given by:

$$Y(t) = F[B(t)K(t)^\phi, A(t)L(t)^\varphi] , 0 < \phi \leq 1; 0 < \varphi \leq 1 \quad (1)$$

Define  $\hat{K}(t) \equiv [B(t)K(t)^\phi]$  as representing effective capital, and  $\hat{L}(t) \equiv [A(t)L(t)^\varphi]$  as representing effective labor. Following Barro and Sala-i-Martin (2004, pp. 27), the production function possesses four characteristics regarding  $\hat{K}(t)$  and  $\hat{L}(t)$ : first, constant returns to scale; second, positive and diminishing returns to both effective factors; third, Inada conditions (Inada, 1963); fourth, both effective factors are

essential, so that if any of them equals zero, no output is produced.

Define the elasticity of output  $Y$  with respect to  $\hat{K}(t)$  and  $\hat{L}(t)$  as  $\gamma_{\hat{K}} \equiv \frac{\partial Y}{\partial \hat{K}} \frac{\hat{K}}{Y}$ ,  $\gamma_{\hat{L}} \equiv \frac{\partial Y}{\partial \hat{L}} \frac{\hat{L}}{Y}$ . Because the production function has constant return to scale,  $\gamma_{\hat{K}} + \gamma_{\hat{L}} = 1$ .

The output elasticity with respect to  $K$  and  $L$  is defined as  $\beta_K \equiv \frac{\partial Y}{\partial K} \frac{K}{Y}$ ,  $\beta_L \equiv \frac{\partial Y}{\partial L} \frac{L}{Y}$ , implying  $\beta_K = \phi \gamma_{\hat{K}}$ ,  $\beta_L = \phi \gamma_{\hat{L}}$ . When  $\phi = \varphi = 1$ , then  $\beta_K + \beta_L = \gamma_{\hat{K}} + \gamma_{\hat{L}} = 1$ , and the function  $F(\cdot)$  exhibits constant returns to scale with respect to  $K$  and  $L$ ; when  $\phi < 1$ ;  $\varphi = 1$ , or  $\phi = 1$ ;  $\varphi < 1$ , then  $\beta_K + \beta_L < 1$ , and the function  $F(\cdot)$  exhibits diminishing returns to scale with respect to  $K$  and  $L$ . Therefore,  $\phi$  and  $\varphi$  are the *return to scale factors with respect to  $K$  and  $L$* , respectively.

Output per effective labor is expressed by  $y(t) \equiv \frac{Y(t)}{\hat{L}(t)} = \frac{Y(t)}{A(t)L(t)^\varphi}$  and effective capital per effective labor is expressed by  $k(t) \equiv \frac{\hat{K}(t)}{\hat{L}(t)} = \frac{B(t)K(t)^\phi}{A(t)L(t)^\varphi}$ . Accordingly, the production function in the intensive form becomes:

$$y(t) = F \left[ \frac{B(t)K(t)^\phi}{A(t)L(t)^\varphi}, 1 \right] = f(k(t)) \quad (2)$$

Factor prices are assumed to equal the respective marginal products of  $K$  and  $L$ , i.e.:

$$\begin{cases} w(t) = \varphi A(t)L(t)^{\varphi-1} [f(k) - kf'(k)] \\ r(t) = \phi B(t)K(t)^{\phi-1} f'(k) \end{cases} \quad (3)$$

Equations (3) show that the return to scale factors  $\varphi$  and  $\phi$  have important influence on factor prices. When they are less than 1, even if  $k$  is constant, a factor's price decreases with the increase of its quantity.<sup>6</sup>

In the general model, the factor supply functions are not specified, but are assumed to depend on the respective factor prices. The price elasticities are the key characteristic of the respective factor supplies, and are given by:

$$\begin{cases} \varepsilon_K \equiv \frac{\dot{K}(t)/K(t)}{\dot{r}(t)/r(t)} \\ \varepsilon_L \equiv \frac{\dot{L}(t)/L(t)}{\dot{w}(t)/w(t)} \end{cases} \quad (4)$$

## 2. The Direction of Technological Progress: Definition

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<sup>6</sup> When the market structure deviates from perfect competition, the above prices will deviate from marginal products, but in the following endogenous growth models with monopolistic competition this would not affect the core conclusion of this paper.



The *direction of technological progress*,  $DTP$ , is the ratio between the rates of capital- and labor-augmenting factors, i.e.

$$DTP \equiv \frac{\dot{B}(t)/B(t)}{\dot{A}(t)/A(t)} \quad (5)$$

The range of  $DTP$  is  $[0, \infty]$ . When  $\dot{B}/B = 0$  and  $\dot{A}/A > 0$  then  $DTP = 0$ , and technological progress is purely labor-augmenting (i.e. Harrod-neutral); when  $\dot{B}/B > 0$  and  $\dot{A}/A = 0$  then  $DTP \rightarrow +\infty$ , and technological progress is purely capital-augmenting (i.e. Solow-neutral); when  $\dot{B}/B = \dot{A}/A > 0$  then  $DTP = 1$ , and technological progress is Hicks-neutral. Figure 1 shows different directions of technological progress.

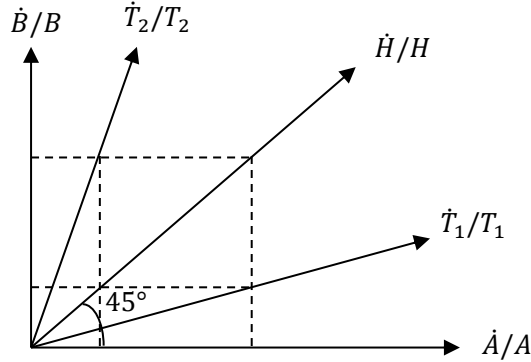


Figure 1: Direction of technological progress

Clearly, the axes represent Harrod-neutral (horizontal) and Solow-neutral (vertical) technological changes. The diagonal  $\dot{H}/H$  represents the location of Hicks-neutral technological changes. The ray  $\dot{T}_1/T_1$  indicates technological progress which tends to be more labor augmenting, while  $\dot{T}_2/T_2$  is more capital augmenting.

Note, that the direction of technological progress is related to the *direction of technology* ( $DT$ ), given by  $DT \equiv B(t)/A(t)$ . Obviously, the direction of technological progress determines the direction of technology, but they are fundamentally different. Specifically, when the direction of technological progress is Hicks neutral, the direction of technology remains unchanged. Otherwise the direction of technology will continuously rise or fall.

### 3. The Determinants of DTP: General Statement

Along a steady-state path the growth rates of  $Y(t)$ ,  $B(t)$ ,  $K(t)$ ,  $A(t)$  and  $L(t)$  are constant. Furthermore,  $k(t) \equiv \frac{B(t)K(t)^\phi}{A(t)L(t)^\phi}$  is also constant, implying:

$$\phi \frac{\dot{K}(t)}{K(t)} + \frac{\dot{B}(t)}{B(t)} = \varphi \frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} \quad (6)$$

In Appendix A we prove the following proposition.

**Proposition 1:** *If the production function of an economy satisfies equation (1) and factor prices equal their respective marginal products, then the steady-state direction of technological progress is given by:*

$$DTP = \frac{(1 + \varepsilon_L)/[1 + (1 - \varphi)\varepsilon_L]}{(1 + \varepsilon_K)/[1 + (1 - \phi)\varepsilon_K]} \quad (7)$$

According to proposition 1, there are two factors that determine the steady-state direction of technological progress: one is the price elasticities of the factor supplies,  $\varepsilon_L$  and  $\varepsilon_K$ ; the other is the factors' return to scale parameters,  $\phi$  and  $\varphi$ . The former reflects the factor accumulation processes, while the latter reflects the production function. Given  $\phi$  and  $\varphi$ , technological progress tends to the factor with the smaller supply elasticity, while it tends to the factor with the smaller return to scale when the factor supply elasticities are given. The relative price (Hicks, 1932) and relative market size (Acemoglu, 2002) do not affect the steady-state direction of technological progress.

For the constant returns to scale (CRS) neoclassical growth model, i.e.  $\phi = \varphi = 1$ , we obtain:

**Corollary 1:** *in the CRS case the direction of technological progress is determined by:*

$$DTP = \frac{1 + \varepsilon_L}{1 + \varepsilon_K} \quad (8)$$

Corollary 1 shows that for the CRS case the steady-state direction of technological progress is determined solely by the relative primary factor supply elasticities and is biased towards the one with the relatively smaller elasticity.

The results so far are based on a very abstract model. We have, as yet, provided no micro-foundation for technological progress. Moreover, we have not established that a steady-state equilibrium exists under some reasonable set of assumptions about individual behavior and the aggregate production function. In the next section, we provide a specific growth model by extending Acemoglu's (2002) framework, in which the direction of technological progress emerges from the optimization problems of enterprises and households.

#### IV. The Determinants of *DTP* in a Specific Growth Model

The Acemoglu (2002) growth model expanded the Romer (1990) technology from one dimension to two. However, it provided only the determinants of the direction of *technology* rather than that of *technological progress*, and that *only* when the steady-state technological progress is Hicks neutral. The current paper extends the Acemoglu model in **two aspects**: one is to allow the production function to admit diminishing returns to scale; the other is to expand the investment elasticities of the factor accumulation processes allowing also the presence of capital-embodied technological progress.

##### 1. The model

We briefly reiterate the Acemoglu (2002) model and emphasize our extensions. The economy consists of two kinds of material factors, and three sectors of production; a final goods sector, an intermediate goods sector and a research and development (R&D) sector. The symbols  $K$ ,  $L$ ,  $S$  represent two kinds of material production factors and “scientists” who specialize in the research and development of new intermediate products, respectively.

The household’s goal is to maximize the discounted flow of utility, given by:

$$U = \int_0^{\infty} \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \quad (9)$$

where  $C(t)$  is consumption at time  $t$ ,  $\rho > 0$  is the discount rate, and  $\theta > 0$  is a utility curvature coefficient of the household.

The household’s periodic budget constraint is given by:

$$C + I_K + I_L = Y = wL + rK + w_S S \quad (10)$$

where the LHS stands for expenditures consisting of consumption and investments  $I_K$  and  $I_L$  into capital and labor, and the RHS is income, obtained from renting out labor at the rate  $w$ , capital at the rate  $r$  and scientists at the rate  $w_S$ .

The final goods sector is competitive, using the following production functions:

$$Y = [\gamma Y_L^{(\varepsilon-1)/\varepsilon} + (1-\gamma) Y_K^{(\varepsilon-1)/\varepsilon}]^{\varepsilon/(\varepsilon-1)}, \quad 0 \leq \varepsilon < \infty \quad (11)$$

where  $Y$  is output and  $Y_L$  and  $Y_K$  are the two inputs, with the factor-elasticity of substitution given by  $\varepsilon$ .

The inputs  $Y_L$  and  $Y_K$  are also produced competitively by constant elasticity of

substitution (CES) production functions using a continuum of intermediate inputs,  $X(i)$  and  $Z(j)$ :

$$Y_L = \left[ \int_0^N X(i)^{\phi\beta} di \right]^{1/\beta} \quad \text{and} \quad Y_K = \left[ \int_0^M Z(j)^{\phi\beta} dj \right]^{1/\beta} \quad (12)$$

where the elasticity of substitution is given by  $v = 1/(1-\beta)$  and  $N$  and  $M$  represent the measure of the two types of the intermediate inputs, respectively. The specification of the production functions extends that of Acemoglu's by introducing the parameters  $\varphi$  and  $\phi$  and assuming that  $0 < \varphi \leq 1$ ,  $0 < \phi \leq 1$ . When  $\varphi = \phi = 1$ , equations (12) degenerate to the form used in Acemoglu (2002). When  $\varphi < 1$  or  $\phi < 1$ , then production of the inputs is characterized by diminishing returns to scale.

The intermediate factors  $X(i)$  are produced by labor, whereas  $Z(j)$  are produced by capital, where the respective production functions are linear:

$$X(i) = L(i) \quad \text{and} \quad Z(j) = K(j) \quad (13)$$

Accordingly,  $Y_L$  and  $Y_K$  represent labor-intensive and capital-intensive inputs respectively. New intermediate inputs are developed by an R&D department. The innovation functions are specified as follows<sup>7</sup>

$$\begin{cases} \dot{N} = d_N N S_N - \delta N \\ \dot{M} = d_M M S_M - \delta M \end{cases} \quad (14)$$

where  $S_N$  and  $S_M$  represent respectively the number of scientists engaged in innovation of the two kinds of intermediate inputs. The total number of scientists is exogenously set at  $S$ , so  $S_N + S_M \leq S$ . Once a new intermediate input is invented, the inventor obtains a permanent patent, as in Romer's (1990) model.

Another important extension of Acemoglu (2002) in this paper is the following specification of the factor accumulation processes:

$$\begin{cases} \dot{K} = q(t) I_K^{\alpha_K} - \delta_K K, \dot{q}(t)/q(t) = g_q \geq 0, \quad 0 \leq \alpha_K \leq 1, \delta_K > 0 \\ \dot{L} = b_L I_L^{\alpha_L} - \delta_L L, \quad b_L > 0, \quad 0 \leq \alpha_L \leq 1, \delta_L > 0 \end{cases} \quad (15)$$

There are two key differences between equations (15) and the analogous Acemoglu specification: the introduction of investment elasticity parameters  $\alpha_K$  and  $\alpha_L$  and the

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<sup>7</sup> The experimental equipment model (Rivera-Natiz and Romer, 1991) can also be used to construct the frontier function of innovation possibilities. It does not change the conclusion but adds some boundary conditions.

inclusion of possible capital-embodied technological progress at the exogenously given rate  $\dot{q}(t)/q(t) = g_q$ .

Two comments are in order concerning these extensions. First, while it may be difficult to empirically estimate  $\alpha_K$  and  $\alpha_L$ , various models have taken a stand on their values. For example, the “standard” neoclassical growth model implicitly sets  $\alpha_K=1$ ,<sup>8</sup> whereas the Malthusian model implicitly sets  $\alpha_L=1$ .<sup>9</sup> Acemoglu’s (2002) assumption that K and L are exogenously given amounts to setting  $\alpha_K$  and  $\alpha_L$  to zero.<sup>10</sup> Irmen (2013) points out theoretically that  $\alpha_K < 1$  may be more reasonable to account for adjustment costs of material capital investment. To capture such possibilities, we assume their values to range from 0 to 1.

Second, the introduction of potential capital embodied technological progress is aimed at accommodating the observed declining trend of the relative price of capital goods (see, e.g., Grossman et al. 2017).

## 2. The market equilibrium

Given the setting of the model, the final goods sector and intermediate goods sector will be in equilibrium when both the final goods firms and the intermediate goods firms maximize their profits and the markets of capital and labor clear. Given the goods market equilibrium, the following two propositions hold:

**Proposition 2:** In the market equilibrium, the final output production function takes the form:

$$Y = [\gamma(AL^\varphi)^{(\varepsilon-1)/\varepsilon} + (1 - \gamma)(BK\phi)^{(\varepsilon-1)/\varepsilon}]^{\varepsilon/(\varepsilon-1)} \quad (16)$$

where  $A \equiv N^{(1-\varphi\beta)/\beta}$  and  $B \equiv M^{(1-\phi\beta)/\beta}$ .

Proof: See Appendix B.

The CES equation (16) is a specific form of the production function (1), with

<sup>8</sup> Under the commonly used assumption of an exogenously growing labor force,  $\alpha_L$  is not specified.

<sup>9</sup> If we let  $L_L \equiv s_L Y$ , then  $\dot{L} = b_L L_L - \delta_L L = s_L b_L Y - \delta_L L$ . This is the labor supply hypothesis of the Malthusian model, where  $s_L$  is endogenously determined by the family’s intertemporal optimization.

<sup>10</sup> Note that when the embodied technological progress is not taken into account,  $q(t)$  is a constant. Therefore, at the steady-state values of  $\dot{K} = 0$  and  $\dot{L} = 0$  we obtain  $K^* = \bar{q}/\delta_K$  and  $L^* = b_L/\delta_L$ .

constant returns to scale to  $AL^\varphi$  and  $BK^\phi$ . With respect to  $L$  and  $K$ , it has constant returns to scale when  $\varphi = 1$  and  $\phi = 1$  and diminishing returns to scale when  $\varphi < 1$  or  $\phi < 1$ .

**Proposition 3:** In the goods market equilibrium, the relative benefits of innovation of capital-intensive and labor-intensive intermediate goods are determined by:

$$\frac{\pi_M}{\pi_N} = \frac{(1 - \phi\beta)\varphi}{(1 - \varphi\beta)\phi} \cdot \frac{r}{w} \cdot \frac{K}{L} \cdot \frac{N}{M} \quad (17)$$

where  $\pi_M$  and  $\pi_N$  represent the monopoly profits of capital-intensive and labor-intensive intermediate goods producers.

Proof: See Appendix C.

Equation (17) shows that for a given ratio of the technology levels, represented here by  $M/N$ , relative invention profits are positively related to the relative factor prices ( $r/w$ ) and the relative factor supplies ( $K/L$ ). Accordingly, a change of the relative price encourages innovations directed at the scarce factor whose price has increased, as suggested by Hicks (1932). Acemoglu (2002) noted that the relative amount of the two factors, ( $K/L$ ), has two countervailing effects on  $\pi_M/\pi_N$ . On the one hand, a higher  $K/L$  causes an increase in  $\pi_M/\pi_N$ , which in turn leads to a technological change favoring the abundant factor (“the market size effect”). On the other hand, a higher  $K/L$  decreases  $r/w$  and  $\pi_M/\pi_N$ , which is the price effect of a change in  $K/L$ . The total effect of a change in  $K/L$  is regulated by the elasticity of substitution  $\varepsilon$  between the two factors. If  $\varepsilon > 1$ , the market size effect dominates the price effect, and increasing  $K/L$  will encourage favoring improvements of the abundant factor. Otherwise, when  $\varepsilon < 1$ , improvements of the scarce factor will be favored.

Based on the above, Acemoglu (2002) proposed that the relative price and market size are the two key factors affecting the direction of technological progress. Notice, however, that holding  $M/N$  fixed implies that these effects are only the static or short-run ones. Specifically, when  $\varepsilon > 1$ , favoring innovation in the capital-intensive

intermediate factor causes  $M/N$  to increase. Equation (17) then shows that a higher  $M/N$  implies an decrease in  $\pi_M/\pi_N$ , discouraging further investment into innovations in the capital-intensive sector. Moreover, equation (17) represents only the demand side of technological change. To get the long-run effects, it is necessary to consider also factors affecting the supply of innovations and material factors, in particular that of  $r/w$  on  $K/L$  and of  $\pi_M/\pi_N$  on  $M/N$ , within a dynamic general equilibrium framework. As will be shown below, in such a context, even if there is a short-run “market size effect”,  $K/L$  and  $M/N$  cannot be *both* continually increasing in the long-run.

### 3. The Steady State

When the goods market and the scientist market are in equilibrium and households maximize their utility, the economy arrives at a steady-state growth equilibrium in which each endogenous variable grows at a constant rate. The following proposition shows that the model has a unique steady-state growth equilibrium.

**Proposition 4:** there exists a unique steady-state growth equilibrium in which the resource allocation is determined by equations (18), and the steady-state growth rates are determined by equations (19).

$$\left\{ \begin{array}{l} S_N^* = \frac{\chi_2(d_M S - \delta) + \chi_1 \delta + \beta(1 - \varphi \alpha_L) \phi g_q}{\chi_1 d_N + \chi_2 d_M} \\ S_M^* = \frac{\chi_1(d_N S - \delta) + \chi_2 \delta - \beta(1 - \varphi \alpha_L) \phi g_q}{\chi_1 d_N + \chi_2 d_M} \\ S_L^* = \frac{(\alpha_L g + \delta_L) \alpha_L \beta \phi^2}{\rho + \delta_L - (1 - \alpha_L - \theta) g} \cdot \frac{\chi_4 d_M}{\chi_3 d_N + \chi_4 d_M} \\ S_K^* = \frac{(\alpha_K g + \delta_K) \alpha_K \beta \phi^2}{\rho + \delta_K - (1 - \alpha_K - \theta) g} \cdot \frac{\chi_3 d_N}{\chi_3 d_N + \chi_4 d_M} \\ S_C^* = 1 - S_K^* - S_L^* \end{array} \right. \quad (18)$$

$$\left\{ \begin{array}{l} \left(\frac{\dot{Y}}{Y}\right)^* = \left(\frac{\dot{I}_L}{I_L}\right)^* = \left(\frac{\dot{I}_K}{I_K}\right)^* = \left(\frac{\dot{C}}{C}\right)^* = g \\ \left(\frac{\dot{K}}{K}\right)^* = \alpha_K g + g_q \\ \left(\frac{\dot{L}}{L}\right)^* = \alpha_L g \\ \left(\frac{\dot{M}}{M}\right)^* = \frac{\beta}{1 - \phi\beta} [(1 - \phi\alpha_K)g - \phi g_q] \\ \left(\frac{\dot{N}}{N}\right)^* = \frac{\beta}{1 - \phi\beta} (1 - \phi\alpha_L)g \\ g \equiv \frac{(1 - \phi\beta)(1 - \phi\beta)[d_M d_N S - (d_N + d_M)\delta] + \phi\beta d_N g_q}{\beta(\chi_1 d_N + \chi_2 d_M)} \end{array} \right. \quad (19)$$

where  $\chi_1 \equiv (1 - \phi\beta)(1 - \phi\alpha_K)$ ,  $\chi_2 \equiv (1 - \phi\beta)(1 - \phi\alpha_L)$ ,  $\chi_3 \equiv (\phi - \phi^2\beta)$  and  $\chi_4 \equiv (\phi - \phi^2\beta)$ .

Proof: See Appendix D.

Equations (18) define the household resource allocations, i.e. that of scientists into the two kinds of innovations  $S_N^*$  and  $S_M^*$ , and that of income into the accumulation processes and consumption, implying two saving rates  $s_K \equiv I_K/Y$ ,  $s_L \equiv I_L/Y$  and a consumption rate  $s_C \equiv C/Y$ . Equations (19) deliver the steady-state growth path at the resource allocation given by equations (18).

While Proposition 1 *assumes* that the general growth model steady-state equilibrium exists, equations (18) and (19) show that in the specific growth model it is in fact the case. Moreover, the steady-state equilibrium is unique.

**Proposition 5:** In the specific growth model, the factor supply elasticities are determined by equation (20) as follows:

$$\left\{ \begin{array}{l} \varepsilon_K = \frac{\alpha_K + g_q/g}{(1 - \alpha_K) - g_q/g} \\ \varepsilon_L = \frac{\alpha_L}{1 - \alpha_L} \end{array} \right. \quad (20)$$

Proof: See Appendix E.

Equations (20) show that the investment elasticities of the factor accumulation



process,  $\alpha_K$  and  $\alpha_L$ , determine the factor supply elasticities,  $\varepsilon_K$  and  $\varepsilon_L$ . When  $0 \leq \alpha_K \leq 1 - g_q/g$ , then  $0 \leq \varepsilon_K \leq \infty$ , and  $0 \leq \varepsilon_L \leq \infty$  when  $0 \leq \alpha_L \leq 1$ .<sup>11</sup>

Substituting equations (20) into equation (7), we obtain

$$DTP = \frac{(1 + \varepsilon_L)/[1 + (1 - \varphi)\varepsilon_L]}{(1 + \varepsilon_K)/[1 + (1 - \phi)\varepsilon_K]} = \frac{(1 - \phi\alpha_K) - \phi g_q/g}{(1 - \varphi\alpha_L)} \quad (21)$$

Since  $A \equiv N^{(1-\varphi\beta)/\beta}$  and  $B \equiv M^{(1-\phi\beta)/\beta}$ , from equation (19) we then get:

$$\begin{cases} (\dot{B}/B)^* = (1 - \phi\alpha_K)g - \phi g_q \\ (\dot{A}/A)^* = (1 - \varphi\alpha_L)g \end{cases} \quad (22)$$

Equation (21) can be also obtained by substituting equations (22) into equation (5) which is the definition of DTP. The resulting equation not only confirms proposition 1, but also gives the concrete determinants of the steady-state direction of technological progress in the specific model.

## V. Applications

### I: Uzawa's Steady-State Theorem

Uzawa's (1961) steady-state theorem challenges growth theory. It holds that in steady-state, technological progress in the neoclassical growth model cannot include capital-augmentation. However, the theorem does not reveal the underlying factors that are responsible for the conclusion. Based on the results of the general and specific models above, it becomes possible to provide the necessary and sufficient conditions required to obtain Uzawa's theorem. As a consequence, the exact conditions under which the steady-state of a growth model can admit capital-augmenting technological are revealed.

#### 1. Necessary and Sufficient Conditions for Uzawa's Steady-State Theorem

By proposition 1, the following corollary, proven in Appendix F, holds:

**Corollary 2 (Uzawa's Steady-State Theorem):** *Suppose the production function of a growth model is characterized by equation (1). Then, along the steady-state equilibrium path technological progress cannot include capital-augmentation (i. e.  $\frac{\dot{B}(t)}{B(t)} = 0$ ) if and only if capital has an infinite supply elasticity (i. e.  $\varepsilon_K = \infty$ ) and the capital return to scale is constant (i. e.  $\phi = 1$ ).*

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<sup>11</sup> It is worth noting that if  $\alpha_K > 1 - g_q/g$  then  $\varepsilon_K < 0$ , which means that the factor supply is negatively associated with its price, contrary to the characteristics of the usual supply function.

## 2. Discussion of Corollary 2

**First**, the traditional neoclassical growth model satisfies the necessary and sufficient conditions of Uzawa's steady-state theorem. In these models the production function is specified as  $Y(t) = F[B(t)K(t), A(t)L(t)]$ , with a capital accumulation process  $\dot{K} = I - \delta K$ . Furthermore, there is no capital-embodied technological progress. This specification implies  $\phi = 1$ ,  $\alpha_K = 1$ ,  $g_q = 0$ . According to equations (20), this implies  $\varepsilon_K = \infty$ . Therefore, the traditional neoclassical growth model cannot include capital-augmenting technological progress in steady-state.

**Second**, while the existing literature has discussed the conditions of Uzawa's steady-state theorem, it has not clearly and completely pointed out the necessary and sufficient conditions underlying its validity.

For example, Acemoglu (2003) and Jones and Scrimgeour (2008) argue that the asymmetry between capital and labor growth processes (endogenous growth of capital and exogenous growth of labor) are the key to explain the theorem, but corollary 2 shows that whether capital-augmenting technological progress may be part of a steady-state equilibrium has nothing to do with labor growth.

Sato (1996) pointed out that the linear specification of the capital accumulation process, that is,  $\alpha_K = 1$ , is necessary for the steady state to exclude capital-augmenting technological progress. However, according to Corollary 2, on the one hand, if  $\phi = 1$ , the specific growth model of Section IV also excludes capital-augmentation even though  $\alpha_K = 1 - \frac{g_q}{g} < 1$ ; on the other hand, even though  $\alpha_K = 1$  and  $\varepsilon_K = \infty$ , technological progress will include capital-augmentation in steady state, provided that  $\phi < 1$  and  $g_q = 0$ .

Grossman et al. (2017) specified the production function as  $Y = F[BK, AL, s]$ , where  $s$  stands for schooling, alongside a capital accumulation process  $\dot{K} = q(t)I - \delta K$ . They argue that  $\dot{s} = 0$  is the necessary condition for the steady-state to exclude capital-augmenting technological progress. But again, on the one hand, as they point out by themselves, if there exists a measure of human capital,  $H(AL, s)$ , such that  $F(BK, AL, s) \equiv \tilde{F}[BK, H(AL, s)]$ , then even if  $\dot{s} > 0$ , the steady-state cannot admit capital-augmenting technological change; on the other hand, by Corollary 2, if  $\phi < 1$  or  $\alpha_K < 1$ , the specific growth model of Section IV will include capital-augmentation in its steady-state even when  $\dot{s} = 0$ . Therefore, the conditions pointed out by Sato (1996) and Grossman et al. (2017) are neither necessary nor sufficient to excluded steady-state capital augmentation.

**Third**, as long as any condition of Corollary 2 is not satisfied, technological progress does include capital-augmentation in steady state.

**(1)  $\phi = 1$ ,  $\varepsilon_K < \infty$ .**

Under these conditions, whatever  $\varepsilon_L$  and  $\varphi$  may be, Proposition 1 implies  $DTP = \frac{(1+\varepsilon_L)/[1+(1-\varphi)\varepsilon_L]}{1+\varepsilon_K} > 0$ , i.e.  $\frac{\dot{B}(t)}{B(t)} > 0$ . In the specific model of Section IV, as long as  $0 < \alpha_K < 1 - g_q/g$ , setting  $\phi = 1$  implies  $0 < \varepsilon_K < \infty$  by equation (20). As a result, from equation (22),  $\frac{\dot{B}(t)}{B(t)} = (1 - \alpha_K)g - g_q > 0$ . Especially, for  $g_q = 0$ ,  $0 < \alpha_K < 1$  is necessary to obtain  $0 < \varepsilon_K < \infty$ . Sato (1999, 2000) and Irmen (2013) fit into this parameter configuration.

**(2)  $\varepsilon_K = \infty$ ,  $\phi < 1$ .**

Here, as long as  $\varepsilon_L$  and  $\varphi$  are within their admissible domains, Proposition 1 implies  $DTP = \frac{(1+\varepsilon_L)/[1+(1-\varphi)\varepsilon_L]}{(1+\varepsilon_K)/[1+(1-\varphi)\varepsilon_K]} > 0$  and  $\frac{\dot{B}(t)}{B(t)} > 0$ . In the model of Section IV with  $\alpha_K = 1 - g_q/g$ , equation (20) yields  $\varepsilon_K = \infty$ . But then, with  $\phi < 1$  equation (22) implies  $\dot{B}/B = (1 - \phi)g > 0$ .

The effect of the return to scale on the steady-state direction of technological progress has been neglected in the existing literatures. Recently, Casey and Horii (2019) actually rendered  $\phi < 1$  by considering land as an input. Therefore, in their case capital augmentation is part of the steady-state equilibrium. Specifically, because they assume  $\alpha_K = 1$  and  $g_q = 0$ ,  $\frac{\dot{K}(t)}{K(t)} = \frac{\dot{Y}(t)}{Y(t)}$  and the ratio of capital to output (K/Y) remains unchanged.

**(3) Reconciling  $g_q > 0$  with  $\frac{\dot{B}}{B} > 0$ .**

Grossman et al. (2017) argue that the United States is on a balanced growth path with falling investment-good prices and that the elasticity of substitution between capital and labor is less than unitary. In their view, growth models should generate steady-states with capital-embodied as well as capital-augmenting technological progress. As mentioned above, they introduce a schooling variable into the production function in order to achieve this goal. However, according to Corollary 2 the same desiderata are obtained without the introduction of an additional factor if  $\varepsilon_K < \infty$  or  $\phi < 1$ . For example, according to equation (22), as long as  $\frac{(1-\phi\alpha_K)}{\phi} > \frac{g_q}{g}$ , then the model of Section IV accommodates both  $g_q > 0$  and  $\frac{\dot{B}}{B} > 0$  along the steady-state

path.<sup>12</sup>

## II: Industrial Revolution

As mentioned above, according to Ashraf and Galor's (2011) empirical work and Kaldor's (1961) styled facts, before the Industrial Revolution technological progress was almost purely land-augmenting, excluding labor-augmentation, and after the Industrial Revolution technological progress was basically labor-augmenting, excluding capital-augmentation. In terms of the above analysis, what are the underlying factors that cause such tremendous differences in the direction of technological progress?

Suppose the production function is characterized by constant returns to scale ( $\phi = \varphi = 1$ ) and no capital-embodied technological progress ( $g_q = 0$ ). If  $\alpha_L = 1$  and  $\alpha_K < 1$ , equation (22) implies  $\frac{\dot{A}}{A} = 0$ ,  $\frac{\dot{B}}{B} > 0$ , and equation (19) implies  $\frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = \frac{\dot{A}}{A} = 0$ ,  $\frac{\dot{Y}}{Y} - \frac{\dot{K}}{K} = \frac{\dot{B}}{B} > 0$ . Interpreting K as “land”, this is consistent with Ashraf and Galor's (2011) empirical study of the pre-Industrial Revolution era. Under the above conditions, equation (20) implies an infinite labor supply elasticity ( $\varepsilon_L = \infty$ ) and a finite land supply elasticity ( $\varepsilon_K < \infty$ ). These generate a long-term stagnation of per capita income, consistent with the situation before the Industrial Revolution. In other words, the abundance of labor caused technological progress to be purely land-augmenting, leading to the Malthusian subsistence-level trap.

Similarly, if  $\alpha_K = 1$  (i.e. capital becomes abundant) and  $\alpha_L < 1$ , the rate of capital-augmenting technological progress is zero and the rate of labor-augmenting technological progress is greater than zero, that is,  $\frac{\dot{B}}{B} = 0$  and  $\frac{\dot{A}}{A} > 0$ . From equations (19), the growth rate of capital is the same as that of output ( $\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K}$ ), keeping the ratio of capital to output (K/Y) unchanged. Moreover, per capita income is increasing ( $\frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = \frac{\dot{A}}{A} > 0$ ). This is consistent with Kaldor's styled facts concerning the era after the Industrial Revolution. The above analysis then implies that the change of the labor supply elasticity from infinite to finite, and that of material elements from a finite supply elasticity of land to an infinite supply elasticity of capital may hold the key to the emergence of the Industrial Revolution.

How these supply elasticities might have changed is not yet fully explained. The

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<sup>12</sup> In fact, Grossman et al. (2017) also get their results by  $\varepsilon_K < \infty$ . However, their capital supply elasticity is not only finite, but also negative, with  $\varepsilon_K = -(1 + \frac{g}{g_q}) < 0$ .

Unified growth theory of Galor (2011) suggests an endogenous transformation mechanism whereby the labor supply elasticity changes from infinite to zero. Therefore, it can explain how technological progress has changed from one that excludes steady-state labor augmentation to one that includes it. Because the theory focuses on the transition from stagnant to continually growing per capita income, it does not consider the change of the supply elasticities of land and capital. As a result, it has not yet explained why technological progress became purely labor augmenting after the Industrial Revolution, as is required to meet Kaldor's stylized facts.

## VI Concluding Remarks

Using a generalized growth model, this paper proves that the determinants of the direction of technological progress in a steady-state equilibrium depends on two aspects: the relative size of returns to scale of different inputs in the production function and the relative size of the supply elasticities of these inputs. A specific version that extends the Acemoglu (2002) model provides micro-foundation for technological progress and explicit formulation of the underlying factors affecting the direction of technological progress. Both the general model and the specific one show that although the relative price (Hicks, 1932) and the relative market size (Acemoglu, 2002) affect the short-term direction of technological progress, they have no impact on that direction in steady-state.

The results of this paper contain two implications for growth theory. On the positive side, by showing that the changing factor supply elasticities may be the key to the shifting direction of technological progress, the paper's results may guide the research concerning the origins of the Industrial Revolution. The United Growth Theory (Galor, 2011) has already highlighted the connection between the changing labor supply elasticity and the Industrial Revolution. However, it is still incomplete since it has not reconciled capital and its enhancement and with Kaldor's stylized facts.

This leads to the second, less positive, implication. At the current state of the art, Kaldor's facts are consistent with growth theories if and only if the steady-state growth path is purely labor augmenting (Uzawa's steady-state theorem). However, this paper shows that the Uzawa's theorem holds if and only if the return to scale of the production function is constant and the capital supply elasticity is infinite. A violation of any of these conditions means that technological progress is no longer purely labor-enhancing in steady state, thereby contradicting Kaldor's stylized facts. This knife-edge result raises the intuitive and empirical question whether these conditions are met in reality,

and whether they can be met over a long period of time. Indeed, it seems plausible that the supply elasticity of capital is not infinite. Moreover, many technologies that were developed since the first Industrial Revolution seem to intuitively fit the notion of capital augmenting technological progress. Since the neoclassical growth model with an infinite elasticity of capital accumulation is still the basic framework for the study of economic growth and other macroeconomic issues, the challenge remains to find alternative formulations which will better reconcile theory with reality.

### Appendix A: The derivation process of equation (7)

Dividing the denominator of equation (5) by the two sides of equation (6) yields:

$$DTP = \frac{\dot{B}(t)/B(t)}{\dot{A}(t)/A(t)} = \frac{\frac{\dot{B}(t)}{B(t)} / \left[ \frac{\dot{B}(t)}{B(t)} + \phi \frac{\dot{K}(t)}{K(t)} \right]}{\frac{\dot{A}(t)}{A(t)} / \left[ \frac{\dot{A}(t)}{A(t)} + \phi \frac{\dot{L}(t)}{L(t)} \right]} = \frac{1 + \phi \frac{\dot{L}(t)}{L(t)} / \frac{\dot{A}(t)}{A(t)}}{1 + \phi \frac{\dot{K}(t)}{K(t)} / \frac{\dot{B}(t)}{B(t)}} \quad (A1)$$

The growth rates of  $r(t)$  and  $w(t)$  in equations (3) are obtained by taking  $k(t)$  as a constant, yielding:

$$\begin{cases} \frac{\dot{w}(t)}{w(t)} = \frac{\dot{A}(t)}{A(t)} + (\phi - 1) \frac{\dot{L}(t)}{L(t)} \\ \frac{\dot{r}(t)}{r(t)} = \frac{\dot{B}(t)}{B(t)} + (\phi - 1) \frac{\dot{K}(t)}{K(t)} \end{cases} \quad (A2)$$

Substituting equations (A2) into equations (4) then yields:

$$\begin{cases} \varepsilon_L = \frac{\dot{L}(t)/L(t)}{\frac{\dot{A}(t)}{A(t)} + (\phi - 1) \frac{\dot{L}(t)}{L(t)}} \\ \varepsilon_K = \frac{\dot{K}(t)/K(t)}{\frac{\dot{B}(t)}{B(t)} + (\phi - 1) \frac{\dot{K}(t)}{K(t)}} \end{cases} \quad (A3)$$

From equations (A3) we obtain:

$$\begin{cases} \frac{\dot{L}(t)}{L(t)} / \frac{\dot{A}(t)}{A(t)} = \frac{\varepsilon_L}{1 + (1 - \phi)\varepsilon_L} \\ \frac{\dot{K}(t)}{K(t)} / \frac{\dot{B}(t)}{B(t)} = \frac{\varepsilon_K}{1 + (1 - \phi)\varepsilon_K} \end{cases} \quad (A4)$$

Substituting equations (A4) into equation (A1) and rearranging implies equation (7):

$$DTP = \frac{(1 + \varepsilon_L) / [1 + (1 - \phi)\varepsilon_L]}{(1 + \varepsilon_K) / [1 + (1 - \phi)\varepsilon_K]}$$

### Appendix B: Proof of Proposition 2.

Letting the final good serve as numeraire, the representative competitive final good producer faces the input prices  $p_L$  and  $p_K$  and selects the respective  $Y_K$  and  $Y_L$  so as to maximize

$$\pi_Y = Y - p_L Y_L - p_K Y_K \quad (B1)$$

subject to the production function (11), yielding the demand functions:

$$\begin{cases} p_K = (1 - \gamma)[\gamma + (1 - \gamma)(Y_K/Y_L)^{(\varepsilon-1)/\varepsilon}]^{1/(\varepsilon-1)}(Y_K/Y_L)^{-1/\varepsilon} \\ p_L = \gamma[\gamma + (1 - \gamma)(Y_K/Y_L)^{(\varepsilon-1)/\varepsilon}]^{1/(\varepsilon-1)} \end{cases} \quad (B2)$$

The representative producers of  $Y_K$  and  $Y_L$  maximize their profits by choosing  $Z(j)$  and  $X(i)$ , given the intermediate input prices  $p_Z(j)$  and  $p_X(i)$ :

$$\begin{cases} \pi_K = p_K Y_K - \int_0^M p_Z(j) Z(j) dj \\ \pi_L = p_L Y_L - \int_0^M p_X(i) X(i) di \end{cases} \quad (B3)$$

subject to their respective production functions (12). This generates the demand functions

$$\begin{cases} Z(j) = (Y_K)^{\frac{1-\beta}{1-\phi\beta}} (\phi p_K / p_Z(j))^{1/(1-\phi\beta)} \\ X(i) = (Y_L)^{\frac{1-\beta}{1-\phi\beta}} (\phi p_L / p_X(i))^{1/(1-\phi\beta)} \end{cases} \quad (B4)$$

The intermediate input producers, who hold the exclusive right to produce their particular type of input, face the prices of the primary inputs and choose, respectively,  $(p_Z(j), K(j))$  and  $(p_X(i), L(i))$  to maximize

$$\begin{cases} \pi_M(j) = p_Z(j) Z(j) - rK(j) \\ \pi_N(i) = p_X(i) X(i) - wL(i) \end{cases} \quad (B5)$$

subject to their technologies (13) and the demand functions (B4).

From the maximization (B5) we obtain:

$$\begin{cases} p_Z(j) = p_Z = r/\phi\beta \\ p_X(i) = p_X = w/\phi\beta \end{cases} \quad (B6)$$

which imply that all intermediate inputs have the same mark-up over marginal cost. Substituting equations (B6) into (B4), we find that all capital-intensive and all labor-intensive intermediate goods are produced in equal (respective) quantities.

$$\begin{cases} Z(j) = (Y_K)^{\frac{1-\beta}{1-\phi\beta}} (\beta \phi^2 p_K / r)^{1/(1-\phi\beta)} \\ X(i) = (Y_L)^{\frac{1-\beta}{1-\phi\beta}} (\beta \phi^2 p_L / w)^{1/(1-\phi\beta)} \end{cases} \quad (B7)$$

By the production functions of the intermediate inputs (11), all monopolists have the same respective demand for labor and capital.

The material factor market clearing conditions imply:

$$\begin{cases} Z(j) = K/M \\ X(i) = L/N \end{cases} \quad (B8)$$



Substituting equations (B8) into (12), we obtain the equilibrium quantities of the labor-intensive and capital-intensive inputs as equations (B9):

$$\begin{cases} Y_L = \left[ \int_0^N X(i)^{\phi\beta} di \right]^{1/\beta} = N^{(1-\phi\beta)/\beta} L^\phi \\ Y_K = \left[ \int_0^M Z(j)^{\phi\beta} dj \right]^{1/\beta} = M^{(1-\phi\beta)/\beta} K^\phi \end{cases} \quad (\text{B9})$$

Substituting equations (B9) into equation (11), we obtain equations (16) as follows:

$$Y = \left[ \gamma (N^{(1-\phi\beta)/\beta} L^\phi)^{(\varepsilon-1)/\varepsilon} + (1-\gamma) (M^{(1-\phi\beta)/\beta} K^\phi)^{(\varepsilon-1)/\varepsilon} \right]^{\varepsilon/(\varepsilon-1)}$$

### Appendix C: Proof of Proposition 3.

Letting  $k \equiv \frac{BK^\phi}{AL^\phi} = \frac{M^{(1-\phi\beta)/\beta} K^\phi}{N^{(1-\phi\beta)/\beta} L^\phi}$ , the factor-intensive production function becomes:

$$f(k) \equiv Y/AL^\phi = \left[ \gamma + (1-\gamma)k^{(\varepsilon-1)/\varepsilon} \right]^{\varepsilon/(\varepsilon-1)} \quad (\text{C1})$$

Using equation (C1), we transform the market prices of the capital-intensive and labor-intensive inputs (B2) into the following forms:

$$\begin{cases} p_K = \frac{\partial Y}{\partial Y_K} = f'(k) \\ p_L = \frac{\partial Y}{\partial Y_L} = f(k) - kf'(k) \end{cases} \quad (\text{C2})$$

Substituting (B8) and (B9) into (B7), we obtain

$$\begin{cases} K/M = \left( M^{(1-\phi\beta)/\beta} K^\phi \right)^{\frac{1-\beta}{1-\phi\beta}} (\beta\phi^2 p_K/r)^{1/(1-\phi\beta)} \\ L/N = \left( N^{(1-\phi\beta)/\beta} L^\phi \right)^{\frac{1-\beta}{1-\phi\beta}} (\beta\phi^2 p_L/w)^{1/(1-\phi\beta)} \end{cases} \quad (\text{C3})$$

Substituting (C2) into (C3) and rearranging, we obtain the market prices of capital and labor:<sup>13</sup>

$$\begin{cases} r = \beta\phi^2 K^{\phi-1} M^{(1-\phi\beta)/\beta} f'(k) \\ w = \beta\phi^2 L^{\phi-1} N^{(1-\phi\beta)/\beta} [f(k) - kf'(k)] \end{cases} \quad (\text{C4})$$

Substituting equation (13) and (B6) into (B5), we obtain:

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<sup>13</sup> The general model assumes that the prices of factor are paid their marginal products, but equations (C4) show that while here these prices fall short of the marginal products, they are still proportional to the latter, so that Proposition 1 is still valid.

$$\begin{cases} \pi_M(j) = (r/\phi\beta - r)Z(j) \\ \pi_N(i) = (w/\varphi\beta - w)X(i) \end{cases} \quad (C5)$$

Substituting (B8) into (C5) yield:

$$\begin{cases} \pi_M = (r/\phi\beta - r)K/M \\ \pi_N = (w/\beta\varphi - w)L/N \end{cases} \quad (C6)$$

The monopoly profit of each producer of an intermediate product is obtained by substituting (C4) into (C6):

$$\begin{cases} \pi_M = (1 - \phi\beta)\phi M^{(1-\phi\beta-\beta)/\beta} K^\phi f'(k) \\ \pi_N = (1 - \varphi\beta)\varphi N^{(1-\varphi\beta-\beta)/\beta} L^\varphi [f(k) - kf'(k)] \end{cases} \quad (C7)$$

From equations (C7) and (C4) we finally obtain equation (17):

$$\frac{\pi_M}{\pi_N} = \frac{(1 - \phi\beta)\varphi}{(1 - \varphi\beta)\phi} \cdot \frac{r}{w} \cdot \frac{K}{L} \cdot \frac{N}{M}$$

#### Appendix D: Proof of Proposition 4.

**First**, consider the market for scientists which determines the supply of innovations. Free-entry into the R&D sector implies that the marginal innovation value of scientists should be equal across technologies. Using the innovation possibilities frontier function (14), this implies

$$d_N N \pi_N = d_M M \pi_M \quad (D1)$$

From the innovation profit equation (C7) we obtain

$$\frac{d_N (1 - \varphi\beta)\varphi}{d_M (1 - \phi\beta)\phi} = \frac{kf'(k)}{f(k) - kf'(k)} \quad (D2)$$

Applying equation (C1) to (D2) yields

$$k^* = \left[ \frac{\gamma d_N}{(1 - \gamma) d_M} \frac{(1 - \varphi\beta)\varphi}{(1 - \phi\beta)\phi} \right]^{\frac{\varepsilon}{\varepsilon - 1}} \quad (D3)$$

Equation (D3) shows that market clearing implies that  $k^*$  is a constant, determined solely by the parameters  $\gamma, d_M, d_N, \beta, \varphi, \phi$  and  $\varepsilon$ .

**Second**, we solve the Euler equations.

Let the Hamiltonian associated with the household optimization problem be:

$$\begin{aligned} H = & U(C)e^{-\rho t} + \lambda_K (q(t)I_K^{\alpha_K} - \delta_K K) + \lambda_L (b_L I_L^{\alpha_L} - \delta_L L) \\ & + \mu [wL + rK + w_S S - C - (I_K + I_L)] \end{aligned} \quad (D4)$$

The first-order conditions are:

$$\begin{cases} C^{-\theta} e^{-\rho t} = \lambda_K \alpha_K q(t) I_K^{\alpha_K - 1} \\ C^{-\theta} e^{-\rho t} = \lambda_L \alpha_L b_L I_L^{\alpha_L - 1} \\ C^{-\theta} e^{-\rho t} = \mu \end{cases} \quad (D5)$$

Taking log-derivatives of both sides of (D5) over time, we obtain

$$\begin{cases} -\theta \frac{\dot{C}}{C} - \rho = \frac{\dot{\lambda}_K}{\lambda_K} + (\alpha_K - 1) \frac{\dot{I}_K}{I_K} + g_q \\ -\theta \frac{\dot{C}}{C} - \rho = \frac{\dot{\lambda}_L}{\lambda_L} + (\alpha_L - 1) \frac{\dot{I}_L}{I_L} \\ -\theta \frac{\dot{C}}{C} - \rho = \frac{\dot{\mu}}{\mu} \end{cases} \quad (D6)$$

The motion equations of  $\lambda$  are:

$$\begin{cases} \dot{\lambda}_K = -\partial H / \partial K = \lambda_K \delta_K - \mu r \\ \dot{\lambda}_L = -\partial H / \partial L = \lambda_L \delta_L - \mu w \end{cases} \quad (D7)$$

Based on (D5) and (D7), we obtain

$$\begin{cases} \dot{\lambda}_K / \lambda_K = \delta_K - r \alpha_K q(t) I_K^{\alpha_K - 1} \\ \dot{\lambda}_L / \lambda_L = \delta_L - w \alpha_L b_L I_L^{\alpha_L - 1} \end{cases} \quad (D8)$$

Using (D8) in (D6), we obtain the Euler equations (D9).

$$\begin{cases} \frac{\dot{C}}{C} = \frac{1}{\theta} \left\{ r \alpha_K q(t) I_K^{\alpha_K - 1} - (\alpha_K - 1) \frac{\dot{I}_K}{I_K} - g_q - \rho - \delta_K \right\} \\ \frac{\dot{C}}{C} = \frac{1}{\theta} \left\{ w \alpha_L b_L I_L^{\alpha_L - 1} - (\alpha_L - 1) \frac{\dot{I}_L}{I_L} - \rho - \delta_L \right\} \end{cases} \quad (D9)$$

**Third**, we solve for the steady state equilibrium.

We first conjecture that there exists a steady-state equilibrium (hereafter SSEP) then verify it indeed exists by explicitly solving for it.

From the budget constraint (10) and the definition of an SSEP, we obtain

$$\frac{\dot{Y}}{Y} = \frac{\dot{I}}{I} = \frac{\dot{I}_L}{I_L} = \frac{\dot{I}_K}{I_K} = \frac{\dot{C}}{C} \quad (D10)$$

Then, according to the primary factor accumulation functions (15), along an SSEP the following must hold:

$$\begin{cases} \frac{\dot{K}}{K} = \alpha_K \frac{\dot{I}_K}{I_K} + \alpha_K \frac{\dot{q}}{q} = \alpha_K \frac{\dot{C}}{C} + \alpha_K \frac{\dot{q}}{q} \\ \frac{\dot{L}}{L} = \alpha_L \frac{\dot{I}_L}{I_L} = \alpha_L \frac{\dot{C}}{C} \end{cases} \quad (D11)$$

From equation (C1) we obtain:

$$Y = N^{(1-\varphi\beta)/\beta} L^\varphi f(k) = M^{\frac{1-\phi\beta}{\beta}} K^\phi f(k)/k \quad (D12)$$

Since  $k$  is constant along the SSEP, from (D12) we get:

$$\frac{\dot{Y}}{Y} = \frac{1-\varphi\beta}{\beta} \frac{\dot{N}}{N} + \varphi \frac{\dot{L}}{L} = \frac{1-\phi\beta}{\beta} \frac{\dot{M}}{M} + \phi \frac{\dot{K}}{K} \quad (D13)$$

Equations (D10), (D11), (D13), together with the innovation possibilities frontier (14), yield:

$$\begin{cases} (1-\varphi\alpha_L) \frac{\dot{C}}{C} = \frac{1-\varphi\beta}{\beta} \{d_N S_N - \delta\} \\ (1-\phi\alpha_K) \frac{\dot{C}}{C} = \phi \frac{\dot{q}}{q} + \frac{1-\phi\beta}{\beta} \{d_M(S - S_N) - \delta\} \end{cases} \quad (D14)$$

From (D14) and  $S_M + S_N = S$ , we obtain the allocation of scientists between the two kinds of intermediate R&D processes:

$$\begin{cases} S_N^* = \frac{\chi_2(d_M S - \delta) + \chi_1 \delta + \beta(1-\varphi\alpha_L)\phi g_q}{\chi_1 d_N + \chi_2 d_M} \\ S_M^* = \frac{\chi_1(d_N S - \delta) + \chi_2 \delta - \beta(1-\varphi\alpha_L)\phi g_q}{\chi_1 d_N + \chi_2 d_M} \end{cases} \quad (D15)$$

where  $\chi_1 \equiv (1-\varphi\beta)(1-\phi\alpha_K)$  and  $\chi_2 \equiv (1-\phi\beta)(1-\varphi\alpha_L)$ .

Combining (D10), (D14) and (D15), we get the growth rates:

$$\left(\frac{\dot{Y}}{Y}\right)^* = \left(\frac{\dot{L}}{L}\right)^* = \left(\frac{\dot{K}}{K}\right)^* = \left(\frac{\dot{C}}{C}\right)^* = g \quad (D16)$$

where

$$g = \frac{1-\varphi\beta}{\beta} \frac{(1-\phi\beta)[d_M d_N S - (d_N + d_M)\delta] + \phi\beta d_N g_q}{\chi_1 d_N + \chi_2 d_M}$$

Substituting (D16) into (D11) and (D13) we obtain:

$$\begin{cases} (\dot{K}/K)^* = \alpha_K g + \alpha_K g_q \\ (\dot{L}/L)^* = \alpha_L g \\ (\dot{M}/M)^* = \frac{\beta}{1-\phi\beta} [(1-\phi\alpha_K)g - \phi g_q] \\ (\dot{N}/N)^* = \frac{\beta}{1-\varphi\beta} (1-\varphi\alpha_L)g \end{cases} \quad (D17)$$

Equations (D16) and (D17) are replicated in equation (19). These equations

confirm that the model has an SSEP. While (D15) shows that there exists also an allocation of scientists which supports the SSEP, it remains to be verified that there exists an appropriate allocation of income into the competing uses as given in equations (18).

Using equations (C4), the Euler equations (D9) can be written as:

$$\begin{cases} \frac{\dot{C}}{C} = \left[ \alpha_K \frac{q(t) I_K^{\alpha_K} Y M^{(1-\phi\beta)/\beta} K^\phi}{K I_K Y} \beta \phi^2 f'(k) - (\alpha_K - 1) \frac{\dot{I}_K}{I_K} - \rho - \delta_K \right] / \theta \\ \frac{\dot{C}}{C} = \left[ \alpha_L \frac{b_L I_L^{\alpha_L} Y N^{(1-\phi\beta)/\beta} L^\phi}{L I_L Y} \beta \phi^2 [f(k) - kf'(k)] - (\alpha_L - 1) \frac{\dot{I}_L}{I_L} - \rho - \delta_L \right] / \theta \end{cases} \quad (D18)$$

Let  $s_K \equiv I_K/Y$ ,  $s_L \equiv I_L/Y$ . Substituting equation (15),  $s_K$ ,  $s_L$ , the definitions of  $k$  and (C1) into (D18) and rearranging, we get:

$$\begin{cases} \frac{\dot{C}}{C} = \left[ \alpha_K \frac{1}{s_K} \frac{kf'(k)}{f(k)} \beta \phi^2 \left( \frac{\dot{K}}{K} + \delta_K \right) - (\alpha_K - 1) \frac{\dot{I}_K}{I_K} - \rho - \delta_K \right] / \theta \\ \frac{\dot{C}}{C} = \left[ \alpha_L \frac{1}{s_L} \frac{[f(k) - kf'(k)]}{f(k)} \beta \phi^2 \left( \frac{\dot{L}}{L} + \delta_L \right) - (\alpha_L - 1) \frac{\dot{I}_L}{I_L} - \rho - \delta_L \right] / \theta \end{cases} \quad (D19)$$

Substituting (D10) and (D11) into (D19), we obtain

$$\begin{cases} \frac{\dot{C}}{C} = \left[ \alpha_K \beta \phi^2 \frac{1}{s_K} \frac{kf'(k)}{f(k)} \left( \alpha_K \frac{\dot{C}}{C} + \delta_K \right) - (\alpha_K - 1) \frac{\dot{C}}{C} - \rho - \delta_K \right] / \theta \\ \frac{\dot{C}}{C} = \left[ \alpha_L \beta \phi^2 \frac{1}{s_L} \frac{[f(k) - kf'(k)]}{f(k)} \left( \alpha_L \frac{\dot{C}}{C} + \delta_L \right) - (\alpha_L - 1) \frac{\dot{C}}{C} - \rho - \delta_L \right] / \theta \end{cases} \quad (D20)$$

Rearranging (D20) yields:

$$\begin{cases} \frac{\dot{C}}{C} = \frac{\rho + \delta_K \{1 - \alpha_K \beta \phi^2 kf'(k) / [s_K f(k)]\}}{\beta \alpha_K^2 \phi^2 kf'(k) / [s_K f(k)] + 1 - \alpha_K - \theta} \\ \frac{\dot{C}}{C} = \frac{\rho + \delta_L \{1 - \beta \phi^2 \alpha_L [f(k) - kf'(k)] / [s_L f(k)]\}}{\beta \alpha_L^2 \phi^2 [f(k) - kf'(k)] / [s_L f(k)] + 1 - \alpha_L - \theta} \end{cases} \quad (D21)$$

Using  $k^*$  in (D3), we obtain:

$$\begin{cases} \frac{kf'(k)}{f(k)} = \frac{(\varphi - \varphi^2 \beta) d_N}{(\varphi - \varphi^2 \beta) d_N + (\phi - \phi^2 \beta) d_M} \\ \frac{f(k) - kf'(k)}{f(k)} = \frac{(\phi - \phi^2 \beta) d_M}{(\varphi - \varphi^2 \beta) d_N + (\phi - \phi^2 \beta) d_M} \end{cases} \quad (D22)$$

Substituting (D16) and (D22) into D(21), we then get:

$$\begin{cases} s_K^* = \frac{(\alpha_K g + \delta_K) \alpha_K \beta \phi^2}{\rho + \delta_K - (1 - \alpha_K - \theta)g} \cdot \frac{\chi_3 d_N}{\chi_3 d_N + \chi_4 d_M} \\ s_L^* = \frac{(\alpha_L g + \delta_L) \alpha_L \beta \phi^2}{\rho + \delta_L + (\theta + \alpha_L - 1)g} \cdot \frac{\chi_4 d_M}{\chi_3 d_N + \chi_4 d_M} \end{cases} \quad (D23)$$

where  $\chi_3 \equiv (\varphi - \varphi^2 \beta)$  and  $\chi_4 \equiv (\phi - \phi^2 \beta)$ .

Let  $s_C \equiv C/Y$  so that:

$$s_C + s_K + s_L = 1 \quad (D24)$$

Substituting (D14) into (D15), we obtain that along in an SSEP,  $s_C$  is given by:

$$s_C^* = 1 - s_K^* - s_L^* \quad (D25)$$

Equations (D15), (D23) and (D25) provide the allocation of scientists and income needed to obtain the SSEP given by equations (18).

**Finally**, notice that the solution process implies that there exists only one allocation of scientists and income that is consistent with an SSEP.

### Appendix E: Proof of Proposition 5

From equation (C4), we obtain that in steady state:

$$\begin{cases} \frac{\dot{w}}{w} = (\varphi - 1) \frac{\dot{L}}{L} + \frac{1 - \varphi \beta}{\beta} \frac{\dot{N}}{N} \\ \frac{\dot{r}}{r} = (\phi - 1) \frac{\dot{K}}{K} + \frac{1 - \phi \beta}{\beta} \frac{\dot{M}}{M} \end{cases} \quad (E1)$$

Substituting equation (19) into (E1) results in:

$$\begin{cases} \frac{\dot{w}}{w} = (1 - \alpha_L)g \\ \frac{\dot{r}}{r} = (1 - \alpha_K)g - g_q \end{cases} \quad (E2)$$

Substituting (E2) and equation (19) into the factor elasticities yields equation (20), that is:

$$\begin{cases} \varepsilon_K \equiv \frac{\dot{K}/K}{\dot{r}/r} = \frac{\alpha_K + g_q/g}{(1 - \alpha_K) - g_q/g} \\ \varepsilon_L \equiv \frac{\dot{L}/L}{\dot{w}/w} = \frac{\alpha_L}{1 - \alpha_L} \end{cases}$$

### Appendix F: Proof of Corollary 2 (Uzawa's Steady-State Theorem)

**First, “If” direction.** Let  $\varepsilon_K = \infty$  and  $\phi = 1$ . Then  $\frac{\dot{B}(t)}{B(t)} = 0$  in a steady-state equilibrium.

**Proof:**

According to proposition 1,

$$DTP = \frac{(1 + \varepsilon_L)/[1 + (1 - \phi)\varepsilon_L]}{(1 + \varepsilon_K)/[1 + (1 - \phi)\varepsilon_K]}$$

If  $0 < \phi \leq 1$  and  $\varepsilon_L < \infty$ , then

$$DTP = \frac{(1 + \varepsilon_L)/[1 + (1 - \phi)\varepsilon_L]}{\infty} = 0$$

If  $\varepsilon_L = \infty$  and  $0 < \phi < 1$ , then by l’hopital’s rule,  $(1 + \varepsilon_L)/[1 + (1 - \phi)\varepsilon_L] \rightarrow \frac{1}{1 - \phi} < \infty$ . Hence, in this case,

$$DTP = \frac{1/(1 - \phi)}{\infty} = 0$$

If  $\varepsilon_L = \infty$  and  $\phi = 1$ , we obtain:

$$DTP = \frac{1 + \varepsilon_L}{1 + \varepsilon_K} = \frac{\infty}{\infty} \neq 0$$

From (A3) we also know that with  $\varepsilon_K = \infty$  and  $\phi = 1$ ,

$$\varepsilon_K = \frac{\dot{K}(t)/K(t)}{\dot{B}(t)/B(t)} = \infty$$

However,  $\dot{K}(t)/K(t)$  must be finite. Hence in this case too  $\frac{\dot{B}(t)}{B(t)} = 0$ .

In sum, when  $\varepsilon_K = \infty$  and  $\phi = 1$ , we obtain that  $\frac{\dot{B}(t)}{B(t)} = 0$  in a steady-state equilibrium.

**Second, “Only If” direction.** Let  $\frac{\dot{B}(t)}{B(t)} = 0$ . Then  $\varepsilon_K = \infty$  and  $\phi = 1$  in a steady-state equilibrium.

**Proof:**

$$\text{If } \frac{\dot{B}(t)}{B(t)} = 0, \text{ then } DTP = \frac{\dot{B}(t)/B(t)}{\dot{A}(t)/A(t)} = \frac{(1 + \varepsilon_L)/[1 + (1 - \phi)\varepsilon_L]}{(1 + \varepsilon_K)/[1 + (1 - \phi)\varepsilon_K]} = 0$$

Since  $\varepsilon_L \geq 0$ , and  $0 < \phi \leq 1$ , the numerator is strictly positive. Therefore, it must be the case that

$$(1 + \varepsilon_K)/[1 + (1 - \phi)\varepsilon_K] = \infty$$

Clearly, if  $\phi < 1$  and  $0 \leq \varepsilon_K < \infty$ , then

$$\frac{1 + \varepsilon_K}{[1 + (1 - \phi)\varepsilon_K]} < \infty$$

If  $\phi < 1$  and  $\varepsilon_K = \infty$ , then by l'hospital's rule  $\frac{1+\varepsilon_K}{[1+(1-\phi)\varepsilon_K]} \rightarrow \frac{1}{1-\phi} < \infty$

If  $\phi = 1$  and  $\varepsilon_K < \infty$ , then

$$\frac{1 + \varepsilon_K}{[1 + (1 - \phi)\varepsilon_K]} = 1 + \varepsilon_K < \infty$$

Therefore,  $\frac{\dot{B}(t)}{B(t)} = 0$  is obtained only if  $\phi = 1$  and  $\varepsilon_K = \infty$ .

**In conclusion,**  $\varepsilon_K = \infty$  and  $\phi = 1$  are necessary and sufficient conditions for  $\frac{\dot{B}(t)}{B(t)} = 0$  in steady state equilibrium.



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